1 The model

We study the spectrum of the random matrix model:

$$Q = \Sigma_1^{1/2} X_1^T X_1 \Sigma_1^{1/2} + \Sigma_2^{1/2} X_2^T X_2 \Sigma_2^{1/2},$$

where $\Sigma_{1,2}$ are $p\times p$ deterministic covariance matrices, and $X_1 = (x_{ij})_{1\leqslant i\leqslant n_1, 1\leqslant j\leqslant p}$ and $X_2 = (x_{ij})_{n_1+1\leqslant i\leqslant n_1+n_2, 1\leqslant j\leqslant p}$ are $n_1 \times p$ and $n_2 \times p$ random matrices, respectively, where the entries x_{ij}, $1 \leqslant i \leqslant n_1 + n_2 \equiv n$, $1 \leqslant j \leqslant p$, are real independent random variables satisfying

$$\mathbb{E}x_{ij} = 0, \qquad \mathbb{E}|x_{ij}|^2 = n^{-1}. \tag{1.1}$$

For now, we assume that the random variables x_{ij} are i.i.d. Gaussian, but we know that universality holds for generally distributed entries. We shall consider the high-dimensional setting such that

$$\gamma_n := \frac{p}{n} \to \gamma, \quad c_n := \frac{n_1}{n} \to c, \quad \text{as} \quad n\to\infty,$$

for some constants $\gamma \in (0,\infty)$ and $c \in (0,1)$.

We assume that $\Sigma_1^{-1/2}\Sigma_2$ has eigendecomposition

$$\Sigma_1^{-1/2}\Sigma_2^{1/2} = ODO^T, \quad D = \operatorname{diag}(d_1, \cdots, d_p).$$

Then by the rotational invariance of Gaussian matrices, we have

$$\widetilde{Q} \overset{d}{=} \Sigma_1^{1/2} O \widetilde{Q} O^T \Sigma_1^{1/2}, \quad \widetilde{Q} := X_1^T X_1 + D X_2^T X_2 D.$$

Thus we study the spectrum of $\widetilde{Q}$ instead. We define $\mathcal{G}(z) := (\widetilde{Q} - z)^{-1}$ for $z \in \mathbb{C}_+$. With some random matrix tools, we have that

$$\mathcal{G}(z) \approx \operatorname{diag}\left(\frac{1}{-z\left(1 + m_3(z) + d_i^2 m_4(z)\right)} \right)_{1\leqslant i\leqslant p} = \frac{1}{-z\left(1+m_3(z) + D^2 m_4(z)\right)}$$

in certain sense. Here $m_{3,4}(z)$ satisfy the following self-consistent equations

$$\frac{n_1}{n}\frac{1}{m_3} = -z + \frac{1}{n}\sum_{i=1}^p \frac{1}{1+m_3+d_i^2 m_4}, \quad \frac{n_2}{n}\frac{1}{m_4} = -z + \frac{1}{n}\sum_{i=1}^p \frac{d_i^2}{1+m_3+d_i^2 m_4} \tag{1.2}$$

When $z \to 0$, we shall have

$$m_3(z) = -\frac{a_3}{z} + \mathrm{O}(1), \quad m_4(z) = -\frac{a_4}{z} + \mathrm{O}(1), \quad a_3, a_4 > 0.$$

Then for $z \to 0$, the equations in (1.3) are reduced to

$$\frac{n_1}{n}\frac{1}{a_3} = 1 + \frac{1}{n}\sum_{i=1}^p \frac{1}{a_3 + d_i^2 a_4}, \quad \frac{n_2}{n}\frac{1}{a_4} = 1 + \frac{1}{n}\sum_{i=1}^p \frac{d_i^2}{a_3 + d_i^2 a_4}. \tag{1.3}$$

First, it is easy to see that these equations are equivalent to

$$a_3 + a_4 = 1 - \gamma_n, \quad a_3 + \frac{1}{n}\sum_{i=1}^p \frac{a_3}{a_3 + d_i^2[(1-\gamma_n) - a_3]} = c_n.$$

Furthermore, we have

$$\begin{aligned}
\mathrm{Tr}(Q^{-1}) &= \lim_{z\to 0} \mathrm{Tr}\left[\Sigma_1^{-1/2} O \mathcal{G}(z) O^T \Sigma_1^{-1/2}\right] = \mathrm{Tr}\left[\Sigma_1^{-1/2} O \left(\frac{1}{a_3 + D^2 a_4}\right) O^T \Sigma_1^{-1/2}\right] \\
&= \mathrm{Tr}\left[\Sigma_1^{-1/2} \frac{1}{a_3 + \Sigma_1^{-1}\Sigma_2 a_4} \Sigma_1^{-1/2}\right] = \mathrm{Tr}\left[\frac{1}{a_3\Sigma_1 + a_4\Sigma_2}\right].
\end{aligned}$$