Rotational Physics

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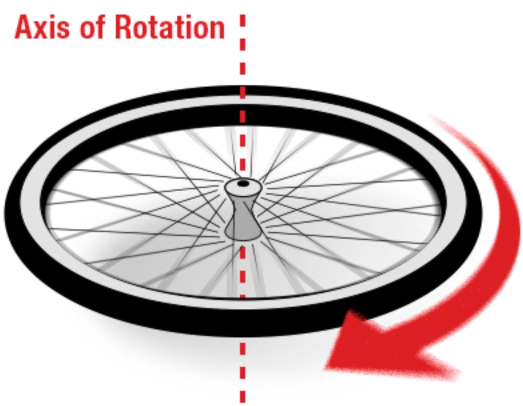
1 Rotational Kinematics

1.1 Rotational Motion

We have learned about linear motion, where we move in a one, two or three-dimensional space. The previously discussed form of motion can be broken down into smaller components of one-dimensional motion. However, we see many everyday examples of another form of motion; ***rotational motion***.

> **Rotational Motion**
> *Rotational motion* is the movement of a body such that every point on the rigid body moves in a circle about an axis of rotation.

For example, imagine we are biking. Each point on our wheels rotates about the axis of rotation (in this case, the bike axle) in its own respective circle. Essentially, think of rotational motion as many (in fact, infinitely many) simultaneous circular motions of particles along the body. It may seem confusing at first, but this fact will lay the foundation needed to approach many rotational motion problems.



1.2 Rotational Counterparts

In linear motion, we had displacement, velocity, acceleration, and time. Well, in rotational motion, we have the rotational (or angular; we will be using these words interchangeably) counterparts; *angular* displacement (θ), *angular* velocity (ω), and *angular* acceleration (α). Time remains the same as it is a scalar. Instead of considering linear motion, we are discussing motion along a circular path, so intuitively we define a respective variable to displacement, velocity or acceleration for motion along a circle.

The kinematic formulas still hold, though we replace each variable with its rotational counterpart as we are moving along a circle rather than in a straight line. The formulas are pictured below:

Fact 1.

1. $\omega = \omega_0 + \alpha t$
2. $\Delta\theta = \dfrac{\omega_0 + \omega}{2} \cdot t$
3. $\Delta\theta = \omega_0 t + \dfrac{1}{2}\alpha t^2$
4. $\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$

Our average angular velocity is simply $\omega_{avg} = \frac{\Delta\theta}{\Delta t}$ and our average angular acceleration is $\alpha_{avg} = \frac{\Delta\omega}{\Delta t}$, similar to the definitions of average velocity and acceleration.

1.2.1 Graphs

Properties of angular variable graphs such as velocity versus time compared to linear variable graphs such as velocity versus time do not change; *the area under an angular velocity vs. ti* As previously said, imagine the angular variables are simply angular counterparts of standard linear variables. There is no reason for any properties of angular variable graphs to change compared to linear variable graphs.

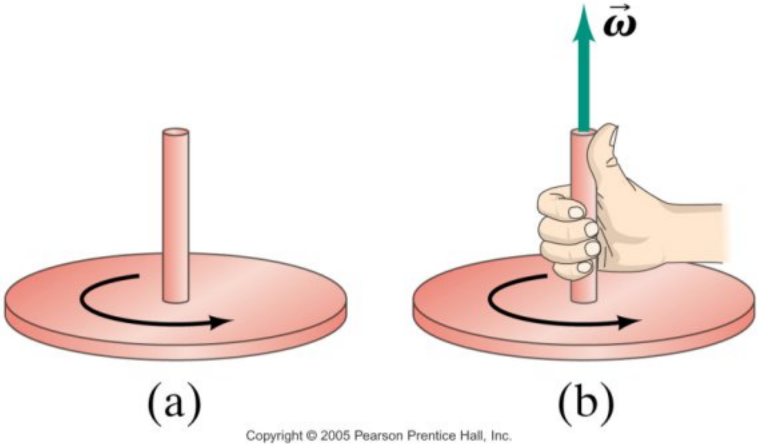
> *Remark 2.* If at any point you are confused in an angular graph problem, you can translate the problem to linear terms. To show this in a rather lucid manner, imagine we were tasked with finding the slope of a θ vs t graph. The linear counterpart of θ is displacement, and the slope of a displacement vs time graph is velocity. Translating back to rotational terms, the slope of a θ vs t graph is the rotational counterpart of velocity, ω.

1.2.2 Direction

In order to find the direction of angular variables, we must use a method known as the ***Right Hand Rule***.

> **Right Hand Rule**
> The *right hand rule* is a method in determining the direction of an angular variable, by curling our fingers about the direction of rotation of the angular variable and determining the direction our thumb is pointing.

This is demonstrated in the following diagram:



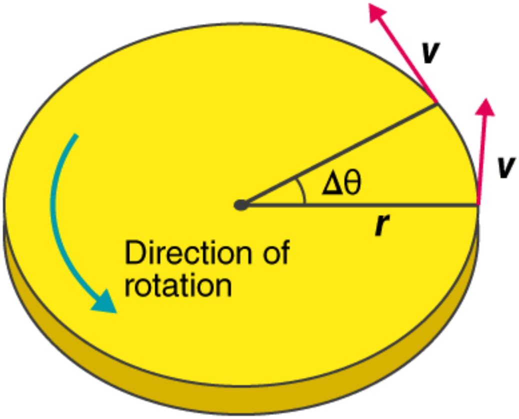
1.3 Relationships

But why?, you may be asking. These angular variables are cute and all, but they mean nothing if we cannot connect them to linear variables in any way. Well, fear not, as we will be discussing the relationships between angular and linear variables in this section.

Claim —

$$v = r\omega$$
$$a = r\alpha$$

Proof. Consider the following diagram:



In a small time Δt, assume we cover an angular displacement of $\Delta\theta$. This means we cover a displacement of $\Delta s = r\Delta\theta$, as θ is in radians. From our definition of angular velocity, we have $\frac{\Delta\theta}{\Delta t} = \omega$ and we know $\frac{\Delta s}{\Delta t} = v$, where Δs is the displacement. Plugging in $\Delta s = r\Delta\theta$, we see

$$v = \frac{\Delta s}{\Delta t} = \frac{r\Delta\theta}{\Delta t} = r\frac{\Delta\theta}{\Delta t} = r\omega.$$

Dividing both sides of the equation by time results in $a = r\alpha$ as well. $\square$

> *Remark 3.* These facts are very useful in problems where one variable is given and the other is not, make sure to keep these in the back of your mind!

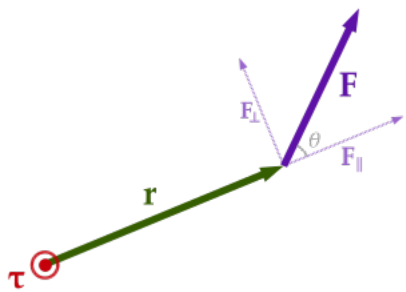
2 Rotational Dynamics

2.1 Torque

Imagine closing a door by giving a quick shove. We apply a force $\vec{\mathbf{F}}$ on the door, and $\vec{\mathbf{F}}$ generates a torque of τ relative to the hinge point, The door rotates around the hinge, eventually closing itself. We have applied a torque on the door, causing it to have angular acceleration. Torque is denoted by τ and is found via

$$\tau = \vec{\mathbf{r}} \times \vec{\mathbf{F}},$$

where $\vec{\mathbf{r}}$ is the vector from the origin to the point where the force $\vec{\mathbf{F}}$ is applied. $\vec{\mathbf{r}}$ is the radius from the axis of rotation and is also called the moment arm.



It is important to remember that torque concerns only the perpendicular component of $\vec{\mathbf{F}}$. So in the above image the torque produced is

$$\tau = \vec{\mathbf{r}} \times \vec{\mathbf{F}}.$$

That means if we apply a force at an angle of θ from $\vec{\mathbf{r}}$, we produce a torque of

$$\tau = \vec{\mathbf{r}} \times \vec{\mathbf{F}} \implies \tau = rF\sin(\theta) = rF_{\perp}.$$

Torque

Torque is rotational equivalent of force, defined as

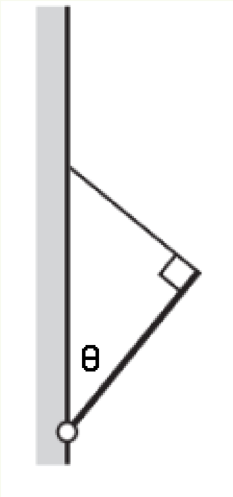
$$\tau = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = rF_{\perp} = rF\sin(\theta).$$

Much like forces, torques can be added up, but the clause is they have to be around the same axis. The sum of all torques on a object is net torque, or τ_{net}.

In a situation given where a body is experiences multiple torques, we must calculate the torques around the same same axis of rotation (or the hinge point).

Example 4

A long rod of mass m and length ℓ is attached to a wall, hinged at its bottom point. A masseless string is tied on the wall, above the rod. The other end of the string is tied to the other end of the rod, such that the rod forms an angle θ with the wall.



What is the tension in the string such that the rod remains still? Express your answer in terms of ℓ, g, θ and m. [a]

[a]Hint: Try conserving torque about the hinge point.

Solution. The gravitational force on the rod can be equated to a single force on the COM with magnitude mg. This force is a distance $\frac{1}{2}\ell$ from the hinge, and is applied at an angle of θ. Therefore the torque it generates is

$$\tau_g = \frac{1}{2}\ell \cdot mgsin(\theta).$$

In order for the net torque to be 0, the torque from the string must be equal in magnitude to τ_g. The tension is applied at a distance of ℓ from the hinge, so we have

$$\tau_g = \tau_T$$
$$\frac{1}{2}\ell \cdot mgsin(\theta) = T \cdot \ell$$
$$\boxed{T = \frac{1}{2}mgsin(\theta)}$$

□

2.2 Moment of Inertia

2.2.1 Introduction

The moment of inertia of an object describes how hard it is to rotate it. An object with a higher moment of inertia will be harder to rotate. This should make sense, an object with larger mass is harder to move so an object with higher moment of inertia would be harder to rotate, intuitively.

Moment of Inertia

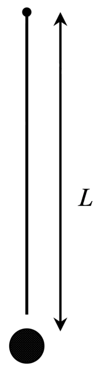
Moment of Inertia is rotational equivalent of mass. It measures how hard it is for an object to rotate, and is defined as

$$I = \sum m_i r_i^2,$$

where m_i and r_i are the mass and distance from the axis of rotation, respectively of an element.

Going forward, the moment of inertia will be referred to as the MOI.

Consider the following system: a pendulum with a massless rod of length L and a small bob with mass M.



The ***moment of inertia*** of an object of mass M at a distance L from the axis of rotation is defined as

$$I = ML^2,$$

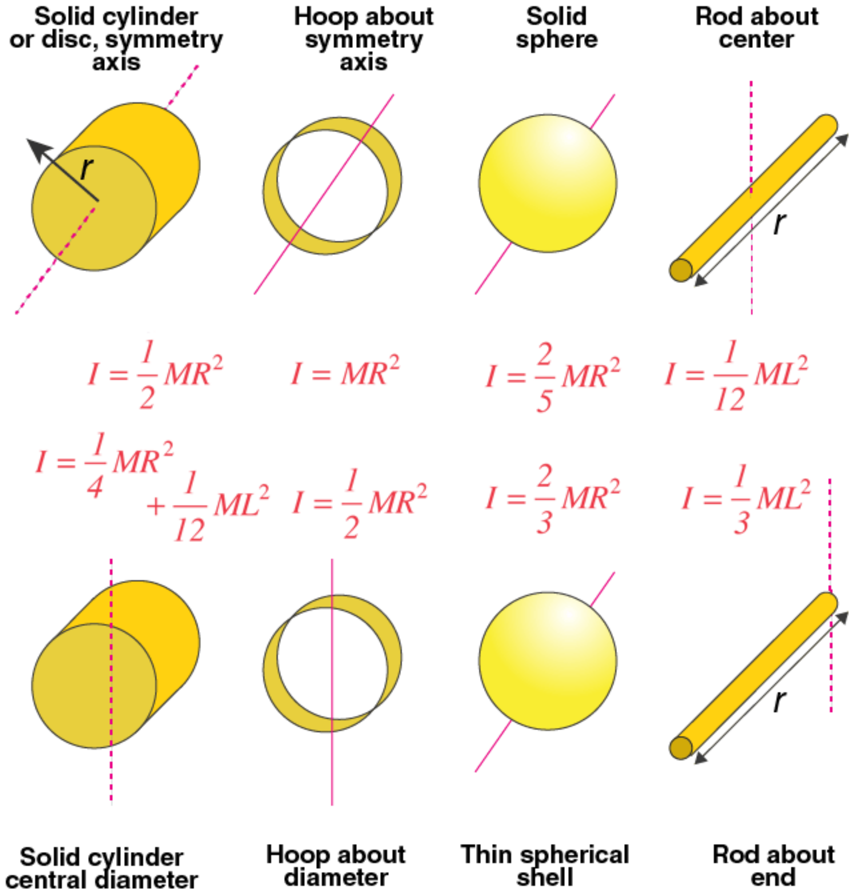
where I is the symbol for the MOI.

2.2.2 Continuous Bodies

Now let us consider an object that isn't as simple, such as a solid disk. Notice, what we have is many masses m_i at varying distances r_i from an axis of rotation. However, we can just sum all the individual values to gives us the MOI. This is defined as

$$I = \sum m_i r_i^2.$$

To calculate the MOI of disks, spheres, and other continuous objects would require calculus. This is beyond the scope of this handout, so here are some general MOIs that you should memorize:



Remark 5. You might have noticed that there are two equations for the MOI of the thin rod, depending on where the axis of rotation is. Remember that the MOI depends on the axis of rotation.

Example 6 (Morin, Problems and Solutions in Introductory Mechanics)
There are four discs with equal masses. Lighter shades indicate lower density, and darker shades indicates higher density. Which of the four has the highest MOI about the central axis?

Solution. The key idea to remember is that the further a mass is from the axis, the larger its MOI is. $\boxed{\text{Choice (d)}}$ has all its mass concentrated at the outer edges, maximizing r for all its mass, therefore maximizing its MOI. □

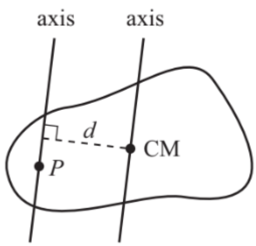
2.3 Axes Theorems

2.3.1 Parallel Axis Theorem

To quickly find the MOI of an object around a certain axis, you can use the parallel axis theorem.

Fact 7 (Parallel Axis Theorem).

$$I_P = I_{CM} + Md^2.$$



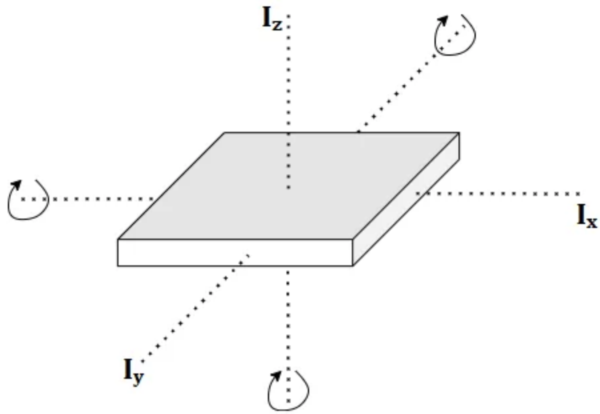
Where I_P is the MOI around an axis of rotation going through a point P (parallel to the central axis), d is the distance from the center of mass to P, M is the mass of the object, and I_{CM} is the MOI of the object around the center of mass.

2.3.2 Perpendicular Axis Theorem

The perpendicular axis theorem relates the MOI's of an object around the three coordinate axes.

Fact 8 (Perpendicular Axis Theorem).

$$I_z = I_x + I_y$$



Remark 9. This theorem only applies to planar (or thin) objects.

This theorem states that the MOI of a planar object about an axis that is perpendicular to the plane the object (I_z) is on is equal to the sum of two MOI's with axes on the plane of the object, that are perpendicular to each other (I_x and I_y).

Example 10
The moment of inertia of a solid disk with radius r and uniform mass passing through the center of the disk (the axis is "piercing" the disk), perpendicular to the plane of the disk, is $\frac{1}{2}mr^2$. Find the moment of inertia of an axis tangent to the disk.

Solution. Let us first find the MOI about the diameter of the disk. From the perpendicular axis theorem we get

$$I_z = \frac{1}{2}mr^2 = I_x + I_y,$$

We know that I_x and I_y are equal since the disk will always be the same no matter how it is rotated, thus $I_x = I_y = \frac{1}{4}mr^2$. Note that a tangent to the disk is parallel to a diameter and is a distance r away from the diameter. Applying the parallel axis theorem gives us an answer of

$$I = \frac{1}{4}mr^2 + mr^2 = \boxed{\frac{5}{4}mr^2}.$$

□

2.3.3 Newton's Rotational Laws

Just like how we have converted linear variables into rotational ones, we can convert Newton's three laws.

Newton's First Rotational Law
An object at rest or in rotation will remain at rest or in rotation unless compelled to change by a torque.

Similar to $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, there is a rotational equivalent.

Newton's Second Rotational Law

$$\tau = I\alpha,$$

where α is the angular acceleration of the body.

The third law can't be as easily translated, but it is what proves the conservation of angular momentum, which will be discussed in the next section.

2.4 Angular Momentum and Energy

Angular momentum is the rotational counterpart of momentum. We believe it deserves its own section as it can be very tricky, it can be conserved in situations where linear momentum isn't!

Angular momentum (L) is defined similar to torque, it is the perpendicular component of the linear momentum multiplied by the distance from the axis of rotation, or

Angular Momentum

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} \implies L = rp\sin(\theta) = rp_{\perp}.$$

2.4.1 Conservation of Angular Momentum

Angular momentum is conserved when there is no external torque acting on the system, much like linear momentum is conserved when there is no external force applied onto the system. This fact may not seem very important at face value, but there are many applications of this regarding external forces. A highlighting example of this fact is shown below.

Example 11 (2020 F=ma)
A ballerina is quickly twirling around a vertical axis of rotation. She releases a pen from her hand. What happens to the angular momentum of the system consisting of the ballerina and the pen when the pen hits the ground?

Solution. Note that gravity and normal both point in the vertical direction, in the same direction as angular momentum (this may be a bit hard to visualize at first, but use the right hand rule to verify this). Therefore, at any point in time they cannot provide any external torque, as the force passes through the vertical direction. This can also be shown mathematically using the definition of torque, we have $\tau = rF\sin(\theta)$ but θ, the angle between the distance and force vectors, is 0, meaning $\tau = 0$. Hence, the angular momentum $\boxed{\text{does not change}}$. □

2.4.2 Formulas

Depending on the type of rotation, the angular momentum has a slightly varying formula.

Fact 12. If the linear momentum stays constant then as previously discussed, $L = rp_\perp = m(rv_\perp)$.

Fact 13. If the object is purely rotating, $L = I\omega$, where the moment of inertia I is about some axis of rotation (and the angular velocity and angular momentum are also about this same axis of rotation).

Fact 14. If the object is rotating and translating simultaneously, $L = I_c\omega + mv_cr$, where I_c and v_c are the moment of inertia and velocity about the center of mass, respectively. Note that this case of motion is not heavily used.

2.4.3 Energy

Rotational kinetic energy is not much different than standard kinetic energy, as energy is a scalar quantity. In fact, we must simply add the rotational energy to the standard kinetic energy of $\frac{1}{2}mv^2$. Thus, total kinetic energy is defined as

Kinetic Energy

$$K = \frac{1}{2}I_c\omega^2 + \frac{1}{2}mv_c^2,$$

where once again I_c and v_c are the moment of inertia and velocity about the center of mass, respectively.

The gravitational potential energy does not change in rotational motion.

Example 15
A softball pitcher rotates their arm of length ℓ at a speed v. If they let go of the softball at some point, what is the maximum possible height above the point of release in terms of ℓ and v? Assume the softball is uniform and it also rotates at an angular speed $v = \ell\omega$.

Solution. The system has an initial kinetic energy of

$$K_i = \frac{1}{2} I_c \omega^2 + \frac{1}{2} m v^2.$$

Since the ball is in uniform circular motion before being released, $v = \ell\omega$. Thus, the ball's initial kinetic energy is

$$K_i = \frac{1}{2}\left(\frac{2}{5} m\ell^2\omega^2\right) + \frac{1}{2} m(\ell\omega)^2 = \frac{7}{10} m\ell^2\omega^2.$$

By the conservation of energy, we have $K_i = U_f$, so we have

$$\frac{7}{10} m\ell^2\omega^2 = mgh \implies h = \frac{7\ell^2\omega^2}{10g} = \boxed{\frac{7v^2}{10g}}.$$

□

2.4.4 Graphs

The area under a force versus time graph was defined to be impulse. Similarly, the area under a torque versus time graph is defined to be angular impulse, or $\Delta L = \Delta p \cdot d$.

The area under a force versus displacement graph was defined to be work. Similarly, the area under a torque versus displacement is also defined to be work, as $W = \Delta E$, which is a scalar.

> *Remark 16.* Once again, if you are confused at any point, you can translate the graph to linear terms and translate it back to rotational terms.