Various Gravitational Field Strengths

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Contents

1 Introduction

This brief handout will serve as an introduction to some basic gravitational field strengths and their derivations. Before we start, we would like to share a very inspirational quote from one of the writers said while writing the handout.

> *"The gravitational field strength of your mom is ∞."*
>
> \- Kevin Zhang

Gravitational Field Strength

The gravitational field strength (GFS) of a point P in a gravitational field is defined as

$$\vec{\mathbf{g}} = \frac{\vec{\mathbf{F}}_{g,P}}{m_P},$$

or essentially the force per unit mass. In the case of a point mass, $g = \frac{GM}{R^2}$.

With that out of the way, we can move onto some more complex configurations.

2 Ring Configurations

Consider a ring of mass M and radius r for the following configurations.

2.1 Point at Center

By considering an element of width dx, assume the GFS of this width is $\vec{\mathbf{g}}$. Considering the diametrically opposite element on the ring gives a GFS of $-\vec{\mathbf{g}}$. Adding these GFS's gives a net GFS of 0 for these two elements. Thus, we can see the net GFS of a point at the center of a ring is simply 0, by considering the elements on a semi-ring and their diametrically opposite counterpart.

2.2 Point on Axis

Suppose the point P is at a distance x away from the center of the ring, and lies on the line perpendicular to the plane of the ring passing through its center.

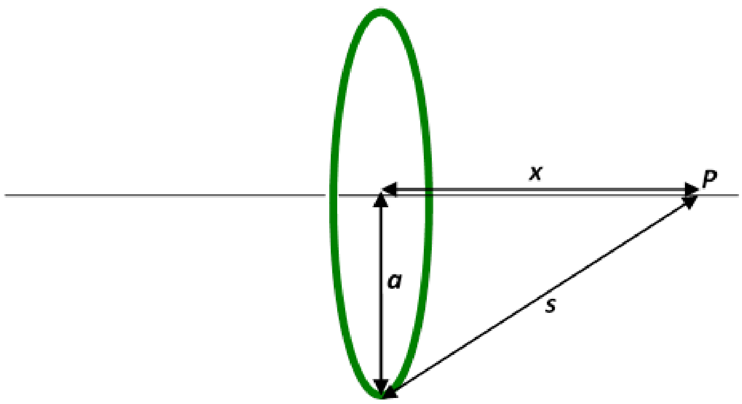


Image credit: Galileo

We consider the GFS from each element along the ring, making an angle θ with the axis of the ring and the line passing through the point P and tangent to the ring at the element. Note that the diametrically opposite element cancels the vertical component of this GFS, so we simply consider its horizontal component, or $dg = \frac{Gdm}{r^2+x^2} \cdot \cos\theta$. Solving for $\cos\theta = \frac{x}{\sqrt{x^2+r^2}}$ with straightforward geometry, we get $dg = 2 \cdot \frac{Gdm}{r^2+x^2} \cdot \frac{x}{\sqrt{r^2+x^2}}$. Integrating gives us

$$\boxed{g = \frac{GMx}{(r^2+x^2)^{3/2}}}.$$

3 Shell Theorem

Isaac Newton stated that

1. A spherically symmetric body applies a gravitational force as though all of its mass were concentrated at a point at its center.
2. If an object is in the interior of a body that is a spherically symmetric shell, the object will experience a net gravitational force of zero from the body.

In his famous book *Principia Mathematica*. These statements hold true for any point which is not under the effect of any other gravitational fields.

3.1 Outside the Shell

A sphere can be represented as an infinite number of shells layered upon each other. If we can prove that if a shell acts like a point mass, then we have proof that a sphere acts like a point mass. Before we look at an entire shell, lets look at a ring of it first.

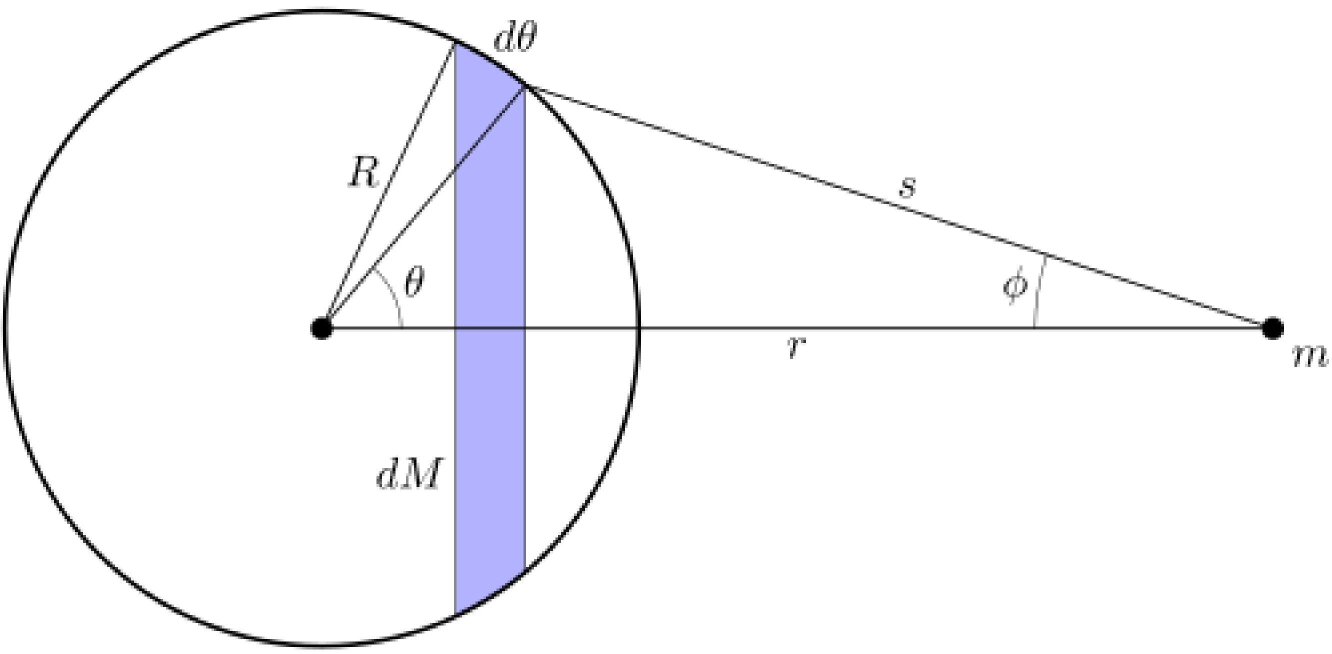


Image credit: Wikipedia

This ring is very thin, with a width of $Rd\theta$. Every point on this ring will be s away from the object. Consider a point O with mass dm_O. From Newton's Law of Gravitation we know the "horizontal" force from O to be

$$dF_O = G\frac{m \cdot dm_O}{s^2}\cos(\phi).$$

The "vertical" force will simply cancel out with the point opposite to O. Summing the forces exerted by all the points on the ring we get]

$$dF = G\frac{m \cdot dM}{s^2}\cos(\phi),$$

where dM is the total mass of the ring. To find dM we will relate it to its volume. Look only at the ring, and cut it such that you can lay it flat. What we are left with is a strip of paper like object. This object has a thickness of t, a length and width of $2\pi R\sin(\theta)$ and $Rd\theta$ (the length comes from the circumference of the ring). The mass of this ring is

$$dM = \rho dV$$
$$= \rho \cdot t \cdot Rd\theta \cdot 2\pi\sin(\theta)$$

Now to prepare for integration, we must eliminate Φ and θ. Draw a mental triangle bound by the ring, s and r. From this, we know

$$\cos(\theta) = \frac{r - R\cos(\theta)}{s}.$$

Now consider the triangle formed by R, r and s. Applying the Law of Cosines,

$$s^2 = R^2 + r^2 - 2Rr\cos(\theta)$$
$$\implies R\cos(\theta) = \frac{r^2 + R^2 - s^2}{2r}$$

Differentiating this equation to eliminate θ, we get

$$\sin(\theta)d\theta = \frac{s}{Rr}dx.$$

Plugging in our values back,

$$\cos(\phi) = \frac{r - \frac{r^2 - R^2 + s^2}{2r}}{s}$$
$$dM = \rho tR2\pi\frac{s}{Rr}dx.$$

Simplifying into

$$dF = G\frac{\rho\pi mtR}{r^2}(\frac{r^2 - R^2}{s^2} + 1)dx$$

Now we can finally integrate over x. This integral ranges from $r - R$ to $r + R$ to cover all the rings in the shell.

$$F = \int dF$$
$$= \int_{r-R}^{r+R} G\frac{\rho\pi mtR}{r^2}(\frac{r^2 - R^2}{s^2} + 1)dx$$
$$= G\frac{\rho\pi mtR}{r^2}\int_{r-R}^{r+R}(\frac{r^2 - R^2}{s^2} + 1)dx$$

$$= G\frac{\rho\pi mtR}{r^2} \cdot 4R.$$

Given the sphere has volume $4\pi R^2$, it has a mass of $M = 4\pi R^2 t\rho$. Therefore the force from the sphere is

$$F = G\frac{Mm}{r^2}$$

or Newton's Gravitational Law for point masses.

3.2 Inside the Shell

The proof of the second statement is the almost the exact same as the proof for the first, but the object is inside. They differ in the integral, where the lower limit becomes $R - r$. This results in the integral equating 0. The object experiences a net force of zero inside the sphere.

4 Infinite Wire

Consider an infinite wire of mass with linear mass density λ kg/m, meaning for every meter of wire, this length contains 1 kg of mass.

Example 1
What is the gravitational field strength at a point which is distance r away from the wire?

Solution.

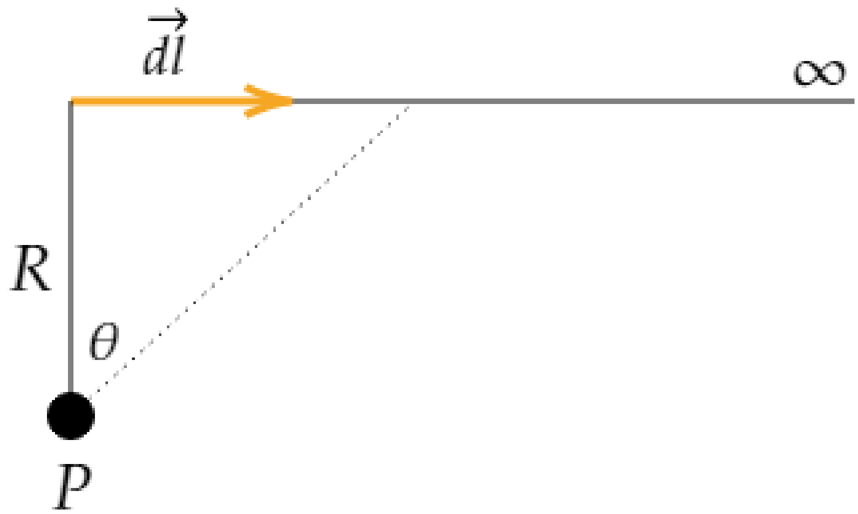


Image credit: Stack Exchange

Consider a point x on the wire that has mass $dm = \lambda dl$ and is a distance $\sqrt{l^2 + r^2}$ from P. We have

$$dg = G\frac{\lambda dl}{l^2 + r^2}.$$

We also know that $l = r\tan(\theta)$ and $\tan^2(\theta) + 1 = \sec^2(\theta)$,

$$\begin{aligned} dg &= G\frac{\lambda r\sec^2(\theta)d\theta}{r^2(\sec^2(\theta))} \\ &= G\frac{\lambda d\theta}{r}. \end{aligned}$$

We are only looking at the component of the force that is perpendicular to the wire. The parallel component is cancelled out, as for each dl that generates a parallel force component there is another point on the other side of the object that produces a similar component that is equal in magnitude but opposite in direction. From this, we get

$$\begin{aligned}g &= 2 \cdot \int_0^{\frac{\pi}{2}} G\frac{\lambda d\theta}{r}\cos(\theta)\\ &= 2G\lambda\frac{\sin(\theta)}{r}\Big|_0^{\frac{\pi}{2}}\\ &= \frac{2G\lambda}{r}.\end{aligned}$$

Thus, the gravitational field strength is $\boxed{g = \frac{2G\lambda}{r}}$. □

5 Infinite Sheet

consider an infinite sheet of area density σ kg$/m^2$.

Example 2
What is the gravitational field strength at a point which is distance ℓ away from the sheet?

We can view the sheet as an infinite number of rings with thickness dr that are perfectly fitted together. Taking a ring out and "unrounding" it, we get a rectangle with area $2\pi r dr$, so each ring has mass $dm = 2\pi r dr\sigma$. Again, we are only only considering the perpindicular component, as the parallel component cancels out. Plugging this into the equation for the GFS of rings,

$$dg = G\frac{dmx}{(x^2+r^2)^{3/2}}$$

$$= G\pi\sigma x\frac{2rdr}{(x^2+r^2)^{3/2}}.$$

Let $u = x^2 + r^2$, meaning $du = 2rdr$. Substituting and integrating,

$$dg = G\pi\sigma x\frac{du}{(u)^{3/2}}$$

$$\int_{x^2}^{\infty} dg = \int_{x^2}^{\infty} G\pi\sigma x\frac{du}{(u)^{3/2}}$$

$$= G\pi\sigma x(-2\frac{1}{\sqrt{u}})\Big|_{x^2}^{\infty}.$$

Flipping the limits and solving,

$$= G\pi\sigma x(2\frac{1}{\sqrt{u}})\Big|_{\infty}^{x^2}$$

$$= G\pi\sigma x(2\frac{1}{x} - 0).$$

Thus, the gravitational field strength is $\boxed{2G\pi\sigma}$.

6 Gauss's Law

Gauss's Law relates the gravitational field to the enclosed mass. Specifically, it states that the ***gravitational flux through any Gaussian surface is proportional to the enclosed mass***. In order to understand what this means, we must first understand what gravitational flux is.

6.1 Gravitational Flux - An Intro

When representing the field visually, we use various lines called field lines to demonstrate the direction and magnitude of the field at various points. In most standard textbooks, flux is defined the number of field lines passing through a surface.

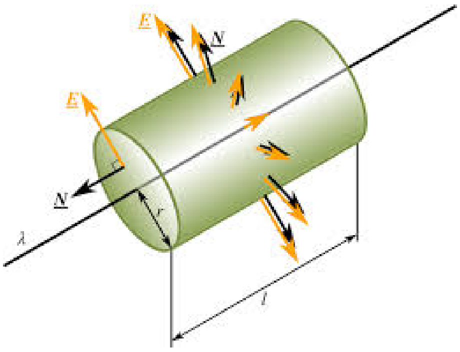


Image credit: Isaac Physics

We can also think of flux as the flow rate of something, if we consider the flow rate of the field (strength) per unit area, we obtain the flux. Let's see how we can use this definition to obtain Gauss's Law.

6.2 Gauss's Law Equation

Using the analogy of flux as a flow rate, we can obtain Gauss's Law. The proof of Gauss's Law itself is slightly beyond the scope of this handout, as this handout simply serves as an introduction *various gravitational field strengths*. In fact, you may be wondering why Gauss's Law relates to the gravitational field strength at all. The connection can be seen in the following (awaited) equation:

Gauss's Law (Integral Form)

$$\oiint \mathbf{g} \cdot \mathrm{d}\mathbf{A} = -4\pi G M_{encl}.$$

This may leave you even more confused. You may be asking, "How can you take the dot product with area, isn't area a scalar?". Fear not, there is a way in which area can be considered as vector, though that is beyond the scope of this handout as well.

The more common form of Gauss's Law that is probably what you will use for most cases is the alternative form:

Gauss's Law (Useful Form)

$$gA = -4\pi G M_{encl}.$$

The negative symbol simply means that the gravitational force is an attractive force, so the flux is negative, and M_{encl} simply means the mass enclosed by the Gaussian surface. This is the most useful form of Gauss's Law as all of the situations that we will be seeing will require the total area that the gravitational field is perpendicular to.

7 Infinite Wire (Revisited)

Once again consider an infinite wire of mass with linear mass density λ kg/m, meaning for every meter of wire, this length contains 1 kg of mass.

Example 3
What is the gravitational field strength at a point which is distance r away from the wire?

Solution. The *hypothetical* Gaussian surface we will use is a cylinder with radius r and height ℓ. Since the gravitational field is radially outward from the wire, we can see the area that the gravitational field acts on the lateral surface area of the cylinder. Also, the mass enclosed is simply $\lambda\ell$. This can be seen in the following diagram:

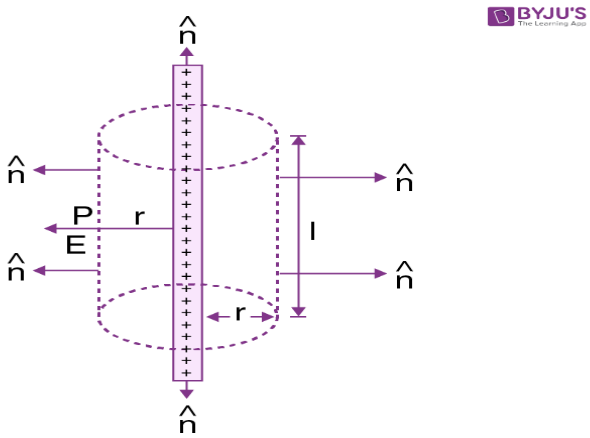


Image credit: Byju's

So, using Gauss's Law, we get:

$$g(2\pi r\ell) = -4\pi G(\lambda\ell)$$
$$\boxed{g = -\frac{2G\lambda}{r}}.$$

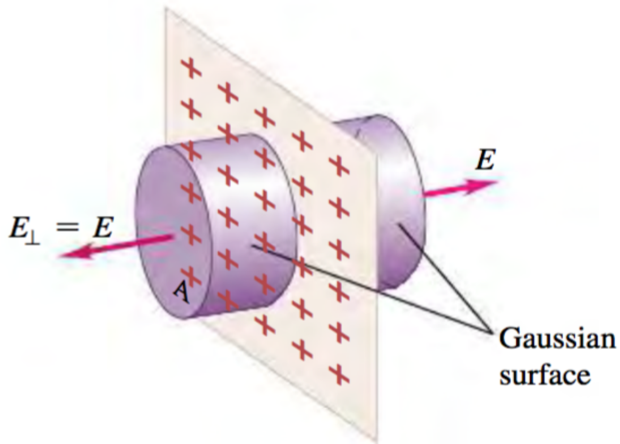
□

8 Infinite Sheet (Revisited)

Again consider an infinite sheet of area density σ kg/m^2.

Example 4
What is the gravitational field strength at a point which is distance ℓ away from the sheet?

This time, the *hypothetical* Gaussian surface we will use is a cylinder oriented perpendicular to the plane of the sheet, with height 2ℓ and radius r. The gravitational field acts perpendicular to the sheet, so we consider only the areas of the "caps" of the cylinders.



The mass enclosed by the cylinder is $\sigma\pi r^2$, and the total area of the caps is $2\pi r^2$. Thus, by Gauss's Law:

$$g(2\pi r^2) = -4\pi G(\sigma\pi r^2)$$
$$\boxed{g = -2G\pi\sigma}.$$

9 Summary

To summarize, we have derived some important gravitational field strengths that can be used in a variety of problems. For a symmetrical situation (generally involving curved surfaces), we can use Gauss's Law to calculate the gravitational field strength at a certain point. For asymmetric situations, we must stick to the standard integration definitions. Knowing both methods is important and vital for solving problems of all types.