ps2-4-solve

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1 (a)

$K(x, z) = K_1(x, z) + K_2(x, z)$ is necessarily a kernel

symmetric:

$$\begin{aligned} K^T &= (K_1 + K_2)^T \\ &= K_1^T + K_2^T \\ &= K_1 + K_2 \\ &= K \end{aligned}$$

PSD:

$$\begin{aligned} z^T K z &= z^T (K_1 + K_2) z \\ &= z^T K_1 z + z^T K_2 z \\ &\geq 0 \end{aligned}$$

2 (b)

$K(x, z) = K_1(x, z) - K_2(x, z)$ isn't necessarily a kernel

symmetric:

$$\begin{aligned} K^T &= (K_1 - K_2)^T \\ &= K_1^T - K_2^T \\ &= K_1 - K_2 \\ &= K \end{aligned}$$

PSD:

$$\begin{aligned} z^T K z &= z^T (K_1 - K_2) z \\ &= z^T K_1 z - z^T K_2 z \end{aligned}$$

K isn't PSD when $K_1 < K_2$

3 (c)

$K(x, z) = aK_1(x, z)$ isn't necessarily a kernel

symmetric:

$$\begin{aligned} K^T &= (aK_1)^T \\ &= aK_1^T \\ &= aK_1 \\ &= K \end{aligned}$$

PSD:

$$\begin{aligned} z^T K z &= z^T a K_1 z \\ &= a z^T K_1 z \end{aligned}$$

K isn't PSD when $a < 0$

4 (d)

$K(x, z) = -aK_1(x, z)$ isn't necessarily a kernel

symmetric:

$$\begin{aligned} K^T &= (-aK_1)^T \\ &= -aK_1^T \\ &= -aK_1 \\ &= K \end{aligned}$$

PSD:

$$\begin{aligned} z^T K z &= z^T (-aK_1) z \\ &= -a z^T K_1 z \end{aligned}$$

K isn't PSD when $a > 0$

5 (e)

$K(x,z) = K_1(x,z)K_2(x,z)$ is necessarily a kernel

symmetric:

$$\begin{aligned}
K^T &= (K_1K_2)^T \\
&= K_2^T K_1^T \\
&= K_2K_1 \\
K^T_{ij} &= \sum_{k=1}^{n} K_{2_{ik}} K_{1_{kj}} \\
&= \sum_{k=1}^{n} K_{1_{jk}} K_{2_{ki}} \\
K^T &= \sum_{k=1}^{n} K_{1_{\cdot k}} K_{2_{k\cdot}} \\
&= K_1K_2 \\
&= K
\end{aligned}$$

PSD:

$$\begin{aligned}
z^TKz &= \sum_i \sum_j z_i K_{ij} z_j \\
&= \sum_i \sum_j z_i K_1(x^{(i)}, x^{(j)}) K_2(x^{(i)}, x^{(j)}) z_j \\
&= \sum_i \sum_j z_i \phi_1(x^{(i)})^T \phi_1(x^{(j)}) \phi_2(x^{(i)})^T \phi_2(x^{(j)}) z_j \\
&= \sum_i \sum_j z_i z_j \sum_k \phi_{1k}(x^{(i)}) \phi_{1k}(x^{(j)}) \sum_l \phi_{2l}(x^{(i)}) \phi_{2l}(x^{(j)}) \\
&= \sum_k \sum_l \sum_i z_i \phi_{1k}(x^{(i)}) \phi_{2l}(x^{(i)}) \sum_j z_j \phi_{1k}(x^{(j)}) \phi_{2l}(x^{(j)}) \\
&= \sum_k \sum_l \big(\sum_i z_i \phi_{1k}(x^{(i)}) \phi_{2l}(x^{(i)}) \big)^2 \\
&\geq 0
\end{aligned}$$

6 (f)

$K(x,z) = f(x)f(y)$ is necessarily a kernel

symmetric:

$$
\begin{aligned}
K_{ij} &= K(x^{(i)}, x^{(j)}) \\
&= f(x^{(i)})f(x^{(j)}) \\
&= f(x^{(j)})f(x^{(i)}) \\
&= K(x^{(j)}, x^{(i)}) \\
&= K_{ji}
\end{aligned}
$$

PSD:

$$
\begin{aligned}
z^T K z &= \sum_i \sum_j z_i f(x^{(i)}) f(x^{(j)}) z_j \\
&= \sum_i z_i f(x^{(i)}) \sum_j z_j f(x^{(j)}) \\
&= \big(\sum_i z_i f(x^{(i)}) \big)^2 \\
&\geq 0
\end{aligned}
$$

7 (g)

$K(x, z) = K_3(\phi(x), \phi(z))$ is necessarily a kernel

symmetric:

$$
\begin{aligned}
K^T &= K_3^T \\
&= K_3 \\
&= K
\end{aligned}
$$

PSD:

$$
\begin{aligned}
z^T K z &= z^T K_3 z \\
&\geq 0
\end{aligned}
$$

8 (h)

let $p(x) = \sum_i a_i x^i$, so $K(x, z) = p(K_1(x, z))$ is necessarily a kernel when $a_i \geq 0$ for any i

symmetric:

$$
\begin{aligned}
K^T &= (p(K_1))^T \\
&= \big(\sum_i a_i K_1^i\big)^T \\
&= \sum_i a_i (K_1^i)^T \\
&= \sum_i a_i K_1^i \\
&= p(K_1) \\
&= K
\end{aligned}
$$

PSD:

$$
\begin{aligned}
z^T K z &= z^T \big(\sum_i a_i K_1^i\big) z \\
&= \sum_i a_i z^T K_1^i z
\end{aligned}
$$

From (e), we know $K_1(x,z)^2$ is a kernel, so $K_1(x,z)^n$ is also a kernel. And, we have $a_i \geq 0$ for any i, then

$$
z^T K z \geq 0
$$

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