An Introduction to Markov Chains

David Zhao

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Introduction

In the year 1906, Russian mathematician Andrey Markov published his first paper introducing a probabilistic model that analyses sequences of events, which would later be designated as *Markov Chains* or *Markov Processes*. His work has been extensively employed in many important fields such as Natural Language Processing (NLP), PageRank, speech recognition, robot localization, etc.

Markov Chain

A Markov chain describes a discrete-time stochastic process that consists of a finite (or infinite) set of states $S = \{s_1, s_2, ..., s_n\}$, which is also known as a *state space*. Initially, the process begins at any one of these states and then successively "steps" from one state to another according to some probability known as a *transition probability*. Formally, if the process is currently in state s_i, then we denote the probability that it transitions to state s_j in one step as p_{ij}.
Furthermore, consider a sequence of random variables $\mathbf{X} = \{X_n \in S | n \in T\}$ where T is a time index set. Then $\mathbf{X}$ is said to be a Markov Chain if, at any given time n, the conditional probability distribution for future states of the process depends only on its present state and not its past states. In mathematical notation, this memoryless property can be expressed as

$$p_{ij} = P(X_{n+1} = s_j | X_n = s_i, X_{n-1}, ..., X_0) = P(X_{n+1} = s_j | X_n = s_i)$$

This is formally known as the *Markov property* or *Markov assumption*

Time Homogeneity

Another important assumption that must be made about the Markov process $\mathbf{X}$ is that the conditional probability distribution of X_{n+1} given X_n is independent of the time n.

$$\forall n \in T, p_{ij} = P(X_{n+1} = s_j | X_n = s_i)$$

In contrast, processes whose transition probabilities actually depend on the time index n are called *time-inhomogenous*. For our purposes, we will only focus on *time-homogeneous* processes.

Transition Probabilities

For a single-step transition, recall that the probability of moving from state s_i to s_j is denoted as p_{ij}. However, this notation quickly becomes strenuous if we wish to determine the evolution of the process after 2 or more steps. For that reason, we introduce a parameter n such that $p_{ij}(n)$ denotes the n-step transition probability, that is, the probability of moving from state s_i to s_j in exactly n steps.

$$p_{ij}(n) = P(X_n = s_j | X_0 = s_i)$$

To illustrate this, consider a Markov chain with state space $S = \{s_1, s_2, s_3\}$ and starting at s_1. If we want to determine the probability of moving to s_3 in exactly 2 steps, then, by the law of total probability, we must consider every possible route from s_1 to s_3.

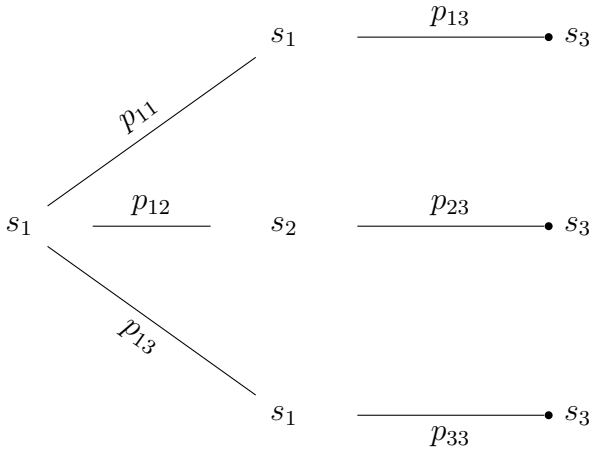


Figure 1: Probability Tree Diagram of Routes From s_1 to s_3

As such, the probability of moving from s_1 to s_3 in 2 steps is

$$p_{13}(2) = p_{11}p_{13} + p_{12}p_{23} + p_{13}p_{33} \tag{1}$$

Notice that the above equation is closely related to a dot product between two probability vectors. To make better use of this observation, let us introduce some useful notation. Let $m = |S|$, then we define the matrix $\mathbf{P} \in \mathbb{R}^{m \times m}$ where the (i, j) entry corresponds to the transition probability p_{ij}. In our example, the *transition matrix* is written as

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{32} & p_{33} \end{pmatrix}$$

Notice in (1) we are dotting the first row of the matrix P with its third column to obtain the probability that the process starting at s_1 finishes at s_3 in 2 steps. We can further generalize this observation for any 2-step transition probability.

$$\begin{aligned} p_{ij}(2) &= P(X_2 = s_j | X_0 = s_i) \\ &= \sum_{k=1}^{3} p_{ik}p_{kj} \\ &= (P^2)_{ij} \end{aligned}$$

It follows that the second power of the transition matrix $\mathbf{P}$ summarizes every 2-step transition probability from s_i to s_j.

$$\mathbf{P}^2 = \begin{pmatrix} p_{11}(2) & p_{12}(2) & p_{13}(2) \\ p_{12}(2) & p_{22}(2) & p_{23}(2) \\ p_{13}(2) & p_{32}(2) & p_{33}(2) \end{pmatrix}$$

Similarly, we can show by induction that the n-step transition probability $p_{ij}(n)$ corresponds to the (i, j) entry of $\mathbf{P}^n$.

$$
\begin{aligned}
p_{ij}(n) &= P(X_n = s_j | X_0 = s_i) \\
&= \sum_{k=1}^{|S|} P(X_n = s_j | X_{n-1} = s_k) P(X_{n-1} = s_k | X_0 = s_i) \\
&= \sum_{k=1}^{|S|} p_{kj} p_{ik}(n-1) \\
&= \sum_{k=1}^{|S|} p_{kj} (\mathbf{P}^{n-1})_{ik} \\
&= (\mathbf{P}^n)_{ij}
\end{aligned}
$$

State Probability Distribution

Suppose a Markov chain $\{X_n : n = 0, 1, 2, ...\}$ has a finite state space $S = \{s_1, s_2, ..., s_k\}$. Then each random variable X_n has a state probability distribution that can be represented using a $1 \times k$ vector. For instance, consider the probability vector $\vec{\pi}$ which contains the starting probability distribution of X_0.

$$
\vec{\pi}^T = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_k \end{pmatrix} = \begin{pmatrix} P(X_0 = s_1) \\ P(X_0 = s_2) \\ \vdots \\ P(X_0 = s_k) \end{pmatrix}
$$

Using the probability distribution of X_0, we can determine the probability distribution of X_1.

$$
\begin{aligned}
P(X_1 = s_j) &= \sum_{i=1}^{|S|} P(X_1 = s_j \cap X_0 = s_i) \\
&= \sum_{i=1}^{|S|} P(X_1 = s_j | X_0 = s_i) P(X_0 = s_i) \\
&= \sum_{i=1}^{|S|} p_{ij} \pi_i \\
&= (\vec{\pi} P)_j
\end{aligned}
$$

Therefore

$$(\vec{\pi}P)^T = \begin{pmatrix} P(X_1 = s_1) \\ P(X_1 = s_2) \\ \vdots \\ P(X_1 = s_k) \end{pmatrix}$$

Likewise, we can use a similar approach to find the probability distribution of X_n given the starting probability distribution of X_0.

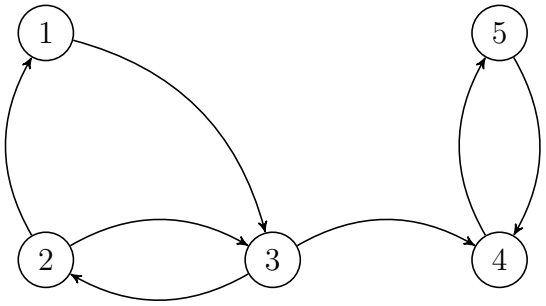
$$\begin{aligned} P(X_n = s_j) &= \sum_{i=1}^{|S|} P(X_n = s_j \cap X_0 = s_i) \\ &= \sum_{i=1}^{|S|} P(X_n = s_j | X_0 = s_i) P(X_0 = s_i) \\ &= \sum_{i=1}^{|S|} p_{ij}(n)\pi_i \\ &= \sum_{i=1}^{|S|} (P^n)_{ij}\pi_i \\ &= (\vec{\pi}P^n)_j \end{aligned}$$

Therefore

$$(\vec{\pi}P^n)^T = \begin{pmatrix} P(X_n = s_1) \\ P(X_n = s_2) \\ \vdots \\ P(X_n = s_k) \end{pmatrix}$$

*State Classification

In any Markov chain, each individual state can either be classified as a *recurrent* or *transient* state. A state s_i is called *recurrent* if the process starting at s_i will eventually return to s_i after some n steps. Conversely, a state s_i is called *transient* if the process starting again at s_i might never return to s_i in the future. Consider the following figure.



In this example, the states s_1, s_2, s_3 are *transient* because if the process starts at any one of these states and moves from s_3 to s_4, then it will never return to any of the transient states located in the left node. However, the states s_4 and s_5 are *recurrent* because the process starting at any one of those states must eventually return to that initial state. More formally, if we define f_{ii} as the probability that a process starting at s_i returns to s_i in some n steps

$$f_{ii} = P(X_n = s_i \,\text{for some}\, n \geq 1 | X_0 = s_i)$$

then s_i is said to be *recurrent* if $f_{ii} = 1$, and *transient* if $f_{ii} < 1$. This means that if a process starts at a *recurrent* state, then there must exist some n steps for which the process returns to that state again. When that happens, the memoryless and time-homogeneous properties ensure that the process will continue to return repeatedly to that recurrent state in the future since $f_{ii} = 1$ every time the process reaches that state. Therefore, the recurrent state will be visited infinitely many times. On the other hand, if the process starts at a *transient* state, then it will continue to return to that state with probability f_{ii} every time it reaches it, until it is no longer able to return anymore with probability $1 - f_{ii}$. Subsequently, we observe that the number of times N that the transient state is visited follows a geometric distribution with "success" probability $1 - f_{ii}$, which is the probability that the process can no longer revisit the transient state s_i. If we count $X_0 = s_i$ as the first visit to s_i, and define N_i as the number of visits to s_i

$$P(N_i = k | X_0 = s_i) = f_{ii}^{k-1}(1 - f_{ii}),\ k \geq 1$$

The expected number of visits to the transient state s_i given that the process starts at s_i can be determined

$$
\begin{aligned}
E[N_i|X_0 = s_i] &= \sum_{k=1}^{\infty} k f_{ii}^{k-1}(1 - f_{ii}) \\
&= (1 - f_{ii}) \sum_{k=1}^{\infty} k f_{ii}^{k-1} \\
&= (1 - f_{ii})(\frac{d}{df_{ii}} \sum_{k=1}^{\infty} f_{ii}^{k}) \\
&= (1 - f_{ii})(\frac{d}{df_{ii}} \frac{f_{ii}}{1 - f_{ii}}) \\
&= (1 - f_{ii}) \frac{1}{(1 - f_{ii})^2} \\
&= \frac{1}{1 - f_{ii}}
\end{aligned}
$$

Although this equation is valid, the value of f_{ii} is fairly difficult to calculate though possible. Alternatively, let I_n be the indicator random variable for the event $X_n = s_i$. In other words, the indicator returns 1 if the event satisfied and 0 if it is not, thereby acting as a "counter" for the number of times the process returns to s_i.

$$
N_i = \sum_{n=0}^{\infty} I_n
$$

Then the expected number of visits to the transient state s_i given that the process starts at s_i can be expressed as such

$$
\begin{aligned}
E[N_i|X_0 = s_i] &= E\left[\sum_{n=0}^{\infty} I_n | X_0 = s_i\right] \\
&= \sum_{n=0}^{\infty} E\left[I_n | X_0 = s_i\right] && \text{By linearity of expectation} \\
&= \sum_{n=0}^{\infty} P(X_n = s_i | X_0 = s_i) \\
&= \sum_{n=0}^{\infty} p_{ii}(n) \\
&= \sum_{n=0}^{\infty} (P^n)_{ii}
\end{aligned}
$$

Therefore, we can use the transition matrix $\mathbf{P}$ to determine the mean number of visits to a transient state.

*Accessibility and Communication

Now that we have defined the two state classes, let us examine their relationship. We say that a state s_j is *accessible* from s_i if there exists some n for which the n-step transition probability $p_{ij}(n) > 0$. Alternatively, if s_j is not accessible from s_i, then $p_{ij} = 0$ for all n. Subsequently, if s_i and s_j are accessible from each other, then they are said to *communicate*, which we denote by $s_i \leftrightarrow s_j$. Communication is considered to be an *equivalence relationship*, meaning that it is *reflexive*, *symmetric*, and *transitive*.

- Reflexive: s_i communicates with itself since s_i is accessible to itself for $n = 0$, i.e. $p_{ii}(0) = 1 > 0$
- Symmetric: If $s_i \leftrightarrow s_j$, then $s_j \leftrightarrow s_i$ by definition of communication
- Transitive: If $s_i \leftrightarrow s_k$ and $s_k \leftrightarrow s_j$, then $s_i \leftrightarrow s_j$. We know by definition of communication that there exists some integers n and m such that $p_{ik}(n) > 0$ and $p_{kj}(m) > 0$. Therefore, for s_j to be *accessible* from s_i, we need to show that $p_{ij}(l) > 0$ for some integer l. Consider the case where $l = n + m$.

$$
\begin{aligned}
p_{ij}(n+m) &= P(X_{n+m} = s_j | X_0 = s_i) \\
&= \sum_{k=1}^{|S|} P(X_{n+m} = s_j \cap X_n = s_k | X_0 = s_i) \\
&= \sum_{k=1}^{|S|} \frac{P(X_{n+m} = s_j \cap X_n = s_k \cap X_0 = s_i)}{P(X_0 = s_i)} \qquad \textit{Baye's Theorem} \\
&= \sum_{k=1}^{|S|} \frac{P(X_{n+m} = s_j | X_n = s_k \cap X_0 = s_i) P(X_n = s_k \cap X_0 = s_i)}{P(X_0 = s_i)} \\
&= \sum_{k=1}^{|S|} P(X_{n+m} = s_j | X_n = s_k \cap X_0 = s_i) P(X_n = s_k | X_0 = s_i) \\
&= \sum_{k=1}^{|S|} P(X_{n+m} = s_j | X_n = s_k) P(X_n = s_k | X_0 = s_i) \qquad \textit{Markov Property} \\
&= \sum_{k=1}^{|S|} p_{kj}(m) p_{ik}(n)
\end{aligned}
$$

By assumption, there exists at least one intermediate state s_k that communicates both with s_i and s_j (i.e. $p_{ik}(n) > 0$ and $p_{kj}(m) > 0$), then the following inequality holds true only for the communicating intermediate states s_k.

$$p_{ij}(n+m) \geq p_{ik}(m)p_{kj}(n) > 0$$

Hence, s_j is accessible from s_i. A similar argument can be made to show that s_i is accessible from s_j.

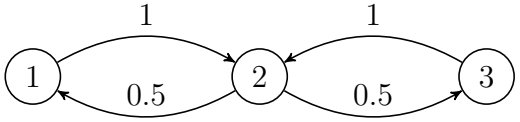
Furthermore, we say that the *communication classes* of a Markov Process are the equivalence classes of states under this relation

Convergence to Equilibrium

By combining individual states in a process, we obtain different types of Markov chains; some of them have stabilizing long term behaviors while others do not. Note that by "stabilizing long term behavior", we mean that the process gradually converges towards a unique steady state distribution, regardless of the initial conditions. This equilibrium distribution will then help us predict the long term proportion of time spent in each state. We will first look at examples of processes that do not have a stable long term behavior.

Periodic Markov Chains

A state in a Markov chain is said to be periodic if the process can only return to that state at multiples of an integer greater than 1. It should also be noted that periodicity is a *class property*, meaning that if one state is periodic, then every other state in its communication class is also periodic. To illustrate, consider the following periodic Markov chain.



If the process starts at state s_1, then we see that it can only return to that state in an even number of steps. The same can be said about s_3, whereas s_2 strictly has a period of 2 steps. Moreover, consider the transition matrix of this process.

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix}$$

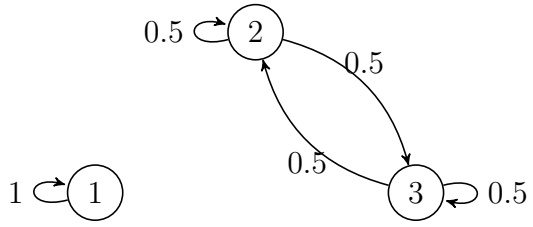
When we take successive powers of the transition matrix, it can be observed that the resulting matrix alternates between $P^{n_{odd}}$ and $P^{n_{even}}$.

$$\mathbf{P}^{n_{odd}} = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix} ; \mathbf{P}^{n_{even}} = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{pmatrix}$$

The n-step transition matrix $\mathbf{P}^n$ does not converge to some equilibrium matrix σ as $n \to \infty$. Therefore, the long term state distribution of a periodic process cannot be constant

Reducible Markov Chains

A Markov chain is said to be *reducible* if it has more than one communication class. This implies that not every state in the state space is able to communicate with each other. To illustrate, consider the following reducible Markov chain.



In this example, there are two communication classes, namely $\{s_1\}$ and $\{s_2, s_3\}$. As a result, the long term behavior of the process depends entirely on the initial state of the process. For instance, if the process starts at s_1, then it will stay there forever, but if it starts either at s_2 or s_3, then the process will stay in that communication class forever. Therefore, the stationary probability distribution of a reducible Markov chain depends on its starting distribution $\vec{\pi}$ (More on this in the next section).

Regular Markov Chains

We now look at Markov chains that do possess an equilibrium distribution, also known as *regular* Markov chains. Contrary to the previous two types of Markov chains we have seen, a regular Markov chain must be *aperiodic*, *irreducible*, and also *positive recurrent*. In other words, this means that the process must only have states that are visited aperiodically, one communication class for the entire state space, and a mean number of visits to every state that is infinite.

One easy way to check if a Markov chain obeys these conditions is by finding some power of the n-step transition matrix $\mathbf{P}^n$ for which the matrix contains only strictly positive entries. As such, for that number n, every n-step transition probability $p_{ij}(n)$ has a strictly positive value, thereby demonstrating that the process is *irreducible* since every state communicates with each other and belongs to a single communication class. Moreover, this method also works to show that the Markov chain is aperiodic since every n-step self-transition probability is greater than 0 (i.e. $p_{ii}(n) > 0$). This means that the process must again be able to perform this self-transition in the next step $p_{ii}(n+1) > 0$, thereby breaking the periodicity of the process. Finally, knowing that the process is irreducible, this implies that every state in the process must be recurrent such that the mean number of visits to every state is infinite. Hence, positive recurrence is a property that is always present in a finite irreducible Markov chain. In brief, regular Markov chains are the only processes for which the steady-state distribution both exists and is nontrivial. We will examine how these equilibrium distributions can be determined in the next section.

Steady-State Distribution

Recall from a previous section that the state probability distribution of X_n given the starting distribution X_0 can be expressed as such

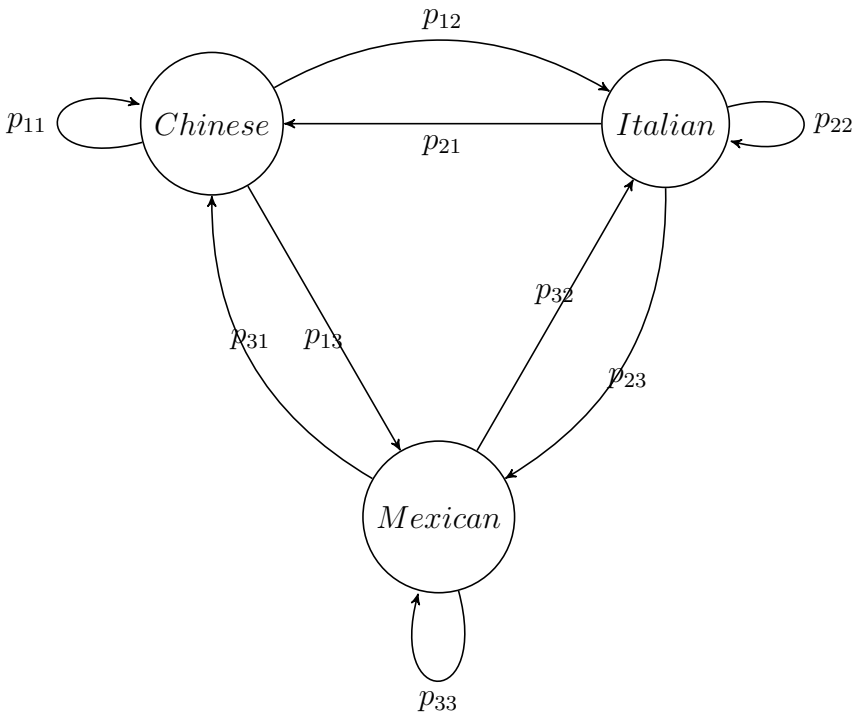
$$(\vec{\pi}P^n)^T = \begin{pmatrix} P(X_n = s_1) \\ P(X_n = s_2) \\ \vdots \\ P(X_n = s_k) \end{pmatrix}$$

As mentioned, for regular Markov chains, we can expect to find a stationary distribution as $n \to \infty$. However, instead of computing large powers of the transition matrix, we can solve the following matrix equation to find the equilibrium distribution $\vec{\pi^*}$

$$\vec{\pi}^* = \vec{\pi}^* P$$
$$\vec{\pi}^*(I_k - P) = \vec{0}$$

where I_k is a k-sized identity matrix. As a result, the steady-state distribution vector is simply a left eigenvector of the transition matrix with corresponding eigenvalue 1.

*Example 1: Restaurants Every day, a hungry customer eats at one of three restaurants, each serving a different cuisine: Chinese, Italian, and Mexican, conveniently denoted as states 1, 2, and 3 respectively. The customer's restaurant choice for the next day is probabilistically determined by the restaurant he visits on the current day (see Figure below).



For this Markov chain to be regular, we will consider the following transition matrix, which is also the default matrix implemented in the Streamlit simulation.

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.6 & 0.3 & 0.1 \\ 0.4 & 0.5 & 0.1 \end{pmatrix}$$

Since the one-step transition matrix already has only strictly positive entries, this process is considered to be regular and therefore it must have an equilibrium distribution.

In the Python code, the theoretical equilibrium distribution is found by computing the left eigenvector of the transition matrix with corresponding

eigenvalue 1, which is then compared to the experimental distribution that is found by computing successive powers of the transition matrix. As the process moves forward in steps, it can be observed that the theoretical and experimental distributions start to match as the initial condition of the process quickly becomes insignificant.

Absorbing Markov Chain

We now begin our study of *absorbing* Markov chains, which are almost polar opposites of *regular* Markov chains. To contextualize, if a process is in state s_i and is unable to leave it, then that state is considered to be an *absorbing* state. Additionally, if it is possible to go from any *non-absorbing* state to an absorbing state in one or more steps, then the entire process is considered to be an *absorbing* Markov chain. Notice that *absorbing* states are also *recurrent*, whereas non-absorbing states are *transient* since the process can be absorbed starting from any non-absorbing state.

Canonical Form

The canonical form is a representation of the transition matrix $\mathbf{P}$ into 4 smaller matrices. Suppose the transition matrix of an absorbing Markov chain has t transient states and r absorbing states. Then the canonical form of the transition matrix $\mathbf{P}$ is

$$P = \left(\begin{array}{c|c} Q & R \\ \hline 0 & I_r \end{array}\right)$$

where Q is a $t \times t$ matrix, 0 is an $r \times t$ zero matrix, R is a $t \times r$ nonzero matrix, and I_r is a $r \times r$ matrix. Since the matrix Q only contains transient states, it summarizes the transition probabilities between non absorbing states. The matrix R contains transient and recurrent states, thereby representing the transition probabilities from non-absorbing states to absorbing ones. Next, we know that it is impossible to go from an absorbing state to a non-absorbing state, hence the 0 matrix. And finally, the identity matrix I_r contains the self-transition probabilities for the absorbing states which are obviously 1.

Moreover, a simple exercise shows that the nth power of the canonical form matrix yields

$$P^n = \left(\begin{array}{c|c} Q^n & R + QR + Q^2R + ... + Q^{n-1}R \\ \hline 0 & I_r \end{array}\right)$$

$$P^n = \left(\begin{array}{c|c} Q^n & \sum_{k=1}^{n} Q^{k-1} R \\ \hline 0 & I_r \end{array} \right)$$

Notice that we obtain the n-step transition probabilities between transient states under Q^n. Even more interestingly, we observe that the top-right matrix expression is closely related to a geometric distribution. Now, consider the n-step transition matrix as n approaches infinity

$$\lim_{n \to \infty} P^n = \left(\begin{array}{c|c} \lim_{n \to \infty} Q^n & \sum_{k=1}^{\infty} Q^{k-1} R \\ \hline 0 & I_r \end{array} \right)$$

First, let us examine the top-left expression (i.e. $lim_{n \to \infty} Q^n$). Since it is possible to go from any non-absorbing state to an absorbing one, this means that the sum of every row in Q must be smaller than 1. As a result, the largest eigenvalue of Q must also be smaller than 1, and therefore $Q^n \to 0$ as $n \to \infty$. In other words, the probability that an absorbing Markov chain will be absorbed is 1.

As for the top-right expression (i.e. $\lim_{n \to \infty} \sum_{k=1}^{\infty} Q^{k-1} R$), it contains the absorbing probabilities of the Markov chain, which will be further discussed in the next section.

The Fundamental Matrix

For any absorbing Markov chain, the *fundamental matrix* denoted N provides the expected number of visits to a non-absorbing state.

$$N = \sum_{n=0}^{\infty} Q^n = (I - Q)^{-1}$$

More specifically, the (i,j) entry of the fundamental matrix N gives the expected number of visits to the transient state s_j given that the process starts at the transient state s_i. This can be shown by starting from the definition of expectation and by defining the indicator random variable I_n for the event $\{X_n = s_j\}$. Recall that a very similar derivation was already shown on page 7.

$$
\begin{aligned}
E[N_j|X_0 = s_i] &= E\left[\sum_{n=0}^{\infty} I_n|X_0 = s_i\right] \\
&= \sum_{n=0}^{\infty} E\left[I_n|X_0 = s_i\right] \\
&= \sum_{n=0}^{\infty} P(X_n = s_j|X_0 = s_i) \\
&= \sum_{n=0}^{\infty} q_{ij}(n) \\
&= \sum_{n=0}^{\infty} (Q^n)_{ij} \\
&= N_{ij}
\end{aligned}
$$

From these above equations, we can further simplify our limiting transition matrix

$$
\lim_{n\to\infty} P^n = \left(\begin{array}{c|c} 0 & NR \\ \hline 0 & I_r \end{array}\right)
$$

If we let B = NR, then the (i,j) entry of matrix B gives the probability that the the process will be absorbed at the absorbing state s_j given that it started at the transient state s_i. To show that the is true, by the law of total probability, we must consider every possible route from s_i to s_j through intermediate transient states s_k, and also for all possible number of steps.

$$
\begin{aligned}
B_{ij} &= \sum_n \sum_k q_{ik}(n) r_{kj} \\
&= \sum_n \sum_k (Q^n)_{ik} r_{kj} \\
&= \sum_k n_{ik} r_{kj} \\
&= (NR)_{ij}
\end{aligned}
$$

Finally, we can even use the fundamental matrix N to determine the expected time to absorption given that the process starts at a transient state s_i. To show this, let t be a column vector such that its entries t_i give the expected time to absorption given that the process starts at s_i. Then

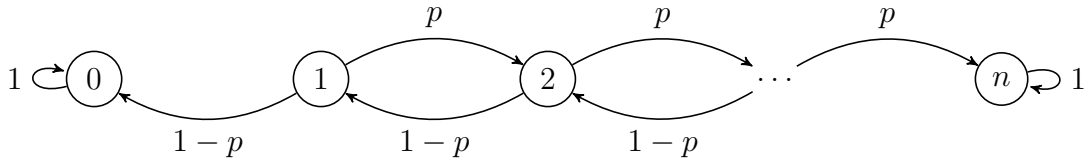
$$t = N1$$

where *1* is a column vector of 1s. To visualize the calculation,

$$t_i = \sum_k N_{ik}$$

Therefore, we are adding the expected visits to all non-absorbing states given that the process starts at s_i, which is also indicative of the total expected number of steps until the process is absorbed.

*Example 2: Gambler's Ruin A gambler has an initial wealth of **k** dollars and bets 1$ every round with probability **p** of winning each round. The gambler keeps playing until his wealth is either emptied or reaches a value **n** (see Figure 2).



The states s_0 and s_n are considered *absorbing* whereas the states $s_1, s_2, ..., s_{n-1}$ are *transient*. Furthermore, the transition probabilities can be summarized using a transition matrix **P**

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1-p & 0 & p & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-p & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1-p & 0 & p & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix}$$

For the sake of simplicity, we will consider the following transition matrix that contains 3 transient states and 2 absorbing states.

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The canonical form of the transition matrix

$$A = \left(\begin{array}{ccc|cc} 0 & 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Therefore

$$\mathbf{Q} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix} ; \mathbf{R} = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \\ 0 & 1/2 \end{pmatrix}$$

The fundamental matrix is given as

$$N = (I - Q)^{-1} = \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix}$$

We now find the t column vector whose entries t_i contain the expected time to absorption given that the process starts at s_i

$$\mathbf{t} = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$$

Now that we have the theoretical results, we can use the Streamlit simulation to produce experimental results for comparison. As initial conditions, we set the target to 4 and the initial wealth to 2. Therefore, the expected number of steps until the process is absorbed is 4 (according to the second entry of the t column vector).

For a sample size of 101, it was experimentally found that the sample mean was 3.9802 and the standard deviation was 2.4248. As an exercise, we will construct a 95% confidence interval for the average number of steps

to absorption. Since the population variance is unknown, we use the t-distribution to find the critical t value: $t_{0.025,100} = 1.984$. Then, we construct the confidence interval

$$\begin{aligned} \overline{x} - t_{0.025,100}\frac{s}{\sqrt{n}} &\leq \mu \leq \overline{x} + t_{0.025,100}\frac{s}{\sqrt{n}} \\ 3.9802 - 1.984\frac{2.4248}{\sqrt{101}} &\leq \mu \leq 3.9802 + 1.984\frac{2.4248}{\sqrt{101}} \\ 3.5015 &\leq \mu \leq 4.4589 \end{aligned}$$

With repeated sampling, we are 95% confident that the true mean number of games that the gambler must play until they lose or win lies between 3.5015 and 4.4589. Since $4 \in$ C.I., we have sufficient evidence to believe that the experimental results match the theoretical ones.

References

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