Assignment 2 - Solutions

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PC3274A

Mathematical Methods in Physics II · November 11, 2025

1 PC3274A Assignment 2 - Solutions

1a: Affine Transformations - Part (a)

We need to show that the curve in the o′-frame takes the form: $\vec{r}'(t) = B^{-1} \cdot A(t) \cdot \vec{r}_0 - B^{-1}(\theta) \cdot \vec{c}$

Solution:

Using the affine structure property: $\Omega(o, \sigma(t)) = \Omega(o, o') + \Omega(o', \sigma(t))$

Rearranging: $\Omega(o', \sigma(t)) = \Omega(o, \sigma(t)) - \Omega(o, o')$

In the o-frame:

- $\Omega(o, \sigma(t)) = \vec{r}(t) = A(t) \cdot \vec{r}_0$
- $\Omega(o, o') = \vec{c} = c^1\vec{e}_1 + c^2\vec{e}_2$

Therefore: $\Omega(o', \sigma(t)) = A(t) \cdot \vec{r}_0 - \vec{c}$

Now we need to express this in the o′-frame basis. The basis transformation is: $\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} = B^T(\theta)\begin{pmatrix} \vec{f}_1 \\ \vec{f}_2 \end{pmatrix}$

where $B(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Any vector $\vec{v} = v^1\vec{e}_1 + v^2\vec{e}_2$ in the o-frame becomes $\vec{v} = v'^1\vec{f}_1 + v'^2\vec{f}_2$ in the o′-frame where: $\begin{pmatrix} v'^1 \\ v'^2 \end{pmatrix} = B^{-1}(\theta)\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$

Applying this transformation: $\vec{r}'(t) = B^{-1}(\theta) \cdot [A(t) \cdot \vec{r}_0 - \vec{c}] = B^{-1}(\theta) \cdot A(t) \cdot \vec{r}_0 - B^{-1}(\theta) \cdot \vec{c}$

✓ **Result proven.**

1b: Affine Transformations - Part (b)

We need to evaluate the tangent vectors and show: $\vec{\vec{V}} = B(\theta) \cdot \vec{\vec{V}}'$

Solution:

In the o-frame: $\vec{\vec{V}} = \frac{d\vec{r}}{dt} = \frac{d}{dt}[A(t) \cdot \vec{r}_0] = A'(t) \cdot \vec{r}_0$

where: $A'(t) = \begin{pmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{pmatrix}$

So: $\vec{\vec{V}} = \begin{pmatrix} -x_0^1 \sin t + x_0^2 \cos t \\ -x_0^1 \cos t - x_0^2 \sin t \end{pmatrix}$

In the o′-frame: $\vec{\vec{V}}' = \frac{d\vec{r}'}{dt} = B^{-1}(\theta) \cdot A'(t) \cdot \vec{r}_0$

To verify the relation: $B(\theta) \cdot \vec{\vec{V}}' = B(\theta) \cdot B^{-1}(\theta) \cdot A'(t) \cdot \vec{r}_0 = I \cdot A'(t) \cdot \vec{r}_0 = \vec{\vec{V}}$

✓ **Relation verified.**

2a: Integral Curves - Part (a)
Given vector field: $\bar{V} = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$ Show that the integral curve through (x_0^1, x_0^2) is: • $x^1(t) = x_0^1 \cos t + x_0^2 \sin t$ • $x^2(t) = -x_0^1 \sin t + x_0^2 \cos t$

Solution:

The integral curves satisfy the system: $\frac{dx^1}{dt} = x^2, \quad \frac{dx^2}{dt} = -x^1$

Taking the derivative of the first equation: $\frac{d^2x^1}{dt^2} = \frac{dx^2}{dt} = -x^1$

This gives us the harmonic oscillator equation: $\frac{d^2x^1}{dt^2} + x^1 = 0$

General solution: $x^1(t) = A\cos t + B\sin t$

Using initial conditions:

- At $t = 0$: $x^1(0) = x_0^1 \Rightarrow A = x_0^1$
- From $\frac{dx^1}{dt} = x^2$: At $t = 0$: $x^2(0) = x_0^2 = B$

Therefore: $x^1(t) = x_0^1 \cos t + x_0^2 \sin t$

From $\frac{dx^1}{dt} = x^2$: $x^2(t) = -x_0^1 \sin t + x_0^2 \cos t$

✓ **Solution verified by substitution into the original system.**

2b: Integral Curves - Part (b)
Show that integral curves are circles of radius $\sqrt{(x_0^1)^2 + (x_0^2)^2}$ around the origin.

Solution:

Computing the distance from the origin: $(x^1(t))^2 + (x^2(t))^2 = (x_0^1 \cos t + x_0^2 \sin t)^2 + (-x_0^1 \sin t + x_0^2 \cos t)^2$

Expanding: $= (x_0^1)^2 \cos^2 t + 2x_0^1 x_0^2 \sin t \cos t + (x_0^2)^2 \sin^2 t \quad + (x_0^1)^2 \sin^2 t - 2x_0^1 x_0^2 \sin t \cos t + (x_0^2)^2 \cos^2 t$

$= (x_0^1)^2(\cos^2 t + \sin^2 t) + (x_0^2)^2(\sin^2 t + \cos^2 t)$

$= (x_0^1)^2 + (x_0^2)^2 = \text{constant}$

✓ **The integral curves are circles with radius $r = \sqrt{(x_0^1)^2 + (x_0^2)^2}$**

3a: Vector Field Compositions - Part (a)
Show that $W \bigcirc V$ does not satisfy the Leibniz rule.

Solution:

For $W \bigcirc V$ to be a vector field, it must satisfy: $(W \bigcirc V)(fg) = f(W \bigcirc V)(g) + g(W \bigcirc V)(f)$

Computing the left side: $(W \bigcirc V)(fg) = W(V(fg)) = W(fV(g) + gV(f))$

Using linearity and Leibniz rule for W: $= W(f)V(g) + fW(V(g)) + W(g)V(f) + gW(V(f))$

The right side would be: $f(W \bigcirc V)(g) + g(W \bigcirc V)(f) = fW(V(g)) + gW(V(f))$

Comparing: The extra terms $W(f)V(g) + W(g)V(f)$ appear on the left side.

✓ **Therefore, $W \bigcirc V$ violates the Leibniz rule and is not a vector field.**

3b: Vector Field Compositions - Part (b)
Show that the Lie bracket $[V, W] = V \bigcirc W - W \bigcirc V$ satisfies the Leibniz rule.

Solution:

$[V, W](fg) = (V \bigcirc W)(fg) - (W \bigcirc V)(fg)$

$= V(W(fg)) - W(V(fg))$

$= V(fW(g) + gW(f)) - W(fV(g) + gV(f))$

Expanding using the Leibniz rule: $= V(f)W(g) + fV(W(g)) + V(g)W(f) + gV(W(f))$ $-[W(f)V(g) + fW(V(g)) + W(g)V(f) + gW(V(f))]$

Rearranging terms: $= f[V(W(g)) - W(V(g))] + g[V(W(f)) - W(V(f))]$ $+[V(f)W(g) - W(f)V(g)] + [V(g)W(f) - W(g)V(f)]$

The bracketed terms cancel due to symmetry considerations, leaving: $= f[V, W](g) + g[V, W](f)$

✓ **The Lie bracket satisfies the Leibniz rule and is therefore a vector field.**

4a: Tensor Products - Part (a)
$(k_1 + k_2) \otimes k_3 = k_1 \otimes k_3 + k_2 \otimes k_3$

Solution:

Proof: For any vectors $v, w \in V$: $[(k_1 + k_2) \otimes k_3](v, w) = (k_1 + k_2)(v) \cdot k_3(w)$ $= [k_1(v) + k_2(v)] \cdot k_3(w) = k_1(v) \cdot k_3(w) + k_2(v) \cdot k_3(w) = (k_1 \otimes k_3)(v, w) + (k_2 \otimes k_3)(v, w)$

✓ **Property proven.**

4b: Tensor Products - Part (b)
$k_1 \otimes (k_2 + k_3) = k_1 \otimes k_2 + k_1 \otimes k_3$

Solution:

Proof: $[k_1 \otimes (k_2 + k_3)](v, w) = k_1(v) \cdot (k_2 + k_3)(w)$ $= k_1(v) \cdot [k_2(w) + k_3(w)]$ $= k_1(v) \cdot k_2(w) + k_1(v) \cdot k_3(w) = (k_1 \otimes k_2)(v, w) + (k_1 \otimes k_3)(v, w)$

✓ **Property proven.**

4c: Tensor Products - Part (c)
$(\alpha k_1) \otimes k_2 = \alpha(k_1 \otimes k_2)$

Solution:

Proof: $[(\alpha k_1) \otimes k_2](v, w) = (\alpha k_1)(v) \cdot k_2(w) = \alpha k_1(v) \cdot k_2(w) = \alpha[k_1(v) \cdot k_2(w)] = \alpha[(k_1 \otimes k_2)(v, w)]$

✓ **Property proven.**

4d: Tensor Products - Part (d)
$k_1 \otimes (\alpha k_2) = \alpha(k_1 \otimes k_2)$

Solution:

Proof: $[k_1 \otimes (\alpha k_2)](v, w) = k_1(v) \cdot (\alpha k_2)(w) = k_1(v) \cdot \alpha k_2(w) = \alpha[k_1(v) \cdot k_2(w)] = \alpha[(k_1 \otimes k_2)(v, w)]$

✓ **Property proven.**

5a: Duality and Metric Tensors - Part (a)
Given $v = \sum_i v^i \frac{\partial}{\partial x^i}$ and $g = \sum_{i,j} g_{ij} dx^i \otimes dx^j$, find $\tilde{v} = \hat{g}(v)$.

Solution:

The dual covector is defined by $\tilde{v}(w) = g(v, w)$ for any vector w.

For $w = \sum_k w^k \frac{\partial}{\partial x^k}$: $g(v, w) = \sum_{i,j} g_{ij} v^i w^j$

Therefore: $\tilde{v} = \sum_j \left(\sum_i g_{ij} v^i\right) dx^j = \sum_j v_j dx^j$

where $v_j = \sum_i g_{ij} v^i$ (lowering the index).

5b: Duality and Metric Tensors - Part (b)
Transform $\tilde{v}$ to the o′-frame.

Solution:

Under the transformation $x^i = \sum_j a^i_j y^j + c^i$: $dx^i = \sum_j a^i_j dy^j$

Therefore: $\tilde{v} = \sum_i v_i dx^i = \sum_i v_i \sum_j a^i_j dy^j = \sum_j \left(\sum_i a^i_j v_i\right) dy^j$

$\tilde{v}' = \sum_j v'_j dy^j$ where $v'_j = \sum_i a^i_j v_i$

5c: Duality and Metric Tensors - Part (c)
Transform vector v to the o′-frame.

Solution:

The basis vectors transform as: $\frac{\partial}{\partial x^i} = \sum_j \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}$

If $\left(b^j_i\right)$ is the inverse of $\left(a^i_j\right)$, then: $v' = \sum_j \left(\sum_i b^j_i v^i\right) \frac{\partial}{\partial y^j} = \sum_j v'^j \frac{\partial}{\partial y^j}$

where $v'^j = \sum_i b^j_i v^i$.

5d: Duality and Metric Tensors - Part (d)
Transform the metric tensor to the o′-frame.

Solution:

The metric tensor transforms as: $g' = \sum_{k,l} g'_{kl} dy^k \otimes dy^l$

where: $g'_{kl} = \sum_{i,j} a^i_k a^j_l g_{ij}$

This is the standard tensor transformation law for a (0,2) tensor.

5e: Duality and Metric Tensors - Part (e)
Evaluate $\tilde{v}' = \hat{g}'(v')$.

Solution:

Using the transformed metric: $\tilde{v}' = \sum_l \left(\sum_k g'_{kl} v'^k\right) dy^l$

Substituting the transformations: $= \sum_l \left(\sum_{k,i,j,m} a^i_k a^j_l g_{ij} b^k_m v^m\right) dy^l$

Using $\sum_k a^i_k b^k_m = \delta^i_m$: $= \sum_l \left(\sum_{i,j} a^j_l g_{ij} v^i\right) dy^l = \sum_l v'_l dy^l$

5f: Duality and Metric Tensors - Part (f)
Yes, the expressions from parts (b) and (e) are the same.

5g: Duality and Metric Tensors - Part (g)
Key Takeaway:

Solution:

Key Takeaway: The metric-induced duality between vectors and covectors is coordinate-independent. The diagram commutes:

```
v   ----F----> v'
|              |
ĝ              ĝ'
↓              ↓
ṽ   ----F----> ṽ'
```

This demonstrates that:
1. The musical isomorphisms (♭ and ♯) are natural operations
2. The metric structure is geometrically intrinsic
3. Affine transformations preserve the duality relationship
4. Physical quantities can be consistently represented as either vectors or covectors

This is fundamental to the covariant formulation of physics, particularly in general relativity where the metric plays a central role in relating different types of tensorial quantities.