INDIAN INSTITUTE OF TECHNOLOGY MADRAS

B

Roll No. | M | M | 1 | 6 | B | 0 | 2 | 3 |

Name : H. Vishal

Total No. of Pages

Quiz I ✓ Quiz II/ Mid-Sem ☐ End-Semester ☐ Make-up ☐ Date : Feb 14, 2020

Semester & Degree : IDDD DS 8th Sem Course No. MA 5470 Part :

Question No.	1	2	3	4	5	6	7	8	9	10
Marks										

11	12	13	14	15	16	17	18	19	20	Total
										14

Answer on both sides of the paper including the space below

1. $Ax = b$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 5 \\ 4 & 1 & 6 \end{pmatrix}$$

$S = \{3, 5, 6\}$

$P = \{1, 2, 3\}$

$$\max\left\{\frac{|a_{p_i,1}|}{|S_{p_i}|}\right\} = \max\left\{\frac{1}{3}, \frac{3}{5}, \frac{4}{6}\right\} = \frac{4}{6}$$

corresponds to $i = 3$

$\Rightarrow P = \{3, 2, 1\}$

$\Rightarrow$ Interchange row 1 and row 3 ?

$$\Rightarrow A \sim \begin{pmatrix} 4 & 1 & 6 \\ 3 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & 1 & 6 \\ 3 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 4 & 1 & 6 \\ 0 & 5/4 & 1/2 \\ 0 & 7/4 & 3/2 \end{pmatrix}$$

$R_2 \leftarrow R_2 - \frac{3}{4}R_1$

$R_3 \leftarrow R_3 - \frac{R_1}{4}$

$$\Rightarrow \max\left\{ \frac{|a_{pi,2}|}{|S_{pi}|} \right\} = \max\left\{ \frac{1}{4}, \frac{7}{24} \right\} = \frac{7}{24}$$

corresponds to $i = 3$

$\Rightarrow$ Interchange row 2 and row 3

$$\Rightarrow A \sim \begin{pmatrix} 4 & 1 & 6 \\ 0 & 7/4 & 1/2 \\ 0 & 5/4 & 1/2 \end{pmatrix}$$

$$\Rightarrow \text{second pivot row } r = \begin{pmatrix} 0 \\ 7/4 \\ 1/2 \end{pmatrix}^t$$

(2) Let Gauss elimination be performed on $A \in \mathbb{R}^{n \times n}$

No. of divisions : n (to find pivot row)

No. of computations on A for row elimination: $n(n-1)$

$\Rightarrow$ total $= n(n-1) + n = n^2$

⇒ Summing over all instances,

Total no of operations on A $= n^2 + (n-1)^2 + (n-2)^2 + \dots 2^2$

$= \sum n^2 - 1$

No. of operations on b: $\sum (n-1) = \frac{n(n-1)}{2}$

No. of operations for back substitution: $\sum n = \frac{n(n+1)}{2}$

⇒ Total operational count for gauss elimination with scaled row pivoting

$$= \sum n^2 - 1 + \sum (n-1) + \sum n$$

$$= \frac{n(n+1)(2n+1)}{6} - 1 + \frac{n^2 - n}{2} + \frac{n^2 + n}{2}$$

$$\frac{(n^2+n)(2n+1)}{6} - 1 + \frac{2n^2}{2}$$

$2n^3 + n^2 + 2n^2 + 1$

$$\sim O\left(\frac{2n^3 + 3n^2 + 1}{6} - 1 + n^2\right)$$

$$\sim O\left(\frac{n^3}{3} + \frac{3n^2}{2}\right)$$

(3)

$$C = \begin{pmatrix} 26 & 12 & 42 \\ 12 & 9 & 22 \\ 42 & 22 & 70 \end{pmatrix}$$

$$\begin{pmatrix} 26 & 12 & 42 \\ 12 & 9 & 22 \\ 42 & 22 & 70 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$\Rightarrow u_{11} = 26, \quad u_{12} = 12, \quad u_{13} = 42$

$l_{21} u_{11} = 12 \Rightarrow l_{21} = \frac{12}{26} = \frac{6}{13}$

$l_{31} u_{11} = 42 \Rightarrow l_{31} = \frac{42}{26} = \frac{21}{13}$

$l_{21} u_{12} + u_{22} = 9$

$\Rightarrow \left(\frac{6}{13}\right)(12) + u_{22} = 9 \Rightarrow u_{22} = \frac{45}{13}$

$l_{21} u_{13} + u_{23} = 22$

$\Rightarrow \left(\frac{6}{13}\right) 42 + u_{23} = 22 \Rightarrow u_{23} = \frac{34}{13}$

$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 70$

$\Rightarrow \left(\frac{21}{13}\right) 42 + l_{32} * \left(\frac{34}{13}\right) + u_{33} = 70$

$$l_{31}u_{12} + l_{32}u_{22} = 22$$

$$\Rightarrow \left(\frac{21}{13}\right)42 + l_{32}\left(\frac{45}{13}\right) = 22$$

$$\Rightarrow l_{32} = \frac{34}{45}$$

$$\Rightarrow \left(\frac{21}{13}\right)42 + \left(\frac{34}{45}\right)\left(\frac{34}{13}\right) + u_{33} = 70$$

$$\Rightarrow u_{33} = \frac{8}{45}$$

$$\Rightarrow D^{1/2} = \begin{pmatrix} \sqrt{u_{11}} & 0 & 0 \\ 0 & \sqrt{u_{22}} & 0 \\ 0 & 0 & \sqrt{u_{33}} \end{pmatrix} = \begin{pmatrix} \sqrt{26} & 0 & 0 \\ 0 & \sqrt{\frac{45}{13}} & 0 \\ 0 & 0 & \sqrt{\frac{8}{45}} \end{pmatrix}$$

$$\Rightarrow \tilde{L} = LD^{1/2} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} \sqrt{26} & 0 & 0 \\ 0 & \sqrt{\frac{45}{13}} & 0 \\ 0 & 0 & \sqrt{\frac{8}{45}} \end{pmatrix}$$

$$\Rightarrow l_{23} = 0, \quad l_{32} = l_{32}\sqrt{\frac{45}{13}}$$

$$= \frac{34}{45}\sqrt{\frac{45}{13}} = \frac{34}{\sqrt{13\times 45}} = 1.4$$

(5)

(4)

$$\begin{pmatrix} 1 & 2 & 6 \\ 9 & 6 & 1 \\ 3 & 7 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 16 \\ 12 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$Dx = (D - A)\,x + b$$

$$\Rightarrow x^{[k]} = (I - D^{-1}A)\,x^{[k-1]} + D^{-1}b$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \left(I - D^{-1}A\right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + D^{-1}b$$

$$I - D^{-1}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 & 6 \\ 9 & 6 & 1 \\ 3 & 7 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 6 \\ 3/2 & 1 & 1/6 \\ 3/2 & 7/2 & 1 \end{pmatrix}$$

$$x_1 + 2x_2 + 6x_3 = 9 \quad -(1)$$

$$\Rightarrow x_1 = 9 - 2x_2 - 6x_3$$

$$9x_1 + 6x_2 + x_3 = 16 \quad -(2)$$

$$\Rightarrow x_2 = \frac{16}{6} - \frac{9}{6}x_1 - \frac{x_3}{6}$$

$$3x_1 + 7x_2 + 2x_3 = 12$$

$$\Rightarrow x_3 = 6 - \frac{3}{2}x_1 - \frac{7}{2}x_2 \quad -(3)$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & -2 & -6 \\ -9/6 & 0 & -1/6 \\ -3/2 & -7/2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 9 \\ 16/6 \\ 6 \end{pmatrix}$$

$$(I - D^{-1}A)$$

$$\Rightarrow x_1^{[1]} = (I - D^{-1}A)\, x^{[0]} + D^{-1}b$$

$$\Rightarrow \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} = \begin{pmatrix} 5 \\ 11/6 \\ 7/2 \end{pmatrix}$$

$$\begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(3)} \end{pmatrix} = \begin{pmatrix} I - D^{-1}A \end{pmatrix} \begin{pmatrix} x^{(1)} \end{pmatrix} + D^{-1}b$$

$$= \begin{pmatrix} -23/6 \\ 775/291 \\ 999/167 \end{pmatrix} \begin{pmatrix} -47/3 \\ -65/12 \\ -95/12 \end{pmatrix}$$

$$\begin{pmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{pmatrix} = \begin{pmatrix} 9 - 2\left(\frac{-65}{12}\right) - 6\left(\frac{-95}{12}\right) \\ \frac{16}{6} - \frac{1}{6}\left(\frac{-9 \times 23}{6} + \frac{999}{167}\right) \\ 6 - \frac{1}{2}\left(\frac{3 \times -23}{6} + 7 \times \frac{775}{291}\right) \end{pmatrix}$$

$$= \begin{pmatrix} -32.21 \\ 2.672 \\ 5.29 \end{pmatrix} = \begin{pmatrix} 67.33 \\ 27.48 \\ 48.45 \end{pmatrix}$$