MA5470 Numerical Analysis

End Semester

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(5)

$$f'(x) = \frac{1}{12\Delta x}\Big(-f(x+2\Delta x) + 8f(x+\Delta x) - 8f(x-\Delta x) + f(x-2\Delta x)\Big)$$

$$= \frac{1}{12\Delta x}\Big(-\cancel{f(x)} - \cancel{2\Delta x f'(x)} - \cancel{2\Delta x^2 f''(x)} - \frac{8}{6}\Delta x^3 f'''(x) + o(\Delta x^4)$$

$$+\cancel{8f(x)} + \cancel{8\Delta x f'(x)} + \cancel{32\Delta x^2 f''(x)} + \frac{512}{6}\Delta x^3 f'''(x) + o(\Delta x^4)$$

$$-\cancel{8f(x)} + \cancel{8\Delta x f'(x)} - \cancel{32\Delta x^2 f''(x)} + \frac{512}{6}\Delta x^3 f'''(x) + o(\Delta x^4)$$

$$+\cancel{f(x)} - \cancel{2\Delta x f'(x)} + \cancel{2\Delta x^2 f''(x)} - \frac{8}{6}\Delta x^3 f'''(x) + o(\Delta x^4)\Big)$$

Leading error term

$$e = \frac{\Delta x^3 f'''(x)}{12\Delta x}\left(2\left(\frac{512}{6}\right) - 2\left(\frac{8}{6}\right)\right) = \frac{168\,\Delta x^3 f'''(\xi)}{12\Delta x}$$

$$= 14\,\Delta x^2 f'''(\xi)$$

(6) Since all polynomials of deg $<=2$ are linear comb of $1, x, x^2$, they can be used as stencils

$$\Rightarrow \quad c_0 + c_1 + c_2 = \int_0^2 1\,dx = 2 \quad - (1)$$

$$c_1 + 2c_2 = \int_0^2 x\,dx = 2 \quad - (2)$$

$$c_1 + 4c_2 = \int_0^2 x^2\,dx = \frac{8}{3} \quad - (3)$$

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$\Rightarrow 2c_2 = \frac{8}{3} - 2 = \frac{2}{3} \Rightarrow c_2 = \frac{1}{3}$

$\Rightarrow c_1 = 2 - \frac{2}{3} = \frac{4}{3}$

$\Rightarrow c_0 = 2 - \frac{1}{3} - \frac{4}{3} = \frac{1}{3}$

$\Rightarrow c_0 = \frac{1}{3}, \quad c_1 = \frac{4}{3}, \quad c_2 = \frac{1}{3}$

(7) Let p be a polynomial of degree at most n that interpolates f at $x_0, x_1, \dots x_n$.

$\Rightarrow$ There exists ξ in (a,b) s.t

$$f(x_n) - p(x_n) = \frac{1}{n!} f^{n}(\xi) \prod_{i=0}^{n-1} (x_n - x_i) = f[x_0, \dots x_n] \prod_{i=0}^{n} (x_n - x_i)$$

$$\Rightarrow f[x_0, \dots x_n] = \frac{1}{n!} f^{n}(\xi)$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^k$$

$$f'(x) = a_1 + 2a_2 x + \dots k a_n x^{k-1}$$

$\vdots$

$$f^{(n)}(x) = (\ ) + (\ ) + \dots k(k-1)\dots(k-n) x^{k-n}$$

but $k < n \Rightarrow f^{(n)}(x) = 0$

$$\Rightarrow f[x_0, \dots x_n] = \frac{1}{n!}(0) = 0$$

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(8) $$y_n - y_{n-2} = \frac{\Delta x}{3}\left(f_n + 4f_{n+1} + f_{n+2}\right)$$

$$y' = f(x, y) \quad , \quad y(x_0) = y_0$$

Comparing with $\sum a_{k-i} y_{n-i} = \sum \Delta x \, b_{k-i} f_{n-i}$,

$$(a_0, a_1, a_2) = (-1, 0, 1) \quad (b_0, b_1, b_2) = \left(\frac{1}{3}, \frac{4}{3}, \frac{1}{3}\right)$$

$$d_0 = \sum a_i = a_0 + a_1 + a_2 = 0$$

$$d_1 = \sum i a_i - b_i = -b_0 + (a_1 - b_1) + (2a_2 - b_2)$$

$$= -\frac{1}{3} + \left(0 - \frac{4}{3}\right) + \left(2 - \frac{1}{3}\right) = 0$$

$$d_2 = \sum \frac{i^2}{2} a_i - i b_i = \left(\frac{a_1}{2} - b_1\right) + (2a_2 - 2b_2)$$

$$= \left(0 - \frac{4}{3}\right) + \left(2 - \frac{2}{3}\right) = 2 - \frac{6}{3} = 0$$

$$d_3 = \sum \frac{i^3}{6} a_i - \frac{i^2}{2} b_i = \left(\frac{a_1}{6} - \frac{b_1}{2}\right) + \left(\frac{4}{3} a_2 - 2b_2\right)$$

$$= \left(0 - \frac{2}{3}\right) + \left(\frac{4}{3} - \frac{2}{3}\right) = 0$$

$$d_4 = \sum \frac{i^4}{24} a_i - \frac{i^3}{6} b_i = \left(\frac{a_1}{24} - \frac{b_1}{6}\right) + \left(\frac{2a_2}{3} - \frac{4}{3} b_2\right)$$

$$= \left(0 - \frac{4}{18}\right) + \left(\frac{2}{3} - \frac{4}{9}\right) = 0$$

$$d_5 = \sum \frac{i^5}{120} a_i - \frac{i^4}{24} b_i = \left(\frac{a_1}{120} - \frac{b_1}{24}\right) + \left(\frac{4a_2}{15} - \frac{2}{3} b_2\right)$$

$$= \left(0 - \frac{4}{72}\right) + \left(\frac{4}{15} - \frac{2}{9}\right) = -\frac{1}{90} \neq 0$$

$\Rightarrow$ Order $= 4$

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(9) $$y_{n+1} - y_n = \Delta x\left(\frac{3}{2}f_n - \frac{1}{2}f_{n-1}\right)$$

$$y_{n+1} = y_n + ahy_n' + bhy_{n-1}' \quad \left(\text{since } y' = f(x,y)\right)$$

Expanding both sides in taylor series about x_0,

$$y(x_0) + \Delta x\, y'(x_0) + \frac{\Delta x^2}{2} y''(x_0) + \frac{\Delta x^3}{6} y'''(x_0) + o(\Delta x^4)$$

$$= y(x_0) + a\Delta x\, y'(x_0) + b\Delta x\left(y'(x_0) - \Delta x y''(x_0) + \frac{\Delta x^2}{2} y'''(x_0) + o(\Delta x^3)\right)$$

$$= y(x_0) + (a+b)\Delta x\, y'(x_0) - b\Delta x^2 y''(x_0) + \frac{b}{2}\Delta x^3 y'''(x_0) + o(\Delta x^4)$$

Equating coefficients,

$$a + b = 1$$

$$-\frac{b}{2} = \frac{1}{2}$$

$$\Rightarrow a = \frac{3}{2}, \quad b = -\frac{1}{2}$$

$$\Rightarrow y_{n-1} - y_n = \Delta x\left(a f_n + b f_{n-1}\right)$$

$$= \Delta x\left(\frac{3}{2}f_n - \frac{1}{2}f_{n-1}\right)$$

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(10) $y'' = f(x, y, y')$ $y(a) = y_a$ $y(b) = y_b$

Consider the IVP

$$y'' = f(x, y, y') \quad , \quad y(a) = y_a, \; y'(a) = t_0$$

We solve this IVP using Euler or RK method and compute $y(b)$

If $y(b) \approx y_b$, we stop, else, we update t by solving

$$y(b, t) - y_b = 0$$

$$t_k = t_{k-1} - \frac{y(b, t_{k-1}) - y_b}{\frac{dy}{dt}(b, t_{k-1})}$$

In secant method $\frac{dy}{dt}(b, t_{k-1}) \approx \frac{y_{k-1} - y_{k-2}}{t_{k-1} - t_{k-2}}$

In Newton's method, we compute derivative $\frac{dy}{dt}$ analytically

$$y''(x,t) = f(x, y(x,t), y'(x,t))$$

$$\frac{\partial y''}{\partial t}(x,t) = \frac{\partial f}{\partial x} \cancel{\frac{dx}{dt}}^{\,0} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial t}$$

$$y(a,t) = \alpha \Rightarrow \frac{\partial y}{\partial t}(a,t) = 0 \quad \& \quad y'(a,t) = 0 \Rightarrow \frac{\partial y'}{\partial t}(a,t) = 1$$

let $z(x,t) = \frac{dy}{dt}(x,t) \Rightarrow z''(x,t) = \frac{\partial f}{\partial y}(x, y(x,t), y'(x,t))\, z(x,t) + \frac{\partial f}{\partial y'}(x, y(x,t), y'(x,t))\, z'(x,t)$

with $z(a,t) = 0$, $z'(a,t) = 1$

Once we know z, $t_k = t_{k-1} - \frac{y(b_{k-1}) - \beta}{z(b, t_{k-1})}$

More accuracy since $\frac{dy}{dt}$ is computed analytically. But at the same time more comp cost as additional IVP is req

(4)

$$F = \begin{bmatrix} x_1 x_2 - x_3^2 - 1 \\ x_1 x_2 x_3 - x_1^2 - x_2^2 - 2 \\ e^{x_1} - e^{x_2} + x_3 - 3 \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}$$

$$\therefore \quad J = \begin{bmatrix} x_2 & x_1 & -2x_3 \\ x_2 x_3 - 2x_1 & x_1 x_3 - 2x_2 & x_1 x_2 \\ e^{x_1} & -e^{x_2} & 1 \end{bmatrix}$$

$$J\Big|_{(1,1,1)} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & -1 & 1 \\ e & -e & 1 \end{bmatrix}$$

$$x^{(k+1)} = x^{(k)} - J^{-1} F$$

$$F\Big|_{(1,1,1)} = \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}$$

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$$J^{-1} = \begin{pmatrix} -0.316 & -0.816 & 0.184 \\ -0.684 & -1.184 & -0.184 \\ -1 & -1 & 0 \end{pmatrix}$$

$$J^{-1} F = \begin{pmatrix} 2.396 \\ 4.604 \\ 4 \end{pmatrix}$$

$$\Rightarrow \quad x^{(k+1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2.396 \\ 4.604 \\ 4 \end{pmatrix} = \begin{pmatrix} -1.396 \\ -3.604 \\ -3 \end{pmatrix}$$

③ $Ax = b$

$$\Rightarrow \quad Qx = (Q - A)x + b$$

$$\Rightarrow \quad x = Q^{-1}(Q - A)x + Q^{-1}b$$

$$\Rightarrow \quad x = (I - Q^{-1}A)x + Q^{-1}b$$

$$\Rightarrow \quad x^{(k+1)} = \underbrace{(I - Q^{-1}A)}_{G} x^{(k)} + \underbrace{Q^{-1}b}_{C} \quad — ①$$

~~Subtract x on both sides~~

$$\cancel{\Rightarrow (x^{k+1} - x) = (I - Q^{-1}A) x^{(k)} + Q^{-1}b - x}$$

$$\|x^k - x\| \le \|I - Q^{-1}A\|^k \|x^{(0)} - x\|$$

$\Rightarrow$ for convergence, $\|I - Q^{-1}A\| < 1$

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Consider $x^{[k]} - x$

$$x^{(k)} - x = (I - Q^{-1}A)\, x^{[k-1]} + \cancel{Q^{-1} b} - (I - Q^{-1}A)\, x - \cancel{Q^{-1} b}$$

$$\Rightarrow x^{(k)} - x = (I - Q^{-1}A)(x^{[k-1]} - x) = \delta(x^{k-1} - x)$$

$$\cancel{= (I - Q^{-1}A)^2 (x^{[k-2]} - x)} \; \cancel{= \delta(x^{[k-1]} - x)}$$

$$\cancel{= \dots (I - Q^{-1}A)^k (x^{(0)} - x)}$$

$$\cancel{\Rightarrow \|x^k - x\| = \|x^{(k)} - x^{[k-1]} + x^{[k-1]} - x^{[k-2]} + \dots x^{(0)} - x\|}$$

$$\cancel{\le \|x^{(k)} - x^{(k-1)}\| + \|}$$

$$\Rightarrow x^k - x = \delta(x^k - x^{k-1} - x^k - x)$$

$$\Rightarrow \|x^k - x\| \le \delta\|x^k - x\| + \delta\|x^k - x^{k-1}\|$$

$$\Rightarrow (1 - \delta)\|x^k - x\| \le \delta\|x^k - x^{k-1}\|$$

$$\Rightarrow \|x^{(k)} - x\| \le \frac{\delta}{1 - \delta}\|x^{(k)} - x^{(k-1)}\|$$

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(1) $Ax = b$

$a_{ij} = 0 \qquad j \leq n-i$

$\Rightarrow i+j \leq n$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & a_{1,n} \\ 0 & 0 & \cdots & 0 & a_{2,n-1} & a_{2,n} \\ \vdots & & \vdots & & \vdots & \\ a_{n,1} & a_{n,2} & \cdots & & & a_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$\Rightarrow \quad a_{1,n} x_n = b_1$. Find x_n

$a_{2,n-1}x_{n-1} + a_{2,n} x_n = b_2$ Find x_{n-1}

$\vdots$

$a_{n,1} x_1 + \cdots \cdots a_{n,n} x_n = b_n$

Pseudo-code on next page

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integer i, n ; real array $(b_i)_{1:n}, (a_{ij})_{1:n \times 1:n}$,

$(x_j)_{1:n}$

$$x_n \leftarrow b_1 / a_{1,n}$$

for $i = 2$ to n do

$$x_{n-i+1} \leftarrow \frac{1}{a_{i,n-i+1}} \left(b_i - \sum_{j=1}^{i-1} a_{i,n-j+1} x_{n-j+1} \right)$$

end for

(2)

$$x^T A = b^T$$

$$(x^T A)^T = b^T$$

$\Rightarrow$

$$A^T x = b \Rightarrow (LU)^T x = b$$

$\Rightarrow$

$$\Rightarrow U^T L^T x = b$$

$$U^T \underbrace{(L^T x)}_{y} = b$$

$\Rightarrow$

First solve $U^T y = b$, for y, then $L^T x = y$

$$U^T y = b$$

(similar to $Ly = b$ since U^T is lower triangular)

$$y = [y_1, y_2, \ldots y_N]^T$$

$$y_i = \left(- \sum_{j=1}^{i-1} (U_{ji} y_j) + b_i \right) \frac{1}{U_{ii}} \quad - (1)$$

forward substitution

$$x_i = \left(y_i - \sum_{j=i+1}^{N} L_{ij} x_j \right) \frac{1}{L_{ii}} \quad \text{(Backward substitution)}$$

integer i, n ; real array $(b_i)_{1:n}$ $(u_{ij})_{1:n \times 1:n}$ $(y_i)_{1:n}$

$y_1 \leftarrow b_1$

for $i = 2$ to n do

$$y_i \leftarrow \frac{1}{U_{ii}}\left(- \sum_{j=1}^{i-1} U_{ji} y_j + b_i \right)$$

end for

Sly, x is obtained as

integer i, n ; real array $(L_{ij})_{1:n \times 1:n}$, $(x_i)_{1:n}$, $(z_i)_{1:n}$

$$x_n \leftarrow \frac{y_n}{L_{nn}}$$

for $i = n-1$ to 1 step -1 do

$$x_i \leftarrow \frac{1}{L_{ii}}\left(- \sum_{J=i+1}^{N} L_{ji} x_j + y_i \right)$$