INDIAN INSTITUTE OF TECHNOLOGY MADRAS

B

Roll No. MM16B023

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Total No. of Pages

Quiz I ✓ Quiz II/ Mid-Sem End-Semester Make-up Date : 21-02-20

Semester & Degree : IDDD DS 8th Sem Course No. MA6270 Part :

Question No.	1	2	3	4	5	6	7	8	9	10
Marks										

11	12	13	14	15	16	17	18	19	20	Total

Answer on both sides of the paper including the space below

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(2)

$$\frac{\partial u}{\partial t} = \frac{1}{3}\frac{\partial^2 u}{\partial x^2}$$

Explicit F-D Scheme

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{3}\frac{1}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

$$\Rightarrow u_{i,j+1} - u_{i,j} = \frac{r}{3}u_{i-1,j} - \frac{2r}{3}u_{i,j} + \frac{r}{3}u_{i+1,j}$$

$$\Rightarrow u_{i,j+1} = \frac{r}{3}u_{i-1,j} + \left(1-\frac{2r}{3}\right)u_{i,j} + \frac{r}{3}u_{i+1,j}$$

$$\bar{u}_{j+1} = A\bar{u}_j$$

where
$$A = \begin{bmatrix} \left(1-\frac{2r}{3}\right) & \frac{r}{3} & 0 & \cdots & 0 \\ \frac{r}{3} & \left(1-\frac{2r}{3}\right) & \frac{r}{3} & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & 0 \;\; \frac{r}{3} & \left(1-\frac{2r}{3}\right) & \frac{r}{3} \\ 0 & \cdots & 0 & \frac{r}{3} & \left(1-\frac{2r}{3}\right) \end{bmatrix}$$

$$A = I_{N-1} + \frac{r}{3} T_{N-1}$$

where $T_{N-1} = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ & & \ddots & & \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}$

eigen values of $T_{N-1} = -4 \sin^2\left(\frac{s\pi}{2N}\right)$ $s = 1, 2, \ldots N-1$

$\Rightarrow$ eigen values of A, $\lambda_s = 1 - \frac{4r}{3} \sin^2\left(\frac{s\pi}{2N}\right)$

For the scheme to be stable,

$$|\lambda_s| < 1$$

$$\Rightarrow \left| 1 - \frac{4r}{3} \sin^2\left(\frac{s\pi}{2N}\right) \right| \leq 1$$

$$\Rightarrow -1 \leq 1 - \frac{4r}{3} \sin^2\left(\frac{s\pi}{2N}\right) \leq 1$$

Right Hand side gives $r > 0$

Left Hand side gives $\frac{4r}{3} \sin^2\left(\frac{s\pi}{2N}\right) \leq 2$

$$\Rightarrow r \leq \frac{3}{2\sin^2\left(\frac{s\pi}{2N}\right)} \Rightarrow r \leq \frac{3}{2}$$

$$\Rightarrow r \in \left(0, \frac{3}{2}\right]$$

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③ $$\frac{\partial U}{\partial t} = \frac{1}{2}\frac{\partial^2 U}{\partial x^2}$$

$$\Rightarrow \frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}\left[\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}\right]$$

$$\Rightarrow F_{ij}(u) = \frac{u_{i,j+1} - u_{i,j}}{k} - \frac{1}{2h^2}\left(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\right)$$

$$= \frac{1}{k}\left[\cancel{u_{i,j}} + k\left.\frac{\partial u}{\partial t}\right|_{i,j} + \frac{k^2}{2}\left.\frac{\partial^2 U}{\partial t^2}\right|_{i,j} + o(k^3) - \cancel{u_{ij}}\right]$$

$$- \frac{1}{2h^2}\Big[\cancel{u_{i,j+1}} - \cancel{h\left.\frac{\partial u}{\partial x}\right|_{i,j+1}} + \frac{h^2}{2}\left.\frac{\partial^2 U}{\partial x^2}\right|_{i,j+1} - \cancel{\frac{h^3}{6}\left.\frac{\partial^3 U}{\partial x^3}\right|_{i,j+1}} + \cdots$$

$$+ \cancel{u_{i,j+1}} + \cancel{h\left.\frac{\partial u}{\partial x}\right|_{i,j+1}} + \frac{h^2}{2}\left.\frac{\partial^2 U}{\partial x^2}\right|_{i,j+1} + \cancel{\frac{h^3}{6}\left.\frac{\partial^3 U}{\partial x^3}\right|_{i,j+1}} + \cdots$$

$$- \cancel{2u_{i,j+1}}\Big]$$

$$= \left.\frac{\partial U}{\partial t}\right|_{i,j} + \frac{k}{2}\left.\frac{\partial^2 U}{\partial t^2}\right|_{i,j} + o(k^2) - \frac{1}{2h^2}\left[h^2\left.\frac{\partial^2 U}{\partial x^2}\right|_{i,j+1} + \frac{h^4}{12}\left.\frac{\partial^4 U}{\partial x^4}\right|_{i,j+1} + o(h^6)\right]$$

$$= \cancel{\left[\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right]_{i,j}}^{\,0} + \frac{k}{2}\frac{\partial^2 U}{\partial t^2} - \frac{h^2}{24}\left.\frac{\partial^4 U}{\partial x^4}\right|_{i,j} + o(k^2, h^4)$$

$$\Rightarrow F_{ij}(u) = \left.\frac{\partial u}{\partial t}\right|_{i,j} - \frac{1}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j+1} + \frac{k}{2}\left.\frac{\partial^2 u}{\partial t^2}\right|_{i,j} - \frac{h^2}{24}\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j+1} + o(k^2, h^4)$$

$$= \left.\frac{\partial u}{\partial t}\right|_{i,j} - \frac{1}{2}\left[\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} + \frac{k}{2}\left.\frac{\partial^3 u}{\partial t\, \partial x^2}\right|_{i,j} + o(k^2)\right]$$

$$+ \frac{k}{2}\left.\frac{\partial^2 u}{\partial t^2}\right|_{i,j} + \frac{k^2}{6}\left.\frac{\partial^3 u}{\partial t^3}\right|_{i,j}$$

$$- \frac{h^2}{24}\left[\left.\frac{\partial^4 u}{\partial x^2}\right|_{i,j} + k\left.\frac{\partial^5 u}{\partial t\, \partial x^4}\right|_{i,j} + o(k^2)\right]$$

$$\Rightarrow f_{ij}(u)$$

$$= \cancelto{0}{\left[\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\right]_{i,j}} + \frac{k}{2}\cancelto{0}{\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^3 u}{\partial t\, \partial x^2}\right)_{i,j}}$$

$$+ o(k^2) - \frac{h^2}{24}\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j} + o(h^4)$$

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$$\Rightarrow T_{ij} = \frac{k^2}{6}\left(\left.\frac{\partial^3 u}{\partial t^3}\right|_{i,j} - \left.\frac{\partial^4 u}{\partial t^2\, \partial x^2}\right|_{ij}\right) - \frac{h^2}{24}\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j} + o(k^3, h^4)$$

$$r = \frac{k}{h^2}$$

$$\Rightarrow T_{ij} = \frac{r^2 h^4}{6}\left(\frac{\partial^3 U}{\partial t^3} - \frac{\partial^4 U}{\partial t^2 \partial x^2}\right)_{i,j} - \frac{k}{r}\frac{\partial^4 U}{\partial x^4}\bigg|_{i,j} + o(k^3, h^4)$$

h and k appear in the Numerator

$\Rightarrow T_{ij} \to 0$ as $h, k \to 0$ for $r = \frac{k}{h^2}$

$$O \sim (k^2, h^2)$$

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(1)

$$\frac{\partial U}{\partial t} = \frac{1}{3}\frac{\partial^2 U}{\partial x^2}$$

$$U(x,0) = 18x(1-x) \quad , t = 0$$

$$\left.\begin{aligned} \frac{\partial U}{\partial x}(0,t) &= U(0,t) \\ U(1,t) &= 0 \end{aligned}\right\} \quad t > 0$$

Crank - Nicolson F-D scheme

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{3}\frac{1}{h^2}\left[\frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{2} + \frac{u_{i-1,j+1} + u_{i+1,j+1} - 2u_{i,j+1}}{2}\right]$$

$$\Rightarrow 2u_{i,j+1} - 2u_{i,j} = r u_{i+1,j} + r u_{i-1,j} - 2r u_{i,j} + r u_{i-1,j+1} + r u_{i+1,j+1} - 2r u_{i,j+1}$$

$$\Rightarrow -r u_{i-1,j+1} + (2+2r) u_{i,j+1} - r u_{i+1,j+1}$$

$$= r u_{i-1,j} + (2-2r) u_{i,j} + r u_{i+1,j}$$

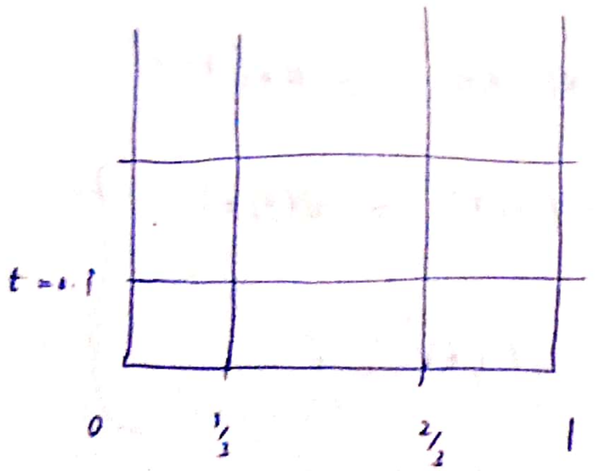
$$\Rightarrow 6u_{i,j+1} - 6u_{i,j} = r u_{i+1,j} + r u_{i-1,j} - 2r u_{i,j}$$

$$+ r u_{i-1,j+1} + r u_{i+1,j+1}$$

$$- 2r u_{i,j+1}$$

$$\Rightarrow -r u_{i-1,j+1} + (2r+6) u_{i,j+1} - r u_{i+1,j+1}$$

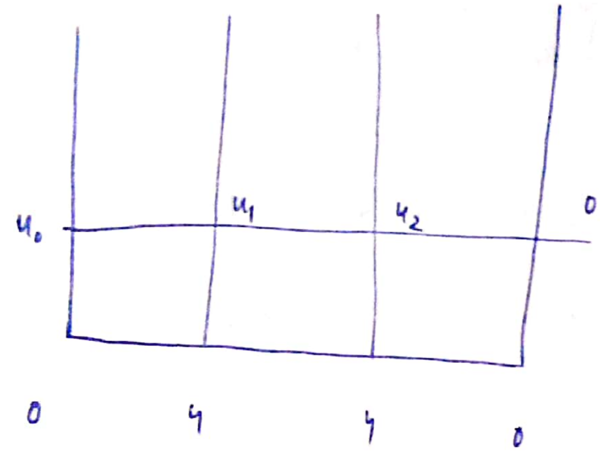
$$= r u_{i-1,j} + (6-2r) u_{i,j} + r u_{i+1,j}$$



$h = \Delta x = \frac{1}{3}, \quad \Delta t = 0.1$

$$\Rightarrow r = \frac{k}{h^2} = \frac{0.1}{(\frac{1}{3})^2} = 0.1(9) = 0.9$$

Can choose this value since scheme is unconditionally stable



$$\frac{u_1 - u_0}{(1/3)} = u_0$$

$$\Rightarrow \quad u_1 - u_0 = \frac{u_0}{3} \quad \Rightarrow \quad u_1 = \frac{4u_0}{3} \qquad \text{— ①}$$

$$r\,u_{i-1,j+1} - (6+2r)\,u_{i,j+1} + r\,u_{i+1,j+1}$$

$$= -r\,u_{i-1,j} - (6-2r)\,u_{i,j} - r\,u_{i+1,j}$$

$$\Rightarrow \quad +0.9u_0 - 7.8u_1 + 0.9u_2 = -0 - 4.2(4) - 0.9(4)$$

$$= -20.4 \qquad \text{— ②}$$

$$0.9u_1 - 7.8u_2 + 0 = -0.9(4) - 4.2(4) - 0$$

$$= -20.4$$

$$\Rightarrow \quad u_1 = \frac{4u_0}{3} \qquad \text{— ①}$$

$$0.9u_0 - 7.8u_1 + 0.9u_2 = -20.4 \qquad \text{— ②}$$

$$0.9u_1 - 7.8u_2 = -20.4 \qquad \text{— ③}$$

$$\Rightarrow \quad 0.9\left(\frac{3}{4}\right) u_1 - 7.8 u_1 + 0.9 u_2 = -20.4$$

$$\Rightarrow \quad -7.125\, u_1 + 0.9\, u_2 = -20.4$$

$$(-) \quad 0.9\, u_1 - 7.8\, u_2 = -20.4$$

$$\Rightarrow \quad -8.025\, u_1 - 6.9\, u_2 = 0$$

$$\Rightarrow \quad 8.025\, u_1 = -6.9\, u_2$$

$$\Rightarrow \quad u_2 = -1.163\, u_1$$

$$\Rightarrow \quad 0.9\, u_1 - 7.8(-1.113\, u_1) = -20.4$$

$$\Rightarrow \quad 9.0714\, u_1 = -20.4$$

$$\Rightarrow \quad u_1 = 2.24$$

$$\Rightarrow \quad u_0 = 2.98, \quad u_2 = 1.68$$

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$t=0.1$	2.98	2.24	1.68	0
	0	4	4	0