am II — PH5170

INDIAN INSTITUTE OF TECHNOLOGY MADRAS

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Roll No. M M 1 6 B 0 2 3

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Quiz I ✓ Quiz II/ Mid-Sem End-Semester Make-up Date : Feb 7, 2020

Semester & Degree : Course No. Part :

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Answer on both sides of the paper including the space below

1 (a)

$$\psi_2 \times \left[-\frac{\hbar^2}{2m} \frac{d^2\psi_1}{dx^2} + V\psi_1 = E\psi_1 \right] \quad -(1)$$

$$\psi_1 \times \left[-\frac{\hbar^2}{2m} \frac{d^2\psi_2}{dx^2} + V\psi_2 = E\psi_2 \right] \quad -(2)$$

Subtracting the 2 eqns,

$$-\frac{\hbar^2}{2m}\left[\psi_2 \frac{d^2\psi_1}{dx^2} - \psi_1 \frac{d^2\psi_2}{dx^2} \right] + V\cancel{\psi_1}\psi_2 - V\psi_1\cancel{\psi_2} = E\psi_1\psi_2 - E\psi_1\psi_2 = 0 \quad -(3)$$

Also, $\frac{d}{dx}\left[\psi_2 \frac{d\psi_1}{dx} - \psi_1 \frac{d\psi_2}{dx} \right]$

$$= \psi_2 \frac{d^2\psi_1}{dx^2} + \cancel{\frac{d\psi_2}{dx} \frac{d\psi_1}{dx}} - \cancel{\frac{d\psi_1}{dx} \frac{d\psi_2}{dx}} - \psi_1 \frac{d\psi_2^2}{dx}$$

$$\Rightarrow -\frac{\hbar^2}{2m} \frac{d}{dx}\left[\psi_2 \frac{d\psi_1}{dx} - \psi_1 \frac{d\psi_2}{dx} \right] = 0$$

$$\rightarrow \frac{d}{dx}\left[\psi_1 \frac{d\psi_2}{dx} - \psi_2 \frac{d\psi_1}{dx}\right] = 0$$

$$\rightarrow \psi_1 \frac{d\psi_2}{dx} - \psi_2 \frac{d\psi_1}{dx} = W \text{ (const)}$$

(b) If the wave f^{ns} are normalizable,

$$\psi \rightarrow 0 \text{ as } x \rightarrow \infty$$

$$\Rightarrow \left(\psi_1 \frac{d\psi_2}{dx} - \psi_2 \frac{d\psi_1}{dx}\right) \rightarrow (0 - 0 = 0)$$

as $x \rightarrow \infty$

But since W is a constant, it must hold for all x

$$\rightarrow W = \psi_1 \frac{d\psi_2}{dx} - \psi_2 \frac{d\psi_1}{dx} = 0$$

(c)

$$\psi_1 \frac{d\psi_2}{dx} = \psi_2 \frac{d\psi_1}{dx}$$

$$\rightarrow \frac{1}{\psi_1}\frac{d}{dx}\left[\psi_1\right] = \frac{1}{\psi_2}\frac{d}{dx}\left[\psi_2\right]$$

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$$\rightarrow \int \frac{d\psi_1}{\psi_1} = \int \frac{d\psi_2}{\psi_2} + C$$

$$\rightarrow \ln \psi_1 = \ln \psi_2 + C$$

$$\Rightarrow \ln\left(\frac{\psi_1}{\psi_2}\right) = C \rightarrow \psi_1 = e^C \psi_2 = k\psi_2$$

where k is a constant

$$\boxed{\psi_1 = k\psi_2}$$

Since ψ_1, ψ_2 are linearly dep, they aren't really unique and therefore the solutions are not distinct

③ a) $$A^{\dagger} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & a & -a \end{pmatrix} = A$$

1

$$B^{\dagger} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & ib \\ 0 & -ib & 0 \end{pmatrix}^T = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

$= B$

Both $\hat{A}$, $\hat{B}$ are Hermitian

$\rightarrow$ They do represent observables

b) ~~No, they cannot be~~ measured ~~simultaneously~~

$$AB = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

$$= \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}$$

$$= \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}$$

$$AB = AB$$

$$\Rightarrow [A, B] = 0$$

↳ null matrix

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Since commutator is zero, the observables can be measured simultaneously

(c)

$$\det(A - \lambda I) = (a-\lambda)(-a-\lambda)(-a-\lambda) = 0$$

2

$\rightarrow \lambda = \underline{-a, -a, a}$ are the eigen value

$$\det \begin{pmatrix} b-\lambda & 0 & 0 \\ 0 & -\lambda & -ib \\ 0 & ib & -\lambda \end{pmatrix} = (b-\lambda)(\lambda^2 + i^2 b^2) = 0$$

$\lambda = b, -b, b$

$$\begin{pmatrix} -a \\ -a \\ a \end{pmatrix} \quad \begin{pmatrix} b \\ b \\ -b \end{pmatrix}$$

$a = 1$, $b = -1$

$\Rightarrow$ the eigen values are same

$\rightarrow$ degenerate

(d)

$$\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = a \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} a\alpha &= a\alpha \\ -a\beta &= a\beta \\ -a\gamma &= a\gamma \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

another vector is $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ why?

For B, it is the same

2

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Graham Schmidt

Speak to me, if needed!

$$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\langle v_1, v_2 \rangle} v_1$$

Something not right!

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - 0 \Rightarrow \tilde{v}_2 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\}$$

② $\hat{T}(a) = \exp\left(\frac{-i\hat{p}\cdot a}{\hbar}\right)$

(a) $[\hat{x}_i, \hat{T}(a)]f$

$$= x_i e^{-(ip\cdot a)/\hbar} f - \left(e^{\frac{-i\left(-i\hbar \frac{d}{dx_i}(x_i)\cdot a\right)}{\hbar}} \right) f$$

$$= x_i e^{-i\left(-i\hbar \frac{d}{dx_i}\right)} - e^{-\hbar} x_i$$

$$= x_i e^{-\hbar \frac{d}{dx_i}} - e^{-\hbar} x_i \, f$$

$[\sim x_1,$

$\hat{p} = (p_1, p_2, p_3)$

$x = (x_1, x_2, x_3)$

$$\left[x_1, e^{-i\frac{\hat{p}\cdot a}{\hbar}} \right]$$

$$= x_1 e^{-\frac{i\hat{p}\cdot a}{\hbar}} - e^{-i\hbar \frac{d}{dx}(x)}$$

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$$[\hat{x}_i, \hat{T}(a)] = x_i e^{-\frac{\hat{p}\cdot a}{\hbar}} - e^{-\frac{i(-i)\hbar}{\hbar}}$$

$$= x_i e^{-\frac{\hat{p}\cdot a}{\hbar}} - e^{-\left[\frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{k}\right][x}$$

$$= x_i e^{-\frac{\hat{p}\cdot a}{\hbar}} - e^{-1}$$

?

b

(b)

$$\langle \bar{\psi} | \hat{x} | \bar{\psi} \rangle$$

$$= c \langle \bar{\psi} | xT - Tx | \bar{\psi} \rangle$$

$$= c \langle \bar{\psi} | x | \bar{\psi} \rangle - c \langle \psi | x | \psi \rangle$$

$$= c\left(\langle \bar{\psi} | \hat{x} | \psi \rangle - \langle \psi | x | \bar{\psi} \rangle\right)$$

$$\langle \bar{\psi} | \hat{x} | \bar{\psi} \rangle$$

?

0

(4) a) $\langle \hat{x} \rangle = \langle \alpha | x \alpha \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \alpha | a_+ + a_- | \alpha \rangle$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\langle a_- \alpha | \alpha \rangle + \langle \alpha | a_- \alpha \rangle \right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\alpha + \alpha^* \right)$$

$$\langle \hat{p}_x \rangle = \langle \alpha | p \alpha \rangle = i\sqrt{\frac{\hbar m\omega}{2}} \langle \alpha | a_+ - a_- | \alpha \rangle$$

$$= i\sqrt{\frac{\hbar m\omega}{2}} \left(\langle a_- \alpha | \alpha \rangle - \langle \alpha | a_- \alpha \rangle \right)$$

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$$= -i\sqrt{\frac{\hbar m\omega}{2}} \left(\alpha - \alpha^* \right)$$

(b)

$$\hat{a} | 0 \rangle = 0$$

↳ ground state

$$\left(\frac{x}{\sqrt{\frac{2\hbar}{m\omega}}} + \frac{ip}{\sqrt{2m\omega\hbar}} \right) | 0 \rangle = 0$$

$$\Rightarrow (m\omega x + ip) | 0 \rangle = 0$$

$$\phi_0(x) = \langle x | 0 \rangle = \langle x | \left(m\omega x + \hbar \frac{d}{dx} \right) | 0 \rangle = 0$$

$$m\omega x \, \phi_0(x) + \hbar \frac{d}{dx} \phi_0(x) = 0$$

$\Rightarrow$

$$\phi_0(x) = A\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

$$\phi_\alpha(x) = A_\alpha H_\alpha\left(x\sqrt{\frac{m\omega}{2\hbar}}\right) e^{-\frac{m\omega x^2}{2\hbar}}$$

$$\cancel{\psi_\alpha(x) = \frac{\langle x|\hat{a}|\alpha\rangle}{\alpha}}$$

$$\cancel{= \alpha\alpha} \qquad \psi_\alpha(x) = \langle x|\alpha\rangle$$

$$= \frac{\langle x|\hat{a}|\alpha\rangle}{\alpha}$$

(1) $c_n = |\psi_n|\alpha\rangle = \frac{1}{\sqrt{n!}}\langle(a_+)^n\psi_0|\alpha\rangle$ $\qquad \psi_0 = |0\rangle$

$$= \frac{1}{\sqrt{n!}}\alpha^n \cancel{\phi}\langle 0|\alpha\rangle = \frac{\alpha^n}{n!}c_0$$

$$1 = \sum_{n=0}^{\infty}|c_n|^2 = |c_0|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}$$

$$\Rightarrow c_0 = e^{-|\alpha|^2/2}$$

$$\Rightarrow \psi_\alpha(x) = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$\hat{x}\sqrt{\frac{2m\omega}{\hbar}} = \alpha + \alpha^*, \quad i\hat{p}\sqrt{\frac{2}{\hbar m\omega}} = \alpha - \alpha^*$$

$$\Rightarrow \psi_\alpha(x) = \sum_{n=0}^{\infty}\frac{\left(\hat{x}\sqrt{\frac{2m\omega}{\hbar}} + i\hat{p}\sqrt{\frac{2}{\hbar m\omega}}\right)^n}{2^n\sqrt{n!}}|n\rangle$$

(5)

$$\Delta A^2 \, \Delta B^2 \geq \frac{1}{4} \left| \langle [\hat{A}, \hat{B}] \rangle \right|^2$$

(a)

$$\Delta S_x^2 \, \Delta S_y^2 \geq \frac{1}{4} \left| \langle [\hat{S_x}, \hat{S_y}] \rangle \right|^2$$

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\chi\rangle = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$|a|^2 + |b|^2 = 1.$$

So without loss of generality,

$$a = \cos\frac{\theta}{2}, \quad b = \sin\frac{\theta}{2} e^{i\phi}$$

$$\rightarrow \chi = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi} \sin\frac{\theta}{2} \end{pmatrix}$$

$$\langle S_x \rangle = \langle \chi^\dagger | S_x | \chi \rangle$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \left(\frac{\hbar}{2}\right) 2 \sin\frac{\theta}{2} \cos\frac{\theta}{2} = \frac{\hbar}{2} \sin\theta$$

$$S_x^2 = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\langle S_x^2 \rangle = \frac{\hbar^2}{4} \quad \text{for spin } \frac{1}{2}$$

Since $\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle$,

$$\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = \frac{3\hbar^2}{4}$$

$$\langle S_y \rangle = \frac{\hbar}{2}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} -i\sin\frac{\theta}{2} \\ i\cos\frac{\theta}{2} \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(-i\sin\frac{\theta}{2}\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\cos\frac{\theta}{2} \right) = 0$$

Also $\langle S_y^2 \rangle = \frac{\hbar^2}{4}$

$$\Rightarrow \Delta S_x^2 = \langle S_x^2 \rangle - \langle S_x \rangle^2$$

$$= \frac{\hbar^2}{4} - \frac{\hbar^2}{4}\sin^2\theta = \frac{\hbar^2}{4}(1 - \sin\theta)$$

$$\Delta S_y^2 = \langle S_y^2 \rangle - \langle S_y \rangle^2 = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4}$$

$$\rightarrow \quad \cancel{\Delta S_x^2 \, \Delta S_y^2 = \frac{\hbar^2}{4}(1-\sin^2\theta)\,\frac{\hbar^2}{4}}$$

$$\cancel{= \frac{\hbar^4}{16}\left(1-\sin^2\theta\right)}$$

$$\langle S_x \rangle = \langle \chi^\dagger | S_x | \chi \rangle$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} e^{i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} + e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \right)$$

$$= \frac{\hbar}{2}\left(\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}\right)\left(2\cos\phi\right)$$

$$S_x^2 = \frac{\hbar^2}{4} I \Rightarrow \langle S_x^2 \rangle = \frac{\hbar^2}{4}$$

$$\Delta S_x^2 = \langle S_x^2 \rangle - \langle S_x \rangle^2$$

$$= \frac{\hbar^2}{4} - \frac{\hbar^2}{4}\sin^2\theta\cos^2\phi$$

$$= \frac{\hbar^2}{4}\left(1 - \sin^2\theta\cos^2\phi\right)$$

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Additional Sheet

$$\Delta S_y^2 = \langle S_y^2 \rangle - \langle S_y \rangle^2 = \frac{\hbar^2}{4} - \langle S_y \rangle^2$$

$$\langle S_y \rangle = \frac{\hbar}{2} \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi} \sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi} \sin\frac{\theta}{2} \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi} \sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} -i e^{i\phi} \sin\frac{\theta}{2} \\ i \cos\frac{\theta}{2} \end{pmatrix}$$

$$= \frac{\hbar}{2} \left[-i e^{i\phi} \sin\frac{\theta}{2} \cos\frac{\theta}{2} + i e^{-i\phi} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \right]$$

$$= \frac{\hbar}{2}\, i\, \frac{\sin\theta}{2} \left[e^{-i\phi} - e^{i\phi} \right]$$

$$= \frac{\hbar}{2}\, i\, \frac{\sin\theta}{\cancel{2}} \left(-\cancel{2} i \sin\phi \right)$$

$$= \frac{\hbar}{2} \sin\theta \sin\phi \left(-i^2 \right) = \frac{\hbar}{2} \sin\theta \sin\phi$$

Not

$$\rightarrow \Delta S_y^2 = \frac{\hbar^2}{4} \left(1 - \frac{\sin^2\theta}{?} \right)$$

$$\Rightarrow \Delta S_x^2 \, \Delta S_y^2 = \frac{\hbar^4}{16}\left(1 - \sin^2\theta \cos^2\theta\right)\left(1 - \sin^4\theta\right)$$

$$= \frac{\hbar^4}{16}\left(1 - t(1-t)\right)\left(1 - t^2\right) \quad \text{where } t = \ldots$$

$$f(t) = (1-t^2)(1 - t + t^2) = 1 - t + t^2 - t^4 + t^3$$

$$f'(t) = 4t^3 + 3t^2 - 1 = 0$$

Product is max when.

$$1 - \sin^2\theta \cos^2\theta = 1 - \sin^4\theta$$

$$\Rightarrow \sin^2\theta \cos^2\theta = \sin^4\theta$$

$$\Rightarrow \tan^2\theta = 1$$

$$\Rightarrow \theta = 45^\circ$$

$$\rightarrow |x\rangle = \cos(45/2)$$

$$\Delta S_x^2 \, \Delta S_y^2 = \frac{\hbar^4}{16}\left(1 - \sin^2\theta \cos^2\phi\right)\left(1 - \sin^2\theta \sin^2\phi\right)$$

Product is max when terms in the bracket are equal

$$\Rightarrow 1 - \sin^2\theta \cos^2\phi = 1 - \sin^2\theta \sin^2\phi$$ why?

$$\Rightarrow \tan\phi = 1 \Rightarrow \phi = 45^\circ$$

$$\Rightarrow \Delta S_x^2 \, \Delta S_y^2 \Big|_{max} = \frac{\hbar^4}{16}\left(1 - \frac{\sin^2\theta}{2}\right)^2$$

this is max when $\theta = 0$

$$\Rightarrow (\theta, \phi) = (0^\circ, 45^\circ) = \left(0, \frac{\pi}{4}\right)$$

$$\Rightarrow \chi = \begin{pmatrix} 1 \\ e^{i\pi/4} \sin 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\phi \rightarrow$ any value

for χ,

$$\Delta S_x^2 = \frac{\hbar^2}{4}, \quad \Delta S_y^2 = \frac{\hbar^2}{4}$$

$$[S_x, S_y] = S_z$$

$$\Rightarrow \cancel{\langle [S_x, S_y] \rangle = |\langle S_z \rangle|^2} = \frac{\hbar^2}{4}$$

$$\Rightarrow \cancel{\frac{1}{4} \left| \langle [\hat{S}_x, \hat{S}_y] \rangle \right|^2} = \frac{\hbar^2}{16}$$

$$\langle S_z \rangle = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar^2}{4}$$

$$\Rightarrow |\langle S_z \rangle|^2 = \frac{\hbar^2}{4}$$

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$\Rightarrow$ equality is satisfied.