Quiz I

From basic quantum mechanics to spin

1. *Absence of degenerate bound states in one spatial dimension:* Two or more quantum states are said to be degenerate if they are described by distinct solutions to the time-independent Schrodinger equation corresponding to the *same* energy. For example, the free particle states are doubly degenerate—one solution describes motion to the right and the other motion to the left. It is not an accident that we have not encountered normalizable degenerate solutions in one spatial dimension. By following the steps listed below, prove that there are no degenerate bound states in one spatial dimension.

   (a) Suppose that there are two solutions, say, ψ_1 and ψ_2, with the same energy E. Using the time-independent Schrodinger equation, show that the quantity
   $$\mathcal{W} = \psi_1 \frac{\mathrm{d}\psi_2}{\mathrm{d}x} - \psi_2 \frac{\mathrm{d}\psi_1}{\mathrm{d}x}$$
   is a constant independent of x. **3 marks**

   (b) Argue that, since the wavefunctions ψ_1 and ψ_2 are normalizable, the quantity $\mathcal{W}$ defined above must vanish. **3 marks**

   (c) If $\mathcal{W} = 0$, integrate the above equation to show that ψ_2 is a multiple of ψ_1 and hence the solutions are not distinct. **4 marks**

2. *The translation operator:* Recall that a wavefunction can be translated by the vector $\boldsymbol{a}$ in three dimensions using the operator
   $$\hat{\mathcal{T}}(\boldsymbol{a}) = \exp -\left(\frac{i\,\hat{\boldsymbol{p}}\cdot\boldsymbol{a}}{\hbar}\right),$$
   where $\hat{\boldsymbol{p}}$ denotes the momentum operator. Let $|\psi\rangle$ be a state vector describing a system and, upon translation, let
   $$|\psi\rangle \to |\bar{\psi}\rangle = \hat{\mathcal{T}}(\boldsymbol{a})|\psi\rangle.$$

   (a) Evaluate the commutation relation: $[\hat{x}_i, \hat{\mathcal{T}}(\boldsymbol{a})]$. **7 marks**

   (b) Using the result of the commutation relation, evaluate $\langle\bar{\psi}|\hat{\boldsymbol{x}}|\bar{\psi}\rangle$ and express it in terms of $\langle\psi|\hat{\boldsymbol{x}}|\psi\rangle$. **3 marks**

3. *Commuting operators and simultaneous eigenkets:* Consider a three-dimensional ket space. If a certain set of orthonormal kets, say, $|1\rangle$, $|2\rangle$ and $|3\rangle$, are used as the base kets, the operators $\hat{A}$ and $\hat{B}$ are found to be represented by the matrices
   $$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -i\,b \\ 0 & i\,b & 0 \end{pmatrix}$$
   with a and b both real.

   (a) Do the operators $\hat{A}$ and $\hat{B}$ represent observables? **1 marks**

   (b) If they do, can the observables be measured simultaneously? **3 marks**

   (c) What is the spectrum of the two operators? Are they degenerate? **2 marks**

   (d) If the $\hat{A}$ and $\hat{B}$ represent observables and, if they can be measured simultaneously, construct a new set of orthonormal kets which are simultaneous eigenkets of both the operators. What are the eigen values of $\hat{A}$ and $\hat{B}$ for these eigenkets? **3+1 marks**

4. *Wave function describing the coherent state:* Earlier, that we had arrived at the wave function $\psi_0(x) = \langle x|0\rangle$ describing the ground state of the oscillator using the following definition of the ground state $|0\rangle$:

$$\hat{a}\,|0\rangle = 0$$

and the position representation of the lowering operator $\hat{a}$. Recall that the coherent state $|\alpha\rangle$ is defined through the relation

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle,$$

where α is a complex number.

(a) Evaluate $\langle\hat{x}\rangle$ and $\langle\hat{p}_x\rangle$ in the state $|\alpha\rangle$. $\boxed{\text{3 marks}}$

(b) Using the above definition of $|\alpha\rangle$, obtain the wave function $\psi_\alpha(x) = \langle x|\alpha\rangle$. Express the wavefunction $\psi_\alpha(x)$ in terms of $\langle\hat{x}\rangle$ and $\langle\hat{p}_x\rangle$. $\boxed{\text{6+1 marks}}$

5. *Maximizing uncertainty:* Recall that, according to the generalized uncertainty principle, the uncertainty associated with two observables represented by the operators $\hat{A}$ and $\hat{B}$ is given by

$$\Delta A^2\,\Delta B^2 \geq \frac{1}{4}\,|\langle[\hat{A},\hat{B}]\rangle|^2,$$

where the uncertainty, say, ΔA^2, is defined as $\Delta A^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$ and the angular brackets as usual represent expectation values evaluated in a given state. Let, as usual, $|+\rangle$ and $|-\rangle$ represent the eigenkets of the $\hat{S}_z$ operator describing a spin-$\frac{1}{2}$ system.

(a) Find the linear combination of the kets $|+\rangle$ and $|-\rangle$ that maximizes the following uncertainty product: $\boxed{\text{7 marks}}$

$$\Delta S_x^2\,\Delta S_y^2.$$

Note: It will be convenient to express the general stated describing the spin-$\frac{1}{2}$ system as follows:

$$|\chi\rangle = \cos(\theta/2)\,|+\rangle + \sin(\theta/2)\,\mathrm{e}^{i\,\phi}\,|-\rangle.$$

(b) Verify explicitly that, for the state $|\chi\rangle$ you have found, the uncertainty relation for the operators $\hat{S}_x$ and $\hat{S}_y$ is not violated. $\boxed{\text{3 marks}}$