PH 5170 End Sem

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(1)
$$\psi(\theta,\phi) = \frac{1}{\sqrt{5}} Y_{1-1} + \sqrt{\frac{3}{5}}\, Y_{10} + \frac{1}{\sqrt{5}} Y_{11}$$

(a)
$$L_+ |l\ m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\ |l\ m+1\rangle$$

$$\Rightarrow L_+ \psi = \frac{1}{\sqrt{5}} \hbar\sqrt{2}\, Y_{10} + \hbar\sqrt{\frac{3}{5}}\sqrt{2}\, Y_{11}$$

$$\Rightarrow \langle \psi | L_+ | \psi \rangle = \hbar \left\langle \frac{1}{\sqrt{5}} Y_{1-1} + \sqrt{\frac{3}{5}} Y_{10} + \frac{1}{\sqrt{5}} Y_{11} \middle| \frac{\sqrt{2}}{\sqrt{5}} Y_{10} + \frac{\sqrt{6}}{\sqrt{5}} Y_{11} \right\rangle$$

$$\Rightarrow \langle L_+ \rangle = \hbar\left(\frac{\sqrt{6}}{5} + \frac{\sqrt{6}}{5}\right) = \frac{2\sqrt{6}}{5}\hbar$$

(b) $-\hbar,\ 0,\ \hbar$

with probabilities $\frac{1}{5}, \frac{3}{5}, \frac{1}{5}$ respectively

(c)
$$L_z = -\hbar$$

$\Rightarrow$ After measurement, particle is in $|1\ -1\rangle$

$$\langle \psi | L_x | \psi \rangle = \left\langle Y_{1,-1} \middle| \frac{L_+ + L_-}{2} \middle| Y_{1,-1} \right\rangle$$

$$= \frac{1}{2}\langle \psi_{1-1} | \sqrt{2}\,\psi_{10} \rangle = 0 \qquad \text{Sly } \langle L_y \rangle = 0$$

$$\langle \psi | L_x^2 | \psi \rangle = \frac{1}{4}\langle \psi_{1-1} | L_+^2 + L_-^2 + L_+L_- + L_-L_+ | \psi_{1,-1} \rangle$$

$$= \frac{1}{4}\langle L_+L_- + L_-L_+ \rangle + 0 + 0 = \frac{1}{2}\langle L^2 - L_z^2 \rangle$$

$$= \left[1(1+1) - (-1)^2\right]\frac{\hbar^2}{2} = \frac{\hbar^2}{2}$$

Sly $\langle L_y^2 \rangle = \frac{\hbar^2}{2}$ $\Rightarrow$ $\Delta L_x = \sqrt{\langle L_x^2 \rangle - \langle L_x \rangle^2} = \sqrt{\frac{\hbar^2}{2} - 0} = \boxed{\frac{\hbar}{\sqrt{2}}} = \Delta L_y$

$$\Rightarrow \Delta L_x \Delta L_y = \frac{\hbar}{\sqrt{2}}\frac{\hbar}{\sqrt{2}} = \frac{\hbar^2}{2} \geq \frac{\hbar}{2}|\hbar| = \frac{\hbar}{2}|L_z|$$

(2)

$$L_i = \epsilon_{ijk} \, x_j \, p_k$$

$$L^2 = L_i L_i = \epsilon_{ijk} \, x_j \, p_k \, \epsilon_{ilm} \, x_l \, p_m$$

$$\Rightarrow L^2 = \left(\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}\right) x_j \, p_k \, x_l \, p_m$$

$$\Rightarrow L^2 = \delta_{jl} \delta_{km} \, x_j \left(x_l p_k - i\hbar \, \delta_{lk} \right) p_m - \delta_{jm} \delta_{kl} \, x_j p_k \left(p_m x_l + i\hbar \, \delta_{lm} \right)$$

$$\Rightarrow L^2 = \delta_{jl} \delta_{km} \, x_j \, x_l \, p_k \, p_m - i\hbar \, \delta_{jl} \delta_{km} \delta_{lk} \, x_j \, p_m$$

$$- \delta_{jm} \delta_{kl} \, x_j \, p_k \left(p_m \, x_l + i\hbar \, \delta_{lm} \right)$$

$$\Rightarrow L^2 = x_i \, x_j \, p_k \, p_m - i\hbar \, x_j \, p_j - i\hbar \, \delta_{jm} \delta_{kl} \delta_{lm} \, x_j \, p_k$$

$$- x_m \, p_m \, \delta_{kl} \, p_k \, x_l$$

$$\Rightarrow L^2 = x_i \, x_j \, p_k \, p_m - i\hbar \, x_j \, p_j - i\hbar \, x_m \, p_m$$

$$- x_m \, p_m \, \delta_{kl} \, p_k \, x_l$$

$$= x^2 p^2 - 2i\hbar \, x \cdot p - (x \cdot p) \left[(x \cdot p) - i\hbar \underbrace{\delta_{kl} \delta_{kl}}_{3} \right]$$

$$= x^2 p^2 - 2i\hbar \, x \cdot p + 3i\hbar \, x \cdot p - (x \cdot p)^2$$

$$= x^2 p^2 - (x \cdot p)^2 + i\hbar \, x \cdot p$$

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③ $\hat{H} = \alpha \hat{S}_1 \cdot \hat{S}_2$

$\Rightarrow H = \frac{\alpha}{2}\left(S^2 - S_1^2 - S_2^2 \right)$

(a) Using CG coeff table,

$$|\tfrac{1}{2}\ \tfrac{1}{2}\rangle_{m_1\, m_2} = \sqrt{\tfrac{3}{4}}\,|2\ 1\rangle_{s_1\, s_2} - \sqrt{\tfrac{1}{4}}\,|1\ 1\rangle_{s_1\, s_2}$$

$$\psi(t) = e^{\frac{i\hat{H}t}{\hbar}}\,|\psi_0\rangle$$

$$= \sqrt{\tfrac{3}{4}}\exp\left(\frac{i\alpha\hbar}{2}\left(2(2+1) - \tfrac{3}{2}\left(\tfrac{3}{2}+1\right) - \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\right)t\right)|2\ 1\rangle$$

$$- \sqrt{\tfrac{1}{4}}\exp\left(\frac{i\alpha\hbar}{2}\left(1(1+1) - \tfrac{3}{2}\left(\tfrac{3}{2}+1\right) - \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\right)\right)|1, 1\rangle$$

$$= \sqrt{\tfrac{3}{4}}\exp\left(\frac{i\alpha\hbar t}{2}\left(\tfrac{3}{2}\right)\right)|2\ 1\rangle - \sqrt{\tfrac{1}{4}}\exp\left(\frac{i\alpha\hbar t}{2}\left(-\tfrac{5}{2}\right)\right)|2\ 1\rangle$$

$$= \sqrt{\tfrac{3}{4}}\,e^{\frac{3i\alpha\hbar t}{4}}\,|2, 1\rangle - \sqrt{\tfrac{1}{4}}\,e^{-\frac{i5\alpha\hbar t}{4}}\,|1, 1\rangle$$

(b) for probability we need to take inner product with $|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{4}}\,|2\ 1\rangle + \sqrt{\tfrac{3}{4}}\,|1\ 1\rangle$

$$\Rightarrow P = \left\langle \sqrt{\tfrac{1}{4}}\exp\left(\frac{3i\alpha\hbar t}{4}\right) + \sqrt{\tfrac{3}{4}}\exp\left(\frac{-5i\alpha\hbar t}{4}\right) \middle| \sqrt{\tfrac{3}{4}}\exp\left(\frac{3i\alpha\hbar t}{4}\right) - \sqrt{\tfrac{1}{4}}\exp\left(\frac{-5i\alpha\hbar t}{4}\right) \right|^2$$

$$= \left| \frac{\sqrt{3}}{4}\exp\left(\frac{3i\alpha\hbar t}{4}\right) - \frac{\sqrt{3}}{4}\exp\left(\frac{-5i\alpha\hbar t}{4}\right)\right|^2 = \left| \frac{\sqrt{3}}{4}\underbrace{\exp\left(\frac{-i\alpha\hbar t}{4}\right)}_{\text{phase}} \frac{\sin(\alpha\hbar t)}{2i}\right|^2$$

$$\Rightarrow P = \frac{(\sqrt{3})^2}{16 \times 4}\sin^2(\alpha\hbar t) = \frac{3}{64}\sin^2(\alpha\hbar t)$$

④ For particle in a box : $\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$

with $$E_n = \frac{n^2\hbar^2\pi^2}{2mL^2}$$

(a) Identical bosons :

The spatial wave f^n has to be symmetric

ground state wave f^n: $\psi^{(0)} = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x_1}{L}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x_2}{L}\right) \cdots \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x_N}{L}\right)$

$$\Rightarrow \psi^{(0)} = \sqrt{\frac{2^N}{L^N}} \sin\left(\frac{\pi x_1}{L}\right) \sin\left(\frac{\pi x_2}{L}\right) \cdots \sin\left(\frac{\pi x_N}{L}\right)$$

$$E^{(0)} = \frac{\hbar^2\pi^2}{2mL^2}\Big(\underbrace{1^2 + 1^2 + \cdots 1^2}_{N \text{ times}}\Big) = \frac{N\hbar^2\pi^2}{2mL^2}$$

(b) Spin $\frac{1}{2}$ fermions :

We have to invoke pauli's Exclusion principle and arrange the particles as follows :

$$\underbrace{|+\rangle\,|-\rangle}_{n=1} \quad \underbrace{|+\rangle\,|-\rangle}_{n=2} \quad \cdots\cdots \quad \underbrace{|+\rangle\,|-\rangle}_{n=N}$$

Evidently, we have to consider cases of N being odd or even

N is even :

$$E^{(0)} = \frac{\hbar^2\pi^2}{2mL^2}\left(2.1^2 + 2.2^2 + \cdots 2\left(\frac{N}{2}\right)^2\right) = \frac{\hbar^2\pi^2}{mL^2}\left(1+4+\cdot\left(\frac{N}{2}\right)^2\right)$$

$$\Rightarrow E^{(0)} = \frac{\hbar^2\pi^2}{mL^2}\left(\frac{\left(\cancel{2}\frac{N}{\cancel{2}}+1\right)\frac{N}{2}\left(\frac{N}{2}+1\right)}{6}\right) = \frac{\hbar^2\pi^2}{mL^2}\,\frac{(N+1)N(N+2)}{24}$$

$$\Rightarrow E^{(0)} = \frac{\hbar^2\pi^2}{mL}\,\frac{N(N+1)(N+2)}{24}$$

$$\psi^{(0)}(x) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(x_1)\chi(s_1) & \psi_1(x_2)\chi(s_2) & \cdots & \psi_N(x_N)\chi(s_N) \\ \psi_1(x_1)\chi(s_1) & \psi_1(x_2)\chi(s_2) & \cdots & \psi_N(x_N)\chi(s_N) \\ \vdots & & & \\ \psi_{N/2}(x_1)\chi(s_1) & \psi_{N/2}(x_2)\chi(s_2) & \cdots & \psi_{N/2}(x_N)\chi(s_N) \\ \psi_{N/2}(x_1)\chi(s_1) & \psi_{N/2}(x_2)\chi(s_2) & \cdots & \psi_{N/2}(x_N)\chi(s_N) \end{vmatrix}$$

<u>N is odd</u>:

$$E^{(0)} = \frac{\hbar^2\pi^2}{2mL^2}\left(2\cdot 1^2 + 2\cdot 2^2 + \cdots 2\left(\frac{N-1}{2}\right)^2 + \left(\frac{N+1}{2}\right)^2\right)$$

$$= \frac{\hbar^2\pi^2}{2mL^2}\left(2\left(1+4+\cdots\left(\frac{N-1}{2}\right)^2\right) + \left(\frac{N+1}{2}\right)^2\right)$$

$$= \frac{\hbar^2\pi^2}{2mL^2}\left(\frac{\cancel{2}\left(\cancel{2}\frac{N-1}{\cancel{2}}+1\right)\frac{N-1}{2}\left(\frac{N-1}{2}+1\right)}{\cancel{6}\,3} + \left(\frac{N+1}{2}\right)^2\right)$$

$$= \frac{\hbar^2\pi^2}{2mL^2}\left(\frac{N}{3}\,\frac{N-1}{2}\,\frac{N+1}{\cancel{2}} + \frac{(N+1)^2}{4}\right) = \frac{\hbar^2\pi^2}{2mL^2}\left(\frac{N(N^2-1)+3(N+1)^2}{12}\right)$$

$$= \frac{\hbar^2\pi^2}{24mL^2}\left(N^3 + 3N^2 + 5N + 3\right)$$

$$\psi^0(x) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(x_1)\chi(s_1) & \psi_1(x_2)\chi(s_2) & \cdots & \psi_N(x_N)\chi(s_N) \\ \psi_1(x_1)\chi(s_1) & \psi_1(x_2)\chi(s_2) & \cdots & \psi_N(x_N)\chi(s_N) \\ \vdots & & & \\ \psi_{\frac{N-1}{2}}(x_1)\chi(s_1) & \psi_{\frac{N-1}{2}}(x_2)\chi(s_2) & \cdots & \psi_{\frac{N-1}{2}}(x_N)\chi(s_N) \\ \psi_{\frac{N+1}{2}}(x_1)\chi(s_1) & \psi_{\frac{N+1}{2}}(x_2)\chi(s_2) & \cdots & \psi_{\frac{N+1}{2}}(x_N)\chi(s_N) \end{vmatrix}$$

(5) (a) $S_x = \frac{\hbar}{2}\sigma_x$, $S_y = \frac{\hbar}{2}\sigma_y$, $S_z = \frac{\hbar}{2}\sigma_z$

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

$$\Rightarrow S^2 = \frac{\hbar^2}{4}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right)$$

We know $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}_{2\times 2}$

$$\Rightarrow S^2 = \frac{\hbar^2}{4}\left(3\mathbb{1}\right) = \frac{3\hbar^2}{4} I_{2\times 2}$$

Evidenty, $|\chi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ is an eigen vector of S^2 with eigen value $\frac{3\hbar^2}{4}$

$\downarrow$

spinor

(b) $S_x = \frac{\hbar}{2}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}$ $S_y = \frac{\hbar}{2}\begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}$ $S_z = \frac{\hbar}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

$$\Rightarrow S^2 = \frac{\hbar^2}{4}\left[\begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_x^2 \end{pmatrix} + \begin{pmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} + \begin{pmatrix} \sigma_z^2 & 0 \\ 0 & \sigma_z^2 \end{pmatrix}\right]$$

$$= \frac{\hbar^2}{4}\left(3\mathbb{1}_{4\times 4}\right) = \frac{3\hbar^2}{4} I_{4\times 4}$$

Evidenty $S^2|\chi\rangle = S^2\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \frac{3\hbar^2}{4}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

$\Rightarrow$ All bi-spinors are eigen vectors of S^2 with eigen value $\frac{3\hbar^2}{4}$