Quiz II

Angular momentum

1. *Commutation relations involving angular momentum:* Evaluate or establish the following commutation relations involving angular momentum:

   (a) $[\hat{L}_i, \hat{x}_j]$, [2 marks]

   (b) $\hat{\boldsymbol{L}} \times \hat{\boldsymbol{r}} + \hat{\boldsymbol{r}} \times \hat{\boldsymbol{L}} = 2\,i\,\hbar\,\hat{\boldsymbol{r}}$, [3 marks]

   (c) $[\hat{\boldsymbol{L}}^2, \hat{\boldsymbol{r}}] = -2\,i\,\hbar\,\hat{\boldsymbol{\Theta}}$, where $\hat{\boldsymbol{\Theta}} = \hat{\boldsymbol{L}} \times \hat{\boldsymbol{r}} - i\,\hbar\,\hat{\boldsymbol{r}}$ [3 marks]

   (d) $[\hat{L}_x^2, \hat{L}_y^2] = [\hat{L}_y^2, \hat{L}_z^2] = [\hat{L}_z^2, \hat{L}_x^2]$. [2 marks]

   Note: $\epsilon_{ijk}\,\epsilon_{ljk} = 2\,\delta_{il}$.

2. *Rotation matrix for $j = 1$:* Consider a system with $j = 1$.

   (a) Construct the matrix describing $\hat{J}_y$. [4 marks]

   (b) Using the matrix representation, show that $\hat{J}_y^3 = \hat{J}_y$. [2 marks]

   (c) As a result, when $j = 1$, show that we can write [4 marks]

$$\exp - \left(\frac{i\,\hat{J}_y\,\alpha}{\hbar} \right) = 1 - i \sin\alpha\, \frac{\hat{J}_y}{\hbar} - (1 - \cos\alpha)\, \frac{\hat{J}_y^2}{\hbar^2}.$$

3. (a) *Energy of a spin-$\frac{3}{2}$ particle:* A spin-$\frac{3}{2}$ particle is described by the Hamiltonian

$$\hat{H} = \frac{\alpha}{\hbar^2} \left(\hat{S}_x^2 + \hat{S}_y^2 - 2\,\hat{S}_z^2 \right) - \frac{\beta}{\hbar}\,\hat{S}_z,$$

   where α and β are real constants. Determine the energy eigen values of the particle. [5 marks]

   (b) *Particle in a central potential:* A particle in a spherically symmetric potential is known to be in the eigenstate $|l, m\rangle$ of the operators $\hat{\boldsymbol{L}}^2$ and $\hat{L}_z$ with eigenvalues $l\,(l+1)\,\hbar^2$ and $m\,\hbar$, respectively.

      i. Evaluate $\langle \hat{L}_x \rangle$, $\langle \hat{L}_y \rangle$ and $\langle \hat{L}_z \rangle$ in the state $|l, m\rangle$. [2 marks]

      ii. Similarly, evaluate $\langle \hat{L}_x^2 \rangle$, $\langle \hat{L}_y^2 \rangle$ and $\langle \hat{L}_z^2 \rangle$ in $|l, m\rangle$. [3 marks]

4. *Expectation value in the singlet state:* Suppose two spin-$\frac{1}{2}$ particles are known to be in the singlet configuration. Let S_{1a} be the component of the spin angular momentum of the first particle in the direction defined by the unit vector $\hat{\boldsymbol{a}}$. Similarly, let S_{2b} be the component of second particle's spin angular momentum in the direction $\hat{\boldsymbol{b}}$. Evaluate the expectation value of the operator $\hat{S}_{1a}\,\hat{S}_{2b}$ in the singlet state. [10 marks]

5. *Addition of spin angular momentum:* Consider a system composed of two particles, one with spin-$\frac{1}{2}$ and another with spin-1.

   (a) List all the allowed spin states of the composite system. [3 marks]

   (b) Express all the spin states of the composite system in terms of the spin states of the individual systems. [7 marks]