End-of-semester exam III

From basics of quantum mechanics to relativistic quantum mechanics

1. *States, measurements and uncertainties:* Consider a system that is initially in the state

$$\psi(\theta, \phi) = \frac{1}{\sqrt{5}}\, Y_{1-1}(\theta, \phi) + \sqrt{\frac{3}{5}}\, Y_{10}(\theta, \phi) + \frac{1}{\sqrt{5}}\, Y_{11}(\theta, \phi),$$

where $Y_{lm}(\theta, \phi)$ are the spherical harmonics.

   (a) Evaluate $\langle \hat{L}_+ \rangle$ in the above state. $\boxed{\text{3 marks}}$

   (b) If $\hat{L}_z$ is measured, what are the values one would obtain and what are the corresponding probabilities? $\boxed{\text{2 marks}}$

   (c) If, after measuring $\hat{L}_z$, one obtains $L_z = -\hbar$, calculate the corresponding uncertainties ΔL_x and ΔL_y as well as their product $\Delta L_x\, \Delta L_y$. $\boxed{\text{5 marks}}$

2. *An operator relation involving angular momentum:* Show that $\boxed{\text{10 marks}}$

$$\hat{\boldsymbol{L}}^2 = \hat{\boldsymbol{x}}^2\, \hat{\boldsymbol{p}}^2 - (\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{p}})^2 + i\, \hbar\, \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{p}}.$$

3. *Interacting spins:* A system of two particles with spins $s_1 = \frac{3}{2}$ and $s_2 = \frac{1}{2}$ is described by the Hamiltonian

$$\hat{H} = \alpha\, \hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2,$$

with α being a given constant. The system is initially (say, at $t = 0$) in the following eigenstate of $\hat{\boldsymbol{S}}_1^2$, $\hat{\boldsymbol{S}}_2^2$, $\hat{\boldsymbol{S}}_{1z}$ and $\hat{\boldsymbol{S}}_{2z}$:

$$\vert s_1\, s_2; m_1\, m_2\rangle = \left\vert \tfrac{3}{2}\, \tfrac{1}{2}; \tfrac{1}{2}\, \tfrac{1}{2}\right\rangle.$$

   (a) Find the state of the system at times $t > 0$. $\boxed{\text{7 marks}}$

   (b) What is the probability of finding the system in the state $\left\vert \tfrac{3}{2}\, \tfrac{1}{2}; \tfrac{3}{2}\, -\tfrac{1}{2}\right\rangle$? $\boxed{\text{3 marks}}$

4. *States and energies of non-interacting bosons and fermions:* Consider a system of N non-interacting, identical particles that are confined to a one-dimensional box with its walls at $x = 0$ and $x = L$. As usual, the potential $V(x)$ is assumed to be zero inside the box and infinity outside. Determine the energy and the wavefunction of the ground state, when the particles are:

   (a) identical bosons, $\boxed{\text{3 marks}}$

   (b) identical spin-$\frac{1}{2}$ fermions. $\boxed{\text{7 marks}}$

5. *Non-relativistic and relativistic spin-$\frac{1}{2}$ particles:* Consider non-relativistic and relativistic spin-$\frac{1}{2}$ particles.

   (a) What is the spin operator $\hat{\boldsymbol{S}}$ describing a non-relativistic spin-$\frac{1}{2}$ particle? What are the eigen values of the corresponding $\hat{\boldsymbol{S}}^2$ operator? Show that *all* spinors are eigen functions of the $\hat{\boldsymbol{S}}^2$ operator? $\boxed{\text{1+1+1 marks}}$

   (b) What is the spin operator $\hat{\boldsymbol{S}}$ describing a *relativistic* spin-$\frac{1}{2}$ particle? What are the eigen values of the corresponding $\hat{\boldsymbol{S}}^2$ operator? Show that *all bispinors* are eigen functions of the $\hat{\boldsymbol{S}}^2$ operator? $\boxed{\text{1+3+3 marks}}$