MM16B023 H. Vishal

PH 5170 - Assignment

①

Spin $= 1 \Rightarrow m_s =$ 0 $|0\rangle$, -1 $|-\rangle$, 1 $|+\rangle$

$$\Psi = \psi(r)\left[\chi_a + \chi_b + \chi_c\right]$$

Following cases are possible : $\Rightarrow$ Spin states have to be symmetric !!

(i) All 3 particles in $|+\rangle$

The spinors have to symmetric for exchange of any 2 particles

Evidently, in this case

$$|\chi\rangle = |+\rangle|+\rangle|+\rangle, \quad s = 3, \ m = 3$$

(ii) 2 particles in $|+\rangle$, one in $|0\rangle$

$$J_-|3\,3\rangle = J_{1-}|+\rangle|+\rangle|+\rangle + J_{2-}|+\rangle|+\rangle|+\rangle + J_{3-}|+\rangle|+\rangle|+\rangle$$

$$\Rightarrow \sqrt{6}\hbar\,|\chi\rangle = \sqrt{2}\hbar\left(|0\rangle|+\rangle|+\rangle + |+\rangle|0\rangle|+\rangle + |+\rangle|+\rangle|0\rangle\right)$$

$$\Rightarrow |\chi\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle|+\rangle|+\rangle + |+\rangle|0\rangle|+\rangle + |+\rangle|+\rangle|0\rangle\right)$$

J_-, J^2 commute

$$\Rightarrow J^2|\chi\rangle = J^2 J_-|+\rangle|+\rangle|+\rangle = J_- J^2|+\rangle|+\rangle|+\rangle$$

$$= 3(3+1)J_-|+\rangle|+\rangle|+\rangle$$

$$\Rightarrow |\chi\rangle = |3, 2\rangle \Rightarrow \text{spin} = 3$$

(iii) All 3 particles in $|-\rangle$

$$|\chi\rangle = |-\rangle|-\rangle|-\rangle$$

$$\Rightarrow s = 3, \quad m = -3$$

(iv) 2 particles in $|-\rangle$, one in $|0\rangle$

similar to (ii), but apply $J_+ = J_{1+} + J_{2+} + J_{3+}$ instead of J_-

$$J_+ |3\ -3\rangle = J_{1+} |-\rangle|-\rangle|-\rangle + J_{2+}|-\rangle|-\rangle|-\rangle + J_{3+}|-\rangle|-\rangle|-\rangle$$

$$\Rightarrow \sqrt{6}\,\hbar\,|\chi\rangle = \sqrt{2}\hbar\big(|0\rangle|-\rangle|-\rangle + |-\rangle|0\rangle|-\rangle + |-\rangle|-\rangle|0\rangle\big)$$

$$\Rightarrow |\chi\rangle = \frac{1}{\sqrt{3}}\big(|0\rangle|-\rangle|-\rangle + |-\rangle|0\rangle|-\rangle + |-\rangle|-\rangle|0\rangle\big)$$

$$[J_+, J^2] = 0 \Rightarrow |\chi\rangle = |3, -2\rangle$$

spin $= 3$

(v) 2 particles in $|0\rangle$, one in $|+\rangle$ spin $= 1$

$$|\chi\rangle = \frac{1}{\sqrt{3}}\big(|0\rangle|0\rangle|+\rangle + |+\rangle|0\rangle|0\rangle + |0\rangle|+\rangle|0\rangle\big)$$

(vi) 2 particles in $|0\rangle$, one in $|-\rangle$ spin $= 1$

$$|\chi\rangle = \frac{1}{\sqrt{3}}\big(|0\rangle|0\rangle|-\rangle + |-\rangle|0\rangle|0\rangle + |0\rangle|-\rangle|0\rangle\big)$$

(vii) 2 particles in $|+\rangle$, one in $|-\rangle$ spin $= 2$

$$|\chi\rangle = \frac{1}{\sqrt{3}}\big(|+\rangle|-\rangle|+\rangle + |-\rangle|+\rangle|+\rangle + |+\rangle|+\rangle|-\rangle\big)$$

(viii) 2 particles in $|-\rangle$, one in $|+\rangle$ spin $= 2$

$$|\chi\rangle = \frac{1}{\sqrt{3}}\big(|-\rangle|-\rangle|+\rangle + |-\rangle|+\rangle|-\rangle + |+\rangle|-\rangle|-\rangle\big)$$

(ix) All three different

$$|\chi\rangle = \frac{1}{\sqrt{6}}\Big[|+\rangle|0\rangle|-\rangle + |-\rangle|+\rangle|0\rangle + |0\rangle|-\rangle|+\rangle + |0\rangle|+\rangle|-\rangle + |-\rangle|0\rangle|+\rangle + |+\rangle|-\rangle|0\rangle\Big]$$

$S_z|\chi\rangle = 0$, but $S^2|\chi\rangle \neq 0$ spin > 2, < 3 (not integer)

Total 10 states possible

(x) $|\chi\rangle = |0\rangle|0\rangle|0\rangle$ spin $= 0$

MM16B023 H. Vishal

② a) $$E_{tot} = \frac{A}{R^2} - \frac{B}{R}$$

$$A = \frac{2\hbar^2}{15\pi M}\left(\frac{9}{4}\pi N q\right)^{5/3}, \qquad B = \frac{3}{5} G N^2 M^2$$

$M \equiv$ mass of nucleon $= 1.674 \times 10^{-27}$ kg

$N \equiv$ no. of electrons per nucleon in sun $= \dfrac{1.989 \times 10^{30}\ \cancel{Kg}}{1.674 \times 10^{-27}\ \cancel{Kg}} = 1.188 \times 10^{57}$

For eqlm radius,

$$\frac{dE_{tot}}{dR} = -\frac{2A}{R^3} + \frac{B}{R^2} = 0 \Rightarrow R = \frac{2A}{B} = \frac{4\hbar}{15M\pi}\left(\frac{9}{4}\pi N q\right)^{5/3}\frac{5}{3GN^2M^2}$$

$$\Rightarrow R = \left[\left(\frac{4}{9\pi}\right)\left(\frac{9\pi}{4}\right)^{5/3}\right]\left(\frac{N^{5/3}}{N^2}\right)\frac{\hbar^2}{GM^3} q^{5/3} = \left(\frac{9\pi}{4}\right)^{2/3}\frac{\hbar^2}{GM^3}\frac{q^{5/3}}{N^{1/3}}$$

$$\Rightarrow R = \left(\frac{9\pi}{4}\right)^{2/3} \frac{(1.055 \times 10^{-34}\ Js)^2\ 1^{5/3}}{(6.673 \times 10^{-11}\ Nm^2 kg^{-2})(1.674 \times 10^{-27}\ kg)^3} (1.88 \times 10^{57})^{-1/3}$$

$$= (1.31 \times 10^{23}\ m)(1.88 \times 10^{57})^{-1/3} = 1.24 \times 10^4\ m = 12.4\ km$$

b) $$E_F = \frac{\hbar^2}{2M}\left(3\pi^2 \frac{Nq}{\frac{4}{3}\pi R^3}\right)^{2/3} = \frac{\hbar^2}{2MR^2}\left(\frac{9\pi N q}{4}\right)^{2/3}$$

$$\Rightarrow E_F = \frac{(1.055 \times 10^{-34}\ Js)^2}{2(1.674 \times 10^{-27}\ kg)(1.24 \times 10^4 m)^2}\left(\frac{9 \times 3.14}{4} \times 1.188 \times 10^{57} \times 1\right)^{2/3}$$

$$\Rightarrow E_F = 8.36 \times 10^{-12}\ J \approx 56 \times 10^6\ eV = 56\ MeV$$

Rest energy of neutron $= Mc^2 = (1.674 \times 10^{-27}\ kg)(3 \times 10^8\ ms^{-1})^2 \approx 939.4$ MeV

Comparing these two, neutron star is non-relativistic

③ (a) Using partial wave analysis,

scattering amplitude $f(\theta) = \sum_{l=0}^{\infty} (2l+1)\, a_l\, P_l(\cos\theta)$

$a_l \rightarrow$ partial amplitude of wave with orbital angular momentum (l)

$$D(\theta) = \frac{d\sigma}{d\Omega} = \underbrace{|f(\theta)|^2}_{f(\theta)\, f(\theta)^*} = \sum_{l}\sum_{l'} (2l+1)(2l'+1)\, a_l\, a^*_{l'}\, P_l(\cos\theta)\, P_{l'}(\cos\theta)$$

Using, $\int_{-1}^{1} P_l(x)\, P_{l'}(x)\, dx = \frac{2}{2l+1}\,\delta_{ll'} \Leftrightarrow \int_0^{\pi} P_l(\cos\theta)\, P_{l'}(\cos\theta)\sin\theta\, d\theta = \frac{2}{2l+1}\,\delta_{ll'}$, ($x = \cos\theta$)

$$\sigma = \left(\int_0^{2\pi} d\phi\right) \sum_{l=0}^{\infty} (2l+1) \sum_{l'=0}^{\infty} (2l'+1)\, a_l\, a^*_{l'} \int_0^{\pi} P_l(\cos\theta)\, P_{l'}(\cos\theta)\sin\theta\, d\theta$$

$$\Rightarrow \sigma = (2\pi) \sum_{l=0}^{\infty} (2l+1) \sum_{l'=0}^{\infty} \cancel{(2l'+1)}\, a_l\, a^*_{l'}\, \frac{2}{\cancel{(2l'+1)}}\,\delta_{ll'}$$

$$\Rightarrow \sigma = 4\pi \sum_{l=0}^{\infty}\sum_{l'=0}^{\infty} (2l+1)\, a_l\, a^*_{l'}\,\delta_{ll'} = 4\pi \sum_{l=0}^{\infty} (2l+1)\, a_l\, a^*_l$$

$$\Rightarrow \sigma = 4\pi \sum_{l=0}^{\infty} (2l+1)\, |a_l|^2$$

(b)

$$\psi(\vec{r},\theta) = e^{ikz} + f(\theta)\,\frac{e^{ikr}}{r}$$

$$= A \sum_{l=0}^{\infty} i^l (2l+1)\, j_l(kr)\, P_l(\cos\theta) + A f(\theta)\,\frac{e^{ikr}}{r}$$

Further expanding,

$$\psi(r,\theta) = \sum_{i=0}^{\infty} i^{l}(2l+1)\left[j_l(kr) + ik\, a_l\, h_l^{1}(kr)\right] P_l(\cos\theta)$$

$$j_l(x) \approx \frac{1}{2x}\left[(-i)^{l+1} e^{ix} + i^{l+1} e^{-ix} \right] \quad \text{For large } x$$

$$h_l^{1}(kr) \approx (-1)^{l+1} \frac{e^{ikr}}{kr} \quad \text{For large } r$$

Therefore, for large r,

$$\psi(r,\theta) \approx A\sum_{l=0}^{\infty}(2l+1)\frac{i^{l}}{2kr}\Big[\underbrace{(-i)^{l+1} e^{ikr}}_{\text{scattered}} + \overbrace{i^{l+1} e^{-ikr}}^{\text{incident}} \Big] P_l(\cos\theta)$$

$$+ \underbrace{A \sum_{l=0}^{\infty} \frac{(2l+1)}{r} a_l\, e^{ikr} P_l(\cos\theta)}_{\text{scattered}} \quad - ①$$

Further simplifying,

$$\psi(r,\theta) \approx A \sum_{l=0}^{\infty} (2l+1)\frac{1}{2ikr}\left(e^{ikr} + (-1)^{l+1} e^{-ikr}\right) P_l(\cos\theta)$$

$$+ A \sum_{l=0}^{\infty} \frac{(2l+1)}{r} a_l\, e^{ikr} P_l(\cos\theta)$$

The nature of the terms mentioned in ① is very imp to compare it with the phase shift eqn:

$$\psi(r,\theta) = A \sum_{l=0}^{\infty} (2l+1) \frac{1}{2ikr}\left(e^{ikr+2i\delta_l} + (-1)^{l+1} e^{-ikr}\right)(P_l \cos\theta) \quad - ②$$

Equating ① and ②, (individual terms of summation)

$$\cancel{A P_l(\cos\theta)}\cancel{(2l+1)}\, a_l \frac{e^{\cancel{ikr}}}{\cancel{r}} = \cancel{A P_l(\cos\theta)}\cancel{(2l+1)}\left[\frac{1}{2ik\cancel{r}}\left(e^{2i\delta_l} - 1\right) e^{\cancel{ikr}} + \cancel{(-1)^{l+1} e^{-ikr}} - \cancel{(-1)^{l+1} e^{-ikr}}\right]$$

$$\Rightarrow a_l = \frac{1}{2ik}\underbrace{\left(e^{2i\delta_l} - 1\right)}_{e^{i\delta_l}\left(e^{i\delta_l} - e^{-i\delta_l}\right)} = \frac{1}{\cancel{2i}k} e^{i\delta_l}\, \cancel{2i} \sin(\delta_l)$$

$$\Rightarrow \quad a_l = \frac{1}{K} e^{i\delta_l} \sin(\delta_l)$$

(c) Substituting a_l in the result obtained using partial wave analysis,

$$f(\theta) = \frac{1}{K} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin(\delta_l) P_l(\cos\theta)$$

~~The additional terms are independent of~~ θ,

~~Therefore setting $a_l = \frac{1}{K} e^{i\delta_l} \sin(\delta_l)$ in the result obtained in (a),~~

$$\cancel{\sigma = 4\pi \sum_{l=0}^{\infty} (2l+1) |a_l|^2 \sin \delta_l \, P_l(\cos\theta)}$$

$$\cancel{\Rightarrow \sigma = 4\pi \sum_{l=0}^{\infty} (2l+1) \frac{1}{K^2} e^{2i\delta_l} \sin^2(\delta_l) P_l(\cos\theta)}$$

From (a),

$$\sigma = 4\pi \sum_{l=0}^{\infty} (2l+1) a_l a_l^* P_l(\cos\theta)$$

$$\Rightarrow \sigma = 4\pi \sum_{l=0}^{\infty} (2l+1) \frac{\cancel{e^{i\delta_l}} \sin(\delta_l)}{K} \frac{\cancel{e^{-i\delta_l}} \sin(\delta_l)}{K} P_l(\cos\theta)$$

$$\Rightarrow \sigma = \frac{4\pi}{K^2} \sum_{l=0}^{\infty} (2l+1) \sin^2(\delta_l)$$

(4) for spherically symmetric potentials,

$$f(\theta) \approx \frac{-2m}{\hbar^2 K} \int_0^\infty r\, V(r) \sin(Kr)\, dr$$

(Born appx.)

$$K = 2k \sin(\theta/2)$$

(a) $$V(r) = \alpha e^{-\mu r}$$

$$\Rightarrow f(\theta) = \frac{-2m\alpha}{\hbar^2 K} \int_0^\infty r e^{-\mu r} \sin(Kr)\, dr$$

$$= \frac{-2m\alpha}{\hbar^2 K} \int_0^\infty \frac{r}{2i} \left[e^{-(\mu - iK)r} - e^{-(\mu + iK)r} \right] dr$$

$$= \frac{-\cancel{2}m\alpha}{\cancel{2}i \hbar^2 K} \left(\frac{e^{-(\mu - iK)r}}{(\mu - iK)^2} (1 + (\mu - iK)r) \Big|_\infty^0 + \frac{e^{-(\mu + iK)r}}{(\mu + iK)^2} (1 + (\mu + iK)r) \Big|_0^\infty \right)$$

$$= \frac{-m\alpha}{i\hbar^2 K} \left(\frac{1}{(\mu - iK)^2} - 0 + 0 - \frac{1}{(\mu + iK)^2} \right)$$

$$= \frac{-m\alpha}{\cancel{i}\hbar^2 \cancel{K}} \left(\frac{(2\mu)(2\cancel{i}\cancel{K})}{(\mu^2 + K^2)^2} \right) = \frac{-4m\alpha\mu}{\hbar^2 (\mu^2 + K^2)^2}$$

$$D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{16 m^2 \alpha^2 \mu^2}{\hbar^4 (\mu^2 + K^2)^4}$$

$$\Rightarrow \sigma = \int_0^\pi \frac{16 m^2 \alpha^2 \mu^2}{\hbar^4 (\mu^2 + K^2)^4} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$\Rightarrow \sigma = \frac{32\pi m^2 \alpha^2 \mu^2}{\hbar^4} \int_0^{\pi} \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) d\theta}{\left(\mu^2 + 4k^2 \sin^2\left(\frac{\theta}{2}\right)\right)^4}$$

$$\Rightarrow \sigma = \frac{32\pi m^2 \alpha^2}{\hbar^4} \mu^2 \frac{1}{\mu^8} \int_0^{\pi} \frac{1}{\left(1+\left(\frac{2k}{\mu}\right)^2 \sin^2\frac{\theta}{2}\right)^4} 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) d\theta$$

Let $\frac{2k}{\mu}\sin\left(\frac{\theta}{2}\right) \equiv x \Rightarrow \cos\left(\frac{\theta}{2}\right) d\theta = \frac{\mu}{k} dx$

$$\Rightarrow \sigma = \frac{32\pi m^2 \alpha^2}{\hbar^4 \mu^6}\left(\frac{\mu}{k}\right)^2 \int_0^{\left(\frac{2k}{\mu}\right)} \frac{x\,dx}{(1+x^2)^4} = \frac{32\pi m^2 \alpha^2}{\hbar^4 \mu^6} \frac{\mu^2}{k^2} \left.\frac{-1}{6}\frac{1}{(1+x^2)^3}\right|_0^{(2k/\mu)}$$

$$\Rightarrow \sigma = \frac{32\pi m^2 \alpha^2}{6\hbar^4 \mu^4 k^2}\left[1 - \frac{1}{\left(1+\frac{4k^2}{\mu^2}\right)^3}\right]$$

(b) $V(r) = \alpha/r^2$

$$\Rightarrow f(\theta) = \frac{-2m\alpha}{\hbar^2 K} \int_0^{\infty} \frac{1}{r} \sin(Kr)\, dr$$

Feynman's trick : Let $I(a) = \int_0^{\infty} \frac{\sin x}{x} e^{-ax} dx$

$$I'(a) = -\int_0^{\infty} \frac{\sin x}{\cancel{x}} e^{-ax} \cancel{x}\, dx$$

$$\Rightarrow I'(a) = -\left[\cancel{\left.\frac{-\sin x\, e^{-ax}}{a}\right|_0^{\infty}} + \int_0^{\infty} \frac{\cos x\, e^{-ax}}{a}\right]$$

$$\Rightarrow I'(a) = -\left[\left.\frac{\cos x\, e^{-ax}}{a^2}\right|_{\infty}^{0} - \int_0^{\infty} \frac{\sin x\, e^{-ax}}{a^2}\right]$$

$$\Rightarrow I'(a)\left(1+\frac{1}{a^2}\right) = -\frac{1}{a^2} \Rightarrow I'(a) = \frac{-1}{1+a^2}$$

$$\Rightarrow I(a) = -\tan^{-1} a + c = \int_0^\infty \frac{\sin x}{x} e^{-ax}\, dx$$

$$a \to \infty \Rightarrow -\frac{\pi}{2} + c = \int_0^\infty 0\, dx = 0 \Rightarrow c = \frac{\pi}{2}$$

$$I(0) = \int_0^\infty \frac{\sin x}{x}\, dx = -\tan^{-1}(0) + \frac{\pi}{2} = \frac{\pi}{2}$$

$$\Rightarrow f(\theta) = \frac{-2m\alpha}{\hbar^2 K} \int_0^\infty \frac{1}{Kr} \sin(Kr)\, d(Kr) = \frac{-\cancel{2}m\alpha}{\hbar^2 K} \frac{\pi}{\cancel{2}}$$

$$\Rightarrow f(\theta) = \frac{-m\alpha\pi}{\hbar^2 K}$$

$$\sigma = \int D(\theta)\, d\Omega = \int |f(\theta)|^2\, d\Omega = \int_0^\pi \frac{\pi^2 m^2 \alpha^2}{\hbar^4 K^2} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$\Rightarrow \sigma = \frac{\cancel{2}\pi^3 m^2 \alpha^2}{\hbar^4 \cancel{4} k^2} \int_0^\pi \frac{\cancel{2}\cancel{\sin\left(\frac{\theta}{2}\right)} \cos\left(\frac{\theta}{2}\right)}{\cancel{4}_2 \sin^{\cancel{2}}\left(\frac{\theta}{2}\right)}\, d\theta$$

$$\Rightarrow \sigma = \frac{\pi^3 m^2 \alpha^2}{\hbar^4 k^2} \int_0^\pi \frac{1}{2} \cot\left(\frac{\theta}{2}\right)\, d\theta$$

$$= \frac{\pi^3 m^2 \alpha^2}{\hbar^4 k^2} \log\left(\sin\frac{\theta}{2}\right)\Big|_0^\pi$$

→ Undefined at $\theta = 0$

⇒ Integral doesn't converge

⑤ (a) $[H, P] = [H, P_1]\,\hat{x} + [H, P_2]\,\hat{y} + [H, P_3]\,\hat{z}$

$[H, P_1] = [\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \beta mc^2, P_1]$

$= [\alpha_1 P_1, P_1] + [\alpha_2 P_2, P_2] + [\alpha_3 P_3, P_1] + [\beta mc^2, P_1]$

$[\alpha_1 P_1, P_1] = [\alpha_1, P_1]\,P_1 + \alpha_1 \cancel{[P_1, P_1]} \to 0$

$[\alpha_1, P_1] = \alpha_1 P_1 - P_1 \alpha_1 = \begin{pmatrix} 0 & \sigma_1 P_1 \\ \sigma_1 P_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & P_1 \sigma_1 \\ P_1 \sigma_1 & 0 \end{pmatrix}$

As $\sigma_1 P_1 = P_1 \sigma_1$, $[\alpha_1, P_1] = 0$

$[\alpha_2 P_2, P_1] = [\alpha_2, P_1]\,P_2 + \alpha_2 \cancel{[P_2, P_1]}^{\,0}$

$= \begin{pmatrix} 0 & \sigma_2 P_1 P_2 \\ \sigma_2 P_1 P_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & P_2 P_1 \sigma_2 \\ P_2 P_1 \sigma_2 & 0 \end{pmatrix} = 0$ as

$P_1 P_2 \sigma_2 = P_2 P_1 \sigma_2$

$\begin{pmatrix} 0 & -iP_1P_2 \\ iP_1P_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -iP_2P_1 \\ iP_2P_1 & 0 \end{pmatrix}$

Sly, $[\alpha_3 P_3, P_1] = 0$

$[\beta mc^2, P_1] = 0$ (evidently true)

Also, H has 'similar' ~~equal~~ contribution from P_1, P_2, P_3 with weights $\sigma_1, \sigma_2, \sigma_3$

$\Rightarrow [H, P_2] = [H, P_3] = 0$ since $[H, P_1] = 0$

$\Rightarrow [P, H] = -[H, P] = -[H, P_1] - [H, P_2] - [H, P_3] = 0$

~~One important observation is that since the momentum operators commute, we can directly work with say $P = P_3\hat{z}$ and if $[A, P_3] = 0$ then $[A, P] = 0$. We will use this in (b) and (c)~~

~~$A = p \cdot \Sigma$~~

(b) Choosing x_3 - axis along the direction of momentum,

$$p \cdot \Sigma = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

$$[p \cdot \Sigma, p_3] = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} p_3 - \frac{1}{2} p_3 \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \sigma_3 p_3 - p_3\sigma_3 & 0 \\ 0 & \underbrace{\sigma_3 p_3 - p_3 \sigma_3}_{0} \end{pmatrix} = 0$$

Sly, $[p \cdot \Sigma, p_2] = [p \cdot \Sigma, p_1] = 0$

$$\Rightarrow [p \cdot \Sigma, p] = [p \cdot \Sigma, p_1] + [p \cdot \Sigma, p_2] + [p \cdot \Sigma, p_3] = 0$$

(c) Considering x_3 as the momentum quantization as in (b),

$$[H, p \cdot \Sigma] = \frac{1}{2}(\bar{\alpha} \cdot \bar{p} + \beta mc^2)\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}(\bar{\alpha} \cdot \bar{p} + \beta mc^2)$$

$$= \frac{1}{2}\begin{pmatrix} 0 & \sigma_3 p_3 \\ \sigma_3 p_3 & 0 \end{pmatrix}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} + \cancel{\frac{mc^2}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}}$$

$$- \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 p_3 \\ \sigma_3 p_3 & 0 \end{pmatrix} - \cancel{\frac{mc^2}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & \sigma_3 p_3 \sigma_3 \\ \sigma_3 p_3 \sigma_3 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & \sigma_3 \sigma_3 p_3 \\ \sigma_3 \sigma_3 p_3 & 0 \end{pmatrix}$$

$$\sigma_3 p_3 \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} p_3 & 0 \\ 0 & -p_3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} p_3 & 0 \\ 0 & p_3 \end{pmatrix}$$

$$\sigma_3 \sigma_3 p_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} p_3 = \begin{pmatrix} p_3 & 0 \\ 0 & p_3 \end{pmatrix}$$

$$\Rightarrow [H, p \cdot \Sigma] = 0$$