INDIAN INSTITUTE OF TECHNOLOGY MADRAS

A

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Quiz I [ ] Quiz II/ Mid-Sem [✓] End-Semester [ ] Make-up [ ] Date : March 6th, 2020

Semester & Degree : IDDD DS 8th Sem Course No. PH5170 Part :

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*Answer on both sides of the paper including the space below*

① (a) $$[\hat{L}_i , \hat{x}_j]$$

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x$$

$$[L_z, x] = [xp_y, x] - [yp_x, x] = i\hbar y$$

$$[L_z, y] = [xp_y, y] - [yp_x, y] = -i\hbar x$$

$$[L_z, z] = [xp_y - yp_x, z] = 0$$

x

z y

Hly, $$[L_x, x] = [L_y, y] = 0$$

$$[L_x, y] = i\hbar z, \quad [L_x, z] = -i\hbar y$$

2

$$\Rightarrow [\hat{L}_i, \hat{x}_j] = i\hbar \epsilon_{ijk} x_k$$

(b) $$\hat{L} \times \hat{r} + \hat{r} \times \hat{L}$$

$$\hat{r} \times \hat{L} = \hat{r} \times (\hat{r} \times \hat{p}) = (\hat{r}.\hat{r})\hat{p} - (\hat{r}.\hat{p})\hat{r} \quad -①$$

$$\hat{L} \times \hat{r} = -(\hat{r}.\hat{r})\hat{p} + (\hat{r}.\hat{p})\hat{r} \quad -②$$

$\Rightarrow$ ① + ② $\Rightarrow$ $\bar{L} \times \bar{r} + \bar{r} \times L = (\hat{r}\hat{p} - \hat{p}\hat{r})\,\hat{r}$

$\Rightarrow$ $\bar{L} \times \bar{r} + \bar{r} \times L = [\hat{r}, \hat{p}]\,\hat{r} = 2i\hbar\,\hat{r}$

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(c) $[\hat{L}^2, \hat{r}]$

$= \hat{L}^2 \hat{r} - \hat{r} \hat{L}^2$

$= \hat{L}[\hat{L}, \hat{r}] + [\hat{L}, \hat{r}]\,\hat{L}$

$= \hat{L}[\hat{r} \times \hat{p}, \hat{r}] + [\bar{r} \times \bar{p}, \hat{r}]\,\hat{L}$

3

$\cancel{[\hat{L}, \hat{r}]} = \cancel{(\hat{r} \times \hat{p})\hat{r} - \hat{r}(\hat{r} \times \bar{p})}$

$\Rightarrow \cancel{[\hat{L}^2, \hat{r}]} = \cancel{LHS = \hat{L}(\hat{r} \times \hat{p})\hat{r} - \hat{r}(\hat{r} \times \hat{p})\hat{L}}$

$\hat{L}[\hat{L}, \hat{r}]_i = \epsilon\,(\bar{r} \times \bar{p})\,i\hbar\,\epsilon_{ljk}\,r_l = \epsilon_{ijk}\,\cancel{\epsilon_{ljk}}$

$= \epsilon_{ijk}\, r_i\, p_j\, i\hbar\,\epsilon_{ljk}\, r_l = + i\hbar\,\delta_{il}\, r_l$

$\{\hat{L}[\hat{L}, \hat{r}] + [\hat{L}, \hat{r}]\,L\}_i = + 2i\hbar\, r_i = -2i\hbar(-r_i) = -2i\hbar\,\hat{\theta}_i$

(d) $\cancel{\hat{L}^2_x, L_y}\ \cancel{[L_x^2 + L_y^2 + L_z^2, L_x^2]} = 0$

$[L_x^2, L_y^2] = L_y[L_x^2, L_y] + \cancel{L_y}[L_x^2, L_y]\,L_y$

$= L_y\left(L_x[L_x L_y] + [L_x, L_y]\,L_x\right) + (L_x[L_x, L_y]) + [L_x, L_y]L_x\, L_y)$

$= \cancel{L_y(L_x L_z \quad L_z L_x)} + \cancel{(L_x L_z \quad L_z L_y)}$

$= L_y L_x [L_x, L_y] + L_y [L_x L_y]\, L_x + L_x [L_x, L_y]\, L_y$

$= i\hbar\left(L_y L_x L_z + L_y L_z L_x + L_x L_z L_y + L_z L_x L_y\right) + [L_x, L_y]\, L_x L_y$

$= 0$ No!

Sly, $[L_x^2, L_z^2] = [L_y^2, L_z^2] = 0$

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② (a) $\hat{J}_y = \dfrac{J_+ - J_-}{2i}$

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \chi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \chi_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$J_z \chi_+ = \hbar \chi_+$, $J_z \chi_0 = 0$, $J_z \chi_- = -\hbar \chi_-$

$J_+ \chi_+ = 0$ $\quad J_+ \chi_0 = \hbar\sqrt{2}\, \chi_+$, $J_+ \chi_- = \hbar\sqrt{2}\, \chi_0$

$J_- \chi_+ = \hbar\sqrt{2}\, \chi_0$, $J_- \chi_0 = \hbar\sqrt{2}\, \chi_-$, $J_- \chi_- = 0$

$J_+$ matrix :

$$\begin{pmatrix} \langle 1\,1 | \hat{J}_+ | 1\,1 \rangle & \langle 1\,1 | \hat{J}_+ | 1\,0 \rangle & \langle 1\,1 | \hat{J}_+ | 1\,{-1} \rangle \\ \langle 1\,0 | \hat{J}_+ | 1\,1 \rangle & \langle 1\,0 | \hat{J}_+ | 1\,0 \rangle & \langle 1\,0 | \hat{J}_+ | 1\,{-1} \rangle \\ \langle 1\,{-1} | \hat{J}_+ | 1\,1 \rangle & \langle 1\,{-1} | \hat{J}_+ | 1\,0 \rangle & \langle 1\,{-1} | \hat{J}_+ | 1\,{-1} \rangle \end{pmatrix}$$

$$= \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Sly, $\quad J_- = \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

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$\rightarrow J_y = \dfrac{J_+ - J_-}{2i} = \dfrac{i\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$$J_y^3 = J_y(J_y^2) \quad (\text{scale } \hbar)$$

2

$$J_y^2 = \frac{-1}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$J_y^3 = \frac{+i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\frac{-1}{2}\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = J_y$$

$$\exp\left(\frac{-i\hat{J}_y\alpha}{\hbar}\right) = 1 - \frac{iJ_y}{\hbar}\alpha - \left(\frac{J_y}{\hbar}\right)^2\frac{\alpha^2}{2} + i\left(\frac{J_y}{\hbar}\right)^3\frac{\alpha^3}{3!} + \left(\frac{J_y}{\hbar}\right)^4\frac{\alpha^4}{4!} - i\left(\frac{J_y}{\hbar}\right)^5\frac{\alpha^5}{5!} + \dots$$

$$= 1 - \frac{iJ_y}{\hbar}\alpha - \left(\frac{J_y}{\hbar}\right)^2\frac{\alpha^2}{2} + i\frac{J_y}{\hbar}\frac{\alpha^3}{3!} + \underbrace{\left(\frac{J_y}{\hbar}\right)^3}_{\frac{J_y}{\hbar}}\left(\frac{J_y}{\hbar}\right)\frac{\alpha^4}{4!} - \dots$$

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$$= 1 - \frac{iJ_y}{\hbar}\left[\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} \dots\right] - \left(\frac{J_y}{\hbar}\right)^2\left[1 - 1 + \frac{\alpha^2}{2} - \frac{\alpha^4}{4!} \dots\right]$$

$$= 1 - \left(\frac{iJ_y}{\hbar}\right)\sin\alpha - \left(\frac{J_y}{\hbar}\right)^2(1 - \cos\alpha)$$

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(3)

$$\hat{H} = \frac{\alpha}{\hbar^2}\left(\hat{S}_x^2 + \hat{S}_y^2 - 2\hat{S}_z^2\right) - \frac{\beta}{\hbar}\hat{S}_z$$

$$j = 3/2$$

$$\Rightarrow \hat{H} = \frac{\alpha}{\hbar^2}\left(\hat{S}^2 - 3\hat{S}_z^2\right) - \frac{\beta}{\hbar}\hat{S}_z$$

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$$\hat{H}|3/2\ m\rangle = \frac{\alpha}{\hbar^2}\left(S^2|3/2\ m\rangle - 3S_z^2|3/2\ m\rangle\right) - \frac{\beta}{\hbar}|3/2\ m\rangle$$

$$= \frac{\alpha}{\hbar^2}\left(\left(\frac{3}{2}\right)\left(\frac{3}{2}+1\right)\hbar^2|3/2, m\rangle - 3m^2|3/2, m\rangle\right) - \beta/\hbar\,|3/2, m\rangle$$

$$= \left(\frac{\alpha}{\hbar^2}\left(\frac{15}{4}\hbar^2 - 3m^2\hbar^2\right) - \frac{\beta}{\hbar^2}m\hbar\right)|3/2, m\rangle$$

$$\Rightarrow E = \alpha\left(\frac{15}{4} - 3m^2\right) - m\beta$$

where $m = -3/2, -1/2, 1/2, 3/2$

$E_{\ell m}$ : $E_{3/2\ 3/2} = -3\alpha - 3/2\,\beta$ | $E_{3/2\ 1/2} = 3\alpha - \frac{\beta}{2}$

$E_{3/2\ -3/2} = -3\alpha + 3/2\,\beta$ | $E_{3/2\ -1/2} = 3\alpha + \frac{\beta}{2}$

b) $L_\pm|lm\rangle \propto |l, m\pm1\rangle$

Since $L_x$, $L_y$ are linear combinations of $L_+$, $L_-$

$\langle L_x\rangle, \langle L_y\rangle = 0$

for a) $\langle lm| L_x | lm\rangle$

$= \langle lm | \frac{L_+ + L_-}{2} | lm\rangle$

$= \frac{1}{2}\langle lm | L_+ + L_- | lm\rangle$

$= \frac{1}{2}\left[ \langle lm | L_+ | lm\rangle + \langle lm | L_- | lm\rangle \right]$

$= \frac{1}{2}\left[ \alpha \cancelto{0}{\langle lm | l, m+1\rangle} + \beta \cancelto{0}{\langle lm | l, m-1\rangle} \right]$

$= 0$

Sly, $\langle lm | L_y | lm\rangle = \frac{1}{2i}\left[ \langle lm | L_+ | lm\rangle - \langle lm | L_- | lm\rangle \right]$

$= \frac{0-0}{2i} = 0$

$\langle L_z\rangle = \langle lm | \hat{L}_z | lm\rangle = m\hbar \langle lm | lm\rangle = m\hbar$

$L_x^2 = \frac{1}{4}\left( L_+^2 + L_+L_- + L_-L_+ + L_-^2 \right)$

$\Rightarrow \langle L_x^2\rangle = \frac{1}{4}\left( L_+L_- + L_-L_+ \right) + 0 + 0$

$= \frac{1}{2}\langle L^2 - L_z^2\rangle = \left[ l(l+1) - m^2 \right]\frac{\hbar^2}{2} = \langle L_y^2\rangle$

$$\langle L_z^2 \rangle = \langle lm | L_z^2 | lm \rangle = m^2\hbar^2 \langle lm | lm \rangle$$

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$$= m^2\hbar^2$$

$$|0\ 0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \quad \text{Singlet state}$$

$$\langle 00 | \hat{S}_a \ \hat{S}_{2b} | 00 \rangle = ?$$

$$\hat{S}_{1a} = \hat{S}_{z1}, \quad \hat{S}_{2b} = \cos\theta\, \hat{S}_{z_2} + \sin\theta\, \hat{S}_{x2}$$

$$\hat{S}_{a1}\ \hat{S}_{b2} |0\ 0\rangle = \frac{1}{\sqrt{2}}\left[\hat{S}_{z1}\left(\cos\theta\, \hat{S}_{z2} + \sin\theta\, \hat{S}_{x2}\right)\right](\uparrow\downarrow - \downarrow\uparrow)$$

$$= \frac{1}{\sqrt{2}}\left[(\hat{S}_z\uparrow)\left(\cos\theta\, \hat{S}_z\downarrow + \sin\theta\, \hat{S}_x\downarrow\right) - \hat{S}_z\downarrow\left(\cos\theta\, \hat{S}_z\uparrow + \sin\theta\, \hat{S}_z\uparrow\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\left(\frac{\hbar}{2}\uparrow\right)\left[\cos\theta\left(-\frac{\hbar}{2}\downarrow\right) + \sin\theta\left(\frac{\hbar}{2}\uparrow\right)\right] - \left(\frac{\hbar}{2}\downarrow\right)\left[\cos\theta\left(\frac{\hbar}{2}\uparrow\right) + \sin\theta\left(\frac{\hbar}{2}\downarrow\right)\right]\right]$$

$$= \frac{\hbar^2}{4}\left[\frac{\cos\theta}{\sqrt{2}}\left(-\uparrow\downarrow + \downarrow\uparrow\right) + \frac{\sin\theta}{\sqrt{2}}\left(\uparrow\uparrow + \downarrow\downarrow\right)\right]$$

$$= \frac{\hbar^2}{4}\left[-\cos\theta\,|0\,0\rangle + \frac{\sin\theta}{\sqrt{2}}\left(|1\,1\rangle + |1\,-1\rangle\right)\right]$$

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where $|1\ 1\rangle = \uparrow\uparrow$, $|1\ -1\rangle = \downarrow\downarrow$

$\rightarrow \langle 0\ 0|\ S_{1x} S_{2x} |0\ 0\rangle$

$$= -\hbar^2 \left(\frac{\cos\theta}{4}\right) \overset{1}{\langle 0\ 0|0\ 0\rangle} + \frac{\sin\theta}{\sqrt{2}} \left[ \overset{0}{\langle 0\ 0|1\ 1\rangle} + \overset{0}{\langle 0\ 0|1\ -1\rangle} \right]$$

$$= -\frac{\hbar^2}{4} \cos\theta$$

where $\theta = \tan^{-1}\left( \frac{\hat{S}_{2x}}{\hat{S}_{2z}} \right)$

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(5)

a) spin → $\frac{1}{2}$ , spin → 1

$$|j_1 + j_2| = \frac{3}{2}$$

$$|j_1 - j_2| = \frac{1}{2}$$

3

$$\Rightarrow \left|\tfrac{1}{2}, 1; \tfrac{3}{2}, \tfrac{3}{2}\right\rangle \qquad \left|\tfrac{1}{2}\, 1; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{1}{2}, 1; \tfrac{3}{2}, \tfrac{1}{2}\right\rangle \qquad \left|\tfrac{1}{2}\, 1; \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{1}{2}, 1; \tfrac{3}{2}, -\tfrac{1}{2}\right\rangle$$

6 states

$$\left|\tfrac{1}{2}, 1; \tfrac{3}{2}, -\tfrac{3}{2}\right\rangle$$

(b)

$$|s\ m\rangle = A\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{3}{2}\ (m-\tfrac{1}{2})\right\rangle + B\left|\tfrac{1}{2}\ (-\tfrac{1}{2})\right\rangle\left|\tfrac{3}{2}\ (m+\tfrac{1}{2})\right\rangle$$

let us replace $\frac{3}{2}$ with a general spin $s_2$

$$|s\ m\rangle = \left[A\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|s_2\ (m-\tfrac{1}{2})\right\rangle + B\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|s_2\ (m+\tfrac{1}{2})\right\rangle\right.$$

$$S^2|s\ m\rangle = \left[(S^{(1)})^2 + (S^{(2)})^2 + 2\left(S_x^{(1)} S_x^{(2)} + S_y^{(1)} S_y^{(2)} + S_z^{(1)} S_z^{(2)}\right)\right]$$

$$= \left(A\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|s_2\ m-\tfrac{1}{2}\right\rangle + B\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|s_2\ m+\tfrac{1}{2}\right\rangle\right]$$

$$= A\Big\{ S^2\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|s_2\ m-\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle S^2\left|s_2\ m-\tfrac{1}{2}\right\rangle$$

$$+\, 2\Big\{ S_x\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle S_x\left|s_2\ m-\tfrac{1}{2}\right\rangle + S_y\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle S_y\left|s_2\ m-\tfrac{1}{2}\right\rangle$$

--- continued

...

$+ S_z |\tfrac12\ \tfrac12\rangle\ S_z|s_2\ m-\tfrac12\rangle$

$+ B \{\ s^2$

Not$^n$:

$|j\ m\rangle = \boxed{\phantom{x}}\ |j m_1\ m_2\rangle$
$+ \boxed{\phantom{x}}\ |m_1'\ m_2'\rangle$

$|\tfrac12, \tfrac12\rangle\quad |\tfrac12\ -\tfrac12\rangle\quad |\tfrac32\ \tfrac32\rangle\quad |\tfrac32\ -\tfrac32\rangle\quad |\tfrac32\ -\tfrac12\rangle\quad |\tfrac32, \tfrac12\rangle$

$|\tfrac32\ \tfrac32\rangle = A$

$= A|\tfrac12\ \tfrac12\rangle + B|\tfrac12\ 1\rangle = \frac{1}{\sqrt2}(\ldots)$

$|\tfrac32\ +\tfrac12\rangle = A|\tfrac12\ 0\rangle + B|0\ \tfrac12\rangle$

$+ C|\ldots\rangle$

$|\tfrac32\ \tfrac32\rangle = |\tfrac12, 1\rangle$

$|\tfrac32\ \tfrac12\rangle = A|\tfrac12, 0\rangle + B|-\tfrac12, 1\rangle$

$= \frac{1}{\sqrt2}\left(|\tfrac12, 0\rangle + |-\tfrac12, 1\rangle\right)$

what are A and B?

$|\tfrac32\ -\tfrac12\rangle = A|-\tfrac12, 0\rangle + B|\tfrac12, -1\rangle = \frac{1}{\sqrt2}\left(|\tfrac12, 0\rangle + |-\tfrac12, 1\rangle\right)$

$|\tfrac32, -\tfrac32\rangle = |-\tfrac12, -1\rangle$

$|\tfrac12, \tfrac12\rangle = A|\tfrac12, 0\rangle + B|-\tfrac12, 1\rangle = \frac{1}{\sqrt2}\left(|\tfrac12, 0\rangle + |-\tfrac12, 1\rangle\right)$

$|\tfrac12\ -\tfrac12\rangle = A|-\tfrac12, 0\rangle + B|\tfrac12, -1\rangle = \frac{1}{\sqrt2}\left(-|-\tfrac12, 0\rangle + |+\tfrac12, -1\rangle\right)$

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