Assignment in lieu of Quiz III

From systems of identical particles to relativistic quantum mechanics

1. *Possible configurations of identical bosons:* Three identical bosons with spin $s = 1$ are in the same orbital states described by the wavefunction $\psi(\boldsymbol{r})$.

   (a) Write down the normalized spin functions for the total system.

   (b) How many independent states are possible?

   (c) What are the possible values of the total spin of the system?

2. *Neutron stars:* Stars that are heavier than the Chandrasekhar limit will not form white dwarfs, but they will collapse further, finally becoming neutron stars under certain conditions. This is due to the fact that, at very high density, inverse beta decay, i.e. $e + p \to n + \nu$, converts virtually all of the protons and electrons into neutrons, liberating neutrinos in the process, which carry off energy. Eventually, the degeneracy pressure of non-relativistic neutrons stabilizes the collapse, just as electron degeneracy does for the white dwarfs.

   (a) Calculate the radius of a neutron star with the mass of the sun.

   (b) Also calculate the neutron Fermi energy, and compare it with the rest energy of a neutron. Is it reasonable to treat such a star non-relativistically?

3. *Partial waves and phase shifts:* Recall that, using the method of partial waves, we had obtained the scattering amplitude to be

$$f(\theta) = \sum_{l=0}^{\infty} (2\,l + 1)\, a_l\, P_l(\cos\theta),$$

   where $P_l(x)$ denotes the Legendre polynomials and a_l was called the l-th partial wave amplitude.

   (a) Using the above result, arrive at the corresponding expression for the differential cross-section and show that total cross-section can be written as

$$\sigma = \sum_{l=0}^{\infty} (2\,l + 1)\, |a_l|^2.$$

   (b) Focusing on a particular l, show that the partial wave amplitude a_l can be expressed in terms of the phase shift δ_l as follows:

$$a_l = \frac{1}{k}\, \mathrm{e}^{i\,\delta_l}\, \sin(\delta_l).$$

   (c) Use this form of a_l to arrive an expression for the total cross-section in terms of the phase shifts δ_l.

4. *Scattering amplitude and cross-section in the Born approximation:* Using the Born approximation, find the scattering amplitude *as well as* the total scattering cross-section of a particle in the following two central potentials: $V(r) = \alpha\, \mathrm{e}^{-\mu\, r}$ and $V(r) = \alpha/r^2$.

5. *Simultaneous diaganolization of $\hat{\boldsymbol{p}}$, $\hat{H}$ and $\hat{\boldsymbol{\Sigma}}$ describing a Dirac particle:* Recall that the Hamiltonian $\hat{H}$ of a Dirac particle is given by

$$\hat{H} = \boldsymbol{\alpha} \cdot \hat{\boldsymbol{p}} + \beta\, m\, c^2,$$

   where $\hat{\boldsymbol{p}}$ is the momentum operator. The quantities α_i with $i = (1, 2, 3)$ and β are the Dirac matrices defined as

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & -\mathcal{I} \end{pmatrix},$$

where σ_i and $\mathcal{I}$ are the 2×2 Pauli matrices and the unit matrix given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Also, note that the spin operator describing the Dirac particles can be expressed in terms of the Pauli matrices $\boldsymbol{\sigma}$ as follows:

$$\hat{\boldsymbol{S}} = \frac{\hbar}{2}\,\boldsymbol{\Sigma},$$

where

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

Show that the commutators $[\hat{\boldsymbol{p}}, \hat{H}]$, $[\hat{\boldsymbol{p}}, \hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{\Sigma}}]$ and $[\hat{H}, \hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{\Sigma}}]$ vanish.