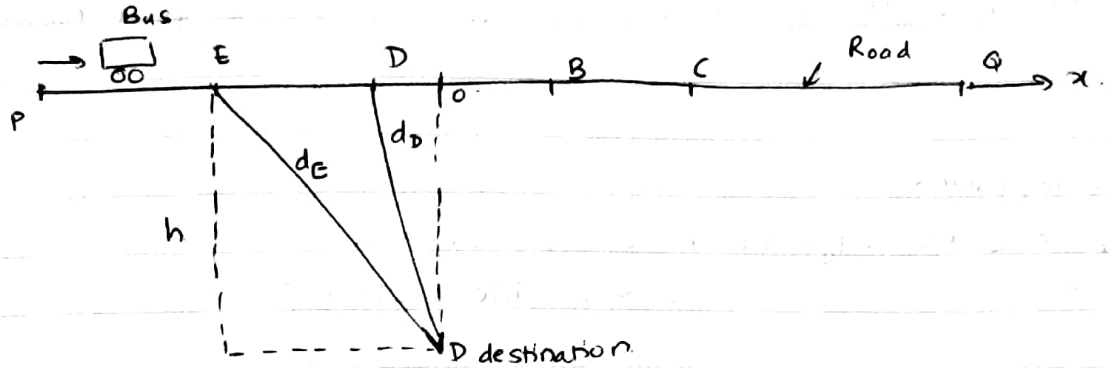
19 01 2018

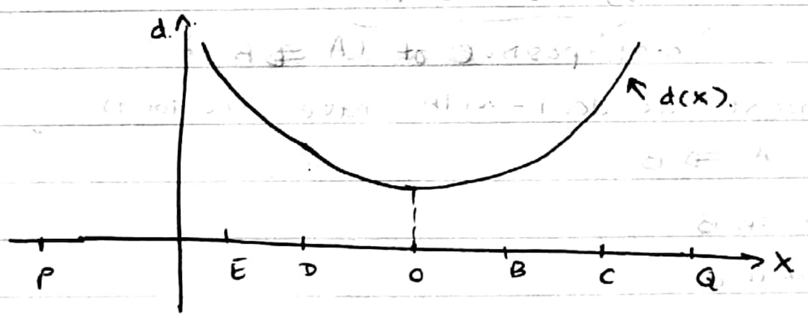
Optimization



* you may get out of the bus at any point. you want to get out from the bus so that you have a minimum distance to the destination. so at what place you are getting out of the bus ?,

Let us denote by d the distance to the D from an arbitrary poin x along the road.

what if $d = d = \sqrt{h^2 + x^2}$



- we have an objective function d(x). objective function is a measure of iritation and goodness. Also, the decision variable is x. and we want to find x from all allowed values, which attain the minimum value of d(x).

Formally optimization problem formulation $\Rightarrow$

minimize	d(x)
subject to	x is ~~on~~ in the line segment PQ.

Q) Are optimization problems very easy ?

ans : not necessarily

- depending on the problem, application They can be extremely difficult.

eg: decision making, application independent, machine learning

So this course is to get the basic theoretical foundations in this area based on what you may enhance your knowledge for

Pre requisites

1. Working knowledge of basic linear algebra
2. " " " Multivariable calculus

Review

1) Implications

$A \Rightarrow B$ If A then B

A is sufficient for B

B is sufficient for A

A only if B

(Not A) or B

$A \Rightarrow B \equiv \underbrace{(\text{Not } B) \Rightarrow (\text{Not } A)}_{\text{contrapositive of } (A \Rightarrow B)}$

Most statement we deal with have the form

$A \Rightarrow B$

prove $A \Rightarrow B$

1) Direct method
2) proof by contraposition
3) proof by contradiction.

2 set definition

$S = \{x \mid x \in \mathbb{R}, x > 0\}$

$\mid$ such that

set of numbers greater than a

$S = \{x \mid x \in \mathbb{R}, x > a\}, a \in \mathbb{R}$

what about

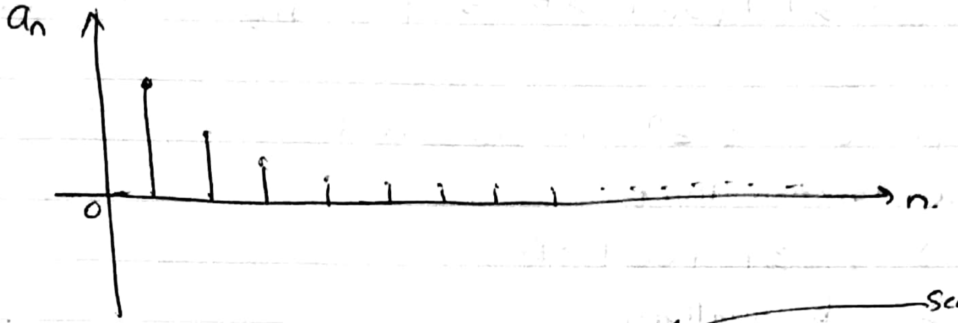
$S = \{x \mid x \in \mathbb{R}, x > a, a \in \mathbb{R}\}$

③ Quantifiers

There exists - $\exists$

For all
For every } $\forall$

e.g.: the sequence $\{1 + \frac{1}{n} = a_n\}_{n \geq 1}$ is bounded



Sequence a_n is bounded

$$(\exists B \geq 0)(\forall n \geq 1)\, |a_n| \leq B$$

③ ~~Quantifier~~ choose $B = 2$

Let $n \geq 1$ be an arbitrary integer.

Then $a_n = 1 + \frac{1}{n} \leq 2 = B$

order of quantifiers - crucial

$$(\forall n \geq 1)(\exists B \geq 0)\, |a_n| \leq B$$

for all $n \geq 1$ There exists a $B \geq 0$ s.t $|a_n| \leq B$

* Let $n \geq 1$ be arbitrary

choose $B = |a_n| + 1$.

Which assure that $|a_n| \leq B$ irrespective of the sequence.

sequence $a_n = \frac{1}{n}$ converge to 0

$\{a_n\}_{n \geq}$ converge to zero

$$(\forall \epsilon > 0)(\exists N \geq 1)(\forall n \geq N)\, |a_n| < \epsilon$$

let $\epsilon > 0$ be arbitrary

function of ϵ.

choose $N = \frac{1}{\epsilon} + 1$ the integer larger than $\frac{1}{\epsilon} \Rightarrow 1$

Let $n \geq N$ be arbitrary

$$|a_n| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

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Logical Negative

mean

$(\exists B \geq 0)(\forall n \geq 1)|a_n| \leq B \Rightarrow \{a_n\}_{n\geq 1}$ is bounded.

$\{(\exists B \geq 0)(\forall n \geq 1)|a_n| \leq B\}$

$= (\forall B \geq 0)(\exists n \geq 1)|a_n| > B \Rightarrow \{a_n\}_{n\geq 1}$ is unbounded

Show that sequence $a_n : 2^n$ is unbounded.

i.e we have to show that

$(\forall B \geq 0)(\exists n \geq 1)|a_n| > B$

let $B \geq 0$ be arbitrary.

choose $n > \log_2(B+1)$

$|a_n| = 2^n > 2^{\log_2(B+1)} = (B+1) > B$

$\therefore a_n$ is unbounded

a

Show that $a_n = (-1)^n$ doesn't ~~chang~~ converge to zero

$\{a_n\}_{n\geq 1}$ convege to zero $\equiv$ ~~$(\forall \epsilon > 0)(\exists N \geq 1)(\forall n \geq N)$~~

$(\forall \epsilon > 0)(\exists N \geq 1)(\forall n \geq N)|a_n| < \epsilon$

$\{a_n\}_{n\geq 1}$ converge to zero $\equiv (\forall \epsilon > 0)(\exists N \geq 1)(\forall n \geq N)|a_n| <$

$\downarrow$

negate

$\{a_n\}_{n\geq 1}$ doesn't converge to zero $\equiv (\exists \epsilon > 0)(\forall N \geq 1)(\exists n \geq N)|a_n| \geq$

We have to show that $a_n = (-1)^n$ conforms to the definition

$(\exists \epsilon > 0)(\forall N \geq 1)(\exists n \geq N)|a_n| \geq \epsilon$

choose $\epsilon = 0.5$

let $N \geq 1$ be arbitrary

choose $n = (N+1)$

$|a_n| = |(-1)^n| = |(-1)^{N+1}| > 0.5 = \epsilon$

show that $a_n = 1 + \frac{1}{n}$ doesn't converge to zero.

choose $\epsilon = \frac{1}{2}$

let $N \geq 1$ be arbitrary.

choose $n = N$

$|a_n| = \left(1 + \frac{1}{N}\right) > 0.5 = \epsilon$

define the set of integers that are divisible by 2 by using quanti-fiers

$S = \{ x \mid x = 2\alpha$ for some integer $\alpha \}$

$S = \{ x \mid \exists\, \alpha \in \mathbb{Z},\ x = 2\alpha \}$

Vectors and matrix

column vector : $\underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}}_{n^{th} \text{ column vector.}} \in \mathbb{R}^n$

The set of column ~~vec~~ n-vectors with real components

- we commonly denote elements of $\mathbb{R}^n$ by lower case letters $x \in \mathbb{R}^n$.
- row vectors : $\underbrace{[a_1 \ a_2 \ \ldots \ a_n]}_{n\text{-row vector}} = a^T$
- vector addition $a+b = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \\ a_n + b_n \end{bmatrix}$

suppose you have $a + x = b$
what is x? $x = (b - a)$.

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A set of vectors $\{a_1, a_2, \dots a_k\}$ is said to be linearly independen[t]

if $\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_k a_k = 0$

$\Rightarrow$ all $\alpha_i = 0$

* $\left(\sum_{i=1}^{k} \alpha_i a_i = 0 \Rightarrow \alpha_i = 0 \text{ for all } i \right)$ $\underbrace{S = \{a_1, a_2, \dots, a_k\} \text{ is linearly independent}}_{P}$

What it means ?

$\underbrace{\text{Not } P}_{}$ Not $\left(\sum_{i=1}^{k} \alpha_i a_i = 0 \Rightarrow \forall i \; \alpha_i = 0 \right.$

S is not linearly independent

$\sum_{i=1}^{k} \alpha_i a_i = 0$ AND $\exists i \; \alpha_i \neq 0$

e.g. (1) $S = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$

(2) $S = \{0, 1\}$ when $1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

<u>Subspace of $\mathbb{R}^n$.</u>

$\exists$ a subset

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Tutorial

4) ii) $b \in \mathbb{Z}$ is divisible by ~~2~~ $\Longrightarrow$ $b \in \mathbb{Z}$ is divisible by 4

Show statement is FALSE.

Give a counter example

~~b is not~~

$b \in \mathbb{Z}$ is not divisible by 2 $\Rightarrow$ $b \in \mathbb{Z}$ is not divisable by 4

take $b = 2$

b is divisible by 2, but it is not divisible by 4.

Statement is FALSE

If you have a system

$Ax = b$

$$A = [a_1 \; a_2 \; a_3 \; \dots \; a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = b$$

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

rank[A] = 3   rank[A|b] = 3

~~all~~ There are 3 independent variables in A

~~has a solution~~

b must reside in the range of A.

all the ~~independent~~ variables in A are independent

~~rank~~

n = no of colums of A

rank (A) = rank (A|b) = n → unique solution

(1) has a solution   rank (A) = rank[A|b]

rank (A) = rank (A|b) < n → infinitely many solutions

(2) No solution   rank (A) < rank [A|b]

$$Ax = b$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2$$

rank (A) = 1

rank (A|b) = 1

based on (1) it has a solution.

it has infinitely many solutions.

$$x_1 + x_2 = 2$$

$$x_1 = \alpha \qquad \alpha \in \mathbb{R}$$

$$x_2 = 2 - \alpha$$

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m = no of rows
n = no of coloumns

Full rank is minimum of m, n

$\square \in \mathbb{R}^{m \times n}$ ← pick the minimum ⟶ full rank.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ ← is not full rank.

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ← full rank

when no of columns > no of rows, ~~the~~ It has infinitely many solutions and rank is lower than n.

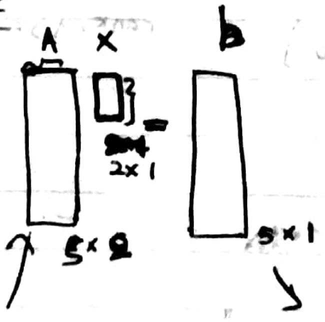
example

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 4 \end{bmatrix}}_{2\times 4} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{4\times 1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

rank(A) = 2 ⇒ full rank.
rank(A|b) = 2
unique solution.

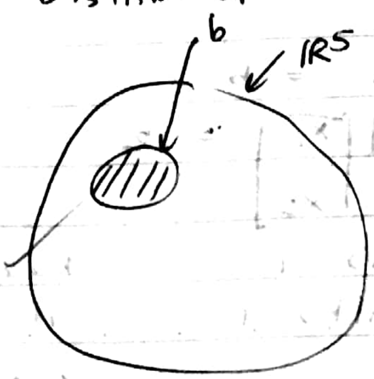
$x_1 = 1$
$x_2 = 2.$
$x_3 = 0$
$x_4 = 0$

Example



~~be~~ ← has a solution if b is in the span of A's vectors

$\in \mathbb{R}^5$

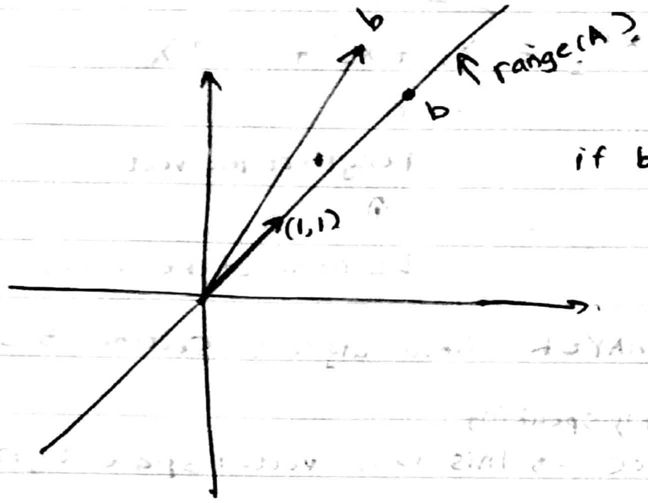


2 vectors in A
$a_1, a_2 \in \mathbb{R}^5$

Span $(a_1, a_2) \subseteq \mathbb{R}^5$

$$\begin{bmatrix}1\\1\end{bmatrix} x = \begin{bmatrix}2\\3\end{bmatrix}$$

$$Ax = b$$



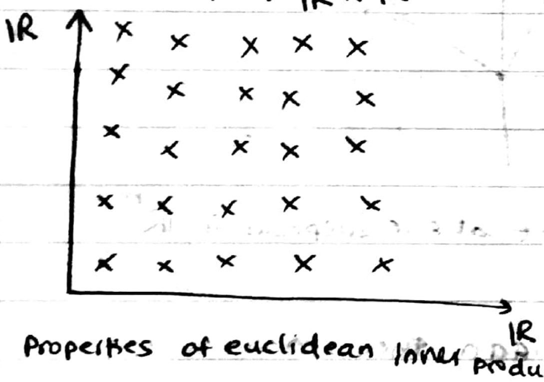
if b is on the line there is a solution to $Ax=b$.

if not we cannot find a solution.

For $(x,y) \in \mathbb{R}^n$ we define the Euclidean inner product by $\langle x, y \rangle \triangleq \sum_{i=1}^{n} x_i y_i = x^T y = y^T x$

inner product if you define as a function,

$f: \mathbb{R} \times \mathbb{R}$ $\qquad$ $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$



if $y = \begin{bmatrix}1\\2\end{bmatrix}$

$x = \begin{bmatrix}2\\3\end{bmatrix}$

$\langle x, y \rangle = 2+6$

$= \underline{\underline{8}} \in \mathbb{R}$

Properties of euclidean inner product

1) positivity

$x \in \mathbb{R}^n$ $\quad \langle x, x \rangle = x^T x = 0 \iff x = 0$ (if and only if)

2) Symmetry $\quad x^T y = y^T x = \langle x, y \rangle = \langle y, x \rangle$

3) Additivity $\quad \langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

4) Homogeneouity : $\langle \gamma x, y \rangle = \gamma \langle x, y \rangle \quad \forall \gamma \in \mathbb{R}$

L_2 norm.

The euclidean norm of a vector $x \in \mathbb{R}^n$ is defined as

$$\|x\|_2 \triangleq \sqrt{x^T x}$$

if $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

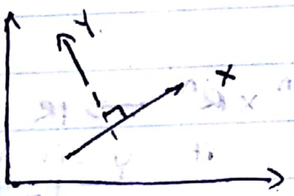
↑ length of the vector

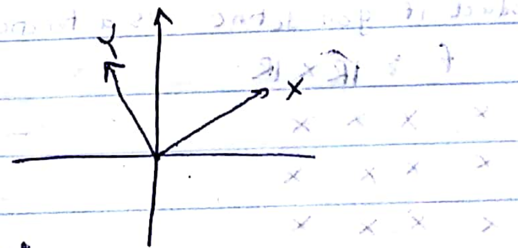
↑ L_2 norm of the vector.

(+) For more details MAYER linear algebra Section 5.3

- Inner product space → Roughly speaking This is a vector space together with a inner product.
- orthonormal → all the vector lengths are one. and ⊥ as well
- orthogonal → all vectors are ⊥ ~~only~~ with each other only.

$\mathbb{R}^2$

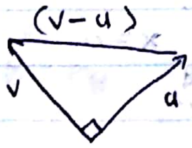




- The ∠ between x and Y is 90°

Visual concept of right angle is not at our disposal in $\mathbb{R}^n$.

Essence is embodied in pythogoriean theorem



Pythogoras.

$$\|u\|^2 + \|v\|^2 = \|u-v\|^2$$

$$\|u\|^2 + \|v\|^2 = (u-v)^T(u-v)$$

$(A+B)^T = A^T + B^T$

property $u^T v = v^T u$

$$\|u\|^2 + \|v\|^2 = (u^T - v^T)(u-v)$$

$$= u^T u - u^T v - v^T u + v^T v$$

$$= \|u\|^2 + \|v\|^2 - 2u^T v$$

$\therefore 2u^T v = 0$

Inner product is zero

Cauchy - Schwarz Inequality

For any $(x, y) \in \mathbb{R}^n$

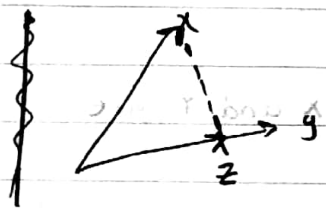
$$|x^T y| \leq \|x\|\|y\| \quad \text{holds}$$

Furthermore, equality holds if and only if
$x = \alpha y$ for some $\alpha \in \mathbb{R}$

$\exists\, \alpha \in \mathbb{R}$ such that $x = \alpha y$

$\exists$ such that.

Proof of cauchy - Schwarz inequality ~~is~~ in the reference.



$\cancel{x}$

$\cancel{z = \alpha y}$

$z = \beta y$ when $\beta \in \mathbb{R}$

$$\beta = \cancel{\frac{x^T y}{\|y\|}} \qquad \cancel{z} = \underbrace{\frac{x^T y}{\|y\|^2}}_{\beta} \cdot y$$

$$\underbrace{\|z\|} \leq \|x\|$$

$$\left\| \frac{x^T y}{\|y\|^2} \cdot y \right\|_2 \leq \|x\|_2$$

$$\cancel{x^T y} \quad \|\beta y\| = |\beta| \|y\|$$

number

$$\frac{x^T y}{\cancel{\|y\|^2}} \cancel{\|y\|} \leq \|x\|$$

$$\frac{x^T y}{\|y\|} \leq \|x\|$$

$$x^T y \leq \cancel{\|x\|\|y\|} \; \|x\| . \|y\|$$

Euclidean norm has following properties

1) Positivity : $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$

length of a vector is zero means that vector is zero

2) Homogenouty : $\|\gamma x\| = \|\gamma\|\|x\|$, $\gamma \in \mathbb{R}$

3) Triangular inequality : $\|x + y\| \leq \|x\| + \|y\|$

ex: $\|x + y\|^2 = (x+y)^T (x+y) = (x^T + y^T)(x + y)$

$= x^T x + \cancel{y^T x +} x^T y + y^T x + y^T y$

$= \|x\|^2 + 2x^T y + \|y\|^2$

$\cancel{x^T y \leq}$ Give a

From cauchy Schwarz

$\langle x^T y \rangle \leq \|x\|\|y\|$

upper bound of $x^T y$ is $\|x\|\|y\|$.

$\|x + y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$

$\|x+y\|^2 \leq (\|x\| + \|y\|)^2$

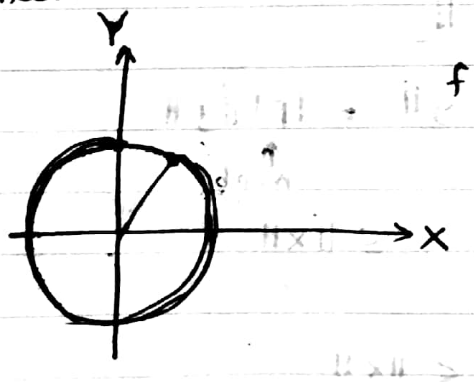
take square root of both sides

$\|x + y\| \leq \|x\| + \|y\|$

There is not restriction for the region x and y are residing.

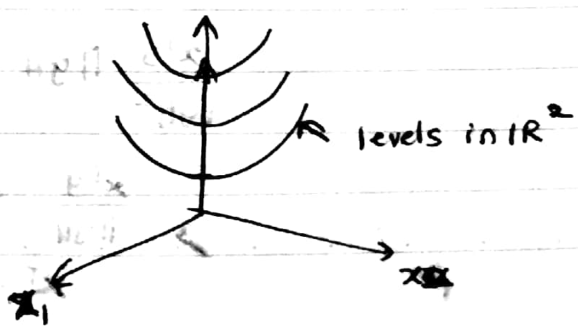
If 2 vectors are orthogonal which are in $\mathbb{R}^n$ ~~are~~

we say that x and y are following pythogorean theorem.

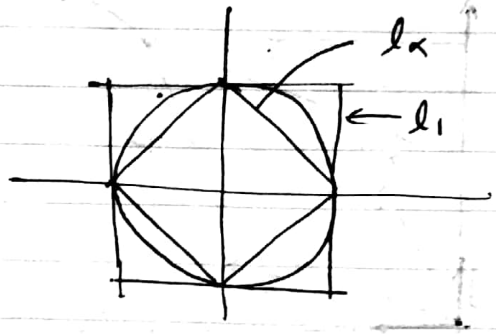


$f(x) = \|x\|^2$

$= \|x_1^2 + x_2^2 + \dots + x_n^2\|$.

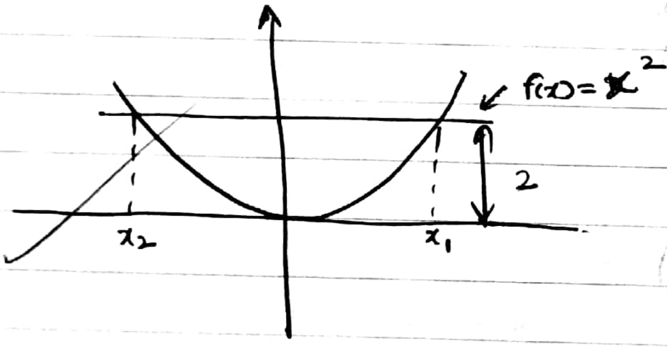


L_1 norm $\triangleq \|x\|_1 = \sum_{i=1}^{n} |x_i|$. $f(x) = \|x\|_1$



l_∞ norm $\triangleq \|x\|_\infty = \max\{|x_1|, |x_2|, \ldots |x_n|\}$

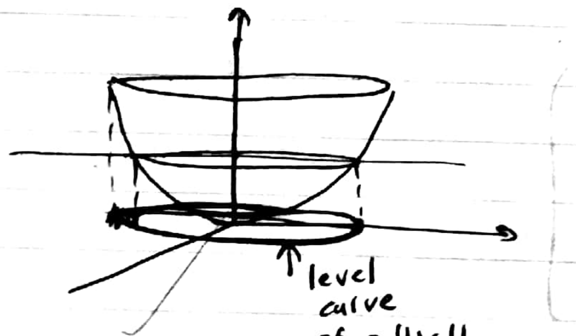
l_p norm $\triangleq \|x\|_p = \left(|x_1|^p + |x_2|^p + |x_3|^p + \ldots + |x_n|^p\right)^{1/p}$

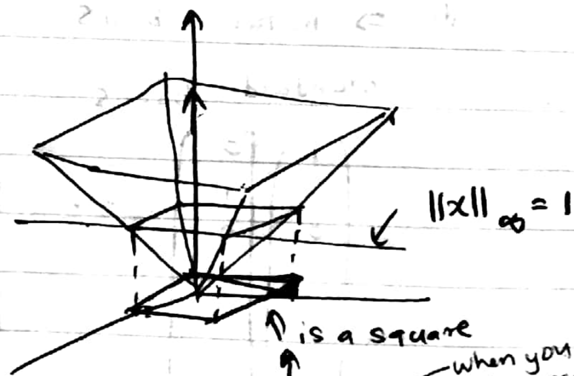


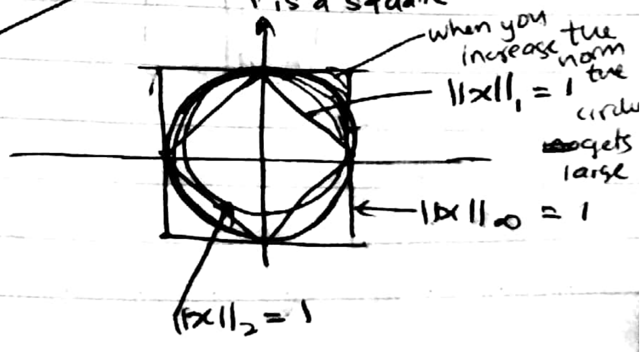
$\{x \mid f(x)=2\} = S_1$

$\{x_2, x_1\}$

$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$





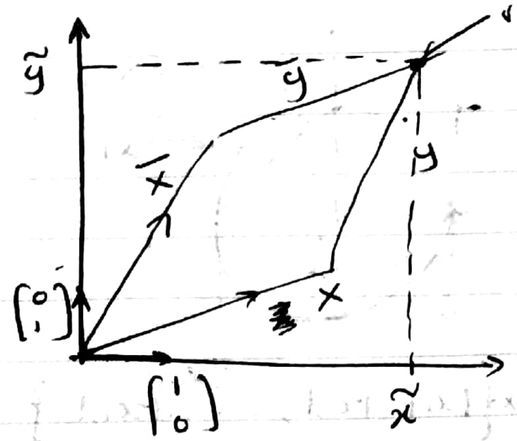


Function $L \quad R^n \to \mathbb{R}^n$ is linear if

1) $L(ax) = a\,L(x) \;\forall x \in \mathbb{R}^n$ and $a \in \mathbb{Z}$

2) $L(x+y) = L(x) + L(y) \;\forall x, y \in \mathbb{R}^n$

vector in $\mathbb{R}^2$



basis 1 $= \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$

basis 2 $= \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

basis 3 $= \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$

$X = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mathbb{R}^n \Rightarrow$ natural basis

standard basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

coordinates of x in the standard basis

$x = x_1 e_1 + x_2 e_2 \quad + x_n e_n$

when $e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ← i^{th} position

$\boxed{f(x) = A.x}$ is linear

can be written easily in ~~the~~ terms of Standard basis coefficients.

Let $\{t_1, t_2, \dots\dots, t_n\}$ be another basis for $\mathbb{R}^n$

$X = \tilde{x}_1 t_1 + \tilde{x}_2 t_2 + \tilde{x}_3 t_3 + \dots + \tilde{x}_n t_n$

these are the coordinates of x in the basis of $\{t_1, t_2, t_3, \dots t_n\}$

$X = \underbrace{(t_1 \quad t_2 \quad t_3 \quad t_4 \quad \dots \quad t_n)}_{T} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \vdots \\ \tilde{x}_n \end{bmatrix}$

This is a matrix of independent basis. The no of coloumns = n and no of rows = n

Thus full rank. → T is invertible.

coordinate basis w.r.t. basis $\{t_1, \dots t_n\}$

$\tilde{x} = T^{-1} X$

Natural basis coordinates.

if a square is fully ranked invertibility is possibility.

T is invertible because $\{t_1 \dots t_n\}$ is a basis for $\mathbb{R}^n$

2×2 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$X = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $t_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $t_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

write X in terms of t_1 and t_2.

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

compute $T^{-1} = \frac{1}{2-1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

$$T^{-1} X = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\tilde{x} = \begin{matrix} 4-3 \\ -2+3 \end{matrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$$

$$X = \boxed{1}\, t_1 + \boxed{1}\, t_2.$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \boxed{1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \boxed{1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

similar matrices

$A \in \mathbb{R}^{n \times n}$ $B \in \mathbb{R}^{n \times n}$ are similar if there exists a non singular matrix T, s.t $A = T^{-1} B T$

$\boxed{\text{non singular}} \longrightarrow$ invertible

$\boxed{\text{Singular}} \longrightarrow$ non invertible.

T^{-1}

$\uparrow$ T must be a square matrix.

Eigen values

$\lambda \in \mathbb{C}$ ($\in$ complex numbers) is an eigenvalue of $A \in \mathbb{C}^{n \times n}$

elements are complex nos

$$L(\lambda) = \det(\lambda I - A) = 0$$

we say λ is an eigen value

det (A) $\neq 0$

$\Longleftrightarrow$ A^{-1} exists
if and only if

$\Updownarrow$

Columns of A are linearly independent

det exists for a square matrix.

① $\Longleftrightarrow \exists$ non zero $v \in \mathbb{C}^n$ such that:

$(\lambda I - A)v = 0$

$\det(A) = 0$

columns of A are dependent

$$\det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = 0$$

~~$\boxed{2} a_1 + (-1)a_2 = 0$~~

$\boxed{2} \cdot a_1 + (-1)a_2 = 0$

$$v = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$Av = 0$$

$\exists$ non zero $w \in \mathbb{C}^n$

s.t $w^T(\lambda I - A) = 0$

Some key points

* if V is an eigen

$\Longleftrightarrow \exists v$

$(\lambda I - A) V = 0$

$\lambda I V = AV$

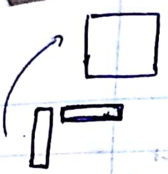
$\lambda V = Av$

$A = aa^T$

$(\lambda I - A)v = 0$

λ

~~$A \cdot A a a^T$~~



$A = aa^T$

$A\,a = a a^T . a$

$A\,a = a(a^T a)$

$A\,a = a\,\|a\|^2$

$A\,a = \|a\|^2\, a \leftarrow$ vector.

vector ↑ ~~vector~~ vector ~~Vector~~

~~The~~

$$A v = \|v\|^2\, v$$

↑ eigen value ↑ eigen vectors

If v is an eigen vector of A with eigen value λ

So is αv for any α in complex $\alpha \in \mathbb{C}$, $\alpha \neq 0$

$A v = \lambda v$ (definition)

Multiply $\bar{v} = \alpha v$

$A(\alpha v) = \lambda(\alpha v)$

$A \bar{v} = \lambda \bar{v}$

eigen values and eigen vectors can be complex.

even when A is real, eigen values λ and eigen vector v can be complex.

in matlab

A = rand (3, 3)

[a, b] = eig (A)

↑ eigen ~~vector~~ Value ↑ eigen vector

When A and λ are real, we can always find an real eigen vector v associated with λ

$Av = \lambda v$ given than $A = R^{m \times n}$

$\lambda \in R$

$v \in \mathbb{C}^n$

$$v = \begin{pmatrix} \alpha + j\beta \\ \gamma + j\delta \end{pmatrix}$$

$$\text{real } v = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}$$

$$\text{Real}(Av) = \text{Real}(\lambda v)$$

$$A(\text{Real } v) = \lambda(\text{Real } v)$$

$$A \underbrace{\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}}_{\bar{v}} = \lambda \underbrace{\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}}_{\bar{v}}$$

$\bar{v}$ is an eigen vector associated with A.

when you use imaginary parts the same thing happens

If A is real and $v \in \mathbb{C}^n$ is an eigen vector associated with λ (eigen value) $\in \mathbb{C}$, then $\bar{v}$ is an eigen vector associated with $\bar{\lambda}$

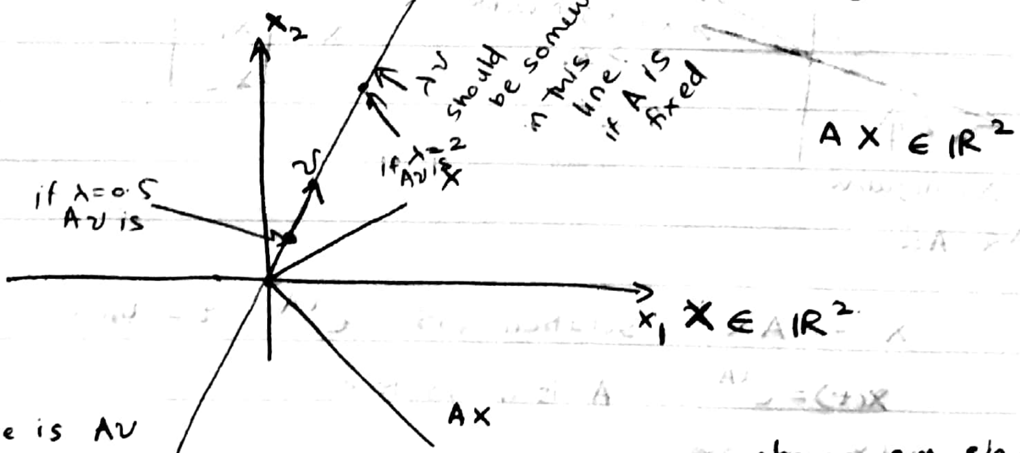
— bar is complex conjugate

$$V = \begin{bmatrix} \alpha + j\beta \\ \gamma + j\delta \end{bmatrix} \qquad \bar{V} = \begin{bmatrix} \alpha - j\beta \\ \gamma - j\delta \end{bmatrix}$$

$$\lambda = a + jb$$
$$\bar{\lambda} = a - jb$$

if you have an eigen vector associated with λ

the conjugate of eigen vector associate with conjugate of λ.



where is Av

if λ is correspondent

choose one eigen vector of A and denote v.

$\lambda \in \mathbb{C}$ $\longrightarrow$ plays a big role in the characterization of

$$\boxed{\dot{x} = Ax}$$

2018.02.03

<u>Question</u>

5 8 3 A

$Ax = \lambda x$

eigen val

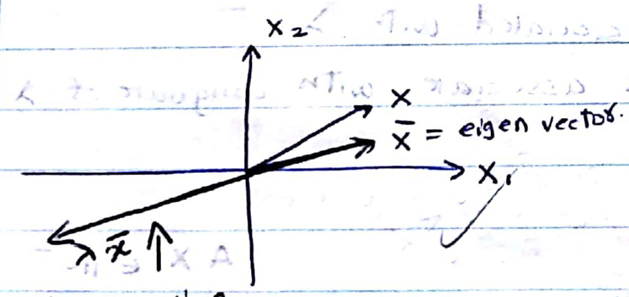
eigen vector.

A is a square matrix (not a fat or skinny matrix)

$A \in \mathbb{R}^{2\times 2}$

$x \in \mathbb{R}^2$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



λ = negative.

$\lambda \bar{x} = A\bar{x}$

$\dot{x} = A\cancel{x}$ solution is e^{tA} t = time.

$x(t) = e^{tA}$ A is a matrix.

$$\frac{dx}{dt} = 2x$$

when λ is more than 1 $\rightarrow$ unstable

when λ s are complex,

$$\begin{bmatrix} \frac{dy}{dt} \\ \frac{dx}{dt} \\ \vdots \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix}$$

$\det(\lambda I - A) = 0$ has distinct n roots

$f(\lambda)$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\det(\lambda I - A) = \left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right) = \begin{pmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{pmatrix} = \lambda^2 - 5\lambda + 5 - 6$$

$$= \underline{\underline{\lambda^2 - 5\lambda - 1}}$$

function of λ

$\lambda^2 - 5\lambda - 1 = 0$ has n distinct roots.

There exists n linearly independent vectors $v_1, v_2, \ldots, v_n$ s.t

$$A v_i = \lambda v_i \quad i = 1, 2, 3, \ldots, n.$$

Theorem 31

- if you have distinct roots you can find λ values which are independent to each other.

Do not skip the proofs in the book. under what method of proof is used.

Go through the proof of $\det(\lambda I - A) = 0$ has distinct n roots, then there exists n linearly independent vectors $v_1, v_2, v_3, \ldots, v_n$

$\det(\lambda I - A) = 0$ v_i are independent if roots are distinct.

A if there are ~~two~~ dependent vectors and $\exists\, x \neq 0$

$A x = 0$

Choice $T = (v_1, v_2 \quad \ldots \quad v_n)^{-1}$ $\quad v_1, v_2, v_3, \ldots, v_n$ are linearly independent

Sufficient condition $\rightarrow$ eigen values of A are distinct.

This is not a necessary condition.

- if A has equal eigen values we can choose eigen vectors to be independent to each other
- can show using contradiction. two eigen values which are equal ~~are~~ are there. You can choose two independent eigen vectors corresponding to the equal eigen value.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ eigen values are $\lambda = 1$, $\lambda = 1$
eigen values are identical

$Ax = \lambda x$

$\lambda = 1$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{matrix} x_1 \\ x_2 \end{matrix}$

eigen vector for Identity Matrix can be anything

$x_1 = x_1$
$x_2 = x_2$

e.g $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 10 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 10 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 10 \end{pmatrix}$ is an eigen vector.

* We can set
$\lambda = 1$ $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda = 1$ $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

when two eigen vectors are independent you cannot say eigen values are distinct.

choice $T = \underbrace{(v_1 \quad v_2 \quad v_3 \quad \dots \quad v_n)}_{\text{linearly independent}}^{-1}$

A is similar to a diagonal matrix

$A \sim B$ T s.t

$TAT^{-1} = B$

$TAT^{-1} = TA\,[v_1 \quad v_2 \quad v_3 \quad \dots \quad v_n]$

$=$

$A\,[B_1 \quad B_2 \quad B_3 \dots \quad B_n]$ if $A \in \mathbb{R}^{m \times n}$
$(AB_1 \quad AB_2 \quad AB_3 \quad AB_n)$ $B \in \mathbb{R}^{n \times L}$

$[A_1 \quad A_2] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ if $A_1 \in \mathbb{R}^{m \times n}$

$A_1 \in \mathbb{R}^{m_2 \times n} \leftarrow X$

$B_1 \in \mathbb{R}^{n \times L}$

* $B_2 \in \mathbb{R}^{n \times L_1} \leftarrow X$

$A_1 \in \mathbb{R}^{m \times n_2} \leftarrow$ possible

Block matrices

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}_{2\times 1} \begin{bmatrix} B_1 & B_2 \end{bmatrix}_{1\times 2} \leftarrow \text{possible.}$$

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \end{pmatrix} = \begin{pmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \end{pmatrix}$$

$$TAT^{-1} = TA\begin{bmatrix} v_1 & v_2 & v_3 & \dots & v_n \end{bmatrix} \qquad v_i \in \mathbb{R}^{n\times 1}$$

$$= T\begin{bmatrix} Av_1 & Av_2 & Av_3 & \dots & Av_n \end{bmatrix}$$

$$= T\begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & & \lambda_n v_n \end{bmatrix}$$

$$= T\begin{pmatrix} v_1 & v_2 \dots & v_n \end{pmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \lambda_3 & \\ & 0 & & \\ & & & \lambda_n \end{bmatrix}_{n\times n}}_{\Lambda}$$

$$= T\,T^{-1}\Lambda$$

$$TAT^{-1} = \Lambda$$

A and Λ are similar. (Λ is a diagonal matrix)

if A has distinct eigen values A is similar to a diagonal matrix

Theorem 32 : pg 28 you have to know to prove

All eigen values of real symmetric ($A = A^T$) matrices are real.

$a_{ij} = a_{ji} \Rightarrow A = A^T$

$a_{ij} = a_{ji}^* \leftarrow$ Hermetian: $\Rightarrow A = A^H$

$(AB)^T = B^T A^T$

$(AB)^H = B^H A^H$

$\langle x, y \rangle = x^H y$

$x, y \; \mathbb{C}^n$

① — $\langle x, Ax \rangle = x^H A x$

$\langle Ax, x \rangle = (Ax)^H x$

$= x^H A^H x$

A is a symmetric matrix $A^H = A$

② — $\langle Ax, x \rangle = x^H A x$ for all x $x \in \mathbb{C}^n$

eigen vectors of A are inside X

$x \in (v_1, v_2 \dots v_n)$

from ① and ②

① = ②

$\langle x, Ax \rangle = \langle Ax, x \rangle$ for all X

instead of X substitude eigen vectors.

let $x = v$ corresponding to eigen value λ

$\langle v, Av \rangle = \langle Av, v \rangle$

$v^H A v = (Av)^H v$

instead of $Av = \lambda v$ can be used

$v^H \lambda v = (\lambda v)^H v$

$\therefore \; = v^H \lambda^H v$

$\lambda^H = \lambda^*$ = complex conjugate of λ

$\underbrace{v^H \lambda v}_{\text{scalar}} = v^H \lambda^* v$

$\lambda v^H v = \lambda^* v^H v$

$\lambda \|v\|^2 = \lambda^* \|v\|^2$ ← for $\|v\| \neq 0$ eigen vectors is non zero

$\lambda = \lambda^*$

$v^H v = \|v\|^2$

$v = \begin{matrix} 1+j \\ 2-j \end{matrix}$

$v^H = (1-j \quad 2+j)$

$v^H . v = \left[(1-j)(2+j)\right] \begin{bmatrix} (1+j) \\ (2-j) \end{bmatrix}$

$v^H v = 1+1+4-1 = 4-3 = 1$

Theorem 3.3 (pg 28)

Any REAL symm $n \times n$ matrix has a set of n eigenvectors that are mutually orthogonal.

$$f(x) = \|Bx\|_2^2 = (Bx)^T(Bx)$$
$$= x^T(B^TB)x$$
$$= x^T A x$$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \end{pmatrix}$

2 1

4 2

you can choose eigen vectors so that they are orthogonal.

$v_1, v_2 \ldots v_n \Rightarrow v_i^T v_j = 0$

$\forall\, i \neq j$

Definition A matrix whose transpose is equal to its inverves is said to be an orthogonal matrix,

given a real symmetric matrix with its associated Eigen ~~values~~ vectors $(v_1, v_2, \ldots v_n)$ In addition v_i are normalized such that $\|v_i\|^2 = 1$

$T = [v_1, v_2, \ldots, v_n]$ is orthogonal.

~~$I \cdot T^{-1} = (v_1 \; v_2 \; v_3 \; v_4 \ldots v_n)$~~ v_1

~~v_2~~

~~$T^T = T$~~

$$T^T T = \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \\ \vdots \\ v_n^T \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1^T v_1 & & 0 \\ & v_2^T v_2 & \\ 0 & & \ddots \\ & & & v_n^T v_n \end{bmatrix}$$

$v_1, v_2, v_3, v_4 \ldots v_n$ are orthogonal. $v_i v_j = 0$

$$= \begin{bmatrix} \|v_1\|^2 & 0 & 0 & 0 & 0 \\ 0 & \|v_2\|^2 & & & \\ & & \|v_3\|^2 & & \\ 0 & & & \ddots & \\ & & & & \|v_n\|^2 \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & 1 \\ 0 & & & \ddots & 1 \end{bmatrix} = I$$

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We say that a linear transformation P is an orthogonal projector on to $\mathcal{V}$ if for all $x \in \mathbb{R}^n$ we have

* $Px \in \mathcal{V}$ and,
* $(I-P)x \in \mathcal{V}^\perp$

<u>meaning of $\mathcal{V}$</u>

$\mathcal{V}$ is a subspace of $\mathbb{R}^n$

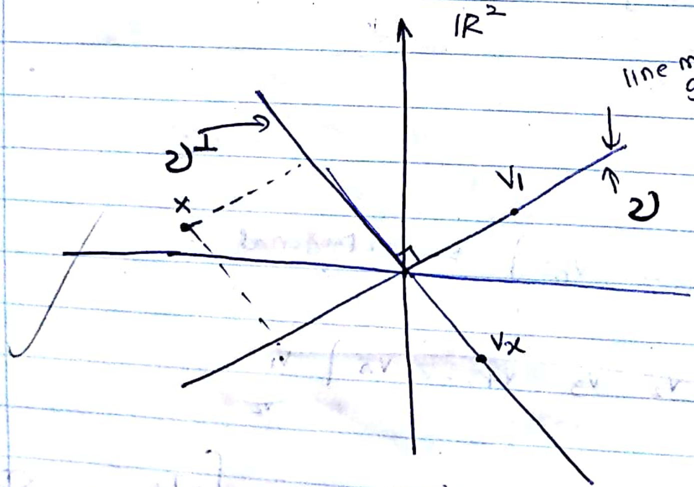
if $x_1, x_2 \in \mathcal{V}$ then $\alpha_1 x_1 + \alpha_2 x_2 \in \mathcal{V}$

for all $\alpha_1, \alpha_2 \in \mathbb{R}$

Zero is included in $\mathcal{V}$. The maximum that x can take are dimensions

$\mathcal{V}$ perp $\rightarrow$ $\mathcal{V}^\perp = \{ x \mid v^T x = 0, v \in \mathcal{V} \}$

if you pick any vector from $\mathcal{V}$ ~~any vector~~ multiplication is zero, it is orthogonal to V



line must go across 0 because subspace include 0

$\mathbb{R}^2$ subspace is a line

$v_1 \perp v_x$

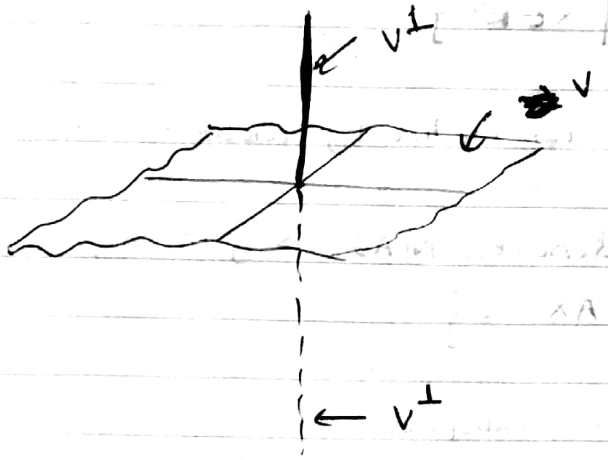
for all v the $\mathcal{V}^\perp$, vs are $\perp$

any general point x can be represented by a vector from $\mathcal{V}$ and $\mathcal{V}^\perp$

$x = \mathcal{V} + \mathcal{V}^\perp$

$x = v + \bar{v}$

$\mathbb{R}^3$ $\quad \mathcal{V}$ = plane of the board.



The $V^\perp$ is not a plane because then the $V^\perp$ plane vectors will be parallel to V which cannot happen. we need $V^\perp$ to be $\perp$ and therefor $V^\perp$ is a line.

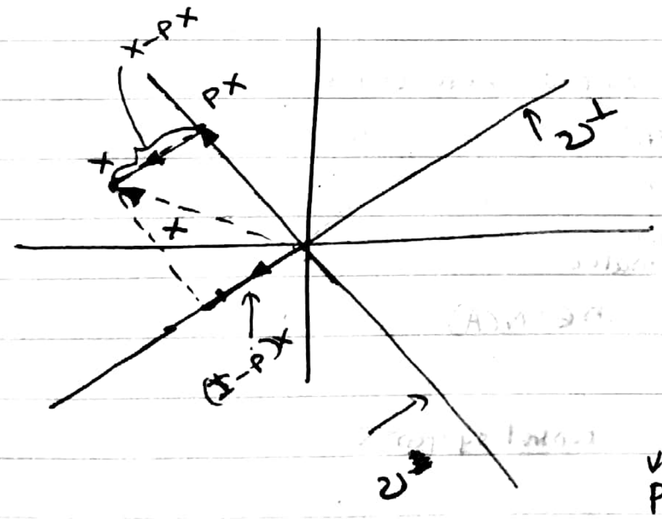
$x = v_1 + v_2$ ← orthogonal projection on to $\mathcal{V}^\perp$ for some $v_1 \in \mathcal{V}$ and $v_2 \in \mathcal{V}^\perp$.

↑ orthogonal projection on to $\mathcal{V}$

~~P is the~~ P is the orthogonal projector on $\mathcal{V}$.

if $Px \in \mathcal{V}$ and

$\cancel{(I-Px)}$. $\cancel{(X-P)x \in \mathcal{V}}$ $\quad (x - Px) \in \mathcal{V}^\perp$



~~X + M =~~

$Px + M = x$

$M = (x - Px)$

$M = (I - P)x$

There is no starting and end point to a vector only a magnitude and direction.

$$\underset{2\times 3}{\overset{\text{plane}}{P}} \, \underset{3\times 1}{x} \leftarrow \mathbb{R}^3$$

$Px = 2\times 1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(*) Range of a matrix A, denoted by $R(A)$ is given by

$$R(A) = \{ AX \mid X \in \mathbb{R}^n \}$$

Take all the points (x) that can be hit by coloumns of A.

(*) Null space of a matrix A denoted $N(A)$ is given by

$$N(A) = \{ X \in \mathbb{R}^n \mid AX = 0 \}$$

Exam

Exercise 1) show that $N(A)$ is a subspace

2) show that $R(A)$ is a subspace

Null space → for some input vectors that output a zero vector

$A^{m \times n}$ input = ~~$m \times n$~~ $n \times 1$

output $= m \times 1$.

Eg $y = Ax_1$

$y = Ax_2$

$A(x_1 - x_2) = 0$ remove y.

$(x_1 - x_2) \in N(A)$

$(x_1 - x_2)$ input vectors output zero vector.

$\therefore (x_1 - x_2)$ is in $N(A)$.

1) Linear measurement system

$y = Ax$ (y: measurements, x: to be estimated)

$y = A(x + n)$ $n \in N(A)$

2) $y = Ax$ ← control inputs (y: destination) control systems

$$\begin{bmatrix} \text{velocity} \\ \text{acceleration} \\ \\ \end{bmatrix} = A \underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{2 \times 4} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{4 \times 1}$$

control systems like $N(A)$ because it enables the selection of a degree of freedom you want read chapter 3 - quadratic forms and matrix norm

Theorem 3.4 (pg 30)

Let $A \in \mathbb{R}^{m \times n}$ be a given matrix ~~whe~~ then

1) $R(A)^{\perp} = N(A^T)$ ← sets

2) $N(A)^{\perp} = R(A^T)$

~~pr~~

(1) proof select an element from $R(A)$ and equate to $N(A^T)$

pick an element of $x \in R(A)^{\perp}$

~~x is perpendicular to R~~

x is a perpendicular ~~vector~~ to $R(A)$

$z^T x = 0$ $\quad z \in R(A)$

$\uparrow$ $\quad z = Ay$ ~~for~~

$R(A)$

$(Ay)^T x = 0$ for all y, $y \in \mathbb{R}^n$ $\quad A \in \mathbb{R}^{m \times n}$

$y^T A^T x = 0$ for all $y \in \mathbb{R}^n$

We need to prove $A^T x = 0$

assume $A^T x \neq 0$ $\quad A^T x = u$

$y^T(A^T x) \neq 0$ $\therefore$ $(A^T x)$ must be zero.

~~$y^T u = 0$~~

~~$u \neq 0$~~

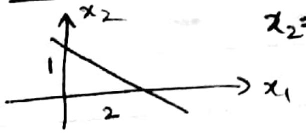
$u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \neq 0$ $\qquad y^T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} =$

given $y^T u = 0$ for all y. $y \in \mathbb{R}^n$ prove that $u = 0$

$\underbrace{y^T u = 0 \ \forall y}_{P} \implies \underbrace{u = 0}_{Q}$ $\quad$ suppose P and Not Q

$(y^T u = 0 \ \forall y$ and $\exists y \neq 0$ s.t $y^T u \neq 0)$

Line



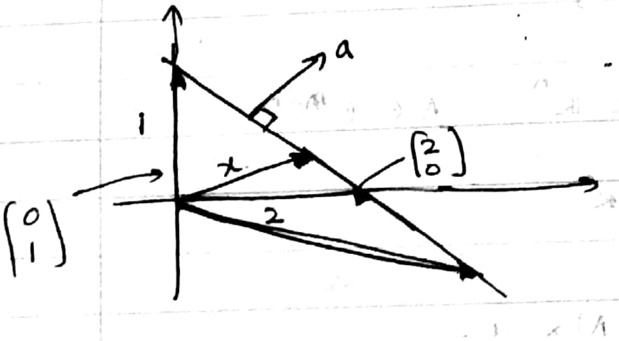
$x_2 = -\frac{1}{2}x_1 + 1$

$x_1 + 2x_2 = 2$

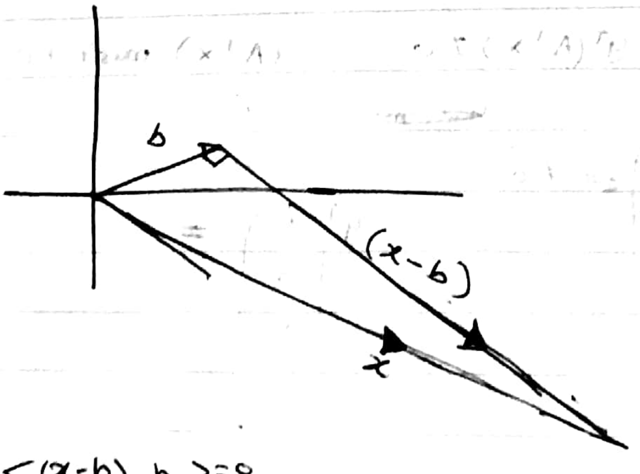
$\underbrace{(1 \quad 2)}_{a^T}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2 \qquad a^T x = 2$

$a = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$l = \{ x \mid a^T x = 2 \}$



There are several vectors satisfying line



$x^T a = a^T x$

normal vector to line

$b = \alpha a$

$\langle (x-b), b \rangle = 0$

$(x-b)^T b = 0$

$(x - \alpha a)^T a = 0$

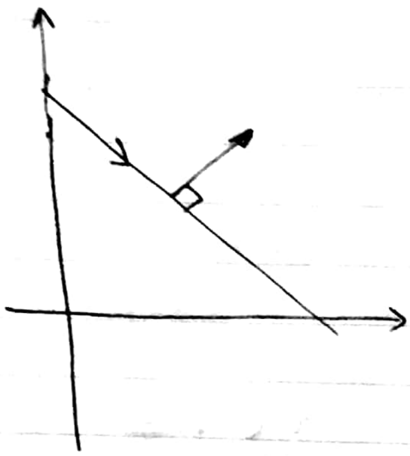
$(\alpha a^T - x^T) a = 0$

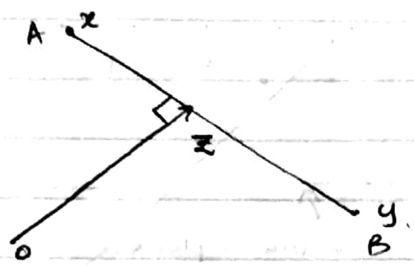
$\alpha \|a\|^2 - a^T x = 0$

$a^T x = \alpha \|a\|^2$

$(1 \quad 2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha \|a\|^2 = 2$

known point





$A = \{ z \mid \alpha x + (1-\alpha) y,\ \alpha \in (0,1) \}$

line segment AB

$A = \{ z \mid \alpha x + (1-\alpha) y,\ \alpha \in \mathbb{R} \}$

$z = y + \alpha(x-y)$ for

$z = \alpha x + (1-\alpha) y$ for

if $\alpha = 1$

$z = x$

$\alpha = 0$

$z = y$

The line can be ~~than~~ formulated by a normal vector or by end points.

<u>hyperplane</u>

$A = \{ x \mid a^T x = v \}$

a: given; v: const

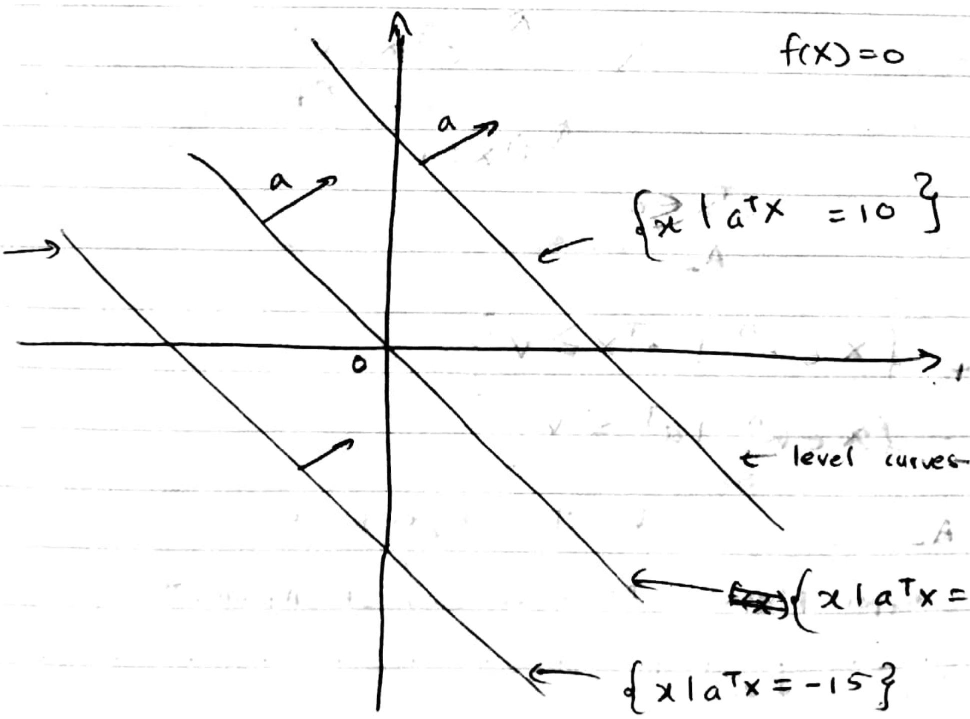
$v = 0$

$a^T x = 0$ at the point $(0,0,0)$

$x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$f(x) = 0$

level curves



$\{ x \mid a^T x = 10 \}$

← level curves of $f(x) = a^T x$

$\{ x \mid a^T x = 0 \}$

$\{ x \mid a^T x = -15 \}$

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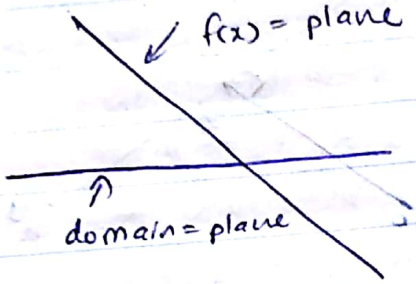
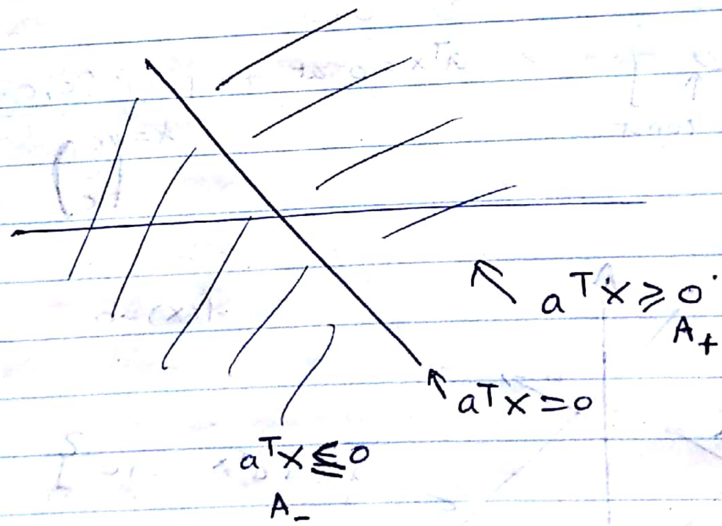


- The height of the ~~plane~~ plane equals the ~~magn~~ ~~magnitude~~ ~~magnitude of~~ distance along $\bar{a}$ direction. which is the 3rd degree of freedom for ~~the~~ $x \in \mathbb{R}^3$ f(x) is a plane and a^T is the normal to plane

$$A = \{ x \in \mathbb{R}^n \mid a^T x = 0 \}$$

$$A = \{ x \in \mathbb{R}^n \mid a^T x \leq 0 \}$$ negative halfspace of A

The negative or positive direction of the plane is determined by a^T direction)



if $A_+ = \{ x \in \mathbb{R}^n \mid a^T x \geq v \}$

$A_- = \{ x \in \mathbb{R}^n \mid a^T \leq v \}$

- $A_+ \cap A_-$ V is the hyper plane.

 = hyper plane is the intersection of A_+ and A_-.

$A^+ \cap A^- = \{ x \mid a^T x = 0 \}$

Homework 1 1.1, 1.2, 1.5

1.1 Construct the truth table for the statement (not B) ⟹ (not A) and use it to show that this statement is equivalent to the statement A ⟹ B

A	B	not A	not B	(not B) ⟹ (not A)	A ⟹ B
F	F	T	T	T	T
F	T	T	F	T	T
T	F	F	T	F	F
T	T	F	F	T	T

~~A ⟹ B~~

(~~not A~~)~~A B~~

not(not B) ∧ (not A)

B ∧ (not A)

1.2) Construct the truth table for the statement "not (A and (not B))" and use it to show that this statement is equivalent to the statement A ⟹ B

A	B	not B	A and (not B)	not (A and (not B))	A ⟹ B
F	F	T	F	T	T
F	T	F	F	T	T
T	F	T	T	F	F
T	T	F	F	T	T

1.5) Suppose that you are shown four cards, laid out in a row. Each card has a letter on one side and a number on the other. On the visible side of the cards are printed the symbols

S 8 3 A

Determine which cards you should turn over to decide if the following rule is true or false: If there is a vowel on one side of the card, then there is an even number on the other side

P = Probability of a vowel on one side of card.

Q = Probability of even number on other side of card.

$P\left(Q/P\right) = ?$

~~cards with a vowel or an even number on one side are 8 and A~~
~~So you should turn those cards~~

- cards with A and 3 printed on one side should be turned over
- Reason : The remaining cards are irrelevant to determine the truth or falsity of the rule.

- The card with 5 is irrelevant because it is not a vowel.

The card with 8 is also not relevant because we are trying to say if ~~an~~ there is vowel on one side the other side is even. We are not going to say if there is an even number on one side there is a vowel on the other side.

~~The card 3~~ By turning A you can directly verify the rule.

By turning 3 you can verify contraposition.

contraposition means not $P \Rightarrow$ not Q.

4) Show that any integer which can be divided by 4 (without a remainder) can also be divided by 2.

~~definition~~ | a number $b \in \mathbb{Z}$ is divisible by 4 $\Rightarrow$ $b \in \mathbb{Z}$ is divisible by 2

you may use either the direct method, proof by contraposition or proof by contradiction.

Moreover show that the converse of the statement above is FALSE, i.e. show that the statement

$b \in \mathbb{Z}$ is divisible by 2 $\Rightarrow$ $b \in \mathbb{Z}$ is divisible by 4. is FALSE.

Here, $\mathbb{Z}$ is the set of integers

① An integer n is divisible by 4 if and only if there exists ~~one~~ an integer k such that $n = 4k$

An integer ~~n~~ is divisible by 2 if and only if there exists an integer k such that $n = 2k$

Assume that n is divisible by 4 but n is not divisible by 2.
①　②nd assumption.

If n is not divisible by 2 then there is no integer k such that
$n = 2k$ — ①

but $n = 4k = 2\times\underbrace{(2k)}_{k_1} = 2k_1$　　$k_1 \in \mathbb{Z}$ — ②

We have shown that $n \neq 2k$ in ①
but $n = 2k_1$ in ②
contradic...

The ~~first second~~ 2nd assumption must be inaccurate

~~∴ n is divisible by 2~~ . ∴ When n is divisible by 4, n is divisible by 2.

④ Assumption 1: ~~Assumption~~ ~~where~~ b is divisible by ~~4~~ 2
Assumption 2: b is ~~divisib~~ ~~not~~ divisble by 4.

if b is divisible by 2, there exists an integer k such that
$b = 2k$　　$k \in \mathbb{Z}$ — ①

if b is divisible by 4, there exists an integer k such that
$b = 4k_1$　　$k_1 \in \mathbb{Z}$ — ②

① ⇒ but $b = 2k = 4\times\left(\frac{1}{2}\right)k$
↑ not $\mathbb{Z}$

~~② ⇒~~ ~~$b = 4k_1$~~

∴ b is not divisible by 4 } contract ~~dict~~
from ② ⇒ $b = 4k_1$

The second assumption must be inaccurate.
~~∴ if when b is divisible by 2 then b is not divisible by 4~~

∴ $b \in \mathbb{Z}$ is divisible by 2 ⇒ $b \in \mathbb{Z}$ is divisible by 4 is FALSE

Take $b = 2$ ~~b is~~ then b is divisible by 2 but not by 4.

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5. Consider the following set of linear equations.

$-x_1 + 3x_2 - 2x_3 = 1$

$-x_1 + 4x_2 - 3x_3 = 0$

$-x_1 + 5x_2 - 4x_3 = 0$

A is not a full rank matrix

$$\underbrace{\begin{bmatrix} -1 & 3 & -2 \\ -1 & 4 & -3 \\ -1 & 5 & -4 \end{bmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{b}$$

~~rank(A) = rank(A ¦ b)~~

~~rank(A) =~~ rank(A) = 3

ran(A ¦ b) = 3

rank (A) = rank(A ¦ B) True.

$b \in$ span{(a_1, a_2, a_3)}

$a_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$ $a_2 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ $a_3 = \begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix}$

if span of $a_1, a_2, a_3 = b$

$$c_1\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + c_3\begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

~~$c_1 = 1$ $c_2 = 1$ $c_3 = 1$~~

$R_1 R_2$

$-c_1 + 3c_2 - 2c_3 = 1$ — (1) ~~no~~ (2)

$-c_1 + 4c_2 - 3c_3 = 0$ — (2)

$-c_1 + 5c_2 - 4c_3 = 0$ — (3)

$$\left[\begin{array}{ccc|c} -1 & 3 & -2 & 1 \\ -1 & 4 & -3 & 0 \\ -1 & 5 & -4 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\left(\begin{array}{ccc|c} -1 & 3 & -2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -2 & -1 \end{array}\right)$$

$\cancel{c_3 \to c_2 + c_3}$

$\cancel{① - ②}$

$\cancel{-c_2 + c_3 =}$

② − ③

$$-c_2 + c_3 = 0$$

$$c_3 = c_2 \quad -③ \; *$$

① − ②

$$-c_2 + c_3 = -1$$

From ③

$$-c_3 + c_3 = -1$$

contradiction

No solutions for c_1, c_2, c_3.

∴ b does not span$[a_1, a_2, a_3]$

$b \notin \text{span}[a_1, a_2, a_3]$

~~There is no solution~~

as rank$(A) \neq$ rank$(A \mid b)$ there are solutions,

as rank(A) = ran$(A \mid b) = 3 \leq 3$ (no of columns) there are ~~infinitely many solutions~~ : has a unique solution.

⑥ Consider the following equation.

$$-x_1 + 2x_2 = 5.$$

Put it in the matrix form $Ax = b$.

$$\underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}_{1\times 2}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{2\times 1} = \begin{bmatrix} 5 \end{bmatrix}_{1\times 1}$$

rank$(A) = 1$

rank$[A \mid b] = 1$

rank(A) = rank$(A \mid b) = 1 < 2$

has infinitely many solutions.

7 Consider the following linear equations

$x_1 + x_2 = 2$

$2x_1 + x_2 = 3$

Put the equations above in the matrix form $Ax = b$ when $x =$

1) What is rank (A)
2) what is rank (A|b)
3) is $b \in$ Range(A)

Note: given a matrix $A = [a_1, a_2 \dots a_k]$ When $a_i \in \mathbb{R}^n$

Range(A) = span $[a_1 \; a_2 \; a_3 \dots a_k]$

more specifically

Range(A) $= \{ Ax \mid x \in \mathbb{R}^k \}$

Does the set of equation above has a solution. Is it unique

1) $$\underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b}$$

rank(A) = 2

rank (A|b) = 2

is $b \in$ span $(a_1 \; a_2)$

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$c_1 = 1 \qquad c_2 = 1$

yes $b \in$ span (a_1, a_2)

rank (A) = rank (A|b) = 2 = no of coloumns of A

$\therefore$ set of equations has a unique solution

Homework 2 ref 1 - 2.1 - 2.10

2.1) Let $A \in \mathbb{R}^{m \times n}$ and rank $A = m$. Show that $m \leq n$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & & \cdots & a_{mn} \end{pmatrix} \qquad \text{rank}(A) = m.$$

using contradiction.

assume $m \geq n$, then the maximum number of linearly independent ~~col~~ rows is n. ~~$A \leq m$~~

~~but rank(A) = m~~

$\therefore$ rank$(A) \leq n$ contradiction

but rank(A) $= m$ (given)

Thus assumption is wrong.

$\therefore$ if $A \in \mathbb{R}^{m \times n}$ and rank $(A) = m$, $m \leq n$

2.2) Prove that the system $Ax = b$, $A \in \mathbb{R}^{m \times n}$ has a unique solution if and only if rank A = rank$(A, b) = n$.

use contradiction.

~~Assume rank (A) < n.~~ It is impossible for rank$(A) > n$ as ~~A has~~ ~~$Ax = b$~~ A ~~will not have solutions if m.~~ ~~only n coloums.~~ A has only n ~~columns~~ coloums.

For $Ax = b$ to have a solution.

rank (A) = rank (A, b) -

Assume rank $(A) < n$ (The rank (A) cannot be greater than n because A has only n coloumns)

then $\exists\, y \in \mathbb{R}^n$ which are solutions to $Ax = b$

$Ay = b$.

also assume ~~that~~ $\exists\, x$ which satisfy $Ax = 0$

as there are coloumns in A which are linear dependent

~~Thus Ax x and y and $x+y$ are both solutions.~~

Thus $x, y, x+y$ are solutions.

$\therefore$ when rank(A) < n there is no unique solution.

$\therefore$ assumption is wrong.

only other possibility is $rank(A) = rank(A|b) = n$

$$rank(A) = rank(A|b) = n.$$

when $rank(A) = n$, assume there are two solutions x and y.

$$Ax = b \quad -(1)$$

$$Ay = b \quad -(2)$$

$(1) - (2)$ $A(x-y) = 0$

as $rank(A) = n$ ~~$A \neq 0$~~ there are no null columns in A

$A \neq 0$

$\therefore x - y = 0$

$x = y$

$\therefore$ there is only one solution. $\Rightarrow$ unique solution

2.3 We know that if $k \geq n+1$, Then the vectors $a_1, a_2, a_3, \ldots, a_k \in \mathbb{R}^n$ are linearly dependent that is there exist scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ such that at least one $\alpha_i \neq 0$ and $\sum_{i=1}^{k} \alpha_i a_i = 0$. Show that if $k \geq n+2$, then there exist scalars $\alpha_1, \ldots, \alpha_k$ such that at least one $\alpha_i \neq 0$ $\sum_{i=1}^{k} \alpha_i a_i = 0$ and $\sum_{i=1}^{k} \alpha_i = 0$

Introduce the vectors $\bar{a}_i = [1, a_i^T]^T \in \mathbb{R}^{n+1}$ $i = 1, \ldots, k$ and use the fact that any $n+2$ vectors in $\mathbb{R}^{n+1}$ are linearly dependent.

$\underline{a}_i = (1, \ldots, a_i^T) \in \mathbb{R}^{n+1}$ $i = 1, \ldots k$ $k \geq n+2$.

Any $n+2$ Vectors in $\mathbb{R}^{n+1}$ are linearly dependent.

$\sum_{i=1}^{k} \alpha_i a_i = 0$ $\alpha_i \neq 0$ at least one for $i = 1, \ldots, k$.

$$\bar{a}_i = \begin{matrix} 1 \\ 2^{T} \\ \vdots \\ a_k^T \end{matrix} \qquad 1, a_i^T = 1, a_{11}\ a_{12}\ a_{13} \dots a_{1n}$$

$$\bar{a}_i = \begin{pmatrix} 1^T \\ a_1^T \\ a_2^T \\ a_3 \\ \vdots \\ a_k^T \end{pmatrix} = \begin{pmatrix} 1 & \dots & 1 \\ a_{11} & & a_{1k} \\ a_{21} & & a_{2k} \\ a_{31} & & a_{3k} \\ \vdots & & \\ a_{k1} & & a_{k(n+1)} \end{pmatrix}$$

$$\sum_{i=1}^{k} \alpha_i \bar{a}_i = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_{11} & & & a_{1k} \\ a_{21} & & & a_{2k} \\ a_{31} & & & a_{3k} \\ \vdots & & & \\ a_{k1} & & & a_{kk} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$> n+2$

Considering first row:

$$\sum_{i=1}^{k} \alpha_i = 0$$

Considering rest of k components.

$$\sum_{i=1}^{k} \alpha_i a_i = 0 \qquad a_i \text{ has } n \text{ components.}$$

$$A\ \bar{a}_i = \begin{pmatrix} 1 & a_{11} & a_{12} & a_{13} & a_{14} & \dots & a_{1n} \\ 1 & a_{21} & a_{22} & a_{23} & a_{24} & & a_{2n} \\ \vdots & & & & & & \\ 1 & a_{k1} & & & & & a_{kn} \end{pmatrix}^T \qquad i = 1, \dots k.$$

$$A = \begin{pmatrix} 1 & 1 & & 1 \\ a_{11} & a_{21} & & a_{k1} \\ a_{12} & a_{22} & & \\ \vdots & \vdots & & \\ a_{1n} & a_{2n} & & a_{kn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

~~for last n components~~

$$\cancel{\sum_{j=1}^{k} \alpha_j a_j = 0}$$

Consider 1^{st} row

$$\sum_{j=1}^{k} \alpha_j = 0$$

consider the next n rows

$$\sum_{j=1}^{k} \alpha_j a_j = 0$$

2.4) Consider an $m \times m$ matrix M that has block form,

$$M = \begin{pmatrix} M_{m-k,k} & I_{m-k} \\ M_{k,k} & O_{k,m-k} \end{pmatrix}$$

Where $M_{k,k}$ is $k \times k$, $M_{m-k,k}$ is $(m-k) \times k$, I_{m-k} is the $(m-k) \times (m-k)$ identity matrix, and $O_{k,m-k}$ is the $k \times (m-k)$ zero matrix.

a) show that $|\det M| = |\det M_{k,k}|$

This result is relevant to the proof of proposition.

b) under certain assumptions, the following stronger result holds

$$\det M = \det(-M_{k,k})$$

identify cases where this is true, and show that it is false in general.

$$\cancel{M = \begin{pmatrix} M_{m-k,k} & I_{m-k} \\ M_{k,k} & O_{k,m-k} \end{pmatrix}}$$

~~M =~~ ~~Multiply~~ M by $\begin{pmatrix} I_{k,k} & \cancel{I_{k,k}} \\ -M_{m-k,k} & \cancel{O_{m-k}} \end{pmatrix}$

need to convert $M \Rightarrow$ diagonal matrix ~~with only $M_{k,k}$ in the non-diagonal~~ ~~space~~

$$M = \begin{bmatrix} M_{m-k,k} & I_{m-k} \\ M_{k,k} & O_{k,m-k} \end{bmatrix} \Rightarrow \begin{bmatrix} I_{k,k} & 0 \\ \cancel{M}\;0 & M_{k,k} \end{bmatrix}$$

$\Downarrow$

$$\underset{4\times4}{\begin{bmatrix} M_{m-k,k} & I_{m-k} \\ M_{k,k} & O_{k,m-k} \end{bmatrix}} \underbrace{\begin{bmatrix} I_{k,k} & 0 \\ -M_{m-k,k} & I_{m-k} \end{bmatrix}}_{P} \; 4\times4.$$

$\Downarrow$

$$\cancel{M \times P} \begin{bmatrix} 0 & I_{m-k} \\ M_{k,k} & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & I_{k,k} \\ I_{m-k} & 0 \end{bmatrix}}_{O}$$

$\Downarrow$

$$\begin{bmatrix} I_{m-k} & 0 \\ 0 & M_{k,k} \end{bmatrix}$$

$$MPO = \begin{bmatrix} I_{m-k} & 0 \\ 0 & M_{kk} \end{bmatrix}$$

det on both sides

$$\det M \times \det P \times \det O = \det \begin{bmatrix} I_{m-k} & 0 \\ 0 & M_{k,k} \end{bmatrix}$$

$$\det P = \begin{bmatrix} I_{k,k} & 0 \\ -M_{m-k,k} & I_{m-k} \end{bmatrix} = \underbrace{\det(I_{k,k})}_{1} \times \underbrace{\det(I_{m-k})}_{1} = 1$$

$$\det O = \begin{bmatrix} 0 & I_{k,k} \\ I_{m-k} & 0 \end{bmatrix} = \cancel{\det} - \underbrace{\det(I_{k,k})}_{1} \times \det \underbrace{I_{m-k}}_{1} = -1$$

$$\therefore \det M \times 1 \times -1 = \det(I_{m-k}) \times \det(M_{k,k}) = 1 \times \det M_{k,k}$$

$$\det M = -\det M_{k,k}$$

magnitude of $\det M = |\det M|$

magnitude of $\det M_{k,k} = |\det M_{k,k}|$

$$|\det M| = |\det(M_{k,k})|$$

b) assume $\det M = \det(-M_{k,k})$

$$\det M = \begin{vmatrix} M_{m-k,k} & I_{m-k} \\ M_{k,k} & O_{k,m-k} \end{vmatrix}$$

$$= |0 - I_{m-k} M_{k,k}|$$

$$= |I_{m-k}||(-M_{k,k})|$$

$$\det M = \det(-M_{kk})$$

Take some examples

$k = 2 \quad m = 3$

$M = M_{1,2}$

$$M = \begin{pmatrix} M_{m-k,k} & I_{m-k} \\ M_{k,k} & O_{k,m-k} \end{pmatrix}$$

1. $m = 2 \quad k = 1$

$$M = \begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix} \quad \text{assume } M_{kk} = 5$$

~~$\det M = -5 = M_{kk}$~~ $\det(-5) = \det$

$\det M = \det(-5) = \det(-M_{kk})$

2. $m = 3 \quad k = 1$, assum $M_{m-k,k} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$M = \begin{pmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{2\times 1} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 5 & (0 \quad 0) \end{pmatrix}$$

$\det M \neq -5$

$\therefore \det M \neq \det(-M_{k,k})$

3. $m = 4 \quad k = 1$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \\ 4 & & & \\ 5 & 0 & 0 & 0 \end{pmatrix}$$

~~$\neq -5$~~ $\det M \neq -5$

$\det M \neq \det(-M_{k,k})$

(4) $m = 4$ $k = 2$ assume $M = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

$$M = \begin{pmatrix} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \\ \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} & & 0 & 0 \\ & & 0 & 0 \end{pmatrix} \Rightarrow \det M = \det(-M_{KK})$$

$\uparrow M_{KK}$

$\therefore$ ~~the~~ $\det(M) = \det(-M_{K,K})$ when $m - k = k$.

$\therefore$ ~~m = 2k~~ when $m = 2k$ $\det(M) = \det(-M_{K,K})$

if $m \neq 2k$ $\det(M) \neq \det(-M_{K,K})$

when $m = 2k$, all sub matrices in M are square and $M_{K,K} = M_{m-k,K}$.

and all sub matrices are of same dimension.

Thus only under this special condition $\det(M) = \det(-M_{K,K})$

for general cases it is false.

2.5) It is well known that for any $a, b, c, d \in \mathbb{C}$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Suppose now that A, B, C, D are real or complex square matrices of the same size. Give a sufficient condition under which

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \cancel{\det}\ AD - BC \text{ is true.}$$

shouldn't this be $\det(AD - BC)$

Sufficient condition :- If at least one of the sub matrices blocks is equal to 0 (zero matrix) the formula holds.

~~eg.~~ ~~Ass~~

~~(1) A = B = 0 or A = C = 0~~ or ~~A = B = C = D = 0~~

~~$\det \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix} = 0$~~

(1) $A = B = 0$, $A = C = 0$, $A = B = C = D = 0$ (coloumn wise, row wise, or entire matrix is zero.)

$$\det \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix} = 0 - 0 \Rightarrow \text{formula holds.}$$

$AB = 0$

$BC = 0$

(2) if $A = D = 0$ or $B = C = 0$ (diagonal matrices are zero)

$$\det \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \overset{\det(AD)}{\cancel{\det(AD)}} - 0 \Rightarrow \text{formula holds}, \; BC = 0.$$

(3) if ~~A=0~~ one ~~mat~~ submatrix is zero, $D = 0$

$$M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \quad \cancel{\det M =}$$

~~post multiply by.~~

determinant of a block triangular matrix is the product of the determinants of its diagonal blocks.

~~to~~ let's convert M to a block triangular matrix using matrices whose det value is 1

$$M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\cancel{\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & I \\ I & I \end{pmatrix}}$$

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \underbrace{\begin{pmatrix} I & I \\ I & 0 \end{pmatrix}}_{L} = \begin{pmatrix} A+B & A \\ C & C \end{pmatrix}$$

~~A+B~~

$$\Downarrow$$

$$\begin{pmatrix} A+B & A \\ C & C \end{pmatrix} \times \underbrace{\begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}}_{K} = \begin{pmatrix} B & A \\ 0 & C \end{pmatrix}$$

~~det Mx~~

$$MLK = \begin{pmatrix} B & A \\ 0 & C \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} I & I \\ I & 0 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}}_{K} = \underbrace{\begin{pmatrix} B & A \\ 0 & C \end{pmatrix}}_{T}$$

~~det~~ ~~MxLK~~ $\det(MLK) = \det(T)$

~~as det(L) = det(k) = 1~~

as det (L) = -1 and det (k) = 1

det ~~(M) det (M)~~ = det (T) det (MLK) = det(M) × -1 × 1 = - det(M)

det (T) = det (B) det(C)

- det M ~~det (M)~~ = det(BC)

- det M = det(BC)

det M = det(-BC)

When one submatrix is zero D = 0

det M = ~~0~~ det(-BC)

= det (0 - BC)

det M = det (A × D - BC) ⇒ formula holds.

D = 0

2.6 Consider the following system of linear equations

$x_1 + x_2 + 2x_3 + x_4 = 1$

$x_1 - 2x_2 - x_4 = -2$

Use theorem 2.1 to check if The system has a solution. Then use the method of theorem 2.2 to find a general solution to the system.

$$\underbrace{\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ \end{array}\begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & -2 & 0 & -1 \end{pmatrix}_{2\times 4}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_{4\times 1} = \underbrace{\begin{pmatrix} 1 \\ -2 \end{pmatrix}_{2\times 1}}_{b}$$

rank (A) = 2 rank(A|b) = 2

∴ rank(A) = rank (A|b) < 4 ~~no of~~

4 = no of coloumns

There are infinitely many solutions

$$Ax = b$$

$$x = A^{-1} b$$

but A^{-1} doesn't exist as A is not square.

$$x_1 + x_2 = -2x_3 - x_4 + 1$$

$$x_1 - 2x_2 = x_4 - 2$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & +1 \end{bmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Take x_1 and x_2 as multiples of x_3 and x_4.

$$x_1 + x_2 = -2x_3 - x_4 + 1$$

$$x_1 - 2x_2 = x_4 - 2.$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} -2x_3 - x_4 + 1 \\ x_4 - 2 \end{bmatrix}$$

assigning arbitrary values d_3 and d_4 to x_3 and x_4

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2d_3 - d_4 + 1 \\ d_4 - 2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \begin{bmatrix} -2d_3 - d_4 + 1 \\ d_4 - 2 \end{bmatrix}$$

$$= \frac{1}{-2-1} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2d_3 - d_4 + 1 \\ d_4 - 2 \end{bmatrix}$$

$$= \begin{bmatrix} +\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} -2d_3 - d_4 + 1 \\ d_4 - 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} \times (-2d_3 - d_4 + 1) + \frac{1}{3}(d_4 - 2) \\ \frac{1}{3}(-2d_3 - d_4 + 1) - \frac{1}{3}(d_4 - 2) \end{bmatrix}$$

$$= \begin{pmatrix} -\frac{4d_3}{3} - \frac{d_4}{3} + \cancel{\frac{2}{3}} - \cancel{\frac{2}{3}} \\ -\frac{2}{3}d_3 - \frac{2}{3}d_4 + \frac{1}{3} + \frac{2}{3} \end{pmatrix} = \begin{bmatrix} -\frac{4d_3}{3} - \frac{d_4}{3} \\ -\frac{2d_3}{3} \ldots \end{bmatrix}$$

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$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{4}{3}d_3 - \frac{d_4}{3} \\ -\frac{2}{3}d_3 - \frac{2d_4}{3} + 1 \\ d_3 \\ d_4 \end{pmatrix}$$

2.7 Prove the seven Properties of the absolute value of a real numbers.

Properties of of absolute Value of a real number a

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

1) $|a| = |-a|$.

~~(2) $-|a| \leq a \leq |a|$.~~

~~(3) $|a+b| \leq |a| + |b|$.~~

(4) $|a|$

(1) $\cancel{|-a| = \begin{cases} -a & \text{if} -a \geq 0 \\ +a & \text{if } a < 0 \end{cases}}$?

(1) $\cancel{|-a| = \begin{cases} -a & \text{if } -a \geq 0 \\ +a & \text{if } -a < 0 \end{cases} = \begin{cases} a & \text{if } a \leq 0 \\ -a & \text{if } a > 0 \end{cases}}$

⇒

(1) $|a| = |-a|$.

Applying the definition

$$|-a| = \begin{cases} -a & \text{if } (-a) \geq 0 \\ -(-a) & \text{if } (-a) < 0 \end{cases} = \begin{cases} -a & \text{if } a \leq 0 \\ a & \text{if } a > 0 \end{cases}$$

$$= \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases} = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases} = |a|.$$

$\therefore$

(2) $\cancel{-|a| \le a \le |a|}$

~~for~~ $a > 0$

$-|a| \le a$ and $a \le |a|$

for $a < 0$

$-\cancel{|a|}$ $a \le |a|$ and $-|a| < a$.

$\therefore -|a| < a \le |a|$

(2) $-|a| \le a \le |a|$

for $a < 0$

$a \le |a|$ as $|a| > 0$

$\cancel{-|a| \le a}$ $|a| = -|a|$

for $a > 0$

$-|a| < a$ as $-|a| \le 0$

$-|a| \le a$

$a = |a|$

considering both cases, in general

$-|a| \le a \le |a|$

(3) $|a+b| \le |a| + |b|$

$$|a+b| = \begin{cases} a+b & \text{if } a+b > 0 \\ -(a+b) & \text{if } (a+b) < 0 \end{cases}$$

$|a|, |b|$.

~~for~~ $\cancel{a+b > 0}$

1 $a > 0$ $\cancel{b > 0}$. 2 $\cancel{a > b > 0}$ $\cancel{a > b}$

1 if $a > 0$ $b > 0$.

$|a+b| = (a+b) = |a| + |b|$.

2 if $a < 0$ $b < 0$

$|a+b| = -(a+b) = \cancel{(a+b)}$ $|a| + |b|$.

3 if $a \ge 0$ $b \le 0$ and $a \ge b$

$\cancel{|a+b|}$ $a+b \ge 0$ $|a+b| = a+b \le |a| + |b|$

~~4 if~~ $\cancel{a \le 0}$ $\cancel{b > 0}$ ~~and~~ $\cancel{b > a}$.

$\cancel{a+b \ge 0}$ $\cancel{|a+b| = a+b \le |a|+|b|}$

4 if $a \ge 0$ $b \le 0$ $b \ge a$

$(a+b) \le 0$ $|a+b| = -(b+a) \le |a| + |b|$.

~~The case~~ $\cancel{a \ge}$ $\cancel{a \le 0}$ $\cancel{b \ge 0}$ ~~and~~ $\cancel{a \ge b}$ ~~is similar to 4~~

~~" "~~ $\cancel{a > 0}$ $\cancel{b \le 0}$ ~~and~~ $\cancel{b \ge a}$ $\cancel{b \le a}$

~~The case $a \le 0$ $b \ge 0$ $a \ge b$ is similar to~~

$a \le 0 \quad b \ge 0 \quad a \ge b$ is similar to 4

$a \le 0 \quad b \ge 0 \quad b \ge a$ is similar to 3

$\therefore \; |a+b| \le |a|+|b|$

(4) $||a|-|b|| \le |a-b| \le |a|+|b|$

$|a-b| \le |a|+|b|$

$|a-b| = |a+(-b)| \le |a|+|-b|$

$|a-b| \le |a|+|b|$ from property 3.

~~$|a| - b$~~ $\quad |-b| = |b|$ from property 1

$||a|-|b|| \le |a-b|$

$$||a|-|b|| = \begin{cases} |a|-|b| & \text{for } |a|>|b| \\ |b|-|a| & \text{for } |a|<|b| \end{cases}$$

~~$|a|$ for $|a|>|b|$~~ $\quad |a| = |a-b+b| \le |a-b|+|b|$ property 3

~~$|a|-|b|$~~ ~~$|a-b| \le |a|+|-b|$~~

~~for $a>b$~~ $\quad |a|-|b| \le |a-b|$ — 1

$|b|-|a| \le |b-a|$.

but $|a-b| = |-(a-b)|$ property 1

$|a-b| = |b-a|$

$\therefore |b|-|a| \le |a-b|$

$-(|a|-|b|) \le |a-b|$ — 2

from 1 and 2

$\therefore \; ||a|-|b|| \le |a-b|$

(5) $|ab| = |a||b|$

~~if a and b are real and~~

① $a > 0 \quad b > 0$

$|ab| = ab = |a||b|$

② $a < 0 \quad b < 0$

$|ab| = (-a)(-b) = |a||b|$

③ $a > 0 \quad b < 0$

$|ab| = a \times (-b) = |a||b|$

④ $a < 0 \quad b > 0$

$|ab| = (-a) \times b = |a||b|$

$\therefore$ ~~for a~~ in general $|ab| = |a||b|$.

(6) $|a| \le c$ and $|b| \le d$ imply that $|a+b| \le \cancel{c+d}\ c+d$.

$|a+b| \le |a| + |b|$ — ② property 3

$|a| \le c$ — ⓐ

$|b| \le d$ ⓑ

ⓐ and ⓑ $\Rightarrow |a| + |b| \le c+d$

② $\Rightarrow |a+b| \le c+d$.

(7) $\cancel{|a| < b \iff -b < a < b}$

$-b \le a \le \cancel{b}$

$\cancel{-|a| \le a \le |a|}$ $\quad \cancel{-|a| \le b \le |a|}$

$\cancel{-|a| \le a \le |a|}$

from Propert 2

$-|a| \le a \le |a|$

$|a| < b$ means

if $a > 0$ $\quad$ if $a < 0$

~~$|a| < b$~~ $a \le b$ $\quad |a| = -a \quad |a| \le b$

$-a \le b$

$a > -b$

Combined $\Rightarrow -b \le a \le b$

(7) from property 2

$-|a| \le a \le |a|$

$|a| < b$

if $a > 0$ $|a| = a$ $\Rightarrow a < b$

if $a < 0$ $|a| = -a$ $\Rightarrow -a < b \Rightarrow a > -b$ } $-b < a < b$

* also $|a| \le b \Rightarrow -b \le a \le b$

(8) $-|a| \le a \le |a|$

$|a| > b$

if $a > 0$ $|a| = a$ $a > b$

if $a < 0$ $|a| = -a$. $-a > b$ $a < -b$ } $a < -b$ or $a > b$

* also $|a| \ge b \Rightarrow a \le -b$ or $a \ge b$

~~2.8 $\langle x, y\rangle_2 = x^T \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} y = (Qx)^T Q y = x^T Q^2 y$~~

~~$Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$~~

~~$Q = Q^T$ Q is non singular $\Rightarrow$ has an inverse~~

2.8 consider the function $\langle -, - \rangle_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\langle x, y\rangle_2 = 2x_1y_1 + 3x_2y_1 + 3x_1y_2 + 5x_2y_2$ where $x = (x_1, x_2]^T$ and $y = [y_1, y_2]$

Show that $\langle -, - \rangle_2$ satisfies conditions 1 – 4 for inner products.

~~Note.~~

1) Positivity $\langle x, x\rangle \ge 0$ $\langle x, x \rangle = 0$ iff $x = 0$

2) Symmetry $\langle x, y\rangle = \langle y, x\rangle$

3) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ Additivity.

4) Homogeneity $\langle \gamma x, y\rangle = \gamma \langle x, y\rangle$ for every $\gamma \in \mathbb{R}$

~~$\langle x, x\rangle_2 = 2x^2 + 3x^2 + 3x^2 + 5x^2 =$~~

~~$\langle x, y\rangle_2 = x^T \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} y = (Qx)^T Q y = x^T Q^2 y$~~

~~$Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$~~

$\begin{pmatrix} 1 & 2 \end{pmatrix} \cancel{\leftrightarrow} \begin{pmatrix} 1 & 2 \end{pmatrix}$

$G = Q^T$

$$\langle x, y \rangle_2 = x^T \underbrace{\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}}_{Q^2} y = (Qx)^T (Qy) = x^T \underbrace{Q^2}_{} y$$

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

1) $\langle x, x \rangle = x^T \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} x = (Qx)^T (Qx) = x^T Q^2 x.$

$= \|Q^2 x\| \geq 0$

$\rightarrow \quad x = 0 \quad \langle x, x \rangle = 0$

(2) $\langle x, y \rangle_2 = (Qx)^T (Qy) = (Qy)^T \cdot Qx = \langle y, x \rangle_2$

(3) $\langle x+y, z \rangle = \cancel{\langle x, z \rangle + \langle y, z \rangle} \; (x+y)^T Q^2 z.$

$= (Qx + Qy)^T Q z$

$= [Q(x+y)]^T Q . z$

$= (x+y)^T Q^T . Q z$

$= (x+y)^T Q^2 z$

$= (x^T + y^T) Q^2 z$

$= x^T Q^2 z + y^T Q^2 z$

$= \langle x, z \rangle_2 + \langle y, z \rangle_2.$

(4) $\langle \gamma x, y \rangle = (\gamma x)^T Q^2 y$

$= x^T \gamma^T Q^2 y.$

$= \gamma^T (x^T Q^2 y) \quad \gamma^T \in \mathbb{R} \Rightarrow \gamma (x^T Q^2 y)$

$= \gamma \langle x, y \rangle_2.$

2.q) Show that for any two vectors $x, y \in \mathbb{R}^n \; \big| \|x\| - \|y\| \big| \leq \|x - y\|.$

~~Ans~~ $x = (x-y) + y \quad ; \quad y = (y-x) + x$

$\cancel{|x| \leq |x-y| +}$

$|(x-y) + y| \leq |x-y| + |y| \quad \Rightarrow \quad \cancel{|x|} - |y| \leq |x-y|$

$|(y-x) + x| \leq |y-x| + |x| \qquad |y| - |x| \leq |y-x|$

$|y| - |x| \leq |x-y|$

$-(|x| - |y|) \le |x-y|$

$\therefore \ ||x|-|y|| \le |x-y|$

$x = (x-y)+y$

from triangle inequality

~~$\|x\| \le x-y$~~ ~~$\|(x-y)\|+$~~ $\|x-y+y\| \le \|x-y\| + \|y\|.$

$\|x\| \le \|x-y\| + \|y\|. \rightarrow \|x\| - \|y\| \le \|x-y\|$ — ①

Similarly for $y = y-x+x.$

$\|y-x+x\| \le \|y-x\| + \|x\|$

$\|y\| - \|x\| \le \|y-x\|.$

$\|y\| - \|x\| \le \|-(x-y)\|$

$-(\|x\| - \|y\|) \le \|x-y\|$ — ②

combining ① and ②

$$\big| \|x\| - \|y\| \big| \le \|x-y\|$$

2.10 ~~norm~~ show norm is a uniformly continous function that is for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x-y\| < \delta$ then $|\|x\| - \|y\|| < \varepsilon$

$\|x-y\| < \delta$

from $|\|x\| - \|y\|| \le \|x-y\| < \delta$

~~with~~ ~~$\delta = \varepsilon$~~ $\therefore$

$|\|x\| - \|y\|| \le \delta$

if $\delta = \varepsilon$

$|\|x\| - \|y\|| \le \varepsilon$

(property)

Homework 3 - 3.1 - 3.22

① $x = x_1 e_1 + x_2 e_2 + x_3 e_3 + \dots + x_n e_n.$

$x' = x_1 e_1' + x_2 e_2' + x_3 e_3' + \dots + x_n e_n'$

$$\begin{pmatrix} e_1' \\ e_2' \\ e_3' \\ \vdots \\ e_n' \end{pmatrix} = T \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

$T' =$

$$x' = (x_1\ x_2\ x_3\ \dots\ x_n) \begin{pmatrix} e_1' \\ e_2' \\ e_3' \\ \vdots \\ e_n' \end{pmatrix} = (x_1\ x_2\ x_3\ \dots\ x_n)\, T \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_n \end{pmatrix}$$

$$x' = T\,(x_1\ x_2\ x_3\ \dots\ x_n) \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_n \end{pmatrix} = Tx$$

① fix a vector in $\mathbb{R}^n$. Let x be the column of the coordinates of the vector w.r.t the basis $\{e_1, e_2, e_3, \dots, e_n\}$ and x' the coordinates of the same vector with respect to to the basis $\{e_1', e_2', e_3' \dots, e_n'\}$. show that $x' = Tx$. where T is the transformation matrix from $\{e_1, e_2, \dots, e_n\}$ to $\{e_1', e_2', \dots e_n'\}$

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Let the vector be $v \in \mathbb{R}^n$

$$v = x_1 e_1 + x_2 e_2 + \dots + x_n e_n \quad \text{①}$$

$$v = x_1' e_1' + x_2' e_2' + x_3' e_3' + \dots + x_n' e_n' \quad \text{②}$$

① = ②
$$x_1 e_1 + x_2 e_2 + \dots + x_n e_n = x_1' e_1' + x_2' e_2' + x_3' e_3' + \dots + x_n' e_n'$$

$$\cancel{\begin{bmatrix} x_1' & x_2' & x_3' & \dots & x_n' \end{bmatrix} * \begin{pmatrix} e_1' \\ e_2' \\ e_3' \\ \vdots \\ e_n' \end{pmatrix}} = \cancel{\begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_n \end{pmatrix}}$$

$$\cancel{\begin{bmatrix} x_1' & x_2' & \dots & x_n' \end{bmatrix} =}$$

$$\begin{bmatrix} e_1' & e_2' & e_3' & \dots & e_n' \end{bmatrix} \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 & \dots & e_n \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} e_1' & e_2' & e_3' & \dots & e_n' \end{bmatrix}^{-1} \begin{bmatrix} e_1 & e_2 & e_3 & \dots & e_n \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

$$\begin{matrix} e_1' \\ e_2' \\ e_3' \end{matrix} = T \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} \qquad \cancel{\begin{bmatrix} e_1' & e_2' & \dots & e_n' \end{bmatrix} = T \begin{bmatrix} e_1 & e_2 & e_3 & \dots & e_n \end{bmatrix}}$$

$$\cancel{T =}$$

$$\cancel{\begin{bmatrix} e_1 & e_2 & e_3 & \dots & e_n \end{bmatrix} T = \begin{bmatrix} e_1' & e_2' & e_3' & \dots & e_n' \end{bmatrix}}$$

$$\cancel{T = \begin{pmatrix} e_1 & e_2 & \dots & e_n \end{pmatrix}^{-1} \begin{pmatrix} e_1' & e_2' & \dots & e_n' \end{pmatrix}}$$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_n \end{pmatrix}$$

inates

x'

he basis

he

$e_n' \}$

$$\cancel{= (e_1\ e_2\ e_3 \ldots e_n)}$$

$$= T\,(e_1\ e_2\ e_3 \ldots e_n)$$

$$[e_1'\ e_2' \ldots e_n']\,T = [e_1\ e_2\ e_3 \ldots e_n]$$

$$T = [e_1'\ e_2' \ldots e_n']^{-1}[e_1\ e_2\ e_3 \ldots e_n]$$

$$v = x_1'e_1' + x_2'e_2' + x_3'e_3' + \cdots + x_n'e_n'$$

$$v = x_1e_1 + x_2e_2 + x_3e_3 + \cdots + x_ne_n$$

$$[x_1'\ x_2'\ x_3' \ldots x_n']\begin{bmatrix} e_1' \\ e_2' \\ e_3' \\ \vdots \\ e_n' \end{bmatrix} = [x_1\ x_2\ x_3 \ldots x_n]\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_n \end{bmatrix}$$

$$[e_1'\ e_2'\ e_3' \ldots e_n']\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_n' \end{bmatrix} = [e_1\ e_2\ e_3 \ldots e_n]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_n' \end{bmatrix} = [e_1'\ e_2'\ e_3' \ldots e_n']^{-1}[e_1\ e_2\ e_3 \ldots e_n]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_n' \end{bmatrix} = T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \qquad x' = Tx$$

3.2) For each of the following cases, find the transformation matrix T from $\{e_1, e_2, e_3\}$ to $\{e_1', e_2', e_3'\}$

a) $e_1' = e_1 + 3e_2 - 4e_3$

$e_2' = 2e_1 - e_2 + 5e_3$

$e_3' = 4e_1 + 5e_2 + 3e_3$

$$\begin{pmatrix} e_1' \\ e_2' \\ e_3' \end{pmatrix} = \begin{pmatrix} 1 & 3 & -4 \\ 2 & -1 & 5 \\ 4 & 5 & 3 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -4 \\ 2 & -1 & 5 \\ 4 & 5 & 3 \end{pmatrix}^{-1} \begin{pmatrix} e_1' \\ e_2' \\ e_3' \end{pmatrix}$$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \underbrace{\begin{bmatrix} 0.6666 & 0.6904 & -0.261 \\ -0.333 & -0.452 & 0.3095 \\ -0.333 & -0.166 & 0.1666 \end{bmatrix}}_{T} \begin{pmatrix} e_1' \\ e_2' \\ e_3' \end{pmatrix}$$

$$T = (e_1' \quad e_2' \quad e_3')^{-1} [e_1 \quad e_2 \quad e_3]$$

$$[e_1' \quad e_2' \quad e_3']_{1\times3} = [e_1 \quad e_2 \quad e_3]_{1\times3} \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 5 \\ -4 & 5 & 3 \end{bmatrix}_{3\times3}}$$

$$[e_1' \quad e_2' \quad e_3'] \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 5 \\ -4 & 5 & 3 \end{bmatrix}^{-1}}_{T} = [e_1 \quad e_2 \quad e_3]$$

$$T = \begin{bmatrix} 0.6666 & -0.333 & -0.333 \\ 0.6964 & -0.452 & -0.1666 \\ -0.261 & 0.3095 & 0.1666 \end{bmatrix}$$

b) $e_1 = e_1' + e_2' + 3e_3'$

$e_2 = 2e_1' - e_2' + 4e_3'$

$e_3 = 3e_1' + 5e_3'$

$$[e_1 \quad e_2 \quad e_3] = [e_1' \quad e_2' \quad e_3'] \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 3 & 4 & 5 \end{bmatrix}}_{T}$$

$$T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 3 & 4 & 5 \end{bmatrix}$$

3. 3) considered 2 bases of $\mathbb{R}^3$ $\{e_1, e_2, e_3\}$ and $\{e_1', e_2', e_3'\}$ whe

$e_1 = 2e_1' + e_2' - e_3'$

$e_2 = 2e_1' - e_2' + 2e_3'$

$e_3 = 3e_1' + e_3'$

Suppose that a linear transformation has a matrix representation in $\{e_1, e_2, e_3\}$ of the form

$y = Ax$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 6 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the matrix representation of this linear transformation in the basis $\{e_1', e_2', e_3'\}$

$y' = Bx'$

$$[e_1' \quad e_2' \quad e_3'] = [e_1 \quad e_2 \quad e_3] \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[e_1' \quad e_2' \quad e_3'] \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}}_{T} = [e_1 \quad e_2 \quad e_3]$$

$$\cancel{T = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

$y' = Ty$ $\qquad\qquad x' = Tx.$

$y' = T(Ax)$

$= (TAT^{-1})x'$

$y' = Bx'$

$B = TAT^{-1}$

$$[e_1 \quad e_2 \quad e_3] = [e_1' \quad e_2' \quad e_3'] \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -10 & -8 \\ -1 & 8 & 4 \\ 2 & -13 & -7 \end{bmatrix}$$

3.4 Consider two bases of $\mathbb{R}^4$ $\{e_1, e_2, e_3, e_4\}$ and $\{e_1', e_2', e_3', e_4'\}$ where

$e_1' = e_1$

$e_2' = e_1 + e_2$

$e_3' = e_1 + e_2 + e_3$

$e_4' = e_1 + e_2 + e_3 + e_4$

Suppose that a linear transformation has a matrix representation in $\{e_1, e_2, e_3, e_4\}$ of the form

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ -3 & 2 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 0 & 3 \end{bmatrix} \qquad y = Ax$$

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Find the matrix representation of this linear transformation in basis $\{e_1', e_2', e_3', e_4'\}$

$\cancel{y = Ax}$ $\quad$ $y' = Ty$ $\quad$ $x' = Tx$.

$\cancel{Ty' = A}$

$y = Ax$

$y' = Bx'$

$Ty = BTx$

$TAx = BTx$.

$y = Ax$

$y' = Ty = TAx = (TAT^{-1})x'$

$B = TAT^{-1}$.

$$[e_1' \quad e_2' \quad e_3' \quad e_4'] = [e_1 \quad e_2 \quad e_3 \quad e_4]\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[e_1 \quad e_2 \quad e_3 \quad e_4] = [e_1' \quad e_2' \quad e_3' \quad e_4']\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$= [e_1' \quad e_2' \quad e_3' \quad e_4']\underbrace{\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{T}$$

$$B = TAT^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 & 0 \\ -3 & 2 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3 & -3 & 6 \\ -2 & 5 & -4 & 7 \\ 1 & 0 & -2 & 6 \\ 1 & -1 & 0 & 3 \end{pmatrix}$$

$$B = \begin{bmatrix} 5 & 3 & 4 & 3 \\ -3 & -2 & -1 & -2 \\ -1 & 0 & -1 & -2 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$

8.5) consider a linear transformation given by the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 5 & 2 & 1 \\ -1 & 1 & 0 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

eigenvectors of $A = \{v_1, v_2, v_3, v_4\}$.

$AV = \lambda V$.

$A(V_1 \; V_2 \; V_3 \; V_4) = [\lambda_1 V_1 \; \lambda_2 V_2 \; \lambda_3 V_3 \; \lambda_4 V_4]$

$$= (V_1 \; V_2 \; V_3 \; V_4) \underbrace{\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}}$$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are eigen values of A

$$(V_1 \; V_2 \; V_3 \; V_4)^{-1} A (V_1 \; V_2 \; V_3 \; V_4) = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} = B$$

$~~A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}~~$ $~~\lambda \to \lambda~~$

~~The basis of the~~

The linear transformation has a diagonal matrix form w.r.t the basis formed by a linearly independent set of eigenvectors of A

~~det (λ - A~~ $\quad AV = \lambda V$

~~det A - λ~~ $\quad |A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 0 & 0 & 0 \\ 1 & 1-\lambda & 0 & 0 \\ 2 & 5 & 2-\lambda & 1 \\ -1 & 1 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$\lambda = 1 \quad \lambda = 2 \quad \lambda = 3 \quad \lambda = -1$

$\lambda = 1$

$$AV = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 5 & 2 & 1 \\ -1 & 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix}$$

$$\left.\begin{aligned} -P_1 &= P_1 \\ P_1 + P_2 &= P_2 \end{aligned}\right\} P_1 = 0$$

$2P_1 + 5P_2 + 2P_3 + P_4 = P_3 \quad$ — ⓐ

$-P_1 + P_2 + 3P_4 = P_4 \implies P_2 + 2P_4 = 0$

$P_2 = -2P_4.$

ⓐ $\implies$

$5 \times (-2P_4) + 2P_3 + P_4 = P_3$

$-10P_4 + P_4 + P_3 = 0$

$P_3 = 9P_4.$

$$V_1 = \begin{pmatrix} 0 \\ -2 \\ 9 \\ 1 \end{pmatrix} //$$

$\lambda = 2$

$$\left.\begin{aligned} -P_1 &= 2P_1 \\ P_1 + P_2 &= 2P_2 \end{aligned}\right\} \Rightarrow \begin{aligned} P_1 &= 0 \\ P_2 &= 0. \end{aligned}$$

$\Rightarrow \cancel{2P_1} + \cancel{5P_2} + \cancel{2P_3} + P_4 = 2P_3$

$P_4 = 0$

$$V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} //$$

$\lambda = 3$

$-P_1 = 3P_1 \quad \Rightarrow P_1 = 0$

$P_1 + P_2 = 3P_2 \quad \Rightarrow P_2 = 0$

$2P_1 + 5P_2 + 2P_3 + P_4 = 3P_3 \quad \Rightarrow P_4 = P_3$

$3P_4 = 3P_4.$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} //$$

$\lambda = -1$

① $\Rightarrow \cancel{-P_1} = \cancel{-P_1}$

② $\Rightarrow P_1 + P_2 = -P_2 \quad \Rightarrow P_1 = -2P_2$

③ $\Rightarrow 2P_1 + 5P_2 + 2P_3 + P_4 = -P_3$

$2 \times -2P_2 + 5P_2 + 3P_3 + P_4 = 0.$

$P_2 + 3P_3 + P_4 = 0$

④ $\Rightarrow -P_1 + P_2 + 3P_4 = -P_4.$

$-\times(-2P_2) + P_2 + \cancel{3P_4}\, 4P_4 = 0$

$3P_2 = -4P_4.$

$P_2 = \frac{-4}{3} P_4.$

$P_1 = -2 \times P_2 = -2 \times \frac{-4}{3} P_4$

$P_1 = \frac{8}{3} P_4$

$P_3 = -\frac{P_2 - P_4}{3} = \frac{\frac{4}{3}P_4 - P_4}{3} = \frac{P_4}{9}$

$$v_4 = \begin{pmatrix} \frac{8}{3} \\ -\frac{4}{3} \\ \frac{1}{9} \\ 1 \end{pmatrix} = \begin{pmatrix} 24 \\ -12 \\ 1 \\ 9 \end{pmatrix}$$ //

$$\text{basis} = \left\{ \begin{pmatrix} 0 \\ -2 \\ 9 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 24 \\ -12 \\ 1 \\ 9 \end{pmatrix} \right\}$$

3.6) I'm not getting the answer in solution manual.
Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the matrix $A \in \mathbb{R}^{n \times n}$. Show That the eigenvalues of the matrix $I_n - A$ are $1-\lambda_1, \ldots, 1-\lambda_n$.

$(I_n - A)x = \lambda x$

$|I_n - A x| = 0$

$|\lambda - A| = 0$

λ

$Av = \lambda v$

$(I - A)v = v - Av$

$= v - \lambda v$

$(I - A)v. = (1-\lambda)v$

$\therefore (I_n - A)$'s eigen values are

$(1-\lambda_1), (1-\lambda_2), (1-\lambda_3), 1-\lambda_4, \ldots, 1-\lambda_n$.

$\pi_{I_n - A}(1-\lambda) = (-1)^n \pi_A(\lambda)$

3.7. Let $\mathcal{V}$ be a subspace show that $\mathcal{V}^{\perp}$ is also a subspace

Let $x, y \in \mathcal{V}^{\perp}$ and $\alpha, \beta \in \mathbb{R}$.

If $\mathcal{V}$ is a subspace of $\mathbb{R}^n$

if $x_1, x_2 \in \mathcal{V}$

$\alpha_1 x_1 + \alpha_2 x_2 \in \mathcal{V}$

$$\mathcal{V}^{\perp} = \{ x \mid v^T x = 0 \quad v \in \mathcal{V} \}$$

Vector from $\mathcal{V} = v$

~~$v = \alpha_1 x_1 + \alpha_2 x_2$~~

$v = \langle x_1, x_2 \rangle$

let $x_3, x_4 \in \mathcal{V}^{\perp}$

$$V^T(\alpha x_3 + \beta x_4) = \alpha V^T x_3 + \beta V^T x_4$$
$$= \alpha \langle V, x_3 \rangle + \beta \langle V, x_4 \rangle$$

$\langle V, x_3 \rangle = 0$ as $V \perp x$

$\langle V, x_4 \rangle = 0$ as $V \perp y$.

$\therefore \alpha x_3 + \beta x_4 \in \mathcal{V}^{\perp}$

3.8 Find the nullspace of

$$A = \begin{bmatrix} 4 & -2 & 0 \\ 2 & 1 & -1 \\ 2 & -3 & 1 \end{bmatrix}$$

nullspace of $A = N(A) = \{x \in \mathbb{R}^3 \mid Ax = 0\}$

$$\begin{pmatrix} 4 & -2 & 0 \\ 2 & 1 & -1 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

~~$R_3 \to R_3 - R_2$~~ ~~R_2~~

$$\cancel{\begin{pmatrix} 4 & -2 & 0 \\ 2 & 1 & -1 \\ 0 & -4 & 2 \end{pmatrix}} \qquad \cancel{A = \begin{pmatrix} 4 & -2 & 0 \\ 2 & 1 & -1 \\ 2 & -3 & 1 \end{pmatrix}}$$

~~$R_2 \to R_3 + R_2$~~

$R_3 \to R_3 - R_2 \qquad R_2 \to R_3 + R_2$

$$\cancel{[A \mid 0] = \left(\begin{array}{ccc|c} 4 & -2 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & -4 & 2 & 0 \end{array}\right)}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} c$$

$-4x_2 + 2x_3 = 0$

$2x_3 = 4x_2$

$x_3 = 2x_2$

$$\begin{pmatrix} 4 & -2 & 0 \\ 2 & 1 & -1 \\ 2 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 4 & -2 & 0 \\ 0 & 4 & -2 \\ 4 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$4x_1 - 2x_2 = 0$

$4x_2 - 2x_3 = 0$

$4x_1 = 2x_2$
$2x_1 = x_2$
$2x_2 = x_3$
$4x_1 = x_3$

$$N(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} c \; ; \; c \in \mathbb{R} \right\}$$

3.9 Let $A \in \mathbb{R}^{m \times n}$ be a matrix. Show that $R(A)$ is a subspace of $\mathbb{R}^m$ and $N(A)$ is a subspace of $\mathbb{R}^n$.

~~R(A)~~ Pf Let $x, y \in R(A)$ $\qquad u, w \in \mathbb{R}^m$

$x = Au \quad , y = Aw \qquad \alpha, \beta \in \mathbb{R}$

$\alpha x + \beta y = \alpha Au + \beta Aw = A(\underbrace{\alpha u + \beta w}_{y}) = Ay$ $\quad y \in \mathbb{R}^m$

$\therefore \alpha x + \beta y \in R(A)$

$\therefore R(A)$ is a subspace of $\mathbb{R}^m$

because vectors of $R(A) \in \mathbb{R}^m$

Let $x, y \in N(A)$

$Ax = 0 \qquad Ay = 0 \quad \Rightarrow x, y \in \mathbb{R}^n$

$\cancel{Ax + Ay = 0}$ $\qquad \alpha Ax + \beta Ay = 0 \qquad \alpha, \beta \in \mathbb{R}$

$\cancel{A(x+y) = 0}$ $\qquad A(\underbrace{\alpha x + \beta y}_{y}) = 0 \qquad y \in \mathbb{R}^n$

$\therefore \alpha x + \beta y \in N(A)$

$\therefore N(A)$ is a subspace of $\mathbb{R}^n$

because of vectors of $N(A) \in \mathbb{R}^n$

can't understand how to do this

3.10 Prove that if A and B are 2 matrices with m rows and ~~N(A)~~ $N(A^T) \subset N(B^T)$ then $R(B) \subset R(A)$

~~dim R~~ $\dim R(M) + \dim N(M^T) = m$

let $v \in R(B)$ ~~$v = Bx$~~ for some x

consider Av

Then $N[(Av)^T] = \{x \mid ABx = 0\}$

$N(A^T) = \{x \mid Ax = 0\}$

A, B matrices of m rows

A ⇒ $A^{m \times n}$

B ⇒ $B^{m \times y}$

~~A and B has m rows and~~

$N(A^T) = \{x \mid A^T x = 0\}$

$N(B^T) = \{x \mid B^T x = 0\}$

$\cancel{x \in R(A)^{\perp} \quad y^T(A^T x) = (Ay)^T x}$

$\cancel{x \in R(A)^{\perp}}$ $\cancel{R(A)^{\perp} = N(A^T)}$

subspace

$\dim R(M) + \dim N(M^T) = m.$

$v \in R(B)$

$v = Bx$

$\cancel{Av}$ $N(A^T) = N[(Av)^T]$

$u \in N(A^T)$ $u \in N(B^T)$

$u^T v = u^T B x = x^T B^T u = 0$

$\dim R(A)$

proof ⇒ A is given then $R(A)^{\perp} = N(A^T)$ and $N(A)^{\perp} = R(A^T)$

$x \in R(A)^{\perp}$ ⇒ $(Ay)^T \cdot x = 0$

$y^T A^T \cdot x = 0$

$A^T \cdot x = 0$

$\therefore \cancel{N(A^T) = x}$ $x \in N(A^T)$

$R(A)^{\perp} \subset N(A^T)$

$\cancel{x \in N(A^T)}$ $\cancel{x(A^T xy)}$

sir's answer $N(A^T) \subset N(B^T) \Rightarrow R(B) \subset R(A)$

1) $N(A^T) = R(A)^{\perp}$

$N(B^T) = R(B)^{\perp}$

2) $\dim\{R(A)\} + \dim\{N(A^T)\} = m$

$\leq 2 \quad \leq 5$

2

$N(A^T) \subset N(B^T)$

$R(B) \subset R(A)$

$m-5$ $m-2$

3.11. Let $\mathcal{V}$ be a subspace show that $(\mathcal{V}^{\perp})^{\perp} = \mathcal{V}$

$v \in V \quad u \in V^{\perp}$

$u^T \cdot v = v^T \cdot u = 0$

$v \in (V^{\perp})^{\perp}$

$(V^{\perp})^{\perp} \subset V$

$\{a_1, a_2, a_3, \ldots, a_k\}$ be a basis for V

$\{b_1, b_2, b_3, \ldots b_k\}$ be a basis for $(V^{\perp})^{\perp}$

$A = \{a_1, \ldots a_k\} \qquad B = \{b_1, b_2 \ldots b_k\}$

$V = R(A)$

$(V^{\perp})^{\perp} = R(B)$

$N(A^T) \subset N(B^T)$

$x \in N(A^T) \longrightarrow x \in R(A)^{\perp} = V^{\perp}$

since $R(A)^{\perp} = N(A^T)$

$(By)^T x = 0 = y^T B^T x \Rightarrow B^T x = 0$

$x \in N(B^T)$

~~3.b~~ ~~w \in~~ ~~w \in W^{\perp}~~

3.12 let $\mathcal{V}$ and W be subspaces. Show that if $\mathcal{V} \subset W$ then $W^{\perp} \subset \mathcal{V}^{\perp}$

let $w \in W^{\perp}$ and y be any element of V.

$$y \in \mathcal{V}$$

since $\mathcal{V} \subset W$.

$y \in W$

$\therefore$ by definition

$w^T \cdot y = 0$

$w \in V^{\perp}$

3.13. let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$. Show that there exist matrices V and U such that $\mathcal{V} = R(V) = N(U)$.

let $r = \dim(\mathcal{V})$

let $v_1, v_2, \ldots, v_r$ be a basis for $\mathcal{V}$ and V the matrix whose i^{th} column is v_i. Then clearly

$$\mathcal{V} = R(V)$$

$$V = \begin{bmatrix} v_1 & v_2 & v_3 & \ldots & v_i & \ldots & v_r \end{bmatrix}$$

$R(V) = AV \in \cancel{R(V)}\ \mathcal{V}$

let $u_1, u_2 \ldots u_{n-r}$ be a basis for $\mathcal{V}^{\perp}$ U is the matrix whose i^{th} row is u_i^T

Then $\mathcal{V}^{\perp} = R(U^T)$

$$U^T = \begin{bmatrix} u_1^T & u_2^T & u_3^T & \ldots & u_n^T \end{bmatrix}$$

$AU^T \in \mathcal{V}^{\perp}$

$R(U^T) \in \mathcal{V}^{\perp}$

$$\mathcal{V} = (\mathcal{V}^{\perp})^{\perp} = R(U^T)^{\perp} = N(U)$$

3.14) let P be an orthogonal projector onto a subspace $\mathcal{V}$. Show that

a) $Px = x$ for all $x \in \mathcal{V}$

b) $R(P) = \mathcal{V}$

$x \in \mathcal{V}$

$x = Px + (I-P)x$

$Px \in V$

$(I-P)x \in \mathcal{V}^{\perp}$

$x = Px + (I-P)x$ → orthogonal decomposition of x.

$x = x + 0$ is also an orthogonal decomposition of x w.r.t. $\mathcal{V}$

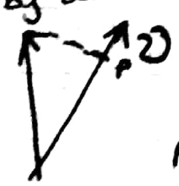
as orthogonal decomposition is ~~only one~~ unique

$x = Px$

b) P is an orthogonal projector onto $\mathcal{V}$

$R(P) \subset \mathcal{V}$ by definition

~~R(P)~~



$R(P) \subset \mathcal{V}$

from a) $x = Px \quad x \in \mathcal{V}$ $\quad R(p) = Px$

$\therefore \mathcal{V} \subset R(p)$

$R(P) = \mathcal{V}$

$-\frac{1}{2}-4$ $\quad -4+\frac{1}{2}$

$\frac{1}{2}+4$ $\quad -\frac{7}{2}$

3.15) Is the quadratic form

$$x^T \begin{pmatrix} 1 & -8 \\ 1 & 1 \end{pmatrix} x$$

positive definite, positive semidefinite, negative definite, negative semi definite, or indefinite.

$$x^T\left[\frac{1}{2}\begin{pmatrix} 1 & -8 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -8 & 1 \end{pmatrix}\right] x.$$

$$x^T \underbrace{\begin{pmatrix} 1 & -7/2 \\ -7/2 & 1 \end{pmatrix}}_{\text{symmetric matrix}} x. \qquad (x_1 \;\; x_2)\begin{pmatrix} 1 & -7/2 \\ -7/2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$(x_1 \;\; x_2)\begin{pmatrix} x_1 - 7/2 x_2 \\ -7/2 x_1 + x_2 \end{pmatrix}$$

leading principal minors are $\Delta_1 = 1$ and $\cancel{\Delta_2 = \frac{49}{4} = \frac{-7\times 7}{2\times 2} = \frac{49}{4}}$

$\Delta_2 = -45/2$ $\therefore$ indefinite.

Therefore quadratic form is indefinite.

$x_1^2 - 7/2 x_1 x_2 - 7/2 x_1 x_2 + x_2^2$

$x_1^2 - 7 x_1 x_2 + x_2^2$

$\cancel{(x_1 \;\;)(x_2 \;\;)}$

$\cancel{(x_1 - x_2)(x_1 + x_2)}$

3.16 Let $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

Show that although all leading principal minors of A are nonnegative A is not positive semidefinite

Theorem → ~~A symmetric matrix is~~ A quadratic form is positive semi definite if and only if all principal minors are non-negative.

Δ_1 s are all nonnegative

~~Δ_2 are all~~

$1 - \frac{49}{4}$ $\quad \frac{45}{4}$

$\Delta_1 = 0 \quad \Delta_2 = 0 \quad \Delta_3 = 0$ → leading principle minors

1, 2×2 non-leading principle minor is < 0

$$\begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} = -4 < 0$$

Quadratic is ~~positive~~ not positive semidefinite because one principal minor is negative

3.17. Consider the matrix

$$Q = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

a) is this matrix positive definite, negative definite or indefinite?

$\Delta_1 = 0 \qquad \Delta_2 = < 0 \qquad \Delta_3 = -(-1) + 1 = 2 > 0$

Indefinite

b) Is this matrix positive definite, negative definite or indefinite on the subspace

$$M = \{x : x_1 + x_2 + x_3 = 0\}$$

$x \in M$

$x_1 + x_2 + x_3 = 0$

$$x^T Q x = (x_1 \quad x_2 \quad x_3)\, Q \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1 \quad x_2 \quad x_3] \left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_2 + x_3 \quad x_1 + x_3 \quad x_1 + x_2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1(x_2 + x_3) + x_2(x_1 + x_3) + x_3(x_1 + x_2)$$

$$= x_1 \times -x_1 + x_2(-x_2) + x_3(-x_3) = -(x_1^2 + x_2^2 + x_3^2)$$

→ negative definite

3.18) For each of the following quadratic forms, determine if it is positive definite, negative definite, positive semidefinite, negative semidefinite or indefinite.

a) $f(x_1, x_2, x_3) = x_2^2$

~~b) $f(x_1, x_2, x_3) = x_1^2$~~

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2^2$$

eigen values are $\lambda = 0, \lambda = 1, \lambda = 0 \rightarrow$ positive semidefinite

b) $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - x_1 x_3$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 2 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$x_1 + x_3 \quad x_2 \quad x + x_3 \quad \underbrace{\qquad}_{A}$

~~eigen values $\Rightarrow \lambda = 1, \lambda = 2, \lambda = 0$ $\rightarrow$ positive~~

$|A - \lambda I| = 0.$

$$\begin{vmatrix} 1-\lambda & 0 & -\frac{1}{2} \\ 0 & 2-\lambda & 0 \\ -\frac{1}{2} & 0 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\{(2-\lambda)\times -\lambda\} + \tfrac{1}{2}(-\tfrac{1}{2})(2-\lambda) = 0$$

$$(1-\lambda)(\lambda^2 - 2\lambda) - \tfrac{1}{4}(2-\lambda) = 0$$

$$(\lambda - 2)\lambda(1-\lambda) - \frac{(2-\lambda)}{4} = 0$$

$$(\lambda - 2)\left(\lambda(\lambda-1) + \tfrac{1}{4}\right) = 0.$$

$$(\lambda - 2)\times\left(4\lambda(\lambda-1) + 1\right) = 0$$

$\lambda = 0$ $\quad = \frac{1}{2}$

$4\lambda^2 - 4\lambda + 1 = 0$

$\lambda = \dfrac{+4 \pm \sqrt{16 - 4\times 4}}{2\times 4}$

$$(1-\lambda)(2-\lambda)\lambda + \frac{1}{4}(2-\lambda) = 0$$

$$(2-\lambda)\,\cancel{(\lambda-\lambda^2)}\left[\lambda(1-\lambda) + \frac{1}{4}\right] = 0$$

$$(2-\lambda)(-4\lambda^2 + 4\lambda + 1) = 0$$

$\lambda = 2$, $\lambda = \frac{1+\sqrt{2}}{2}$, $\lambda = \frac{1-\sqrt{2}}{2}$

>0 >0 <0

~~Posi~~ indefinite

c) $f(x_1, x_2, x_3) = x_1^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}_{1\times 3} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{A}{}_{3\times 3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + x_2 + x_3 & x_1 + x_3 & x_1 + x_2 + x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

~~$A - \lambda I$~~

$x_1^2 + x_3^2$

$x_1^2 + \cancel{x_1x_2} + \cancel{x_1x_3} + \cancel{x_1x_2} + \cancel{x_3x_2} + \cancel{x_1x_3} + \cancel{x_2x_3} + x_3^2$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

~~$(1-\lambda)[-\lambda(1-\lambda) - 1] - [1(1-\lambda) - 1] + (1+\lambda) = 0$~~

~~$(1-\lambda)(\lambda^2 - \lambda - 1) + \lambda + 1 + \lambda = 0$~~

~~$(1-\lambda)(\lambda - 1)\lambda^2$~~

$(\lambda-1)$

$(1-\lambda)(-\lambda\times(1-\lambda)-1)-\{(1-\lambda)-1\}+(1+\lambda)=0$

$(1-\lambda)(\lambda^2-\lambda-1)+\lambda+1+\lambda=0$

$\cancel{\lambda^2}-\lambda^3-\cancel{\lambda}+\cancel{\lambda^2}-1+\cancel{\lambda}$

$\cancel{-\lambda^3+2\lambda^2+\lambda^2+\lambda-\lambda-1+\lambda+1+\lambda=0}$

$\cancel{-\lambda^3-2\lambda^2+2\lambda=0}$

$\cancel{\lambda^3+2\lambda^2-2\lambda=0}$

$\cancel{\lambda(\lambda^2+2\lambda-2)=0}$

$\cancel{(\lambda\quad)(\quad)}$

$\cancel{\lambda=-1+\sqrt{3}\qquad\lambda=-1-\sqrt{3}}$

$\cancel{\lambda=0}$

$-\lambda^3+2\lambda^2-\cancel{\lambda}+\lambda+\cancel{\lambda}+\lambda=0$

$-\lambda^3+2\lambda^2+2\lambda=0$

$\lambda^3-2\lambda^2-2\lambda=0$

$\lambda(\lambda^2-2\lambda-2)=0$

$\underset{>0}{\lambda=1+\sqrt{3}}\qquad\underset{<0}{\lambda=1-\sqrt{3}}\qquad\underset{=0}{\lambda=0}$ are the eigenvalues of A

thus quadratic form is indefinite.

3.19) Find a transformation that brings the following quadratic form into the diagonal form

$$f(x_1,x_2,x_3)=4x_1^2+x_2^2+9x_3^2-4x_1x_2-6x_2x_3+12x_1x_3$$

$$\underbrace{\begin{pmatrix}x_1 & x_2 & x_3\end{pmatrix}}_{x}\underbrace{\begin{bmatrix}4 & -2 & +6\\ -2 & 1 & -3\\ +6 & -3 & 9\end{bmatrix}}_{Q}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

$x=x_1e_1+x_2e_2+x_3e_3$ $\quad e_1,e_2,e_3$ is the basis for $\mathbb{R}^3$

V_1,V_2,V_3 another basis for $\mathbb{R}^3$

$\cancel{x=x_1V_1+x_2V_2+x_3V_3}$

$x=\bar{x}_1V_1+\bar{x}_2V_2+\bar{x}_3V_3\qquad x=V\tilde{x}$

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polynomial / quadratic.

$$f(x) = x^T Q x$$
$$= (V\tilde{x})^T Q (V\tilde{x})$$
$$\tilde{x}^T \underbrace{(V^T Q V)}_{\tilde{Q}} \tilde{x}$$

$$\tilde{Q} = \begin{pmatrix} \tilde{q}_{11} & \tilde{q}_{12} & \tilde{q}_{13} \\ \tilde{q}_{21} & \tilde{q}_{22} & \tilde{q}_{23} \\ \tilde{q}_{31} & \tilde{q}_{32} & \tilde{q}_{33} \end{pmatrix}$$

$$v_1 = \alpha_{11} e_1$$
$$v_2 = \alpha_{21} e_1 + \alpha_{22} e_2$$
$$v_3 = \alpha_{31} e_1 + \alpha_{32} e_2 + \alpha_{33} e_3$$

$\cancel{q_{ij} = v_i Q v_j}$

$$v_i Q e_j = 0 \qquad j < i$$
$$v_i Q e_i = 1 \qquad j = i$$

$$\tilde{Q} = \begin{pmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

$i = 1$

$$v_1^T Q e_1 = (\alpha_{11} e_1)^T Q e_1 = \alpha_{11} \underbrace{(e_1^T Q e_1)}_{q_{11}} = \alpha_{11} q_{11} = 1$$

$$\alpha_{11} = \frac{1}{q_{11}} = \frac{1}{\Delta_1} = \frac{1}{4} = 0.25$$

$\cancel{\alpha_{22}(e_2^T Q e_2) = 1}$

$\cancel{\alpha_{22} q_{22} = 1}$

$\cancel{\alpha_{22} = \frac{1}{1} = 1.}$

$$v_1 = \alpha_{11} e_1 = \begin{pmatrix} 0.25 \\ 0 \\ 0 \end{pmatrix}$$

for $i = 2$

~~case i=2~~

$$v_2^T Q e_1 = 0$$
$$v_2^T Q e_2 = 1$$

$v_2 Q e_1 = 0$

$(\alpha_{21} e_1 + \alpha_{22} e_2)^T Q e_1 = \alpha_{21}(e_1^T Q e_1) + \alpha_{22}(e_2^T Q e_1) = 0$

$= \alpha_{21} q_{11} + \alpha_{22} \cdot q_{21}$

$= \alpha_{21} \times 4 + \alpha_{22} \times (-2) = 0 \quad -(1)$

$v_2 Q e_2 = 1.$

$(\alpha_{21} e_1 + \alpha_{22} e_2)^T Q e_2 = \alpha_{21}(e_1^T Q e_2) + \alpha_{22} \underbrace{e_2^T Q e_2}_{q_{22}} = \phi.$

$= \alpha_{21}(-2) + \alpha_{22} \times 1 = 1 \quad (2)$

~~solving (1) and (2)~~

$\cancel{\alpha_{21}(-2)+2)=1}$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{21} \\ \alpha_{22} \end{pmatrix}$$

$D = 4 - 4 \quad 0$

$V_2^T Q e_2 = 0$ should be satisfied instead of $V_2^T Q e_2 = 1$ so that the system will have a solution.

from (1) $2\alpha_{21} = \alpha_{22}.$

$$\begin{pmatrix} \alpha_{21} \\ \alpha_{22} \end{pmatrix} = \begin{pmatrix} 1/2 \; \alpha_{22} \\ \alpha_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \phi \end{pmatrix} \alpha_{22}.$$

α_{22} is an arbitrary real numbers

$$V_2 = \alpha_{21} e_1 + \alpha_{22} e_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} a. \qquad a \in \mathbb{R}$$

case $i=3$

$V_3^T q e_1 = 0 \quad V_3^T Q e_2 = 0 \quad V_3^T Q e_3 = 1$

$V_3 = \alpha_{31} e_1 + \alpha_{32} e_2 + \alpha_{33} e_3$

$\alpha_{31} q_{11} + \alpha_{32} q_{21} + \alpha_{33} q_{31} = 0 \quad -(1)$

$\alpha_{31} q_{12} + \alpha_{32} q_{22} + \alpha_{33} q_{32} = 0$

$-2\alpha_{31} + \alpha_{32} + 3\alpha_{33} = 0$

$\alpha_{32} = +3\alpha_{33} + 2\alpha_{31}$

$\alpha_{31} q_{13} + \alpha_{32} q_{23} + \alpha_{33} q_{33} = 1$

$$\underbrace{\begin{pmatrix} q_{11} & q_{21} & q_{31} \\ q_{12} & q_{22} & q_{32} \\ q_{13} & q_{23} & q_{33} \end{pmatrix}}_{M} \begin{pmatrix} \alpha_{31} \\ \alpha_{32} \\ \alpha_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

as D of $M = 0$ non invertible.

$\alpha_{32} q'_{22} =$

$\alpha_{32} = -\alpha_{33} q_{31} - \alpha_{33} q_{11}$

$= -6\alpha_{33} - 4\alpha_{33}$

Therefore assume

$v_3^T Q e_3 = 0.$

$$\begin{pmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \\ 6 & -3 & 9 \end{pmatrix} \begin{pmatrix} \alpha_{31} \\ \alpha_{32} \\ \alpha_{33} \end{pmatrix} = \begin{bmatrix} \alpha_{31} \\ 2\alpha_{31} + 3\alpha_{33} \\ \alpha_{33} \end{bmatrix}$$

$\alpha_{31}, \alpha_{33} \in \mathbb{R}$ arbitrary real numbers

$$v_3 = \alpha_{31} e_1 + \alpha_{32} e_2 + \alpha_{33} e_3 = \begin{bmatrix} b \\ 2b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3a \\ a \end{bmatrix}$$

$$= \begin{bmatrix} b \\ 2b + 3c \\ c \end{bmatrix}$$

b, c arbitrary real numbers

$$V = \begin{bmatrix} \frac{1}{4} & \frac{a}{2} & b \\ 0 & a & 2b+3c \\ 0 & 0 & c \end{bmatrix}$$

Q

↑ Transformation that bring quadratic to diagonal.

3.20 Consider the quadratic form

$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 5x_3^2 + 2\varepsilon x_1 x_2 - 2x_1 x_3 + 4x_2 x_3$.

Find the values of the parameter ε for which this quadratic form is positive definite.

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & \varepsilon & -1 \\ \varepsilon & 1 & 2 \\ -1 & 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

leading minors must be >0

$D_1 = 1 > 0$

$D_2 = 1 - \varepsilon^2 > 0$:

$\varepsilon^2 - 1 < 0 \qquad -1 < \varepsilon < 1 \quad (1)$

$(\varepsilon - 1)(\varepsilon + 1) < 0$

$$D_3 = (5-4) - \varepsilon(5\varepsilon+2) - 1(2\varepsilon+1) > 0$$

$$\cancel{1} - 5\varepsilon^2 - 2\varepsilon - 2\varepsilon - \cancel{1} > 0$$

$$-(5\varepsilon^2 + 4\varepsilon) > 0$$

$$5\varepsilon^2 + 4\varepsilon < 0$$

$$\varepsilon(5\varepsilon+4) < 0$$

$$\cancel{\varepsilon < 0 \quad 5\varepsilon + 6 > 0}$$

$$\cancel{\varepsilon > -\tfrac{6}{5}}$$

$$\cancel{\varepsilon < -1.2}$$

$$\cancel{\varepsilon > \left(-\tfrac{4}{5}\right) \text{ or}}$$

$\varepsilon < 0 \quad 5\varepsilon+4 > 0$	$\varepsilon > 0 \quad (5\varepsilon+4) < 0$
$\varepsilon < 0 \quad \varepsilon > -\frac{4}{5}$	$\varepsilon > 0 \quad \varepsilon < -\frac{4}{5}$
$\varepsilon < 0 \quad \varepsilon > -0.8$	$\underbrace{\varepsilon > 0 \quad \varepsilon < -0.8}_{\text{both don't happen}}$

$$\varepsilon \in (-0.8, 0)$$

$$\underbrace{\varepsilon \in (-0.8, 0) \text{ and } \varepsilon \in (-1, 1)}_{\text{both satisfying range}}$$

$$\varepsilon \in (-0.8, 0)$$

$$\varepsilon \in (-4/5, 0)$$

3.21) Consider the function $\langle \cdot, \cdot \rangle_Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $\langle x, y \rangle_Q = x^T Q y$ where $x, y \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix. Show that $\langle \cdot, \cdot \rangle_Q$ satisfies conditions 1 to 4 for inner products. ~~(see~~

Q is symmetric

$$Q = Q^T$$

$$Q^{1/2} Q^{1/2} = (Q^{1/2} Q^{1/2})^T = (Q^{1/2})^T (Q^{1/2})^T > 0$$

$$\langle x, x \rangle = x^T Q x \quad x^T (Q^{1/2})^T (Q^{1/2}) x$$

$$= (Q^{1/2} x)^T (Q^{1/2}) x = \|Q^{1/2} x\|^2 \geq 0$$

$x = 0$ ~~($\because Q^{1/2} > 0$)~~ since $Q^{1/2}$ is non singular.

~~positive definite~~.

$Q^{1/2} > 0$

~~There is an inverse~~

2) $\langle x, y \rangle_Q = x^T Q y = [redacted] (Qy)^T x = y^T Q^T x = y^T Q \cdot x$.

$= \cancel{y^T Q x} \langle y, x \rangle_Q$.

3) ~~$\langle x+y, z \rangle_Q = (x+y)^T Q z = (Qz)^T (x+y)$~~

~~$= z^T Q^T (x+y)$~~

~~$= z^T Q (x+y)$~~

~~$= \langle z, (x+y) \rangle_Q$.~~

~~$= z^T Q x + z^T Q \cdot y$~~

$= \langle z, x \rangle_Q + \langle z, y \rangle_Q$

$\langle x+y, z \rangle_Q = (x+y)^T Q z$

$= x^T Q z + y^T Q z$

$= \langle x, z \rangle_Q + \langle y, z \rangle_Q$.

4) $\langle \gamma x, y \rangle_Q = (\gamma x)^T Q y = [redacted] x^T \gamma Q y = \gamma x^T_Q y = \gamma \langle x, y \rangle_Q$.

3.22) Consider the vector norm $\|\cdot\|_\infty$ on $\mathbb{R}^n$ given by $\|x\|_\infty = \max_i |x_i|$ where $x = [x_1, \dots, x_n]^T$. Define the norm $\|\cdot\|_\infty$ on $\mathbb{R}^m$ similarly. Show that the matrix norm induced by these vector norms is given by

$$\|A\|_\infty = \max_i \sum_{k=1}^{n} |a_{ik}|,$$

Where a_{ij} is the $(i,j)^{th}$ element of $A \in \mathbb{R}^{m \times n}$.

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$$\|A\|_\infty = \max\{\|Ax\|_\infty : \|x\|_\infty = 1\}$$

first show that

$$\|A\|_\infty \le \max_i \sum_{k=1}^{n} |a_{ik}| \; ; \; \|x\|_\infty = 1$$

$$\|Ax\|_\infty = \max_i \left| \sum_{k=1}^{n} a_{ik} x_k \right|$$

$$\|Ax\|_\infty \le \max_i \sum_{k=1}^{n} |a_{ik}||x_k|$$

$\cancel{\max = 1}$ $\cancel{\max = 1}$ $\max_i \sum x_k = \|x\|_\infty = 1$

$$\|A\|_\infty \le \max_i \sum_{k=1}^{n} |a_{ik}|$$

~~For~~ To show $\|A\|_\infty = \max_i \sum_{k=1}^{n} |a_{ik}|$

need to show $\tilde{x} \in \mathbb{R}^n$ $\|\tilde{x}\|_\infty = 1$ ~~so that~~

$$\cancel{\|A\tilde{x}\|_\infty = \max_i \sum_{k=1}^{n} |a_{ik}|}$$

$$\sum_{k=1}^{n} |a_{jk}| = \max_i \sum_{k=1}^{n} |a_{ik}|$$

$$\tilde{x}_k = \begin{cases} |a_{jk}| / a_{jk} & \text{if } a_{jk} \ne 0 \\ 1 & \text{otherwise} \end{cases}$$

$$\|\tilde{x}\|_\infty = 1$$

for $i \ne j$

$$\left| \sum_{k=1}^{n} a_{ik} \tilde{x}_k \right| \le \sum_{k=1}^{n} |a_{ik}| \le \max_i \sum_{k=1}^{n} |a_{ik}| = \sum_{k=1}^{n} |a_{jk}|$$

$$\left| \sum_{k=1}^{n} a_{jk} \tilde{x}_k \right| = \sum_{k=1}^{n} |a_{jk}|$$

$$\|A\tilde{x}\|_\infty = \max_i \left| \sum_{k=1}^{n} a_{ik} \tilde{x}_k \right| = \sum_{k=1}^{n} |a_{jk}| = \max_i \sum_{k=1}^{n} |a_{ik}|$$

$A = \begin{bmatrix}1\\2\end{bmatrix}_{2\times 1}$

$\dim\{R(A)\} = 1$

$\dim\{N(A^T)\} = 1$

$\{x \mid A^T x = 0\} = N(A^T)$

$\{x \mid \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = 0\}$

$\{x \mid x_1 + 2x_2 = 0\} \subseteq \mathbb{R}^2$.

$\dim\{R(A)\} + \dim\{N(A^T)\} = 1 + 1 = 2.$

$R(A) \subseteq \mathbb{R}^m$

$N(A^T) \subseteq \mathbb{R}^m$

$Ax = A_{2\times 1}\, x_{1\times 1} = \begin{bmatrix}\ \end{bmatrix}_{2\times 1}$

$R(A) \oplus R(A)^\perp = \mathbb{R}^m.$

↑ plane　　↑ vector $\perp$ to plane.

2018.02.18

if $H^+ = \{x \mid a^T x \leq b\}$

$H^- = \{x \mid a^T x > b\}$

$H^+ \cap H^- = \emptyset$

Linear variety is a set ~~for~~ of the form.

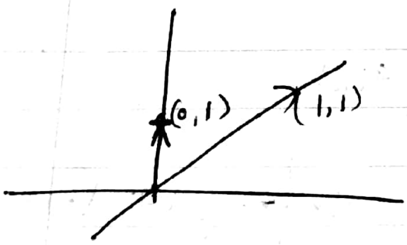
$\mathcal{A} = \{x \mid Ax = b\}$; $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$

$\mathcal{A} = \emptyset$

$\mathcal{A} = \{x \mid \begin{bmatrix}1\\1\end{bmatrix} x = \begin{bmatrix}0\\1\end{bmatrix}\}$

A is the range space

$\mathcal{A} = \emptyset$ because $\begin{bmatrix}0\\1\end{bmatrix}$ cannot be formed from $\begin{bmatrix}1\\1\end{bmatrix}$



A is a single value when $Ax = b$ when x has a unique solution

A is full rank & $\text{rank}(A) = \text{rank}(A|b)$

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$\{ x \mid Ax = b \}$ $A \in \mathbb{R}^{m \times m}$ $b \in$

e.g. $Ax = b$

$$\begin{pmatrix}1\\1\end{pmatrix} x = \begin{pmatrix}3\\3\end{pmatrix}$$

$b \in R(A)$?

If $\dim(N(A)) = r$; We say That the linear variety has $\dim(r)$

$N(A) = \{ x \mid Ax = 0 \}$

if $\dim(N(A)) = 0 \Rightarrow$ There is no Null space. $\Rightarrow$ A is full rank.

A is full coloumn rank.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}_{3\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 5 & 2 \\ -2 & 6 & 4 \end{bmatrix}_{2\times 3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1}$$

$Ax = b$

$(1 \; 1)\begin{pmatrix}x_1\\x_2\end{pmatrix} = 2$

$x_1 + x_2 = 2$

$$\begin{bmatrix} 1 & 2 \end{bmatrix}_{1\times 2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{2\times 1} = x_1 + 2x_2$$

$$\boxed{A}_{3\times 5}$$

$\text{Rank}(A) = 3$

Rows constitute of independent Vectors

$R(A) = \mathbb{R}^3$ A spans $\mathbb{R}^3$ because 3 coloumns are independent.

$\dim(N(A)) =$

$A \in \mathbb{R}^{m \times n}$

$$\dim\{R(A)\} + \dim\{N(A^T)\} = m.$$

inverse.

$$\dim \{R(A^T)\} + \dim\{N(A)\} = n.$$

$$\underbrace{\dim\{R(A^T)\}}_{3} + \underbrace{\dim\{N(A)\}}_{2} = 5.$$

B

5×3

$B = A^T$

A

3×5

$R(A^T) \subseteq \mathbb{R}^5$

$\dim(R(A^T)) = 3$

if Rank (A) $= 1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

only one coloumn is independent.

Let $\hat{x}$ be an particular solution of $Ax = b$

$A\hat{x} = b$ $\rightarrow$ if $\dim(N(A)) = 0$ $\longrightarrow$ only one solution.

~~has~~ unique

$\Downarrow$ full rank (A)

$A\hat{x} = b$ has infinite solution

if $\dim(N(A)) = 2$

more than 0 means infinite solutions

$\exists\, v$ such that

$Av = 0$

$A[\hat{x} + v] = b \Rightarrow \underbrace{A\hat{x}}_{b} + Av = b$

$$\mathcal{A} = \{\hat{x} + v \mid v \in \mathcal{N}(A)\}$$

where $\hat{x}$ is a particular sol^n

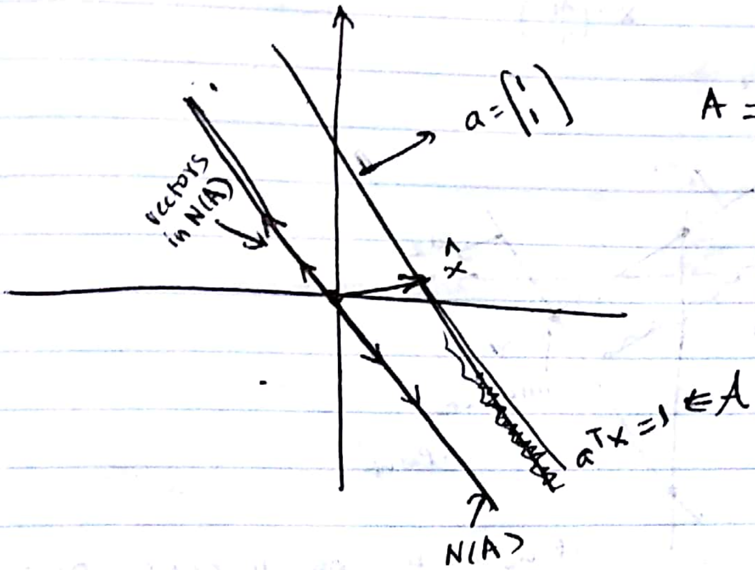
just one point.

$\dim(N(A)) = 2 - 1 = 1$

$\dim(R(A^T))$

$A = \begin{pmatrix} 1 & 1 \end{pmatrix}_{1\times 2}$ rank $(A) = 1$

$A^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



$\mathcal{A} = \{x \mid Ax = 1\}$

$N(A) = \{x \mid Ax = 0\} = \{x \mid a^Tx = 0\}$

line passing through origin

(*) When you add all null space vectors, it will span all the particular solutions.

Can you interprete $Ax = b$ as a intersection of hyperplanes?

$\{x \mid Ax = b\}$

$\cancel{A}^T x = b \qquad a^Tx = b \qquad x \in \mathbb{R}^n$

$a \in \mathbb{R}^{n\times m}$

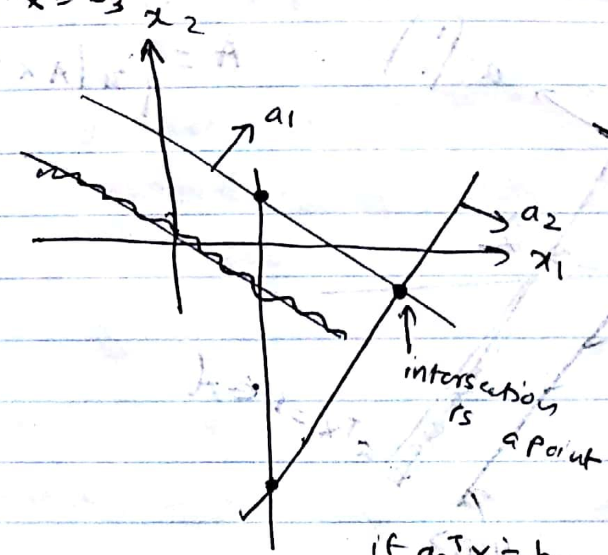
$A = \begin{pmatrix} a_1^T \\ a_2^T \\ \\ a_n^T \end{pmatrix} \cancel{a_2^T} \quad \cancel{a_n^T} \in \mathbb{R}^{n\times m}.$

$\mathcal{A} = \left\{ x \;\middle|\; \begin{pmatrix} a_1^T \\ a_2^T \\ a_3^T \\ a_n^T \end{pmatrix}_{n\times m} \underset{m\times 1}{x} = \begin{pmatrix} b_1 \\ b_2 \\ \\ b_n \end{pmatrix}_{n\times 1} \right\} \Rightarrow \mathcal{A} = \{x \mid a_i^T x = b_i \; ; \; i = 1, 2, 3, \ldots n\}$

$a_1^T x = b_1$ $\qquad x \in \mathbb{R}^2$

$a_2^T x = b_2$ $\qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$

$a_3^T x = b_3$



if $a_3^T x \doteq b_3$ no unique solution.

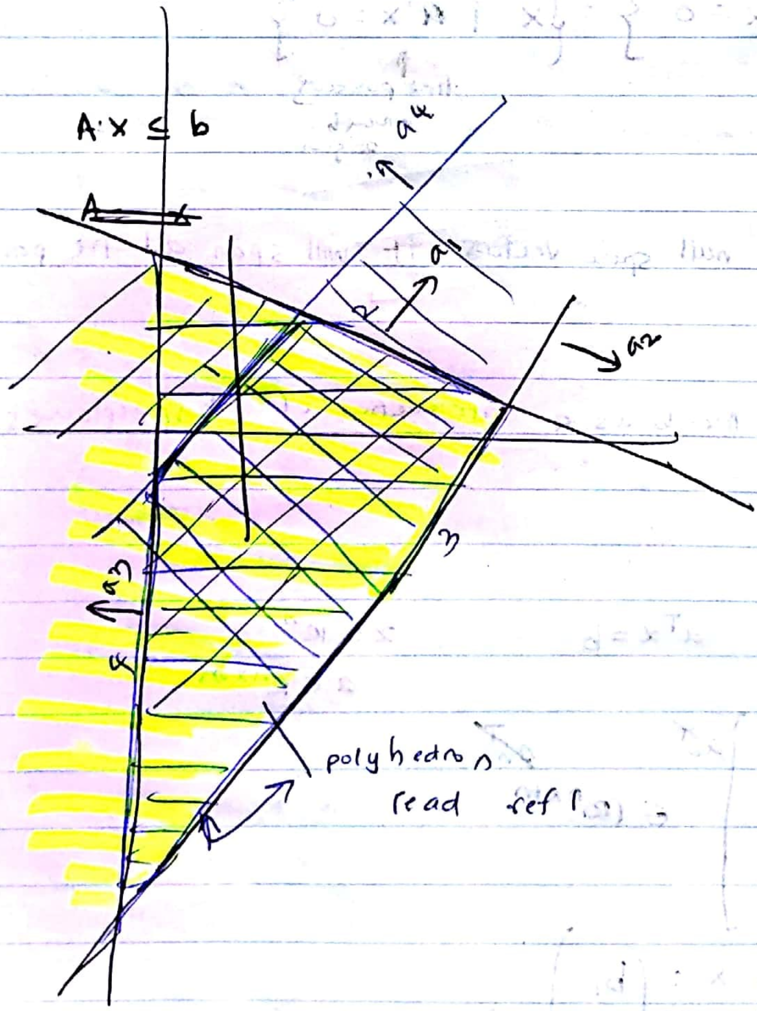
1.5) $A \cdot x \leq b$

$a_1^T x \leq b_1$

$a_2^T x \leq b_2$

$a_3^T \leq b_3$

$a_4^T \leq b_4$.



convex sets

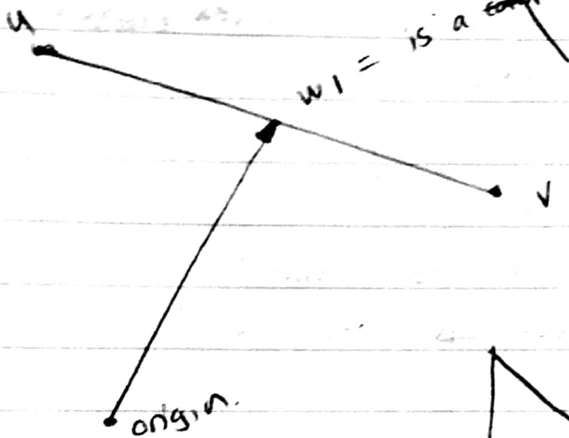
Recall a line segment between 2 points u and v is

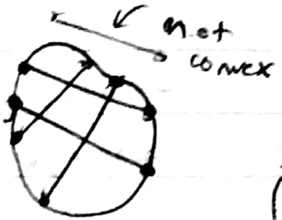
$$\{ w \in \mathbb{R}^n \mid w = \alpha u + (1-\alpha) v, \ \alpha \in [0,1] \}$$

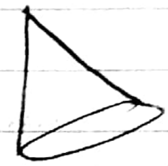
parameterize

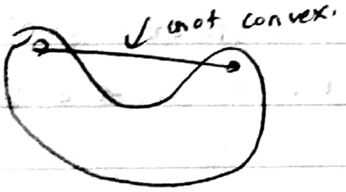
w_1 = is a convex ~~complex~~ combination between points u and v

convex $\rightarrow$ α and $(1-\alpha)$









$$B = \{ x \mid \|x\|_2^2 \le 1 \}$$

disc

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

if $B = \{ x \mid \|x\|_2^2 = 1 \}$

circle $\rightarrow$ not convex.

if B is a sphere

not convex

if it is a ball it is convex

A set $\mathcal{A} \subset \mathbb{R}^n$ is convex if

$\forall$ u, v $\in \mathcal{A}$ the line segment between u & v is in $\mathcal{A}$

A
- subspace
- hyperplane
- half space
- linear variety

$\mathbb{R}^n$

} convex planes

Read theorem 4.1

- Intersection of convex sets $\Rightarrow$ is convex
- Union of convex sets $\Rightarrow$ not convex

→ not sure

A point x in a convex set $\mathcal{A}$ is said to be an EXTREME point if there are no two distinct points u & v $\in \mathcal{A}$ such that $x = \alpha u + (1-\alpha) v$ for some $\alpha \in \mathbb{R}$

Neighbourhood

A neighbourhood of a point $x \in \mathbb{R}^n$ is the set

$$\{ y \in \mathbb{R}^n \mid \|y - x\|_2 < \epsilon \} \quad \epsilon > 0$$

any point

We call this set a ball with radius ϵ and center x

$\mathcal{A}$

There exists a neighbourhood of X which is strictly inside X

Interior point

for all points inside $\mathcal{A}$ there is a neighbourhood which is still in the space

interior point

A point $X \in S'$ is said to be an interior point of the set S if the set S contain some neighbourhood of X

The border points in the boundary are not interior points

← boundary not interior.

← ~~to~~ boundary not in set.

all points in the space ~~[illegible]~~ ~~[illegible]~~ can create a neighbourhood which is inside boundary.

Boundary point

A point X is said to be a boundary point of S if every neighbourhood of X contains(1) a (may be) point in S and(2) a (may be) point ~~[illegible]~~ not in S

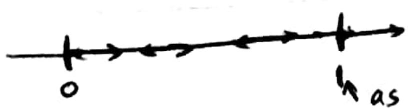
$B\{x \mid \|x-z\|_2 < 5\}$ ← boundary is not in set

$B = \{x \mid \|x\|_2^2 = 5\}$ ← boundary is the set

$B = \{x \mid \|x\|_2^2 \leq 5\}$ boundary has the set

$S = (0,1)$ ← ~~[illegible]~~ boundary not in set S

boundary = 0 and 1



$S = [0,1]$ ← boundary in set S

boundary = 0 and 1.

1 has ↔ ~~[illegible]~~ is not inside S

1 is a boundary point

in $\mathbb{R}^1$ the ball is 1 dimensional

$\mathbb{R}^2$ " 2 "

$\mathbb{R}^3$ " " "

← ~~[illegible]~~ sphere

* Boundary: set of all boundary points.
* open sets: a set S is said to be OPEN if it contains a neighbourhood of each of its points
* closed set: a set is closed if it contains its boundary in the set

S closed $\Leftrightarrow$ $\bar{S}$ is open.

$S = \{1, 2\}$

$\bar{S} = (-\infty, 1) \cup (2, \infty)$

$S = [1, 2] \rightarrow$ closed.

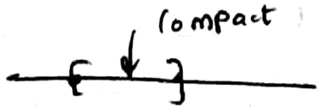
* bounded set

Bounded sets: A set ~~is~~ that is contained in a ball of finite radius is said to be bounded.

* compact set

A set in $\mathbb{R}^n$ is compact if it is both closed and bounded.

Compact



Sequences

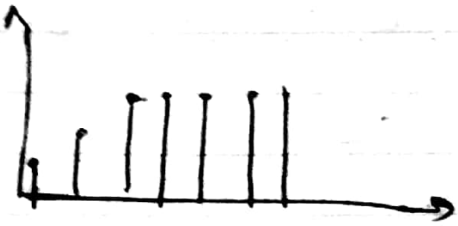
can be real vectors.

• Sequence of real numbers can be viewed as a set of numbers $\{x_1, x_2, x_3, x_4, \dots, x_k, \dots\}$ often denoted as $\{x_k\}$, $\{x_k\}_{k=1}^{\infty}$, $\{x_k\}_{k \in \mathbb{Z}^+}$

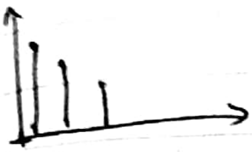
$x_k \in \mathbb{R}^n$

• $x_k = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} \text{distane} \\ \text{velocity} \end{pmatrix}$

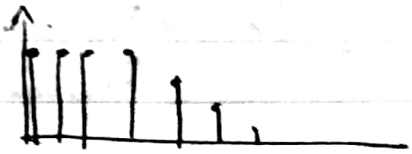
~~increasing~~
increasing



non- decreasing



decreasing

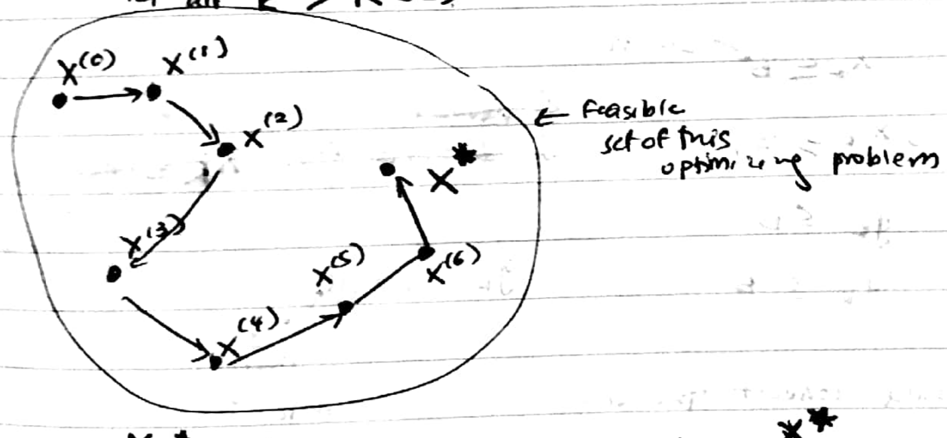


non increasing

<u>limit of a sequence $\{x_k\}$ is x^*</u>

$$\forall \epsilon > 0, \exists K(\epsilon) \in \mathbb{Z}^{*+} \text{ s.t } k > K(\epsilon) \Rightarrow \|x_k - x^*\| < \epsilon$$

meaning: The sequence lies in a ball with radius ϵ with a center x^* for all $k > K(\epsilon)$



$x^* = \lim\limits_{k \to \infty} x_k$ or simply $x \to x^*$

- Every sequence has a limit x

counter example $\Rightarrow x_k = (-1)^k \longrightarrow$ doesn't converge.

A sequence that has a limit is called a convergent sequence

$$\|x_k - x^*\| < \epsilon$$

any norm can be put

normally $\|x_k - x^*\|_2$ is used.

distance.

Theorem 5.1 : convergence sequence has one limit.

read the proof

Bounded sequence : A sequence is bounded, $\exists$ $B \geq 0$, $\forall k \in \{1,2,3,\dots\}$,

$$\|x^*\| < B$$

any norm

compact sets are → bounded.

Theorem 5.2 : Every convergent sequence is bounded.

check the proof

Bounded above → $x_k \leq B$ ← scalar

Bounded below → $x_k \geq B$

~~normal~~ ~~defined for~~ ~~$x_k \in \mathbb{R}$~~ ~~real number~~

$y_k \leq B$

$y_k \leq B$

$y_k = \|x_k\|$ any norm.

- we should convert the vectors to a constant.

let $\{y_k\}$ is a sequence where $y_k \in \mathbb{R}$

$\{y_k\}$ bounded above → $\forall k$ $y_k \leq B$

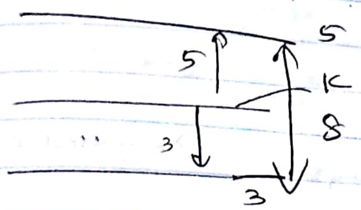
$\{y_k\}$ bounded below → $\forall k$ $y_k \geq B$

If y_k is bounded above by 5

y_k is bounded below by 3

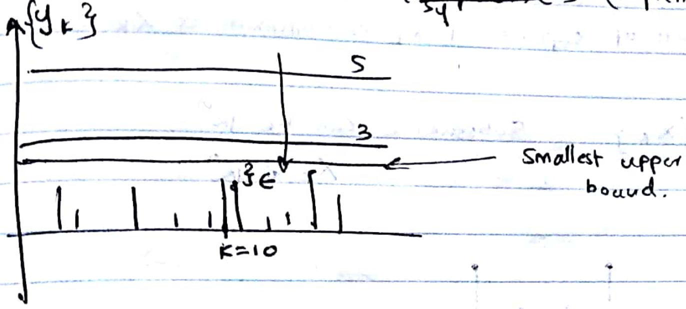
This means y_k is bounded

for all k $\|y_k\| < B$ $\exists$ B, (for all)



~~$y_k < \epsilon$~~

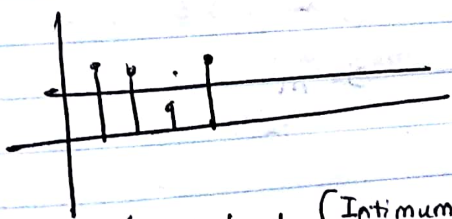
least upper bounded ~~(supreme)~~ (supreme)



• ~~the~~ B is the least upper bound of $\{y_k\}$ if

$(y_k \le B$ for all $k)$ and $(\forall \epsilon > 0 \ \exists K \in \{1,2,3 \dots\}$ s.t $y_k > (B-\epsilon))$

~~$y_k > (B-\epsilon)$~~



greatest lower band [Intimum]

$(y_k \ge R$ for all $k)$ and $\forall \epsilon > 0 \ \exists K \in \{1,2,3 \dots\}$ s.t $y_k < B + \epsilon$

~~Ex~~ supreme of $\{(-1)^k\} = +1$

intimum of $\{(-1)^k\} = -1$

sup of $\{\sin(2^k)\} = 1$

intimum of $\{\sin(2^k)\} = -1$

limit of Supreme $\{X_k\}$

limit of infimum $\{X_k\}$

X_k is a real number.

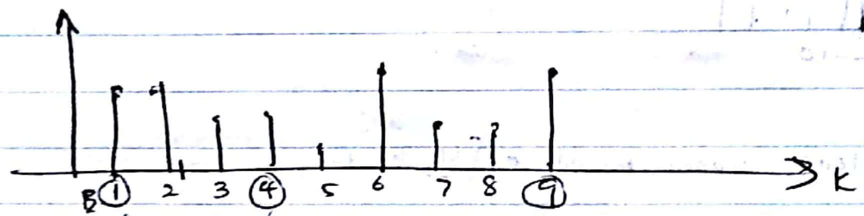
lim inf $\{X_k\}$ } always

lim sup $\{X_k\}$ } ~~always~~ exists

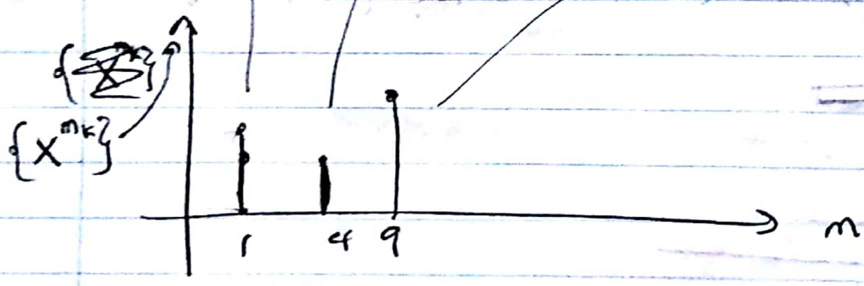
limit of supreme of X_k and infimum of X_k exists

<u>Subsequence of $\{X_k\}$</u>   Subsequence can be $\mathbb{R}^n$

$X_k \in \mathbb{R}^n$



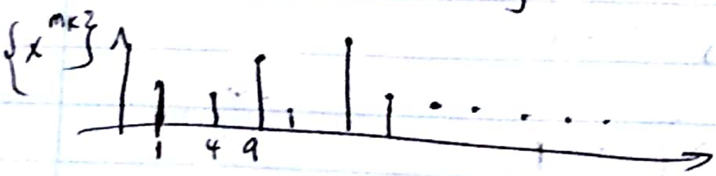
You select a subset of k.



<u>Theorem 5.4</u>

consider a sequence with limit X^*. then any subsequence of $\{X^k\}$ also converge to X^*

You have to keep picking.



$$\lim_{k \to \infty} x_k = x^*$$

limit of subsequence $\rightarrow$ $\lim_{k \to \infty} x_{m_k} = x^*$

of $x_{m_k} \rightarrow x^*$ also

K is an integer

$x_k = e^{-k\,m}$

$\lim_{k\to\infty} x_k = \lim_{k\to\infty} e^{-k} = 0$

~~$\lim_{k\to\infty} x_{mk} = \lim_{k\to\infty} e^{-mk} = 0$~~

m_k = even integers.

$m_1 = 2$

$m_2 = 4$

$m_3 = 6$

$\vdots$

~~r_k = odd~~

$$\lim_{k\to\infty} e^{-m_k} = \lim_{k\to\infty} e^{-2n^k} = 0$$

any integer

r_k = odd.

$r_1 = 1$

$r_2 = 3$

$r_3 = 5$

<u>Theorem</u> if sequence doesn't converge the subsequences will have different ~~lim~~ ~~to~~ limits/bounds.

Theorem 3.6 (Rudit)

R.G. Bartle P. 70.

(b) part

<u>Theorem</u>

Every bounded sequence contains a convergent subsequence

$x_k = (-1)^k \rightarrow$ bounded sequence

~~$x_0 \in Z, x$~~ ~~Assume $Mx = c$~~

~~assume $Ax_0 = b$~~ ~~$N(M) = \{x \mid Mx = 0\}$~~

~~there exists a space S_0,~~ ~~$S_0 = N(A) = \{x \mid Ax = 0\}$~~

X = S is a linear variety $\quad Y = \alpha x + (1-\alpha) y \in S \quad x, y \in S, \ \alpha \in \mathbb{R}$

show $X \Longleftrightarrow Y$

1) $\Rightarrow$ Linear variety is a set of the form

$$S = \{ x \mid Ax = b \} \quad A \in \mathbb{R}^{m \times n} \quad b \in \mathbb{R}^n$$

if $x, y \in S$

$Ax = b$ — ①

$Ay = b$ — ②

multiply ① $\times \alpha$ also $\quad \alpha A x = \alpha b$ — ③

multiply ② $\times (1-\alpha)$ $\quad (1-\alpha) A y = (1-\alpha) b$ — ④

③ + ④

$$\alpha A x + (1-\alpha) A y = \alpha b + (1-\alpha) b$$

$$A [\underbrace{\alpha x + (1-\alpha) y}_{z}] = b$$

$\therefore \quad Az = b$

$\therefore \ z \in S$

$\Leftarrow$ ~~Assume~~ there exists a A such that

$$N(A) = \{ x_0 \mid A x_0 = 0 \} = S_0 \quad — ①$$

$x_0 \in S_0$

assume $\quad Ax = b \quad$ for $S - S_0$

$$\therefore S - S_0 = \{ x_1 \mid A x_1 = b \} \quad — ②$$

$x \in S - S_0$

② + ① $\quad S - S_0 + S_0 = \{ x_1 + x_0 \mid A x_1 + A x_0 = b \}$

$$S - S_0 + S_0 = \{ x_1 + x_0 \mid A(x_1 + x_0) = b \}$$

$$S = \{ z \mid Az = b \} \quad . \ z \in S \qquad S \text{ is a linear variety}$$

4.2. Show that the set $\{x \in \mathbb{R}^n : \|x\| \le r\}$ is convex, where $r > 0$ is a given real number and $\|x\| = \sqrt{x^T x}$ is the euclidean norm of $x \in \mathbb{R}^n$.

$A = \{x \in \mathbb{R}^n : \|x\| \le r\}$

Let $m, n \in A$

to show that A is a convex set, we need to show that m, n follow

$\alpha m + (1-\alpha) n \in A$ rule. $\alpha \in [0,1]$

$\therefore \|\alpha m + (1-\alpha) n\| \le r$

$\|\alpha m + (1-\alpha) n\|^2 \le r^2$

take $p = \|\alpha m + (1-\alpha) n\|^2$

$$= [(\alpha m) + (1-\alpha) n]^T (\alpha m + (1-\alpha) n)$$
$$= [(\alpha m)^T + (1-\alpha) n^T] (\alpha m + (1-\alpha) n)$$
$$= (\alpha m^T + (1-\alpha) n^T)(\alpha m + (1-\alpha) n)$$
$$= \alpha^2 m^T m + (1-\alpha)^2 n^T n + \alpha(1-\alpha) m^T n + (1-\alpha)\alpha n^T m$$

as $m^T n = n^T m$

$$= \alpha^2 \|m\|^2 + (1-\alpha)^2 \|n\|^2 + 2\alpha(1-\alpha) m^T n$$

as α : $0 \le \alpha \le 1$ and $\|m\|^2 < r^2$

$\alpha^2 \|m\|^2 \le \alpha^2 r^2$ (1)

as α : $0 \le (1-\alpha) \le 1$ $\|n\|^2 < r^2$

$(1-\alpha)^2 \|n\|^2 \le (1-\alpha)^2 r^2$ (2)

from Cauchy ~~Schwartz~~ schwartz inequality

$m^T n \le \|m\| \|n\|$

$m^T n = \langle m, n \rangle = \|m\| \|n\| \cos\theta$

when $\theta = 90°$

$\langle m, n \rangle_{max} = \|m\| \|n\|$ occurs

$\therefore \langle m, n \rangle \le \|m\| \|n\|$

$\|m\|$

2

(1) + (2)

4.3 Show

(linea

Let

to s

m, n

Take

$\|m\| \leq r$ and $\|n\| \leq r$

$\|m\|\|n\| \leq r^2$

$2(1-\alpha)\alpha \|m\|\|n\| \leq 2(1-\alpha)\alpha \cancel{\|m\|\|n\|}\, r^2$ — ③

① + ② + ③

$\therefore \alpha^2 \|m\|^2 + (1-\alpha)^2 \|n\|^2 + 2(1-\alpha)\alpha \|m\|\|n\| \leq$
$$\underbrace{\alpha^2 r^2 + (1-\alpha)^2 r^2 + 2(1-\alpha)\alpha r^2}_{r^2}$$
$1+\cancel{\alpha^2}-\cancel{2\alpha}$ $\quad$ $\cancel{2\alpha}-\cancel{2\alpha^2}$

$\therefore p \leq r^2$

$\therefore A$ is a convex set

4.3 Show that for any matrix $A \in \mathbb{R}^{m \times n}$ and vector $b \in \mathbb{R}^m$ the set (linear variety) $\{x \in \mathbb{R}^n ; Ax = b\}$ is convex.

$$\beta = \{x \in \mathbb{R}^n ; Ax = b\}$$

Let $m, n \in \beta$

to show that β is a convex set, we need to show that m, n follow

$\alpha m + (1-\alpha) n \in \beta$ rule. $\quad \alpha = [0,1]$

$$A[\alpha m + (1-\alpha)n] = b.$$

Take $P = A(\alpha m + (1-\alpha)n)$

$P = \alpha Am + (1-\alpha) An \qquad \alpha \in \mathbb{R}$

but

$Am = b$, $An = b$

$P = \alpha b + (1-\alpha)b = b$

$P = b$

$\therefore \beta$ is a convex set

4.4) Show that the set $\{x \in \mathbb{R}^n : x \geq 0\}$ is convex (where $x \geq 0$ means that every component of x is nonnegative.

~~let m,n ∈~~

$$\gamma = \{x \in \mathbb{R}^n : x \geq 0\}$$

let $m, n \in \gamma$ $\Rightarrow$ each component of m, n are nonnegative

to show that γ is a convex set, we need to show that m, n follow

$\alpha m + (1-\alpha)n \in \gamma$ rule where $\alpha \in [0,1]$

$$\alpha m + (1-\alpha)n \geq 0$$

$m = [m_1, m_2, \ldots m_n]^T$ $n = [n_1, n_2, n_3 \ldots n_n]^T$

take $P = \alpha m + (1-\alpha)n$

as $m \geq 0$

as $0 < \alpha < 1$ $\alpha m \geq 0$ — (1)

$\cancel{(1-\alpha)n}$ $n \geq 0$

as $0 < (1-\alpha) < 1$ $(1-\alpha)n \geq 0$ (2)

(1) + (2) $\alpha m + (1-\alpha)n \geq 0$

$P \geq 0$

$\therefore$ γ is a convex set

take $P = \alpha m + (1-\alpha)n$ $P \in \mathbb{R}^n$

consider ~~taking~~ i^{th} component ~~only of~~ of m and n.

$m_i \geq 0$

$\alpha m_i \geq 0$ $(\because 0 < \alpha < 1)$

$n_i \geq 0$

$(1-\alpha)n_i \geq 0$ $(\because 0 < 1-\alpha < 1)$

$\alpha m_i + (1-\alpha)n_i \geq 0$

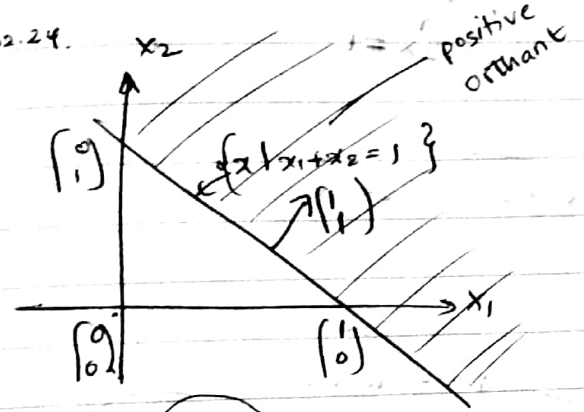
$p_i \geq 0$ (in general any compount of p is greated than 0)

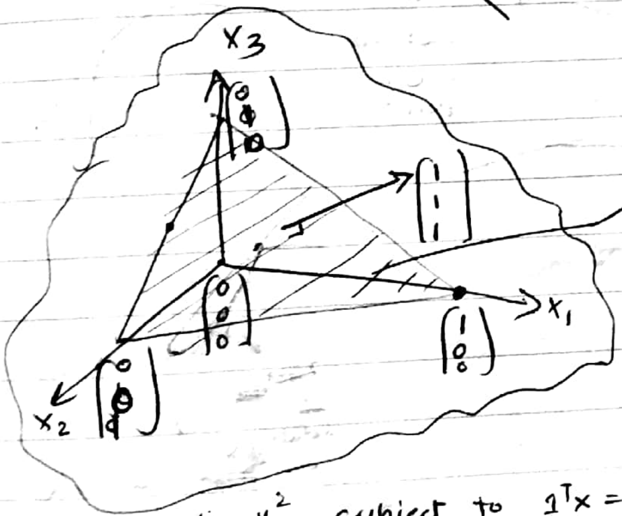
$\therefore$ $p \geq 0$ (~~##~~ p is non negative all compounts)

$\therefore$

- γ is a convex set

2018.02.24.





$S = \{x \mid x_1 + x_2 + x_3 = 1\}$

$S = \{x \mid 1^T x = 1\}$

above S surface $\rightarrow$ positive orthant

minimize $\|x\|^2$ subject to $1^T x = 1$

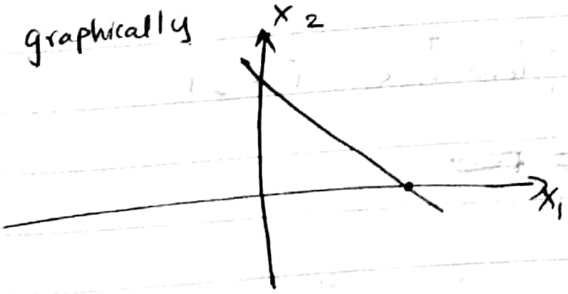
hyperplane $\Rightarrow$ level curve which correspond to $f(x) = c^T x$

- minimize $\|x\|_2^2$ = minimize length square

$\|$ ~~$x_1^2 + x_2^2$~~ $\|$ = $(x_1^2 + x_2^2)$

$x \in \mathbb{R}^2$

- solve graphically



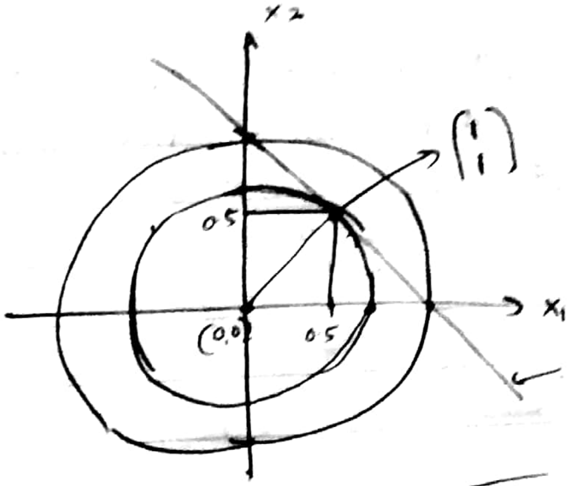
$\|x\|^2 = x_1^2 + x_2^2$

$S = \{x \mid c^T x = 1\}$

$f(x) = c^T x$

$S = \{x \mid f(x) = 1\}$

$\uparrow$ level curve

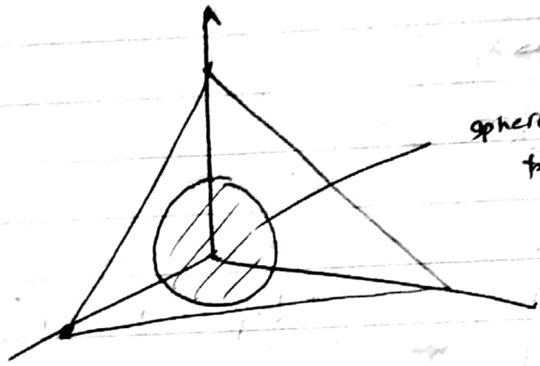


$\|x\|_2^2 = x_1^2 + x_2^2 = f(x)$

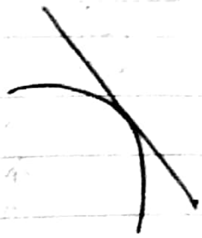
minimize $f(x)$
assume $f(x) = 0$

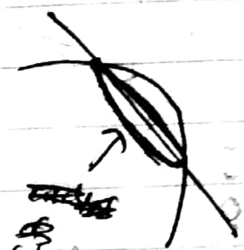
$x_1 = 0 \quad x_2 = 0$

$r = \sqrt{0.5^2 + 0.5^2} = \sqrt{0.25}$



sphere which touch only one point



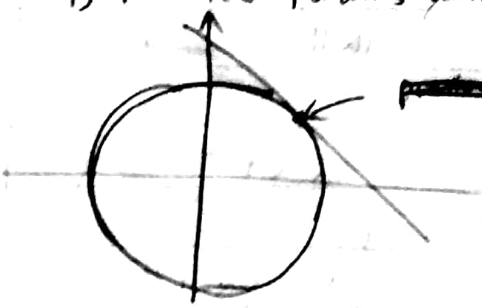


solutions are in a circle

minimize $\|x\|^2 = x_1^2 + x_2^2 = f(x)$ subject to $1^T x \le 1$ $x_1 = 0, x = 0$

$1^T x \ge 1$

1) minimize radius and subject to level curve $1^T x \ge 1$



same as 1st case as the constraint of minimizing radius is there.

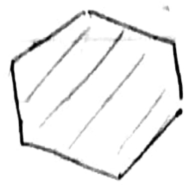
$\|x\|^2 = \alpha_1 x_1^2 + \alpha_2 x_2^2$ = ellipse

level curve is the intersection points of hyper plane and the $f(x)$

~~Minimize~~

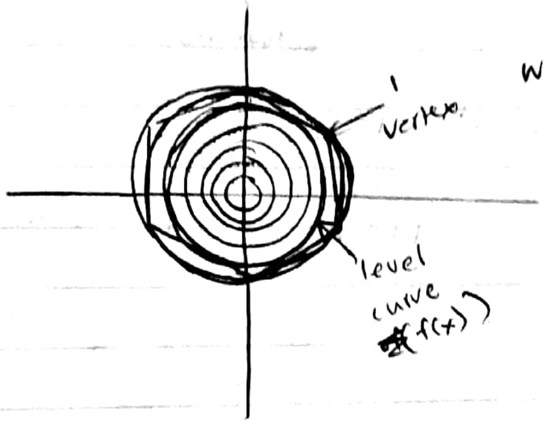
maximize $\|x\|_2^2$

subject to



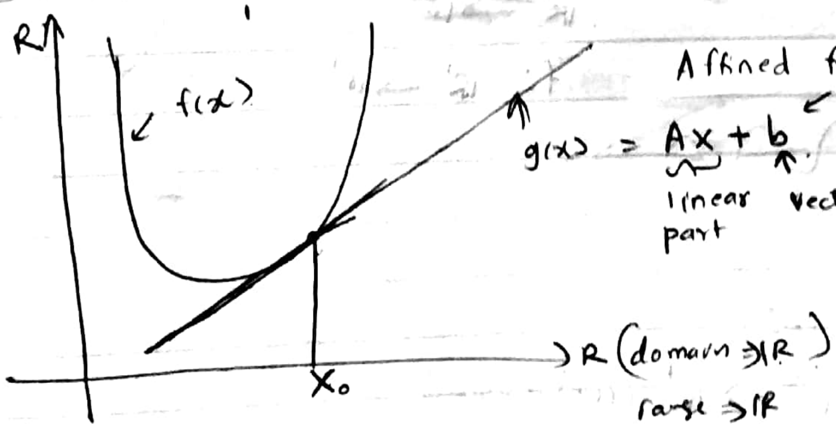
→ corresponding to $f(x)$

level curves → circles → maximum → until the critical point in feasible area



When a convex level curve is subjected to polyhedron the optimum solution is one vertex

calculus

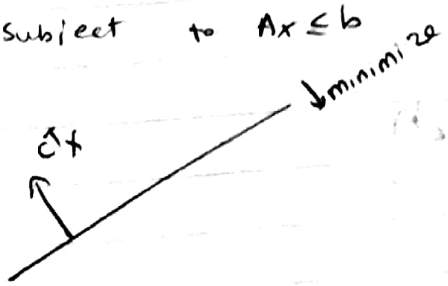


Affined function

$g(x) = \underbrace{Ax}_{\text{linear part}} + \underbrace{b}_{\text{vector}}$

minimize $c^T x$

subject to $Ax \le b$



given a function $f : \mathbb{R}^n \to \mathbb{R}^m$

and a point $x_0 \in \mathbb{R}^n$

find an affine approximation function

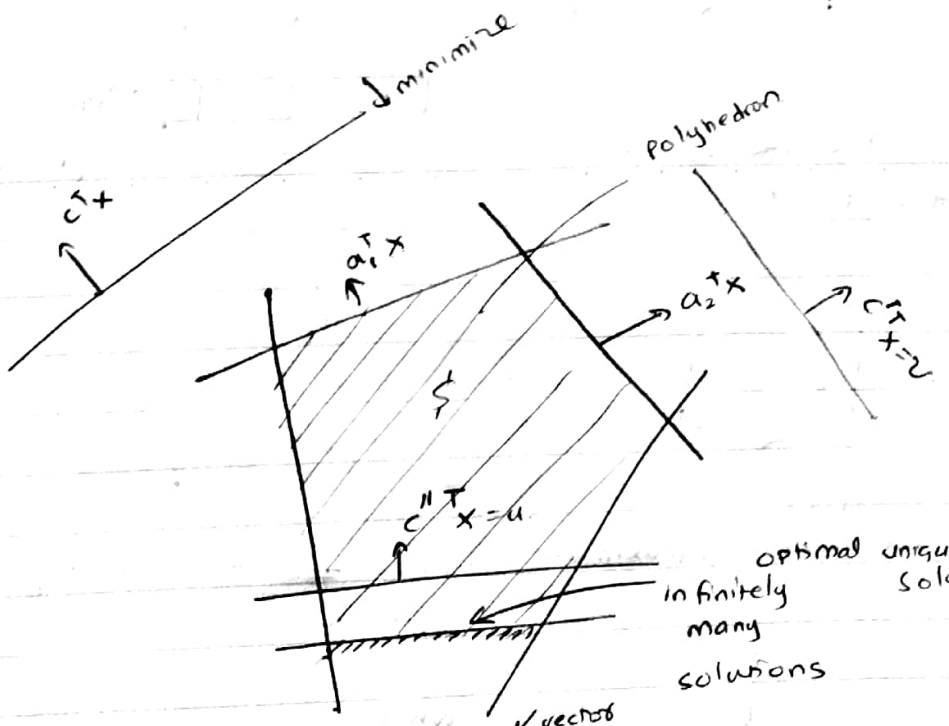
$A : \mathbb{R}^n \to \mathbb{R}^m$ That

approximate f near x_0

$A : \mathbb{R}^1 \to \mathbb{R}^1$

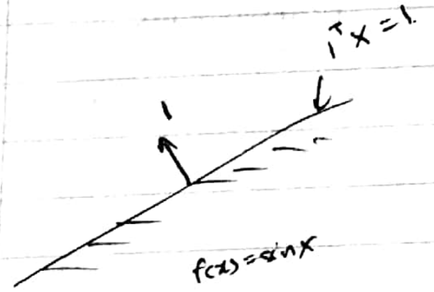
read lec note on affine function

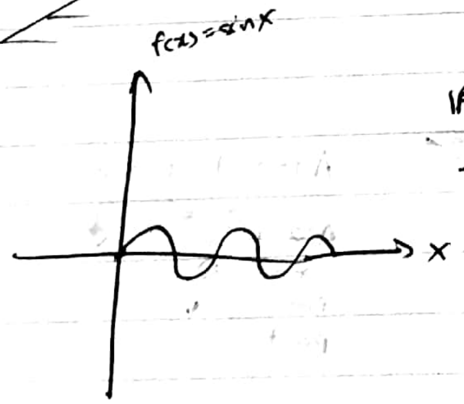
$A(x) = L(x - x_0) + f(x_0)$ (1)



vector
minimize $\|x\|_2$ subject to $1^T x = 1$ scalar

$1^T = [1\ 1\ 1\ \dots\ 1]$ $\quad 1^T x = \sum_{i=1}^{n} x_i$





$\mathbb{R}^1 \to \mathbb{R}$

$f: \mathbb{R}^1 \to \mathbb{R}^1$

$f(x) = x_1^2 + x_2^2$

$f: \mathbb{R}^2 \to \mathbb{R}^1$ ($f(x)$ is a scalar)

$f(x) = \|x\|^2$

$x \in \mathbb{R}^n$

$f: \mathbb{R}^n \to \mathbb{R}^1$

$f(x) = \begin{bmatrix} \|x\|^2 \\ \|x\|^2 + 2 \end{bmatrix}$ $\quad x \in \mathbb{R}^n$

$f: \mathbb{R}^n \to \mathbb{R}^2$

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~~t~~ $A(x) = L(x - x_0) + f(x_0)$

derivative matrix denoted $Df(x)$ is the only candidate can be used to chacterize L given in ①

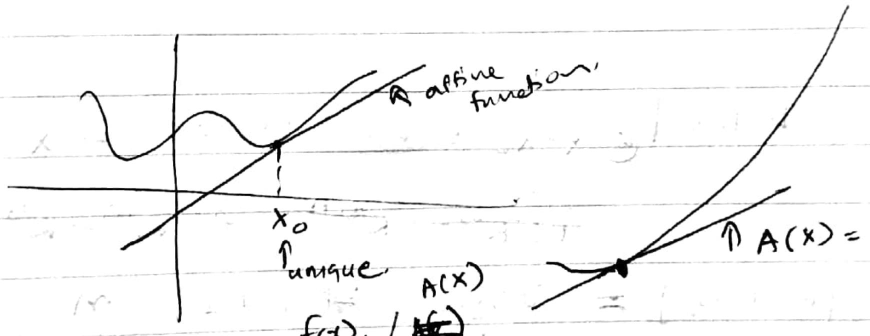
derivative is used to approximate of an affine function at a given point to a function $f(x)$

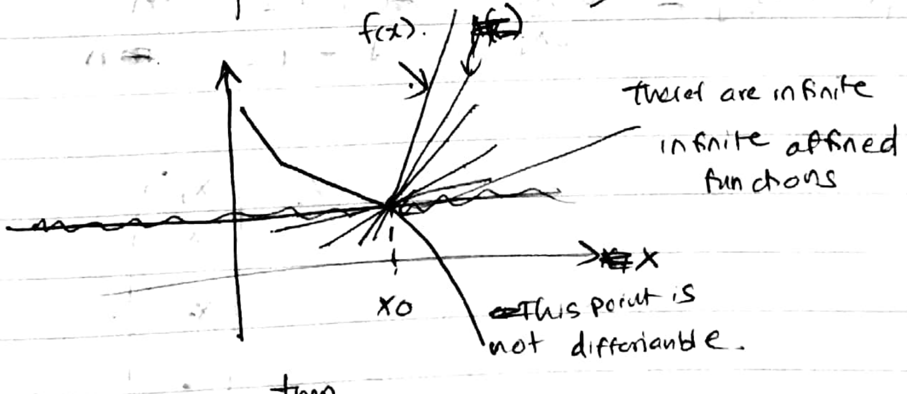
Jacobian matrix

Derivative matrix (have an idea about differentiability)

Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ and x_0 is an interior point of of domain of f, the derivative (or Jacobian) of f of x_0 is the matrix $Df(x)|_{x = x_0}$

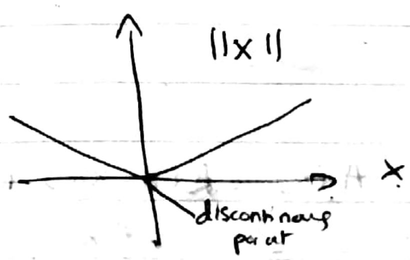
(if you have sharp edges in f(x) f(x) is not differentiable for all x)





if differentiable $\Rightarrow$ then continous

if continous $\not\Rightarrow$ then differentiable



$f(x) = x^2+1$ is not differentiable in $S=[0,1]$
because at the boundary it is not different

$f(x) = x^2+1$ is not differentiable in $S=(0,1]$

has a boundary in the set S
right and left limits are not equal at the boundary
not continuous at boundary

$f(x) = x^2+1$ is differential in $S=(0,1)$

no boundary

left and ~~[illegible]~~ right limits can be made equal at the 0, 1 points
~~[illegible]~~

$D f(x)\big|_{@ x=x_0} = D f(x)$ evaluated at $x = x_0$

~~$f: R^m \to R^n$~~ $f: R^m \to R^n$

$$[D f(x)] = \frac{\partial f_i(x)}{\partial x_j} \quad \begin{array}{l} i = 1,2,\dots,m \\ j = 1,2,\dots,n \end{array}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_j \\ x_n \end{pmatrix} \qquad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_i(x), \\ \cancel{f_m(x)} \\ f_m(x) \end{pmatrix}$$

$A(x) = L(x-x_0) + f(x_0)$

instead $L \to D f(x)$

$A(x) = \underset{\mathbb{R}^m}{Df(x)}\,\underset{\mathbb{R}^n}{(x-x_0)} + \underset{\mathbb{R}^m}{f(x_0)}$

$Df\big|_{@ x=x_0} = D f(x_0)$

$$\boxed{Df}$$
$\uparrow$
$\mathbb{R}^{m\times n}$

$$A(x) = \cancel{Df(x_0)}\ \overset{\text{matrix}}{Df(x_0)}\ \overset{\text{vector}}{(x-x_0)} + f(x_0)$$

$J : \mathbb{R}^n \to \mathbb{R}^m \longrightarrow$ a matrix ($\mathbb{R}^{m\times n}$) $\Rightarrow$ transform $x \in \mathbb{R}^n$ to $f \in \mathbb{R}^m$

$\downarrow$

$Df(x_0)$

$$f(x) = 1^T x \qquad x \in \mathbb{R}^2 \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad f = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Df(x) = \frac{\partial f_i(x)}{\partial x_j}$$

$$= \left[\frac{\partial(x_1+x_2)}{\partial x_1} \quad \frac{\partial(x_1+x_2)}{\partial x_2} \right] = \begin{bmatrix} 1 & 1 \end{bmatrix}_{1\times 2}$$

$$\cancel{\frac{\partial(x_1+x_2)}{\partial x_2}}$$

first $[Df(x)]_{1\times 2} \in \mathbb{R}^{m\times n}$
$\mathbb{R}^{1\times 2}$
$m=1$
$n=2$

sic $\Rightarrow$ $$Df(x) = \left[\frac{\partial(1^T x)}{\partial x_1} \quad \frac{\partial(1^T x)}{\partial x_2} \right] = \begin{bmatrix} 1 & 1 \end{bmatrix} = 1^T$$

$$f(x) = \begin{pmatrix} 1^T x \\ x_1^2 + x_2^2 \end{pmatrix} \qquad f(x) \in \mathbb{R}^2, \quad x \in \mathbb{R}^2$$

$(Df(x))_{2\times 2}$

$$Df(x) = \begin{pmatrix} \frac{\partial 1^T x}{\partial x_1} & \frac{\partial (1^T x)}{\partial x_2} \\ \frac{\partial (x_1^2+x_2^2)}{\partial x_1} & \frac{\partial (x_1^2+x_2^2)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2x_1 & 2x_2 \end{pmatrix} =$$

$$f(x) = c^T x$$
$$Df(x) = c^T$$

gradient

When f is real valued. the derivative $Df(x)$ is a $1 \times n$ matrix

$$\left(\left. \begin{array}{l} f: \mathbb{R}^n \to \mathbb{R} \\ f: \mathbb{R}^5 \to \mathbb{R} \\ f: \mathbb{R}^{10} \to \mathbb{R} \end{array} \right\} \text{real valued function} \to \text{return real value.} \right)$$

$Df(x)$ is a row vector. Its transpose is called the gradient of the function f.

i. e, $D^T f(x) = \underbrace{\nabla f(x)}$ notation for gradient of f at X

$f(x) = c^T x$

$Df(x) = D^T f(x) = (c^T)^T = c$

(gradient is there for real valued functions only)

Affined of a real valued function.

$A(x) = Df(x_0).(x - x_0). + f(x_0)$

$A(x) = \underbrace{\nabla^T f(x_0)(x - x_0) + f(x_0)}$ ← FIRST order approximation of function at $x = x_0$.

for real valued

$\nabla f(x_0) = D^T f(x_0)$

$\nabla^T f(x_0) = Df(x_0)$

$f(x) = 2x$

gradient = 2

$f(x) = c^T x$

~~gradient~~ gradient = $c = \nabla f(x) = D^T f(x)$

$(Df(x) = c^T)$

$f(x) = \underbrace{x^T}_{1 \times n} \underbrace{P}_{\mathbb{R}^{n \times n}} \underbrace{x}_{n \times 1} + q^T x + v.$ compute gradient

R^1

$f: \mathbb{R}^n \to R^1$

v, P, q are constant matrices

$f: \mathbb{R}^n \to \mathbb{R}^1$

$(\nabla f(x))$ $1\times n$

$\nabla f(x) = \begin{bmatrix} \\ \\ \end{bmatrix}_{n\times 1}$

$f(x) = x^T P x + q^T x + v$

$\nabla f(x) = \cancel{2xP}\ 2Px + q \Rightarrow$ just like $f(x) = 2x^2 + 5x$

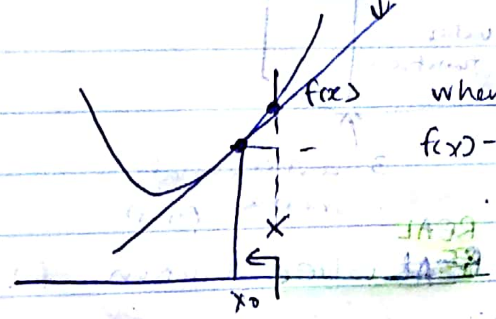
$f'(x) = 4x + 5$

first order approximation of at $x = x_0$

~~f(x) very~~ $x \approx x_0$

if x is very close to x_0

$A(x)$ is very small



when $x \approx x_0$

$f(x) - A(x) \approx 0$

when it is second order $f(x) - A(x)$ is very very small

$f: \mathbb{R}^n \to \mathbb{R}^m$

$g: \mathbb{R}^n \to \mathbb{R}^p$

$h: \mathbb{R}^n \to \mathbb{R}^p$

$f(x) = g(f(x))$

$f'(x) = g'\, f(x)\ f'(x)$

$h(x) = g(f(x))$

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $f(x) = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$ $g = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_p \end{bmatrix}$

$h = \begin{bmatrix} h_1 \\ \\ h_p \end{bmatrix}$

$f(x) = \cancel{g f(x)}\ g(f(x))$
$= g'(f(x)) f'(x)$

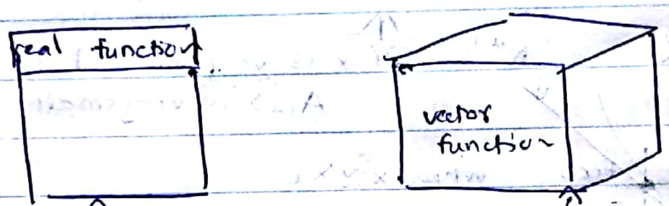
$h(x) = g(f(x))$

$$\nabla h(x) = \underbrace{D[g(f(x))]}_{p \times n} \quad [D f(x)]_{m \times n}$$

$\approx g'(f(x))$ $\quad \approx f'(x)$

gradient $\rightarrow$ vector value

Second derivative of real valued functions



2 dimensional matrix form

3 dimensional matrix form

The second derivative of a REAL valued function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ also called HESSIAN matrix of f.
Hessian matrix of f at x an interior point of domain f denoted by $\nabla^2 f(x)$ is given by

$$\nabla^2 f(x) = \frac{\partial^2 f(x)}{\partial x_i x_j} \qquad i = 1, 2, \ldots n \quad j = 1, 2, \ldots n.$$

$\partial x_i x_j$

differentiable $\rightarrow$ second derivative should exist

hessian $\rightarrow$ characterize $\rightarrow$ 2nd derivative $\rightarrow$ real valued $f(x)$

$f(x) = c^T x$

$\nabla f(x) = c \leftarrow \in \mathbb{R}^n$

$$\nabla^2 f(x) = \begin{bmatrix} 0 & 0 & \\ \vdots & & \\ 0 & \cdots & \end{bmatrix}_{n \times n} \quad \text{zero matrix}$$

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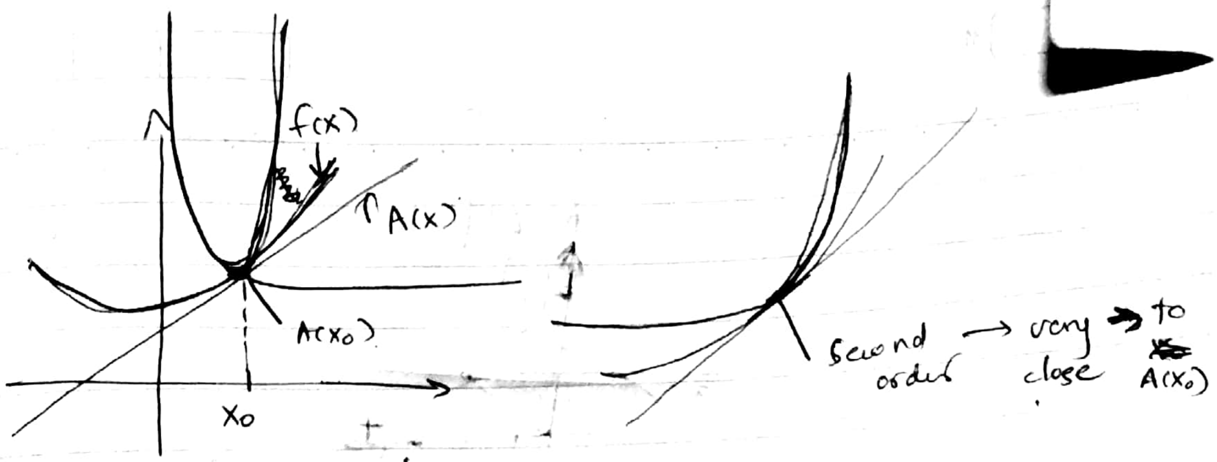


$$Q(x) = f(x_0) + \underbrace{\nabla f^T(x_0)(x-x_0)}_{\text{linear}} + \underbrace{\tfrac{1}{2}(x-x_0)^T \nabla^2 f(x_0)(x-x_0)}_{\text{quadratic form}}$$

↑ second order approximation.

$$f(x) = \tfrac{1}{2}x^T P x + q^T x + v$$

$$\nabla f(x) = \tfrac{1}{2}\, 2Px + q = Px + q \Rightarrow \mathbb{R}^n \to \mathbb{R}$$

$n\times n$ ... $n\times 1$ ↑ $n\times 1$

$$\nabla^2 f(x) = P$$

$$f(x) = 2x_1^2 + 3x_2^2$$

$$\nabla f(x) = \frac{\partial(2x_1^2 + 3x_2^2)}{\partial x_1} \quad \frac{\partial(\;"\;)}{\partial x_2}$$

$$\nabla f(x) = \begin{pmatrix} 4x_1 \\ 6x_2 \end{pmatrix}$$

Gradient at $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\nabla f(x_0) = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

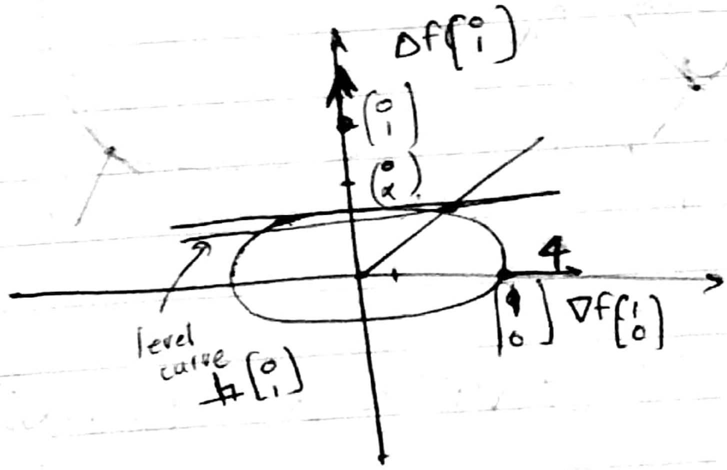
$f(x) = 0$

$2x_1^2 + 3x_2^2 = 0$ $\quad x_1 = 0,\ x_2 = 0.$

$f(x) = 1$

$2x_1^2 + 3x_2^2 = 1$

dimension of subspace = no of independent vectors needed to span the space



24
40
58
71
95

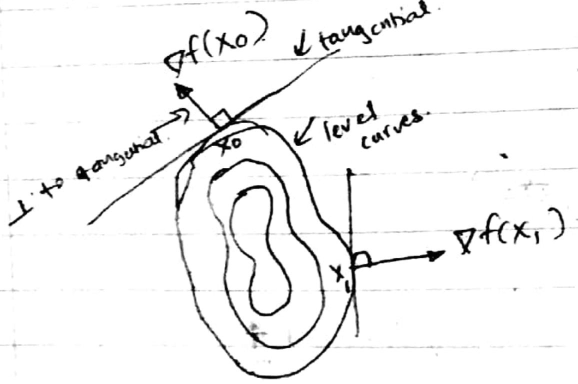
$2x_1^2 + 3x_2^2$

big, should be small

$f(x) = 2$ at $x = [1, 0]$

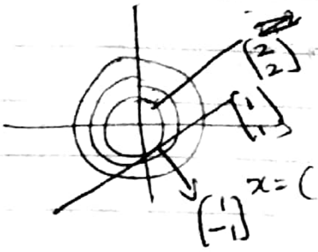
$\nabla f(x)\big|_{@\, x = \begin{bmatrix}1\\0\end{bmatrix}} = \begin{bmatrix}4\\0\end{bmatrix}$

$x = \begin{bmatrix}0.5\\0.25\end{bmatrix}$ $\quad \nabla f = \begin{matrix}4\times 0.5\\6\times 0.25\end{matrix} \quad \begin{bmatrix}2\\0.5\end{bmatrix}$



The direction of gradient ⇒ ~~the~~ the direction level curve increase height

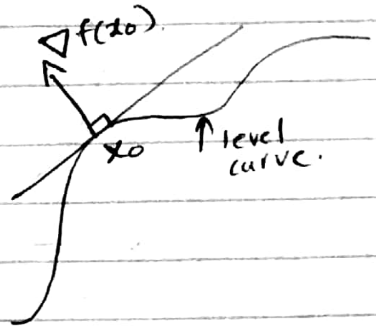
$f(x) = x_1^2 + x_2^2 \qquad \nabla f(x) = \begin{bmatrix}2x_1\\2x_2\end{bmatrix}$



$x = (1, 1) \qquad \nabla f(x) = \begin{pmatrix}2\\2\end{pmatrix}$

Theorem 5.7 (Ref 1 , P 10).

The vector $\nabla f(x_0)$ is orthogonal to the tangent vector ~~to~~ of an arbitrary smooth curve passing through x_0 on the level set deteimined by $f(x)=f(x_0)$

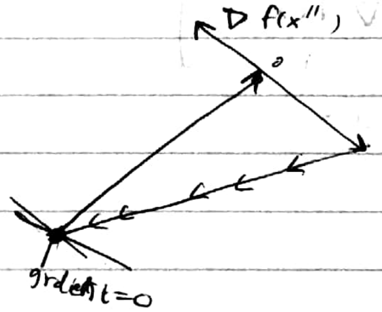


a hyper plane defined by c at x_0

$= c^T x_0$

hyperplane defined by $\nabla f(x_0)$ that pass through x_0

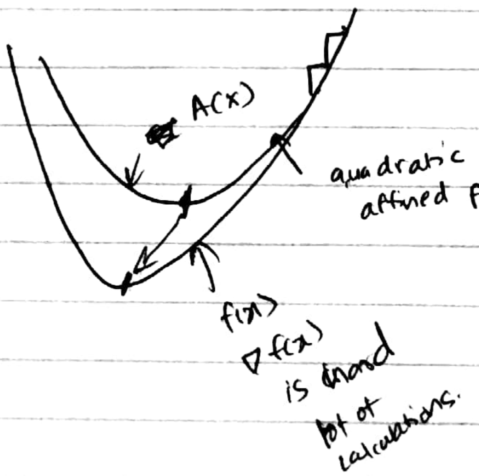
$$\nabla f^T(x_0).\, x = b.$$



$\cancel{x^{(1)} - x^{(0)} = \nabla f(x^{n+1})\ \nabla f(x^{n+1})}$

$$x^{(1)} = x^{(0)} - \alpha(\nabla f(x^{n+1}))$$

Unconstrained minimization.



quadratic affined funⁿ

$A(x) = x^T Q x + q^T x + r$

$\downarrow$

$\nabla f x = 2Qx + q = 0$

find x That $\nabla fx = 0$

$Qx = b$

HomeWork:

1. Taylor Theorem ~~Big o~~ last chapter.
2. big-oh $O(g(x))$
3. little-oh $o(h(x))$

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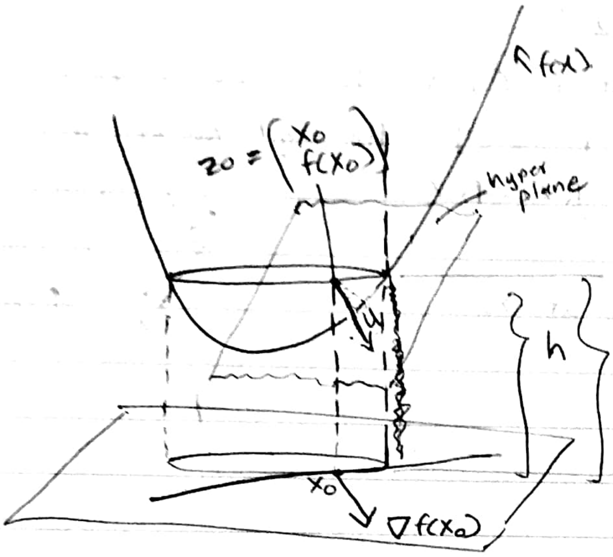
real valued

$f : \mathbb{R}^n \to \mathbb{R}$



affine approx of f(x) at x_0

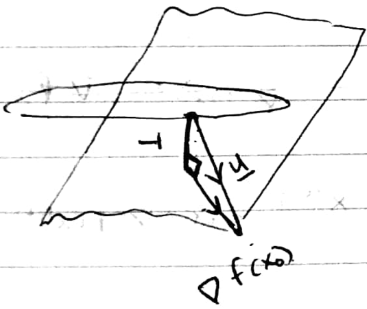
$u \in \mathbb{R}^{n+1}$

$z_0 = \begin{pmatrix} x_0 \\ f(x_0) \end{pmatrix}$ $\quad z_0 \in \mathbb{R}^{n+1}$

$\therefore u \in \mathbb{R}^{n+1}$

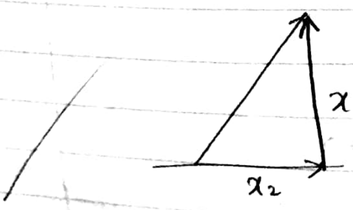
$u = \begin{bmatrix} \nabla f(x_0) \\ -1 \end{bmatrix}$

coordinates



$u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

coordinates.



Homework

5.5 Consider $f(x) = x_1 x_2 / 2$ $\quad g(s,t) = [4s+3t, 2s+t]^T$

evaluate $\frac{\partial}{\partial s} f(g(s,t))$ and $\frac{\partial}{\partial t} f(g(s,t))$

$$g(s,t) = \begin{bmatrix} 4s+3t \\ 2s+t \end{bmatrix} \qquad f(g(s,t)) = \frac{(4s+3t)(2s+t)}{2}$$

$$\frac{\partial}{\partial s} f(g(s,t)) = \frac{\partial}{\partial s}\left(\frac{8s^2 + 10st + 3t^2}{2}\right)$$

$$= \frac{1}{2}\left(8 \times 2s + 10t + 0\right)$$

$$= \frac{16s + 10t}{2} = 8s + 5t$$

$$D f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} & \frac{\partial f(x)}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial (x_1 x_2/2)}{\partial x_1} & \frac{\partial (x_1 x_2/2)}{\partial x_2} \end{bmatrix}$$

$$D f(x) = \begin{bmatrix} x_2/2, & x_1/2 \end{bmatrix}$$

$D f(x)$

$$\frac{\partial}{\partial s} f(g(s,t)) = D f$$

$$\frac{\partial}{\partial s} f(g(s,t)) = D f(g(s,t)) (\partial g(s,t))$$

$$= \cancel{\frac{1}{2}\begin{bmatrix} 2s+3t \\ 4s+3t \end{bmatrix}} \times \cancel{\frac{3}{2}\begin{bmatrix} 4 \\ 2 \end{bmatrix}} = \frac{1}{2}[2s+t, 4s+3t] \times \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$= 8s + 5t$$

$$D(g(s,t)) = \frac{\partial}{\partial s}(g(s,t)) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

(20) Let $f(x) = \log \sum_{i=1}^{m} \exp(a_i^T x + b_i)$ $\quad x \in \mathbb{R}^n$ $\quad$ comput $D f(x)$

$$= \log \sum_{i=1}^{m} e^{(a_i^T x + b_i)}$$

$$\cancel{g(y) = e^{a_i^T y + b_i}}$$

$$g(y) = \log \sum_{i=1}^{m} \exp(y)$$

$$y = (a_i^T x + b_i)$$

$$f(x) = \log \sum_{i=1}^{m} \exp\left(a_i^T x + b_i\right) \quad x \in \mathbb{R}^n$$

$a_i \in \mathbb{R}^n.$

$a_i^T x = Ax$

$i = 1, 2$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\cancel{A=} \; a_i^T = \begin{pmatrix} a_1^T \\ a_2^T \\ \\ a_m^T \end{pmatrix} = A \quad a_i \in \mathbb{R}^n \quad A \in \mathbb{R}^{m \times n}.$$

$=$

$$f(x) = \log \sum_{i=1}^{m} \exp\underbrace{(Ax + b_i)}_{y \in \mathbb{R}^m} = \log \sum_{i=1}^{m} \exp(y_i) = g(y)$$

$$\cancel{\nabla f(x) = \nabla g(y) x}$$

$$\cancel{Dgf} \quad Df(x) = D(g(y))\, D(y)$$

$$= \frac{1}{\sum_{i=1}^{M} \exp(y)} \left(\exp(y)\right) \times D(y)$$

$$= \frac{1}{\sum_{i=1}^{m} \exp(y)} \begin{pmatrix} \exp(y_1) \\ \exp(y_m) \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_m \end{pmatrix}$$

$$= \begin{pmatrix} a_1^T x + b_1 \\ a_2^T x + b_2 \\ \\ a_m^T x + b_m \end{pmatrix}$$

~~Take $A\,x+b_i = h(x)$~~

Take $AY + b_i = h(y)$.

~~$f(x)$~~ Take $f(y) = \log \sum_{i=1}^{m} \exp(h(y))$

$$\nabla f(y) = D\left[f(y)\right] D\left[\exp\left[h(y)\right]\right)$$

$$D\left[f(y)\right) = \frac{1}{\sum_{i=1}^{m} \exp(h(y))}$$

~~$D\left[\exp(h(y))\right] = \exp(h(y))$~~

$$\cancel{D(h(y))} = D\left(\exp(h(y))\right) = \exp\left[h(y)\right] \times A^T$$

$$D\left[\exp h(y)\right) = D \begin{bmatrix} \exp(a_1^T x + b) \\ \exp(a_2^T x + b) \\ \vdots \\ \exp(a_m^T x + b) \end{bmatrix}$$

$$= \begin{pmatrix} \exp(a_1^T x + b) \cdot a_1 \\ \exp(a_2^T x + b) \cdot a_2 \\ \\ \exp(a_m^T x + b) \quad a_m \end{pmatrix}$$

$$= \exp(a_i^T x + b) \times \left[a_1 \quad a_2 \quad a_3 \ldots \quad a_m\right]$$

$$D \exp\left[h(y)\right) = \exp(a_i^T + b) \qquad A^T$$

$$\nabla f(y) = \frac{1}{\sum \exp h(y)} \quad \exp(a_i^T + b)\, A^T$$

$$= \frac{1}{\sum \exp a_i^T + b} \quad A^T \exp(a_i^T + b)$$

$$A^T \exp(a_i^T + b)$$

$$= \frac{1}{A^T \exp(a_i^T + b)}$$

17.03.2018

Least Square

linear $f(x) = Ax$

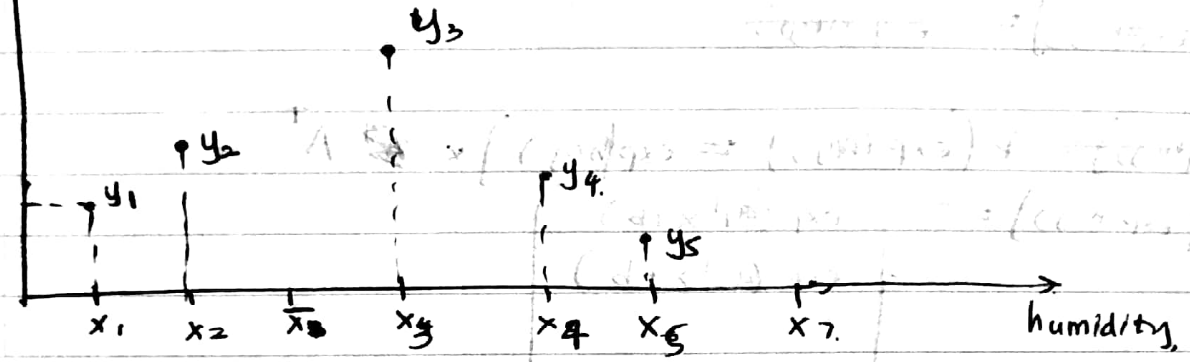
Affined $f(x) = Ax + c$
$\uparrow$ intercept

Quadratic form matrix must be square and symmetric.

Quadratic: $x^T A x + q^T x + r$

affine: $a_1 x + a_2$

quadratic: $a_1 x^2 + a_2 x + a_3$



positive semidefinite.

$A \geq \iff x^T A x \geq 0$
$\forall x$

$x^T A x$

$\left(\frac{A+A^T}{2}\right) = B$

$x^T B x = x^T\left(\frac{A+A^T}{2}\right) x$

$= x^T \frac{A}{2} x + x^T \frac{A^T}{2} x = (x^T A x + x^T A^T x)\, \frac{1}{2}$

$= \cancel{x^T a}$

$\cancel{x^T \frac{A}{2} x = a}$ $\quad x^T A x = a$ (scalar)

$\cancel{a^T = x^T A}$ $\quad a^T = x^T A x$ (same scalar)

$a = a^T$

Date: ....../....../......

$$x^T B x = \frac{a + a^T}{2} = \frac{x^T A x + x^T A x}{2} = x^T A x$$

- non symmetric matrix, can be replaced with a symmetric matrix
- positive semidefiniteness properties can be shown using symmetric quadratic matrix.

$$\begin{bmatrix} f(x_1) = \begin{bmatrix} x_1^2 & x_1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1 x_1^2 + a_2 x_1 + a_3 \\ f(x_2) = \begin{bmatrix} x_2^2 & x_2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1 x_2^2 + a_2 x_2 + a_3 \\ f(x_5) \quad \begin{bmatrix} x_5^2 & x_5 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \\ f(x_5) \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \\ x_4^2 & x_4 & 1 \\ x_5^2 & x_5 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

mismatch

$e_1 = \underset{\text{measured}}{y_1} - \underset{\text{estimate}}{f(x_1)}$

$e_2 = y_2 - f(x_2)$

$e_3 = y_3 - f(x_3)$

$e_5 = y_5 - f(x_5)$

$$= (y_1 - f(x_1))^2 + (y_2 - f(x_2))^2 + (y_3 - f(x_3))^2 + \dots + (y_5 - f(x_5))^2$$

least square error $= \sum_{i=1}^{5} [y_i - f(x_i)]^2$

- when error is minimized iteratively the values for a_1, a_2, a_3 also change each time.

least square error $= \|Xa - y\|_2^2$

$$Xa = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \\ f(x_5) \\ \cancel{f(x_5)} \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

$$(Xa - y) = \begin{bmatrix} f(x_1) - y_1 \\ f(x_2) - y_2 \\ f(x_3) - y_3 \\ f(x_4) - y_4 \\ f(x_5) - y_5 \end{bmatrix}$$

$$\|Xa - y\|_2^2 = (f(x_1) - y_1)^2 + (f(x_2) - y_2)^2 + (f(x_3) - y_3)^2 + (f(x_4) - y_4)^2 + (f(x_5) - y_5)^2$$

$$= \sum_{i=1}^{5} [y_i - f(x_i)]^2 = \text{least square error}$$

~~$ax^2 + bx + c$~~

minimize $\|Xa - y\|_2^2$

optimize variable

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

optimized ~~variable~~ a $= a^*$

$f(x) = a_1^* x^2 + a_2^* x + a_3^*$

$(AB)^T = B^T A^T$

X is not symmetric.

$$\begin{aligned}
\|Xa - y\|_2^2 &= (Xa - y)^T (Xa - y) \\
&= [(Xa)^T - y^T].(Xa - y) \\
&= (Xa)^T Xa - (Xa)^T y - y^T Xa + y^T y \\
&= a^T \underbrace{X^T X}_{B} a - a^T X^T y - y^T Xa + y^T y \\
&\cancel{= a^T B} \\
&= a^T Ba - 2a^T X^T y + d.
\end{aligned}$$

$a^T = a_1 \quad a_2 \quad a_3$

$X^T X =$

$a^T X^T y = p$ (scalar)

$p^T = y^T X a.$ (same scalar)

$p = p^T$

$= \cancel{a^T Ba - 2(Xa)^T y + d}$

$= a^T Ba - 2y^T Xa + y^T y$

$= a^T Ba - 2(X^T y)^T a + d$

$\|Xa - y\|^2 = a^T Ba - \underbrace{2(X^T y)^T}_{q = 2X^T y} a + d.$ ← quadratic of a

$B = X^T X$

$B^T = X^T X = B$

$\therefore$ B is symmetric.

$g(a) = a^T B a - 2(x^T y)^T a + d.$

$\{a \mid g(a) = 1\}$

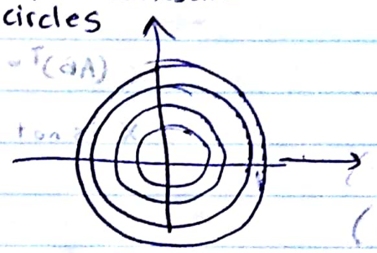
$\{a \mid a^T B a + q^T a + d = 1\}$

$q = 0 \quad B = I$

level curves are circles

$a^T a = 1$

$a_1^2 + a_2^2 = 1$



level curves of $g(a)$ are ellipses.

$\cancel{2a^2 + 3a + 10}$

$\cancel{2\left[a^2 + \tfrac{3}{2}a + 5\right]}$

$\cancel{2\left[\left(a + \tfrac{3}{4}\right)^2 - \tfrac{9}{16} + 5\right]}$

if $\cancel{g(a)}\ P(a) = a^2 + 4a + 10 = (a+2)^2 - 4 + 10 = (a+2)^2 + 6$ ← complete square form.

$g(a)_{min} = 6$ at $a = -2$.

$(a+2)^2 = (a+2)^T (a+2)$

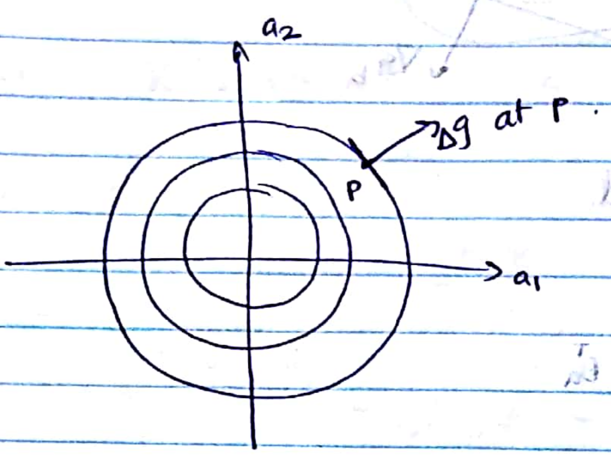
$g(a) = (a - Dr)^T (a - Dr) + 1.$ ← positive

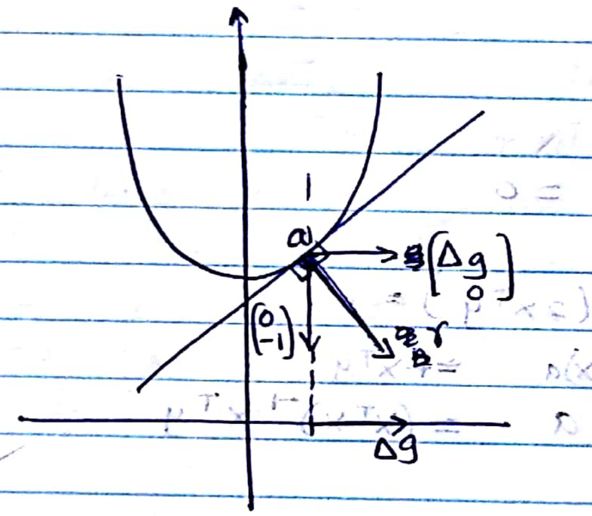
minimum of $g(a) = D$ $a = Dr$

a: vector coordinate; Dr: vector.

differential gradient is perpendicular to level curve

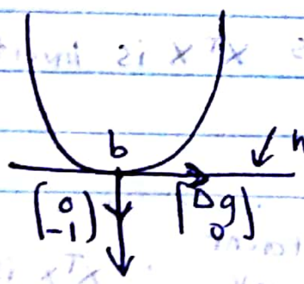
$\nabla g(a)$





- a normal vector $r = \begin{pmatrix}\Delta g\\0\end{pmatrix} + \begin{pmatrix}0\\-1\end{pmatrix} = \begin{pmatrix}\Delta g\\-1\end{pmatrix}$
- $\begin{pmatrix}\Delta g\\0\end{pmatrix}$ is ~~the~~ a vector <u>not the point a.</u>
- $\begin{pmatrix}\Delta g\\0\end{pmatrix}$ is the $\perp$ vector to the level curve at a.

at This point

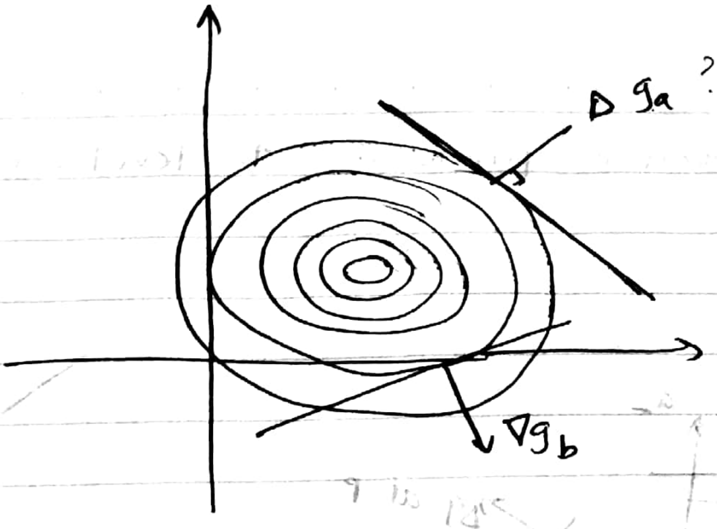


normal to hyperplane $= \begin{pmatrix}0\\-1\end{pmatrix}$

$\Delta g = 0$

$\begin{pmatrix}\Delta g\\0\end{pmatrix}$ vector is $\perp \begin{pmatrix}0\\-1\end{pmatrix}$

~~The~~ when hyperplane is in this position



$g(a) = \frac{1}{2} a^T B a + q^T a + d$

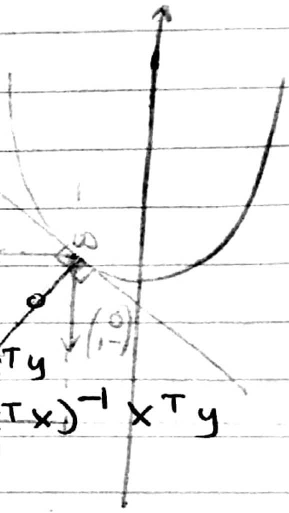
$\nabla g(a) = 0$

$\nabla(a^T B a) = \frac{1}{2} \times 2 B^T a$

$\nabla(q^T a) = q$

$\nabla(d) = 0$

$\nabla g(a) = 2B^T a + q = 0$



$2(X^T X) a + (2X^T y) = 0$

$(X^T X) a = X^T y$

$a = (X^T X)^{-1} X^T y$

$M a = b$

If M is square and ~~invert~~ invertible

There is a unique solution ; $a = M^{-1} b$

$X^T X =$

$$X = \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \\ x_4^2 & x_4 & 1 \\ x_5^2 & x_5 & 1 \end{pmatrix}$$

$X^T X$ is a square matrix

is $X^T X$ is invertible.

provided x_1, x_2, x_3, x_4, x_5 are different

$X \Rightarrow$ full rank $\therefore X^T X$ is a

$$x^T = \begin{pmatrix} x_1^2 & x_2^2 \\ x_1 & x_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \end{pmatrix}$$

↙ full rank

$$= \begin{pmatrix} x_1^4 + x_2^2 & x_1^3 + x_2^3 & x_1^2 + x_2^2 \\ x_1^3 + x_2^3 & x_1^2 + x_2^2 & x_1 + x_2 \\ x_1^2 + x_2^2 & x_1 + x_2 & 1 + 1 \end{pmatrix}$$

$$a = (\cancel{X^T X} \quad (X^T X)^{-1} X^T y$$

minimize $\|AX - b\|_2^2$ X is the decision variable

Least square $X_{LS} = (A^T A)^{-1} A^T b$

↑ A is full rank.

A =  m×n

minimize

$\|Ax - b\|$ A is a fat matrix $b \in \mathbb{R}^m$

$$A = \begin{bmatrix} | & | & | & | \\ a_1 & a_2 & a_3 & a_n \\ | & | & | & | \end{bmatrix}$$

coloumns of A

$a_1, a_2, a_3 \ldots a_n$

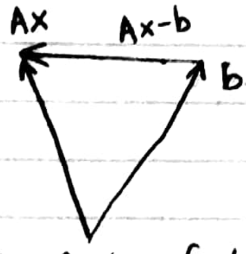
$\boxed{\text{FAT}}$

coloumns of A are vectors of $\mathbb{R}^n$

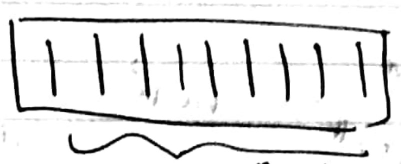
$Ax = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$

When ~~Range~~ b is in the range of A, $AX = b$

Then $g^* = 0$ because g^* is the distance between Ax and b vectors



When A is fat, What Is g^*



$n > m$

$b \in \mathbb{R}^m$

if there are m linearly independent coloumns → $g^* = 0$

exam → if A is skinny

a_n ⇒ linearly dependent on other a_i s.

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$$A = \begin{bmatrix} | & | & | & | & | & & | \\ a_1 & a_2 & a_3 & a_4 & | & & a_n \\ | & | & | & | & | & & | \end{bmatrix}$$

When

$$(X^T X)a = X^T y$$

when $X^T X$ is not invertible

$X^T X$ is having linearly dependent vectors

There are infinitely many solutions to a.

When you have linearly dependent vectors

$$A = \begin{bmatrix} | & | & | & & | \\ a_1 & a_2 & a_3 & \dots & a_n \\ | & | & | & & | \end{bmatrix} \quad (a_1 + a_2)$$

A becomes non invertible.

A has a null space

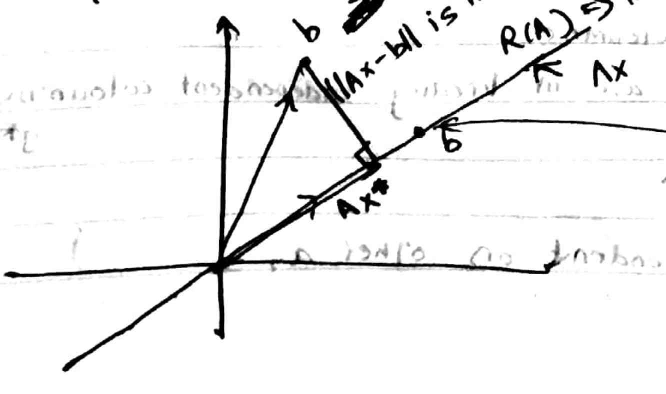
A has infinitely many solutions

$$\underbrace{A^T A}_{\substack{\text{need to see whether} \\ A^T A \text{ is full rank,} \\ \text{rank deficient}}} x = A^T b$$

min. $\|Ax - b\|$ $\quad b \in \mathbb{R}^2$

If $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$



Least square error $= 0$

$b \in R(A)$

Ax is a vector

- If you have redundancies in A you can neglect those rows or coloumns
- LS always has solutions

$$A^T A X = A^T b$$

$$A^T b \in R(A^T A)$$ → try to prove → exam

~~quad~~ LS reduce to $A^T A X = A^T b$ (not $AX = b$)

$A^T b \in R(A^T A)$ because of That LS always have solutions

exam - LS questions will be there.

$$Bb \in R(BB^T)$$

~~$B = A(b_1 \; b_2 \; \dots \; b_n) \; b$~~

$$BB^T = \begin{pmatrix} b_1 & b_2 & b_3 & \dots & b_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

$$(BB^T) = \begin{pmatrix} b_1^2 \\ b_2^2 \\ \vdots \\ b_n^2 \end{pmatrix}$$

$$(BB^T)x = \cancel{b_1 \; b_2 \; b_3} \; Bb$$

$$(BB^T)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 & \dots & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix}$$

Linear Programming

A lined objective
linear constraint

① minimize $c^T x$

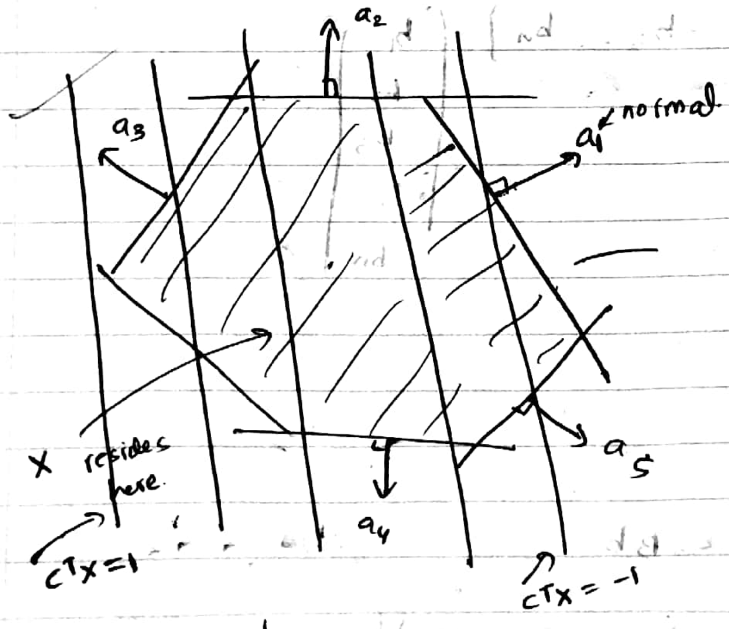
such that $Ax \le b$

$\{x \mid Ax \le b\}$ $\quad x \in \mathbb{R}^2$

$$A = \begin{pmatrix} a_1^T \\ a_2^T \\ a_3^T \\ \vdots \\ a_n^T \end{pmatrix} \qquad \text{Minimize } c^T x = f(x)$$

$a_1^T x \le b_1$

$a_2^T x \le b_2$



- You have to ~~make~~ max $c^T x$ smaller as possible until you come to the feasible domain of x.

you have to go in the (−) gradient direction.

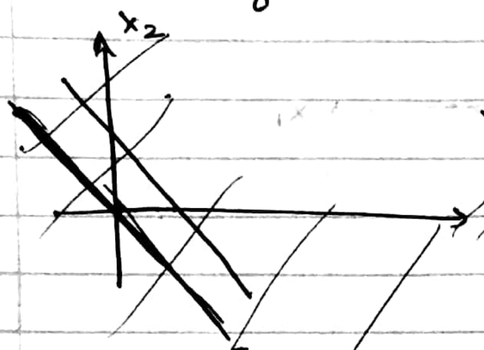
The gradient of $c^T x$ is c

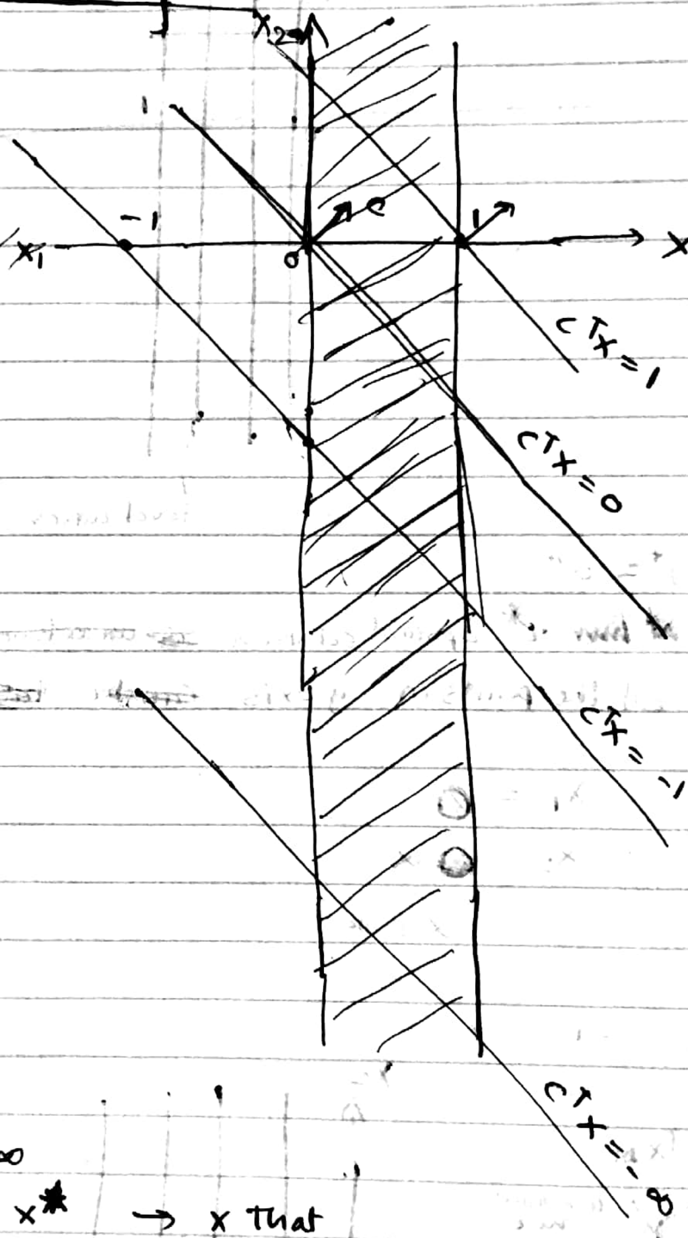
also the gradient of the level curve $c^T x = -1$ is also c

minimize $c^T x = g(x)$

such that $0 \le x_1 \le 1$ $\qquad c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$





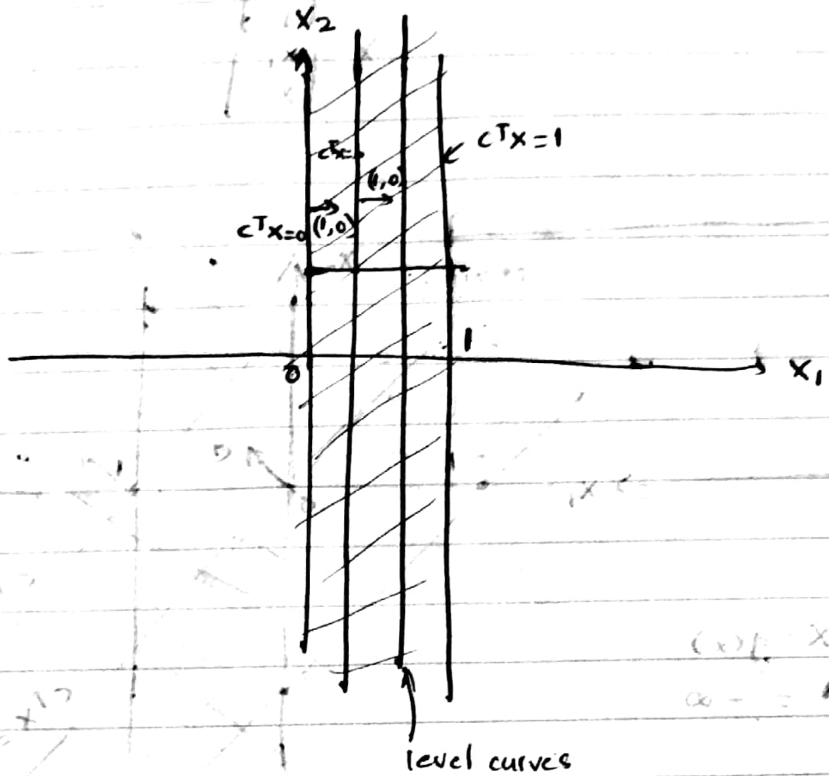
min $c^T x = g(x)$

$g^* = -\infty$

optimal value $g^* = -\infty$

optimal solution $\rightarrow x^* \rightarrow$ x that gives optimum value g^*

~~linear~~ programming



$g^* = 0$

[redacted] x^* optimal solution ~~is undefined~~ $\Rightarrow$ infinitely many solutions

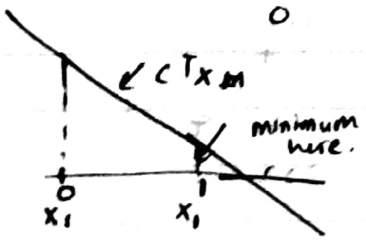
all the points in y axis ~~can be~~ ~~tes~~ are optimal solutions.

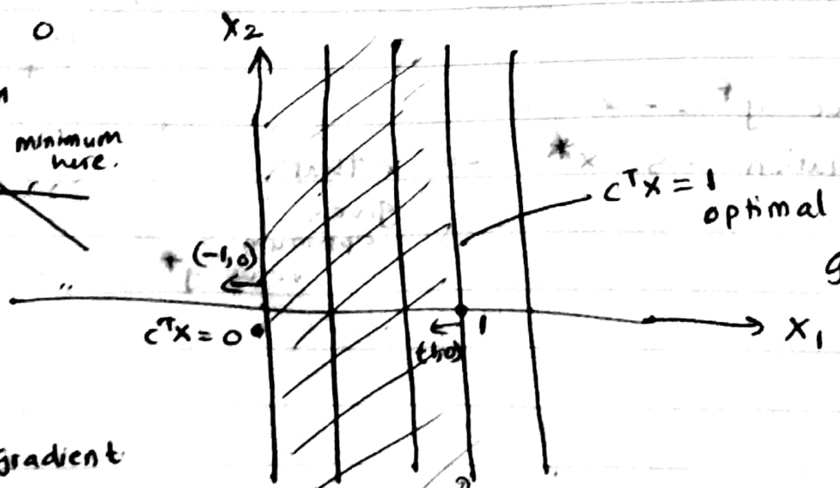
$x_1 = 0$

$x_2 = \alpha$

$\alpha \in \mathbb{R}$

$c = \begin{matrix} -1 \\ 0 \end{matrix}$





$c^Tx=1$ optimal value $= 1$

$g^* = 1$

c is the gradient

function is decreasing in the opposite direction.

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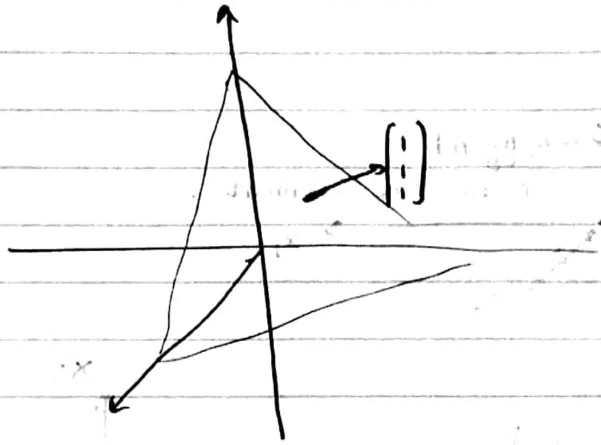
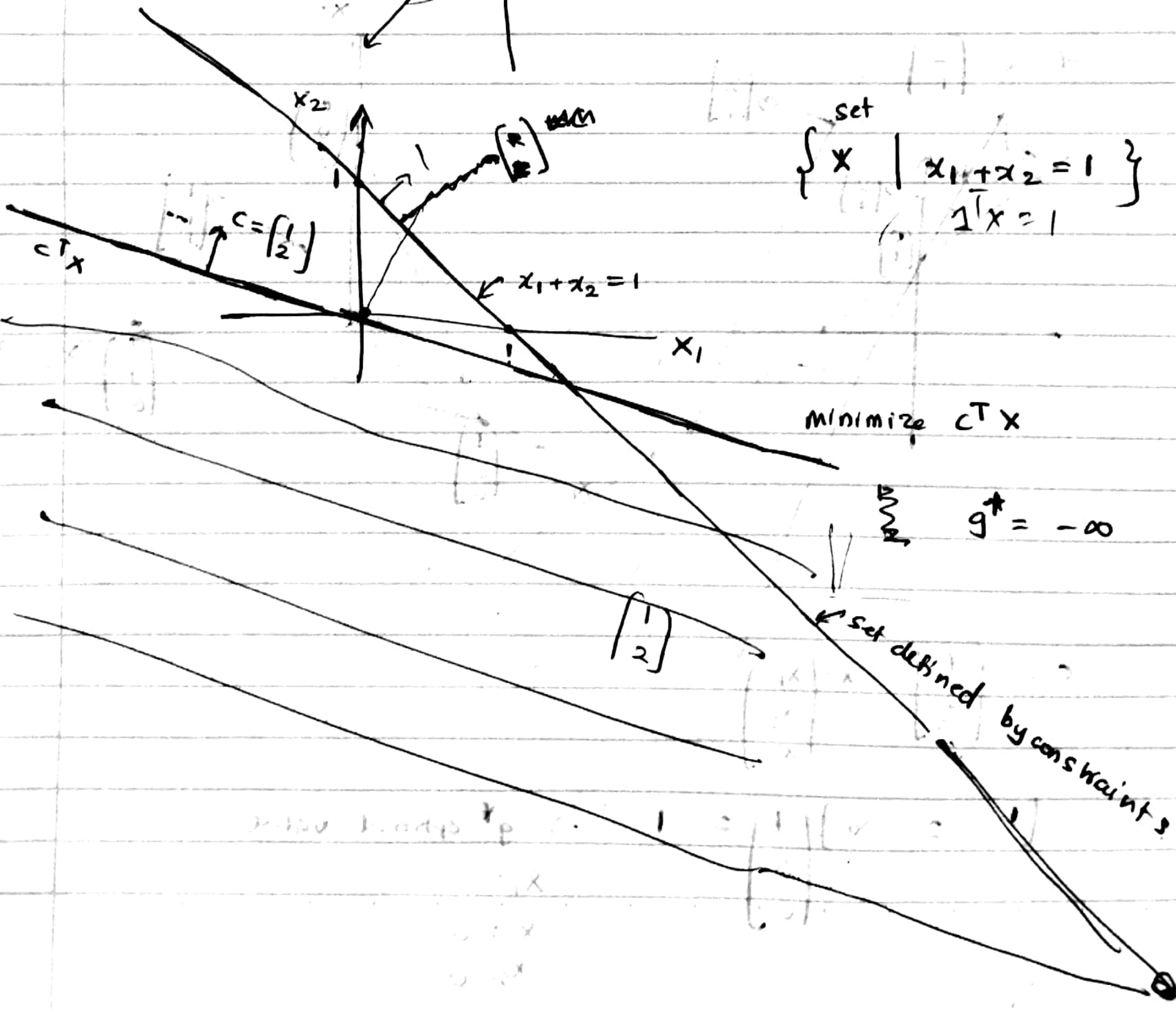
optimal solution ~~is the~~ $\Rightarrow$ infinitely many solutions along

$$x_1 = 1$$
$$x_2 = \alpha$$
$$\alpha \in \mathbb{R}$$

minimize $c^T x$ s.t. $1^T x = 1$ $\quad c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$



$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



set $\{ x \mid x_1 + x_2 = 1 \}$
$1^T x = 1$

minimize $c^T x$

$\Rightarrow$ $g^* = -\infty$

optimal solutions → cannot be defined.

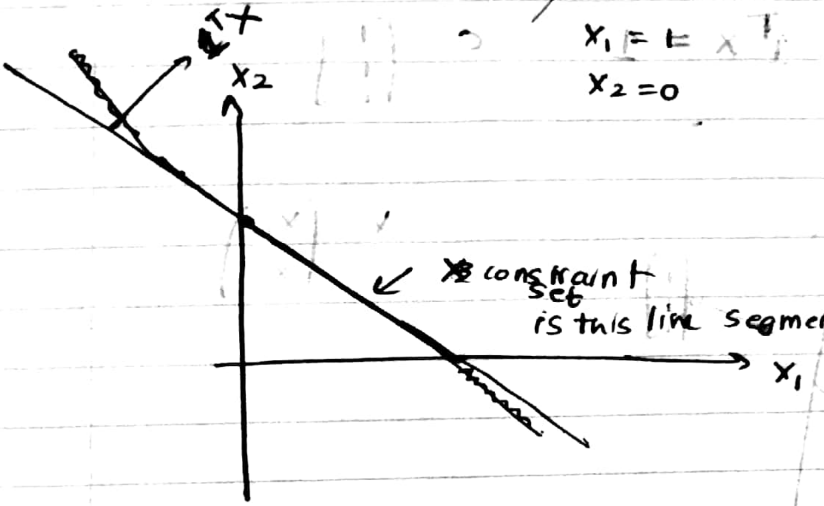
when $x \geq 0$ find minimum $c^T x$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1$

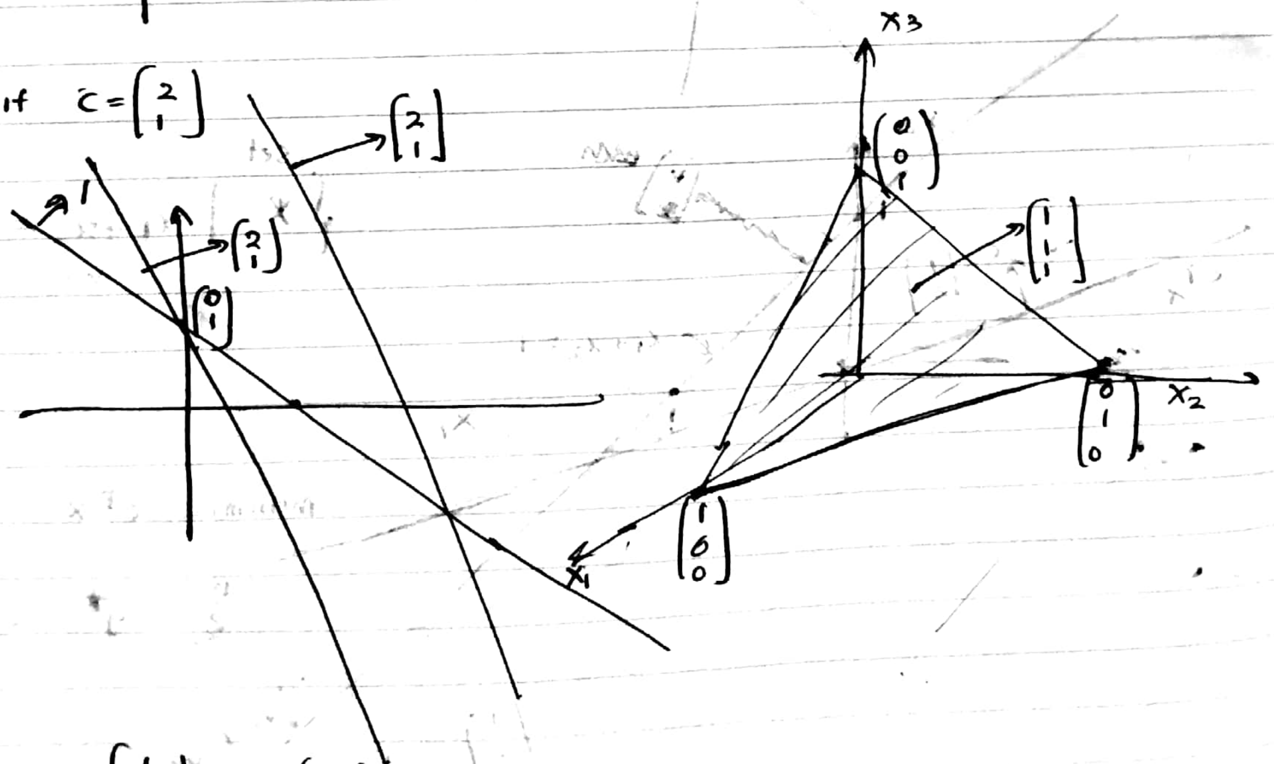
$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 + 0 = 1$

$x_1 = 1$

$x_2 = 0$



if $c = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



$c = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$ → g^* optimal value

$x_1 = 1$

$x_2 = 0$

$x_3 = 0$

~~when~~ we have to compute $c^T x$ at corner points,

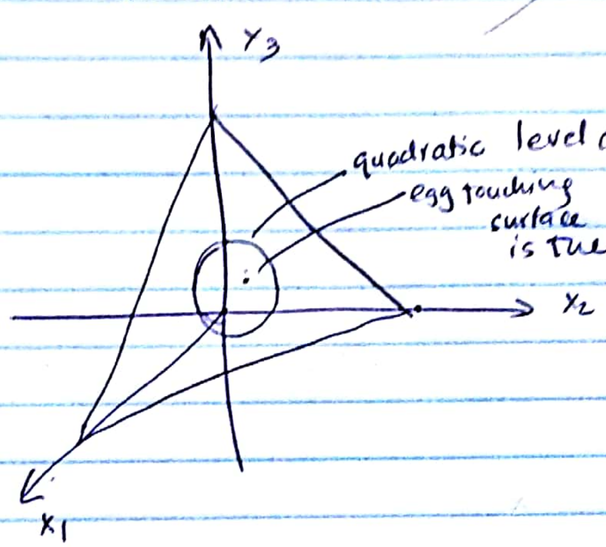
$$\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = 2 \qquad \begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = 3$$

for $c \in \mathbb{R}^n$

$$g^* = \min_{v_i} \left(c^T v_i\right)$$

$$v_i = [0 \; 0 \; 0 \ldots 1 \ldots 0]^T$$

For n dimensions $\longrightarrow$ n corner points.



is the minimum ~~leve~~ function.
The eggs more than that are ~~not~~ ~~the~~ more than minimum

minimize $\underbrace{f_0(x)}_{\text{objective function}}$ s.t $\underbrace{f_i(x) \le b_i}_{\text{constraint.}}$ $i = 1, 2, 3, \ldots m.$

$x \in X$

$$X = \{ x \mid f_i(x) \le b_i \quad i = 1, 2, 3, \ldots, m \}$$

decision variable $\rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

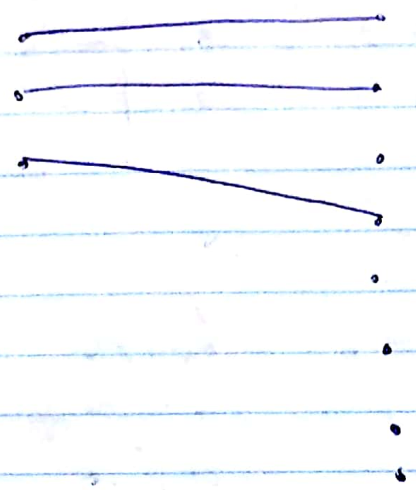
~~to~~ ask ~~S~~ Someone

3 persons given 8 Tasks

Date: ..... / ..... / .....

2

3 persons 8 Tasks



3 persons → 8 tasks

x_{ij}

minimum number of tasks per person 2.

$$x_{ij} = \begin{cases} 1 & \text{if } i \text{ assigned to } j \\ 0 & \text{otherwise} \end{cases}$$

③ maximize $\sum_i \sum_j c_{ij} x_{ij} = c^T x$

① $\sum_j x_{ij} = 1$

$a_i^T x = 1$ $\quad a_i = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix}$ (8, 8, 8) $\quad j = 1, 2, \dots, 8$.

$a_i \in \mathbb{R}^{24}$

③ $\sum_j x_{ij} \geq 2$

$q_j^T x = 1$ $\quad j = 1, 2, 3, \dots, 8$

$$x = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ \vdots \\ x_{18} \\ x_{21} \\ \vdots \\ x_{38} \end{bmatrix}$$

$$X = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ \vdots \\ x_{18} \\ \hline x_{21} \\ \vdots \\ x_{28} \\ \hline x_{31} \\ \vdots \\ x_{38} \end{bmatrix}$$

minimize $c^T x$

s.t : $a_i^T x = 1$ $\quad j = 1, 2, \dots 8$

$a_j^T x \geq 2$ $\quad j = 1, 2, 3, \dots 8$

$x_{ij} \in \{0, 1\}$

$a_i^T x \geq 1$

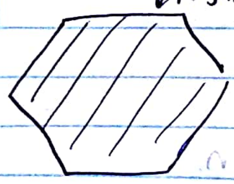
$a_i^T x \leq 1$

? $Ax \geq b$

s.t $Ax \geq b$

$x_{ij} \in \{0, 1\}$

Polyhedron.

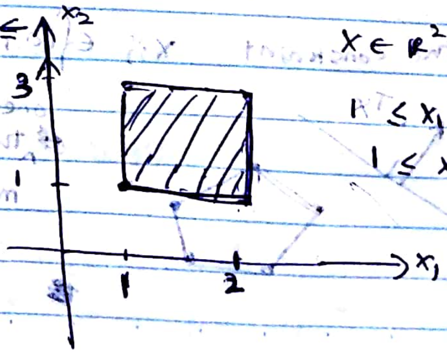


$x \in \mathbb{R}^2$ $\qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq x \leq \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$x \in \mathbb{R}^2$

$1 \leq x_1 \leq 2$

$1 \leq x_2 \leq 3$



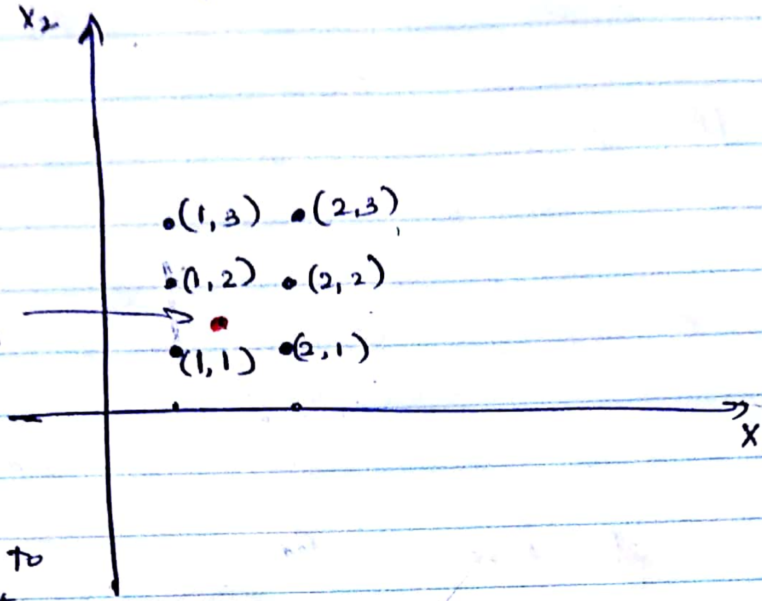
$1, 2, \dots, 8$.

$i \in \mathbb{R}^{24}$

$x_1 \in \{1, 2,\}$

$x_2 \in \{1, 2, 3\}$

$x_{12} \in \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

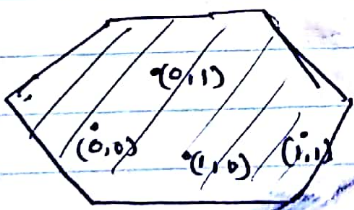


if you get a point like this you can round off to the closest feasible point

$x_{ij} \in \{0,1\}$

$x_{ij} \in \{(0,0), (0,1), (1,0), (1,1)\}$

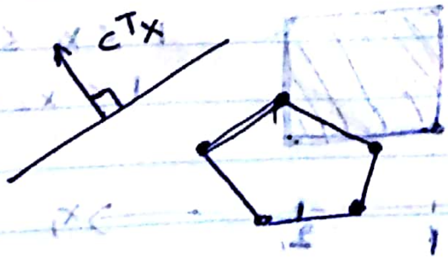
substitute $c^T x$ at points and find the value.



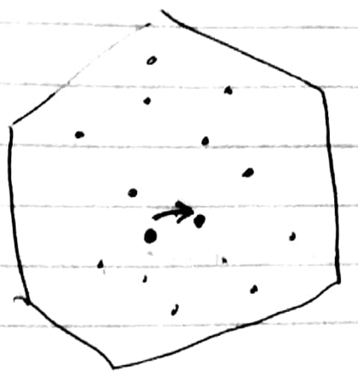
~~For some reasons all the corners of the polyhedron are integer vectors~~

~~If the constraint~~

if the constraint $x_{ij} \in \{(0,1)\}$ is not given.



one of the corner points of the polyhedron must be solutions for the minimiz value of $c^T x$.



$$\begin{pmatrix} 0.1 \\ 0.001 \\ 0.1 \\ 1.7 \\ 2.1 \\ 0.02 \\ \vdots \end{pmatrix} = x$$

↙ from linear programming find x as this vector.
This ~~vector~~ integer ~~can be~~ close to this vector can be found.

① First find the closest integer vector

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0.09 \\ 1.1 \\ 1.3 \end{bmatrix}$$

② check $Ax \geq b$

↙ The estimated integer vector would be outside polygon. Therefore should check $AX \geq b$

③ check $c^T X$ and find the minimized.

* $(A^TA)^{-1}A^Ty$ ← least square → delay is on invertibility of the matrix

$\min \|Ax-y\|_2^2$

$A \backslash y$

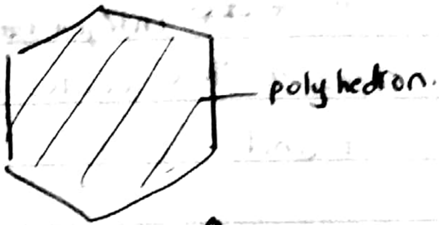
* linear programming /convex

Big 'o' notation → $O(n)$ ↑ order n algorithm. $O(2^n)$ order ↑ 2^n ~~order~~ algorithm

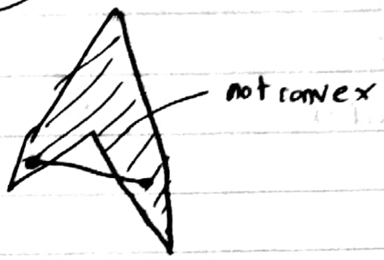
~~npha~~ nphard – nondeterministic polynomial time problem
↑ can't find solution.

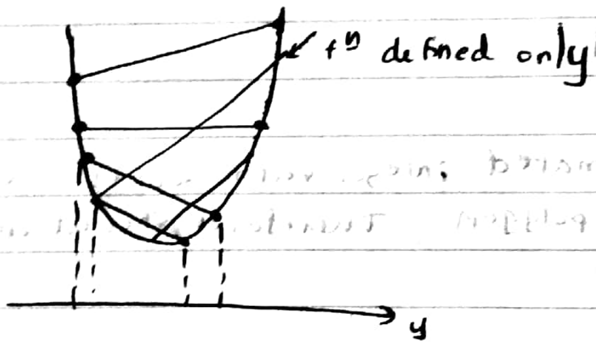
convex optimization

minimize $f_0(X)$ s.t. $X \in \underbrace{\mathcal{X}}_{\text{convex}}$

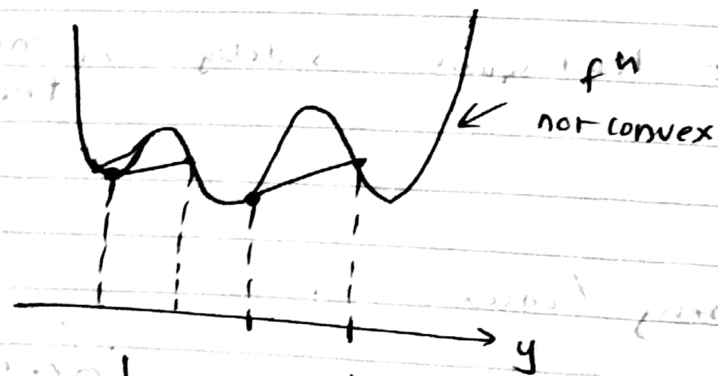


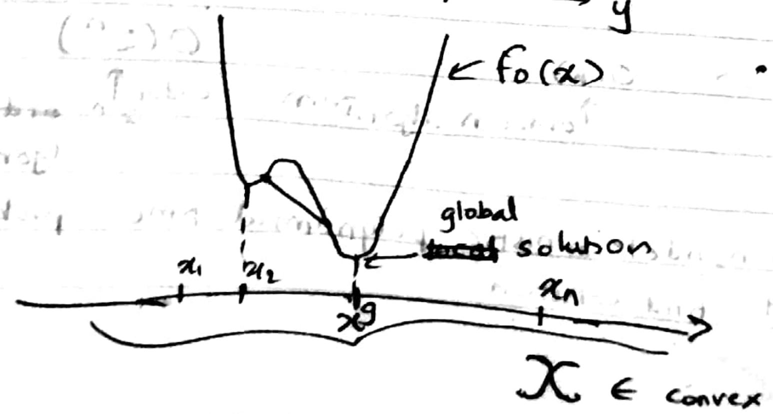
f_0 should be convex.





for $(y_1, y_2) \in Y$, if the the line segment $y_1 y_2$ is above the function, then f^n is convex





• When you plug global solution x^g you can get the smallest for $f_0(x)$

x_2 and x^g are local solutions

ball around x.

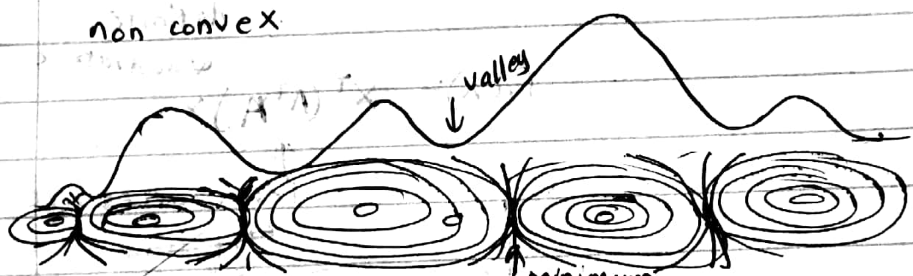
$$\{x \mid \|x-c\|^2 \in \epsilon\}$$

- When you move around x local point and find $\|x-c\|^2$ then ~~the~~ it is bigger than ϵ then it is a local point
- around local minimum the function is 'curved up'

$f_0(x_1, x_2)$

convex functions level curves are curved.

non convex



when minimize $f_0(x)$ s.t $x \in \mathcal{X}$ using convex method

minimize $f_0(x)$

s.t
$$\begin{cases} f_1(x) \le b_1 \\ f_2(x) \le b_2 \\ g_1(x) = d_1 \\ g_2(x) = d_3 \end{cases}$$

$$\mathcal{X} = \{x \mid f_1(x) \le b_1,\ f_2(x) \le b_2,\ g_1(x) = d_1,\ g_2(x) = d_3\}$$

$$\boxed{\begin{matrix} a_1^T x \le b_1 & c_1^T x = d_1 \\ a_2^T x \le b_2 & c_2^T x = d_2 \end{matrix}}$$ ← ~~affined~~

$\mathcal{X}$

can be written as half spaces.

This is an intersection of half spaces

↓ ~~is~~ is a polyhedron.

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~~$c^T x = d$ can be wr~~

$c^T x = d$ can be written as an intersection of hyperplanes

$c^T x \le d$ & $c^T x \ge d$

- intersection of hyperplanes $\Rightarrow$ is a polyhedron.

how to prove $\mathcal{X}$ is convex?

$$X = \left\{ x \;\middle|\; \begin{array}{l} f_1(x) \le b_1 \\ f_2(x) \le b_2 \end{array} \right\}$$

$f_1(x)$ and $f_2(x)$ are convex

$\Rightarrow f_1, f_2$ level curves are convex

- $f_1(x)$ is like a bowl.

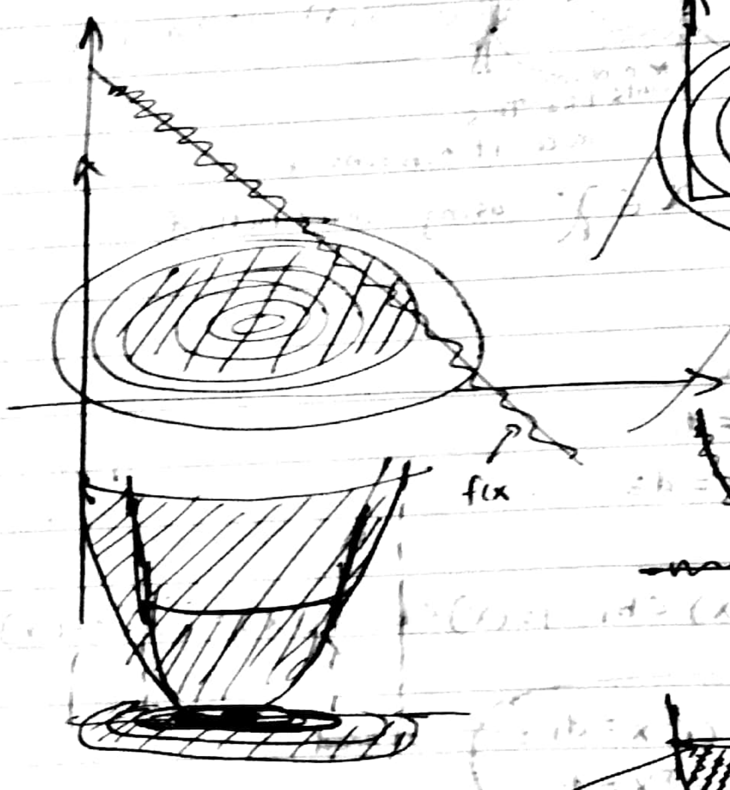
$f_1(x) = x^T A x$

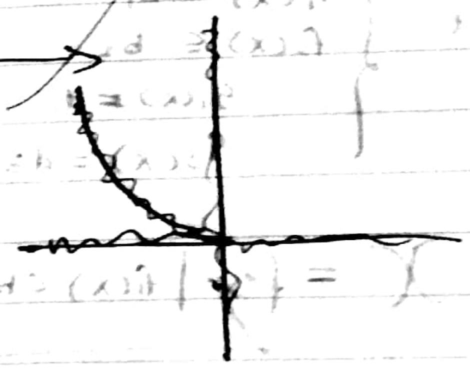
symmetric positive definite Quadratic

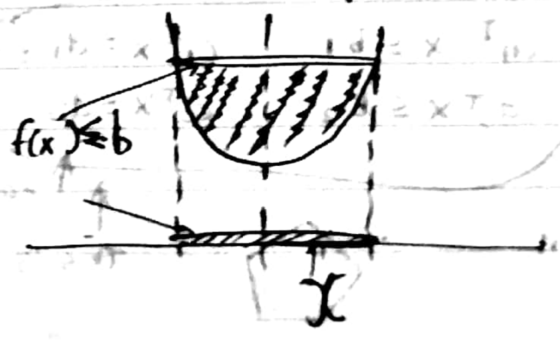
$\{x \mid f(x) \le b_1\}$

$f_1(x) = x^T (A^T A) x$

level curves are ~~ellipsoids~~ ellipses







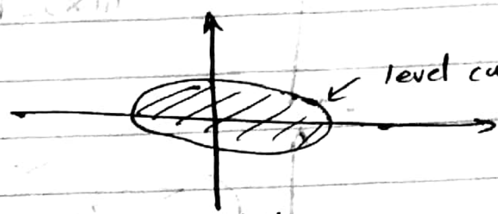
$Ax = b$
quadratic forms
positive semidefinite
infinite
} exam

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$x^T B x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x^T B x = x_1^2 + 2x_2^2$$

$$\{x \mid x^T B x \le 2\} = \{x \mid x_1^2 + 2x_2^2 \le 2\}$$



level curve, with origin O

$x^T B x + q^T x + \gamma$ ← start from a height

$$g = (x-a)^T B(x-a) + \cancel{(x-a)}\, q^T(x-a) + \gamma.$$

if $(x-a)^2 + \gamma$.

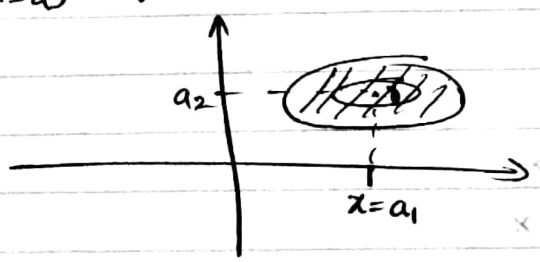
at $x = a$ $g(a)$ becomes γ.

at $x = a$ height is γ

Saddle $x^T \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} x$ → positive indefinite.

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ → positive semidefinite.

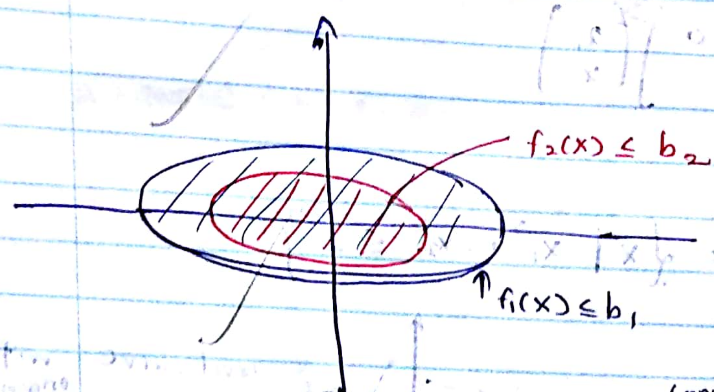
$g(x=a) = \gamma$.

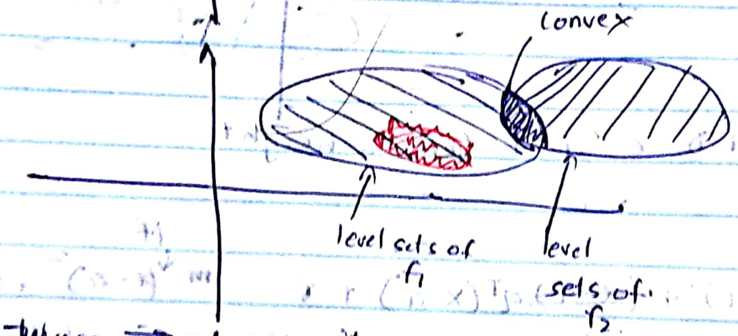


at $x = a$ the ellipse height is γ

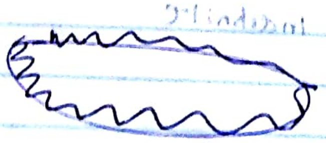
- given any A show that A^TA is positive semidefinite.
- sublevel set of quadratic is ellipse → convex.
- intersection of 2 convex sets is convex.

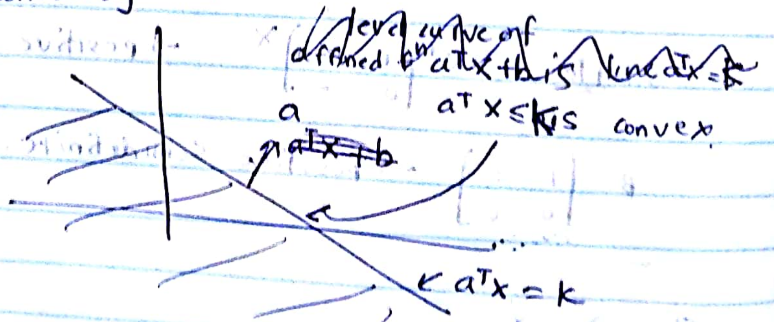
When $f_1(x)$ and $f_2(x)$ are convex $\mathcal{X}$ is convex





2-things → set convexity
function convexity





$\mathcal{X}$ = is convex if → f_1 convex
f_2 convex
g_1 affined
g_2 affined.

$\min.\ f_0(x)$

s.t $f_i(x) \leq 0$ ← convex

$Ax = b$ ← affined function.

$$\begin{bmatrix} a_1^T x = b_1 \\ a_2^T x = b_2 \\ \vdots \\ a_3^T x = b_m \end{bmatrix}$$

convex → problem
→ function.
→ set.

is $Ax \leq b$ convex ?

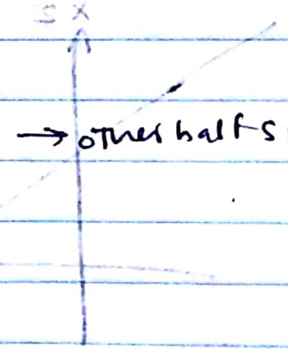
$$\begin{bmatrix} a_1^T x \leq b_1 \\ a_2^T x \leq b_2 \\ a_3^T x \leq b_3 \\ a_m^T x \leq b_m \end{bmatrix}$$

→ $q^T x \leq 2$. ← ~~hyperplane~~ halfspace

↑ function.

$q^T x \geq 2$.

↑ function → other halfspace.



~~is $Ax = b$ is convex ?~~

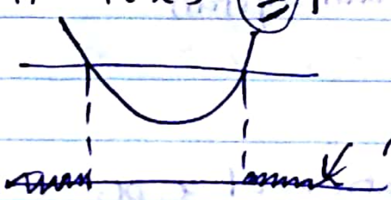
~~$f(x) = x^2$~~

if larger not convex

if $f(x) \geq 1$

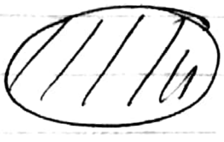
must be less or equal than

$f(x) \leq 1$



not convex

quadratic $x^T A x + q^T A + r$



$\{x \mid f_i(x) \leq 1\}$

$\{x \mid f_i(x) \geq 1\}$

affined is linear

affined → various connected

convex function.

affined

not affined. e.g. quadratic

negative half space

sublevel - convex — convex

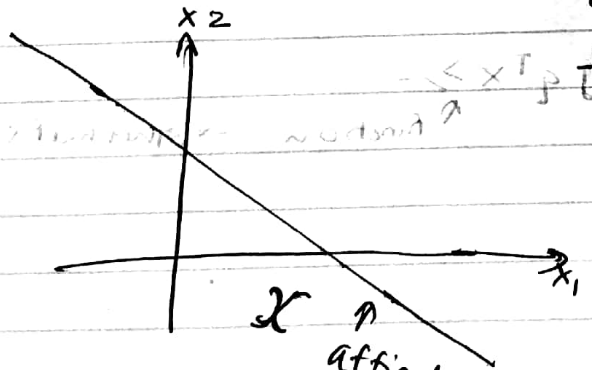
Super level - convex ← positive half space — not convex

level curve - convex — not convex

projection from domain

convex sets - lecture $\boxed{2}$ - (1) affined sets.

contains the line through any ~~distinct~~ two distinct points in the set.



affined and convex

$$\mathcal{X} = \{x \mid Ax = b\}$$

solution of linear system

$$\underbrace{x_1 \quad x_2}_{\alpha x_1 + \beta x_2}$$

$z = \alpha x_1 + (1-\alpha) x_2$ convex combination of x_1, x_2

$A x_1 = b$

$A x_2 = b$

$A(\alpha x_1 + \beta x_2) = \alpha A x_1 + \beta A x_2$

$\alpha b + (1-\alpha) b = b$

$\alpha x_1 + \beta x_2 \in \mathcal{X}$

x_1, x_2 arbitrary

$\alpha x_1 + (1-\alpha)x_2$ → when it is said as affined combination of x_1 and x_2

then the ~~functio~~ line goes beyond x_1 and x_2 points

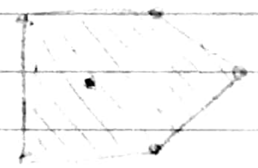
prove that $\mathcal{X} = \{x \mid Bx = 0\}$ ✓ null space of B is convex.

1) Take a convex combination of x_1, x_2 $\quad \mathcal{X} \in \mathbb{R}^2$

$B(\alpha x_1 + (1-\alpha)x_2) = 0 \quad \mathcal{X}$ is convex.

2) show $\mathcal{X}$ is affined.

$\alpha x_1 + (1-\alpha)x_2 =$



- convex combination of

$x_1, x_2, x_3 \ldots x_K$

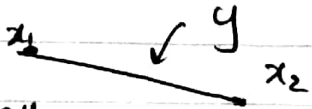
$z = \sum_{i=1} \theta_i x_i$ with

$\sum_{i=1}^{K} \theta_i = 1$ and $\theta_i \geq 0$

- find θ that the Σ of θ is 1 and θ_i are greater than 0

When showing convexity, pick 2 points and take $\alpha x_1 + (1-\alpha)x_2$ combination satisfy the set given.

$\mathcal{X} = \{x_1, x_2\}$



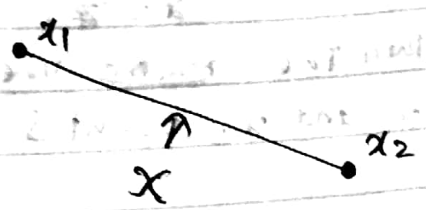
all convex combinations of x_1, x_2

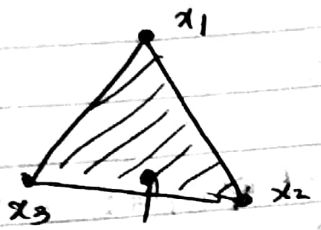
convex hull of $\mathcal{X} = y$

hull → frame
exterior of an object

convex hull → line segment between x_1 and x_2

convex hull $= \mathcal{X}$

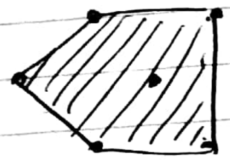




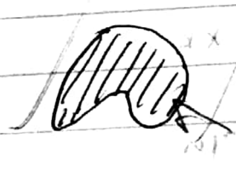
convex hull of x_1, x_2, x_3

$\mathcal{X} = \{x_1, x_2, x_3\}$

convex combination of.



convex hull



convex hull
all possible
convex combinations
of given points

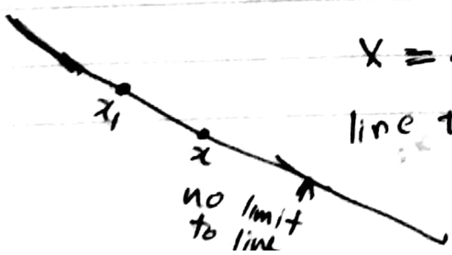
convex hull

Affine combination of

$x_1, x_2, x_2, \ldots x_k$

$$z = \sum_{i=1}^{k} \theta_i x_i \quad \text{with} \quad \sum_{i=1}^{k} \theta_i = 1.$$

What is the affined ~~[illegible]~~ hull of caligraphy x



$X = \{x_1 \; x_2\}$

line that passes through these points.

affined hull covers ~~all the~~ the plane that contain these points.

- affined hull → linear combination of these 3 points

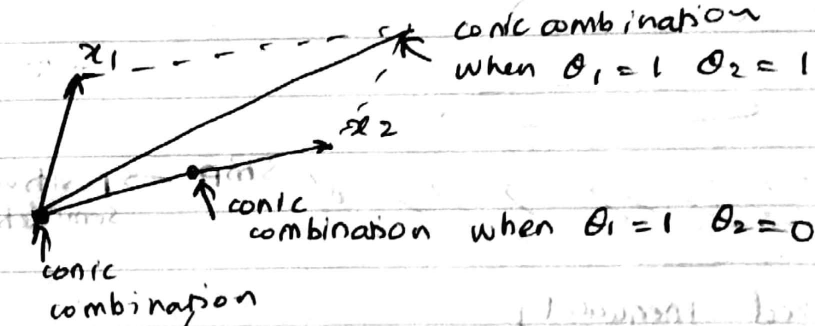
$x \in \mathcal{X}$

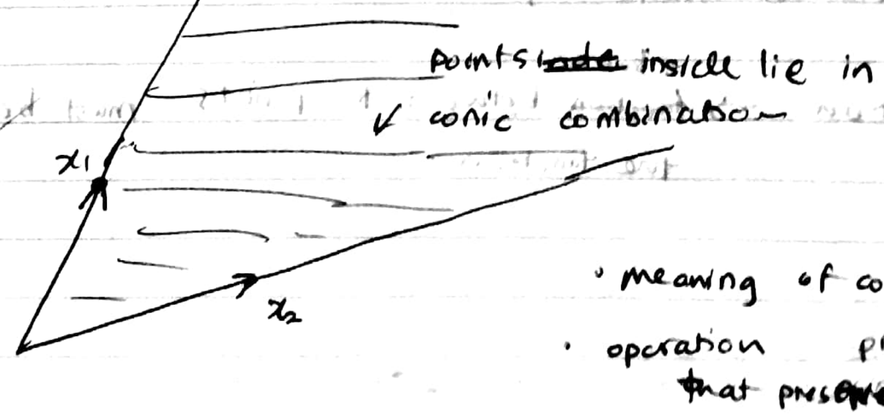
convex cone

(1) convex combination $\xrightarrow{\text{define}}$ convex hull → convex → has limit.
(2) affined combination $\xrightarrow[\text{define}]{}$ affined hull → convex → no limit
(3) conic combination → convex hull → convex → has limit

conic combination ~~the~~ of two points x_1, x_2

$\theta_1 x_1 + \theta_2 x_2$
$\theta_1 \geq 0 \quad \theta_2 \geq 0$





- meaning of convex functions
- operation preserve that ~~preserve~~ preserve convexity

ellipsoid combination

$$\mathcal{X} = \{ x \mid (x - x_c) P^{-1} (x - x_c) \le 1 \}$$

positive definite

quadratic

- $\mathcal{X}$ is convex

if P^{-1} is positive semidefinite.

$x^T P x > 0 \quad \forall x \rightarrow$ positive definite (without equality)

$x^T P x \geq 0 \quad \forall x \rightarrow$ positive semidefinite

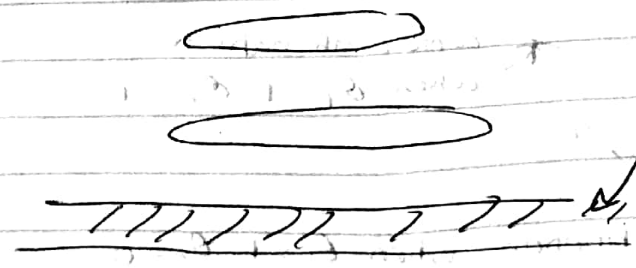
$x^T P x \geq 0 \quad \forall x.$

at least one of eigen value is 0

Positive definite → always quadratic function → curved up.

positive semifinite → not curved up → in direction of eigen vector corresponding to eigen value = 0

positive definite.
level curve circle / ellipse



strip → positive semidefinite

read upto generalized inequality.
listen

convex function → ~~function~~ line between two points must be above the function.

Date: 2018/03/24

Second order cone.

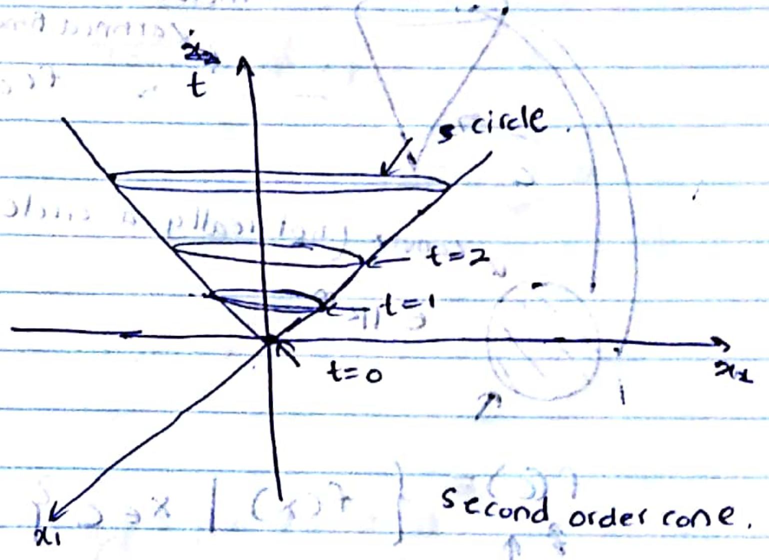
$[x, t] = \begin{bmatrix} x \\ t \end{bmatrix}$ $\quad x \in \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad t \in \mathbb{R}$

$S = \{ (x, t) \mid x \in \mathbb{R}^2, \|x\|_2^2 \leq t \}$

$\|x\|_2^2 = \cancel{x_1^2 + x_2^2 + x_3} \quad x_1^2 + x_2^2$

$\|x\|_2^2 = x_1^2 + x_2^2 \leq t$

$S = \begin{bmatrix} x_1 \\ x_2 \\ t \end{bmatrix}$ height



$t = 0 \quad x_1 = 0 \quad x_2 = 0$

$t = 1$ is a circle

- Positive semidefiniteness is defined for a matrix
- The matrix must be symmetric. (which implies square)

Positive semidefinite cone

$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; X \succeq 0 \right\}$ $\quad y^T X y \geq 0, \forall y$

positive semidefinite

$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}$ symmetric

$S^n \rightarrow$ symmetric matrices of $n \times n$

$S^n_+ \rightarrow$ positive semidefinite

$S^n_{++} \rightarrow$ positive definite

$X \in S$

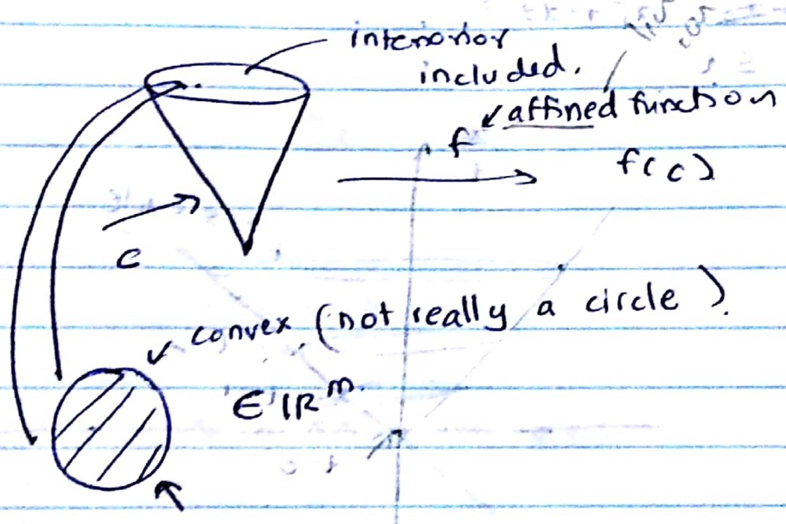
positive everywhere $\quad X \succ 0$

Operations that preserve convexity of a convex set

Affined f^n $\quad f : \underbrace{R^n \to R^m}_{Ax + b}$

$x \in \mathbb{R}^n$

$A \in \mathbb{R}^{m \times n}$

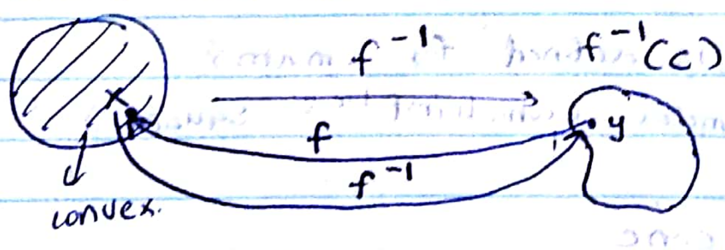


Second order cone → has interior

$f(C) = \{ f(x) \mid x \in C \}$

This set is convex.

When a convex set is given and an affined function f exist



$x \in C$

$f(y) = x$

$x \in C$

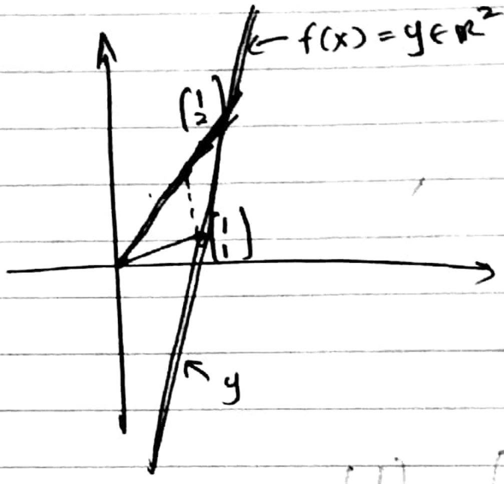
$$f^{-1}(c) = \{ y \mid \underbrace{x = f(y),\ x \in C}_{f(y) \in C} \}$$

- ~~Inverse~~ When you map a convex set with an inverse affine f^n the resultant is convex.

$$f(x) = \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \leftarrow \text{affined function}$$

$$A^{-1}\left(y - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = f^{-1}(y) \leftarrow \text{inverse Affined function exists}$$

$$f(x) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftarrow \text{inverse affined function does not exist.}$$



$$f(x) = \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix}} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

rank $= 2 \rightarrow$ ~~range(X) = R~~ range $\in \mathbb{R}^2$

$$y \rightarrow \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X$$

$$X = \frac{1}{(4-6)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 4-6 \\ -3+3 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

solnⁿ $\Rightarrow X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

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$$A \qquad A^{+}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix}_{2\times 3} \quad \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix}_{3\times 2}$$

(AA^T) exist. $(AA^T) = 2\times 2$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 + 1 + 1 = 3$$

$$\underbrace{A\left(A^T(AA^T)\right)^{-1}}_{I}$$

$$x = A^T(AA^T)^{-1}\left\{\underset{\substack{\uparrow \\ y}}{y} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$f(x) = A\left[A^T(AA^T)^{-1}\left\{y - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}\right] + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$f(x) = AA^T(AA^T)^{-1}\left[y - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = y$$

$$\therefore \; x = \cancel{AA^TAA} \; A^T(AA^T)^{-1}\left\{y - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$ is the sol^n

y is known in this instant

$$? \quad f(x) = \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} \atop \begin{bmatrix} 0 & 0 \end{bmatrix}\right] + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \end{bmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= I + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Sol^n $\therefore \; x = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \atop \begin{bmatrix} 0 & 0 \end{bmatrix}\right)$

$\boxed{\text{FAT}}$

↓ full rank

Right inverse $A^T(AA^T)^{-1}$

Skinny $\rightarrow$ full rank

A

Left inverse

↓

$(A^TA)^{-1}A^T$

$(A^TA)^{-1}A^T$

$\boxed{\begin{pmatrix}x\\t\end{pmatrix} \in C}$ $\quad C = \{(x,t) \mid x \in \mathbb{R}^2, \|x\| \le t\}$ $\rightarrow$ SOC second order cone

find the image of C under $f(z) = \begin{bmatrix} Az+b \\ c^Tz+d \end{bmatrix}$

$A \in \cancel{\mathbb{R}^{3\times 4}}\ \mathbb{R}^{4\times 3}$

$$C \xrightarrow{f} f(C)$$

$$c = \begin{pmatrix}x_1\\x_2\\t\end{pmatrix} \quad x_1^2 + x_2^2 < t$$

$$f(c) = \begin{bmatrix} A\begin{pmatrix}x_1\\x_2\\t\end{pmatrix} + b \\ c^T\begin{pmatrix}x_1\\x_2\\t\end{pmatrix} + d \end{bmatrix}$$

image (range) $\Rightarrow \left\{ f(z) = \begin{bmatrix} Az+b \\ c^Tz+d \end{bmatrix} \;\middle|\; z \in C \right\}$

② find the inverse image of C under f^{-1}

$$f(z) = \begin{bmatrix} Az + b \\ c^T z + d \end{bmatrix} \qquad A \in \mathbb{R}^{2\times 3}$$

$$C \xrightarrow{f^{-1}} f^{-1}(c) = y$$

$$f(y) = c = \begin{pmatrix} x_1 \\ x_2 \\ t \end{pmatrix}$$

$$f(z) = \underbrace{\begin{bmatrix} A \\ c^T \end{bmatrix}}_{B} z + \begin{bmatrix} b \\ d \end{bmatrix}$$

$$z = B^T(BB^T)^{-1}\left[f(z) - \begin{pmatrix} b \\ d \end{pmatrix}\right]$$

$$c = B^T(BB^T)^{-1}\left[f(c) - \begin{pmatrix} b \\ d \end{pmatrix}\right]$$

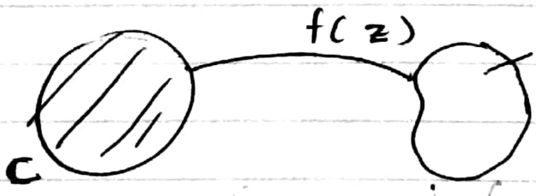
$$\cancel{B^T(BB^T)^{-1}} \quad \cancel{A}\begin{pmatrix} x_1 \\ x_2 \\ t \end{pmatrix} +$$

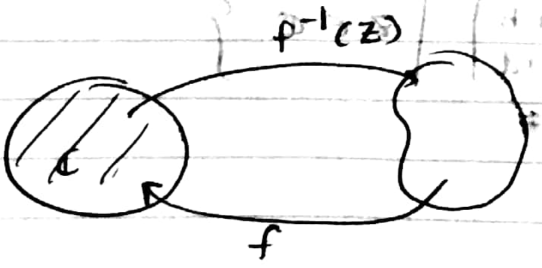
$$\boxed{A \in R^{2\times m}} \left\{ z \,\middle|\, \begin{pmatrix} Az + b \\ c^T z + d \end{pmatrix} \in C \right\} = \left\{ z \,\middle|\, \|Az + b\|_2 \le (c^T z + d) \right\}$$

?

$z \in \mathbb{R}^m$ A and c are different here

$A \in \mathbb{R}^{2\times 3}$





$y \in C$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in C$$

$$\sqrt{y_1^2 + y_2^2} \leq y_3$$

minimize $\underbrace{c^T x + t}_{\text{object}}$ s.t $\|Ax+b\|_2 \leq t$. is this convex?

is affined.

$Ax+b = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + b$

$\uparrow$

inverse of a SOC under an affined function

SOC (C) $\xrightarrow{f(C)}$

affined mapping

$$\begin{bmatrix} A\cancel{x} + b \\ \cancel{c^Tx+d}\ t \end{bmatrix}$$

$\{x \mid \|Ax+b\|_2 \leq t\}$

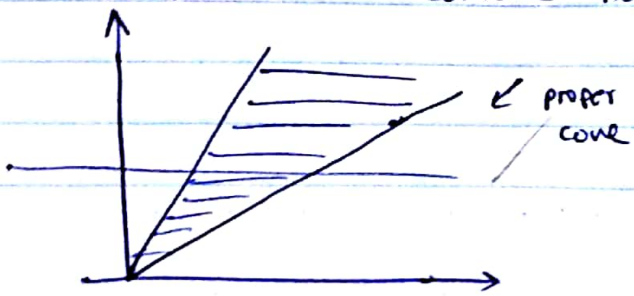
- minimize $(\lambda+1)x + 2$.

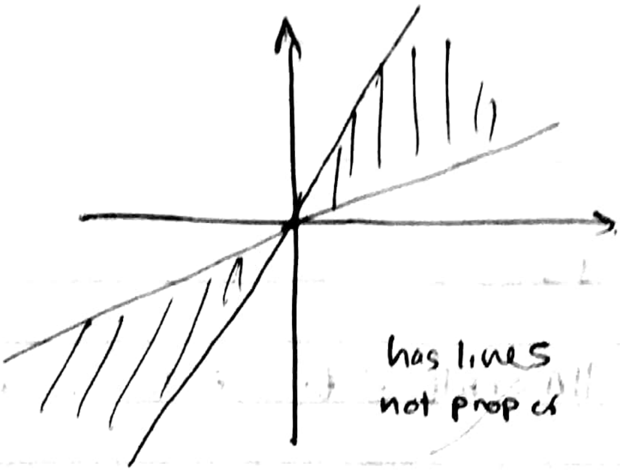
when $\lambda = -1$ optimum value $= 2$.

when $\lambda \neq -1$ optimum value $\to -\infty$

Proper Cone

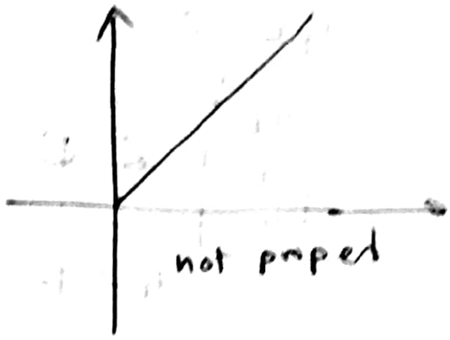
1) K is closed $\to$ cone contains its boundary
2) K is solid $\to$ has non empty interior.
3) K is pointed. $\to$ contains no lines. (has line segments





has lines
not proper

not proper



not proper

no interior
because
This is a line
no amount in $\mathbb{R}^3$

read relative interior $\rightarrow$ in boyd

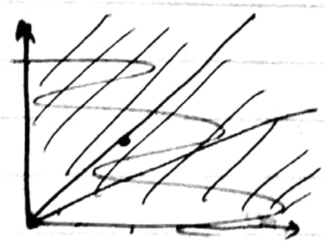
$$K = \{ x \in \mathbb{R}^n \mid x_i \geq 0 \ \forall i \}$$

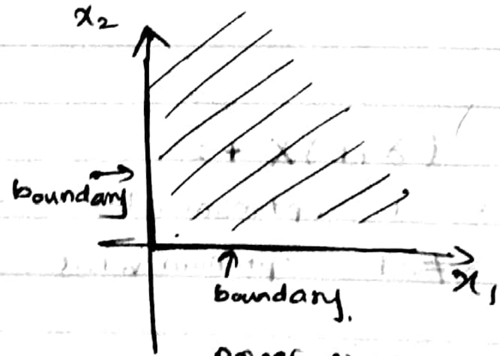
for $n = 2$

$$K = \{ x \in \mathbb{R}^2 \mid x_i \geq 0 \ \forall i \}$$

non negative orthant

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



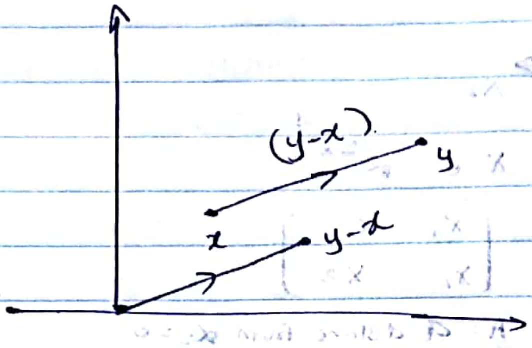


proper cone

- no lines
- pointed
- contains boundary

~~$x \geq y$~~ $\mathbb{R}^n_+$ $\rightarrow$ non negative orthoant

$x \leq_K y \iff (y-x) \in F.$



$\therefore (y-x) \succeq_K 0$

$x <_K y$

$(y-x) \in \text{int}(K)$

interior of K

cone

$X \in S^n$ $\quad$ S = symmetric matrix

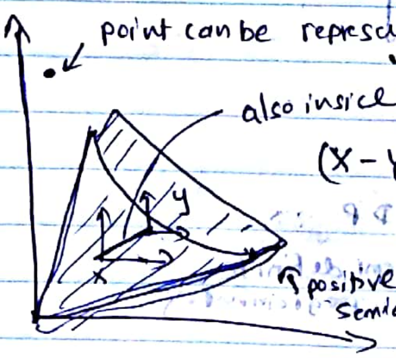
$X \geq_K Y$ $\quad$ K = positive semidefinite cone

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; x \geq 0 \right\} = \{ X \mid X \geq 0 \} = \{ X \mid a^T X a \geq 0 \}$$

$(X-Y) \in K$

$a^T (X-Y) a \geq 0$

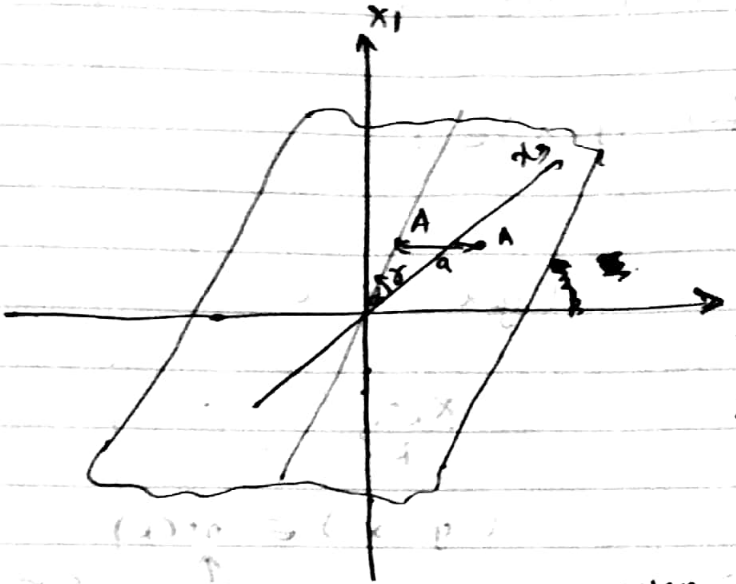
point can be represented by a symmetric matrix



also inside

$(X-Y)$

positive semidefinite cone.



$$A = \begin{bmatrix} r\sin\alpha \\ 0 \\ r\cos\alpha \end{bmatrix}$$

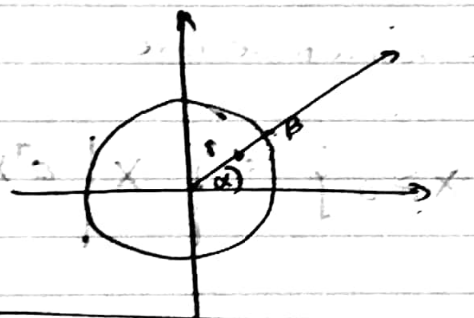
$X \in \mathbb{R}^{2\times 2}$

$$\begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}$$

when $A = a$ distance from $x_2 = 0$

$$\cancel{A =} \begin{bmatrix} \cancel{r\sin\alpha} \\ \cancel{r\cos\alpha} \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} r\sin\alpha \\ a \\ r\cos\alpha \end{bmatrix}$$



maximize r

$a = ?$

$a = 0?$

$\alpha =$

$\sin\alpha \rightarrow$ taylor

$\alpha \rightarrow$ change

$$\boxed{\begin{array}{l} A = \begin{matrix} r\cos\alpha \\ r\sin\alpha \end{matrix} \\ \text{maximize } r \\ \quad \text{subject to } \begin{matrix} r\cos\alpha \\ r\sin\alpha \end{matrix} \in \text{disk} \end{array}}$$

In cone

maximize a

$$\begin{bmatrix} r\sin\alpha \\ a \\ r\cos\alpha \end{bmatrix} \in K_{\text{cone}}$$

SDP

Semi definite programming

$X \succeq_K 0$

s.t $\begin{bmatrix} r\sin\alpha & a \\ a & r\cos\alpha \end{bmatrix} \succeq 0$ positive semidefinite.

* if A is given (A^TA) positive semidefinite
* if X is given which is positive semidefinite all eigen values are non negative
* " " " " positive definite " " " are positive.

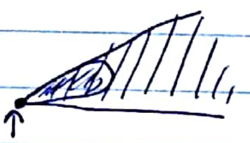
• A → positive definite, quadratic form ~~x^TAx~~ always has curvature up (curved up)
↓
sublevel sets are ellipsoids

$A \in S^n_{++}$ (positive definite), $x^TAx > 0$

$\{x \mid f(x) \le \alpha\}$

A → positive semidefinite, quadratic form → sublevel sets ↓ can contain ~~both~~ cylinders ~~and ellipsoids~~ ~~but~~ not ~~ellipsoids~~

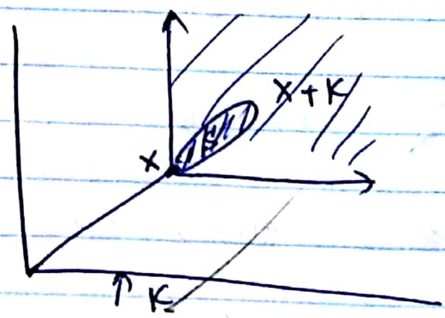
↓ at least one of the eigen values zero
↓ one eigen vector is 0



Minimum point

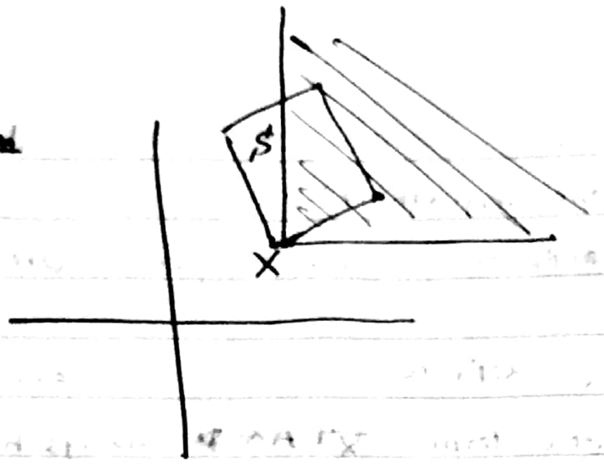
$x \in S$

minimum element of a set S, w.r.t $\preceq_K$.



$S \subseteq x + K$.

x is the minimum point w.r.t $\preceq_K$

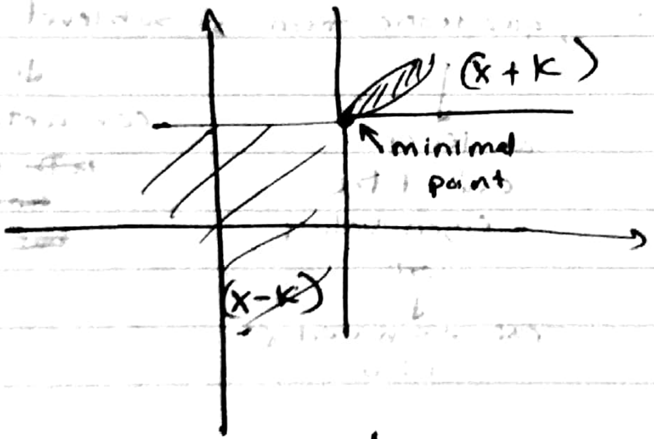


cannot find minimum element becaus S not a subset of

$S \not\subseteq x+K$

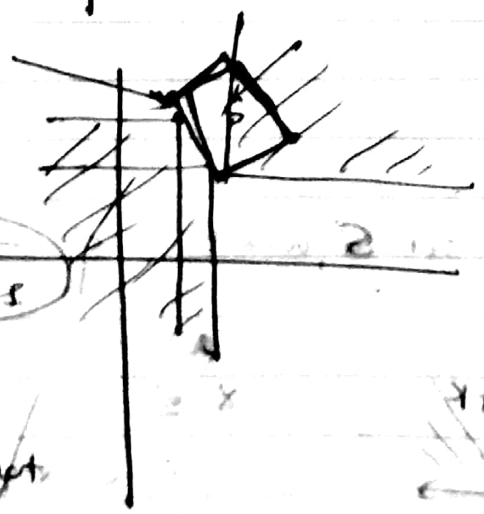
$x \in S$ is a minimal element of S w.r.t $\preceq_K$

$\iff (x-K) \cap S = \{x\}$

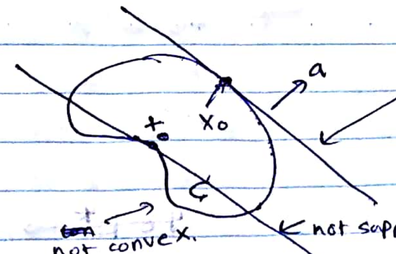


all the boundary points away from x+K are minimal values

but the minimal values cannot be compared



supporting hyperplane to s.t C at boundary x_0



$A = \{x \mid a^T x = a^T x_0\}$

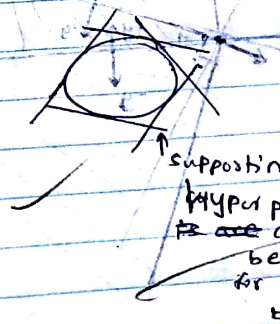
$a^T x \leq a^T x_0 \quad \forall\, x \in C$ and $a \neq 0$

if the set is convex, there is a supporting hyper plane for all boundary points

C convex $\Rightarrow \exists$ a supporting hyper plane at every boundary point of C

- Converse is not true.

circle is not convex. (contain no interior)

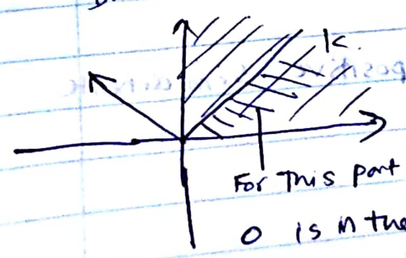


disk is convex but circle (without interior) is not convex.

Dual cone

Dual cone K is given by $K^* = \{y \mid y^T x \geq 0$ for all $x \in K\}$

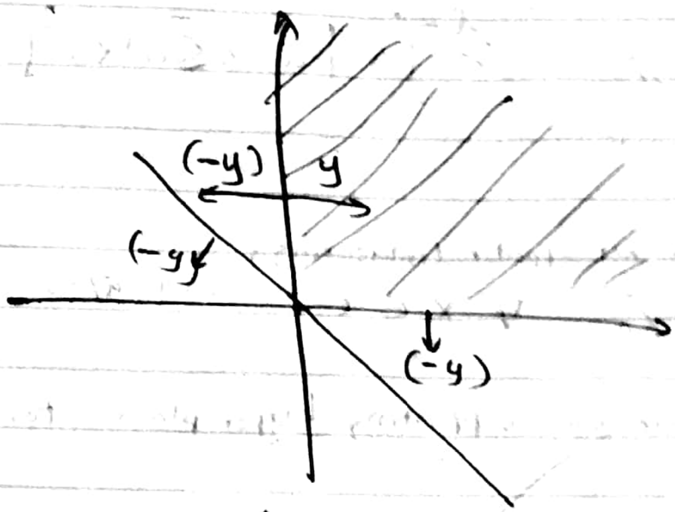
angle between x and y should be below 90°



For this part not satisfied

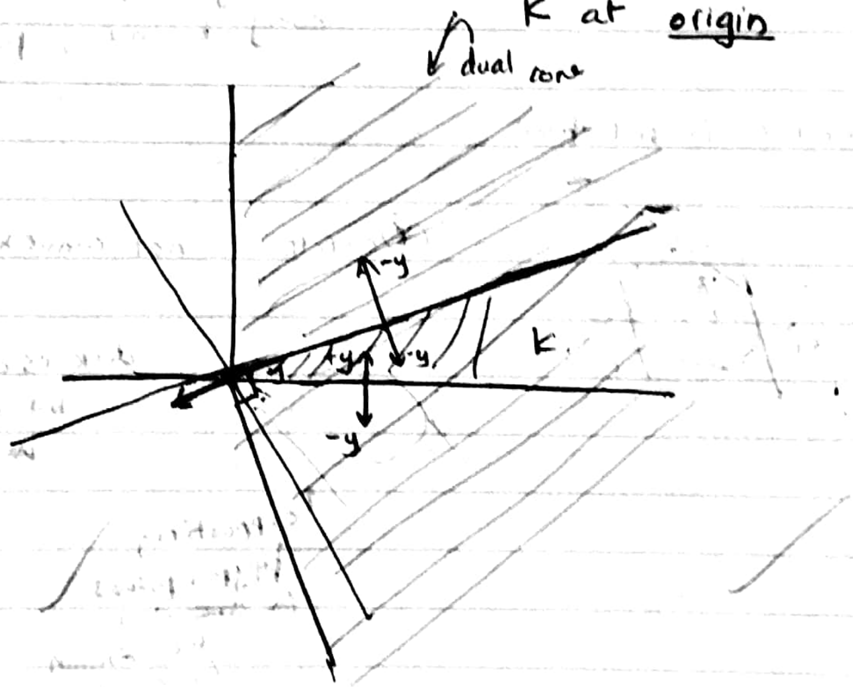
0 is in the dual cone.

$K^* = K$
$\downarrow$
R^n_+ positive orthant itself is the dual cone



~~$y \in K^* \iff$~~

$y \in K^* \iff (-y)$ is the normal of the hyperplane that support K at <u>origin</u>



$$[S_+^n]^* = \{ Y \mid \text{Trace}[XY] \geq 0 \}$$

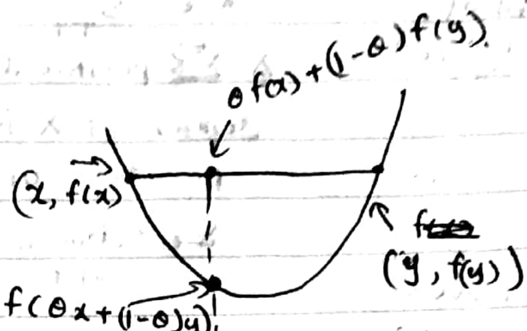
for all $X \in S_+^n$

* dual cone of positive definite ~~matrix~~ cone is the positive semidefinite cone itself

$\text{Trace}(XY) \geq 0$ for all $X \succeq 0$ $\Leftrightarrow$ $Y \succeq 0$

S^n_+

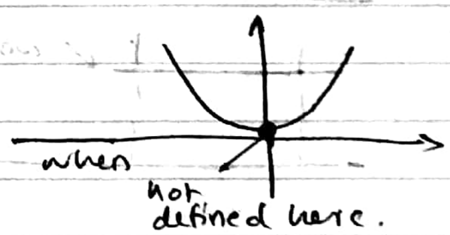
↑ y positive semi definite



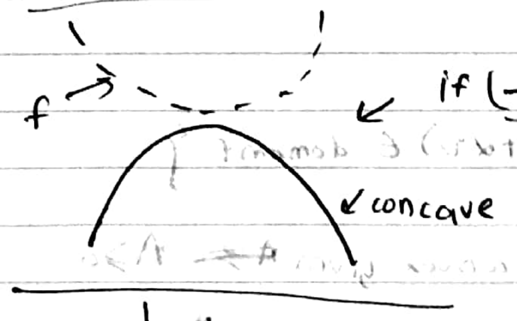
1) fun^n is convex if $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$

2) $[\theta x + (1-\theta)y] \in$ ~~[x, y]~~ domain of f

when the function is not defined at some points



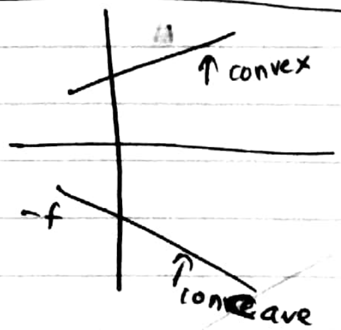
domain f ~~should be~~ is convex



if $(-f)$ is convex, f is concave

$ax + b \rightarrow$ convex

$e^{ax} \rightarrow$ convex



$f(X) = \log(\det(X))$ $\quad$ ~~$X \in$~~ ~~$X \succeq 0$~~ $X \succ 0$

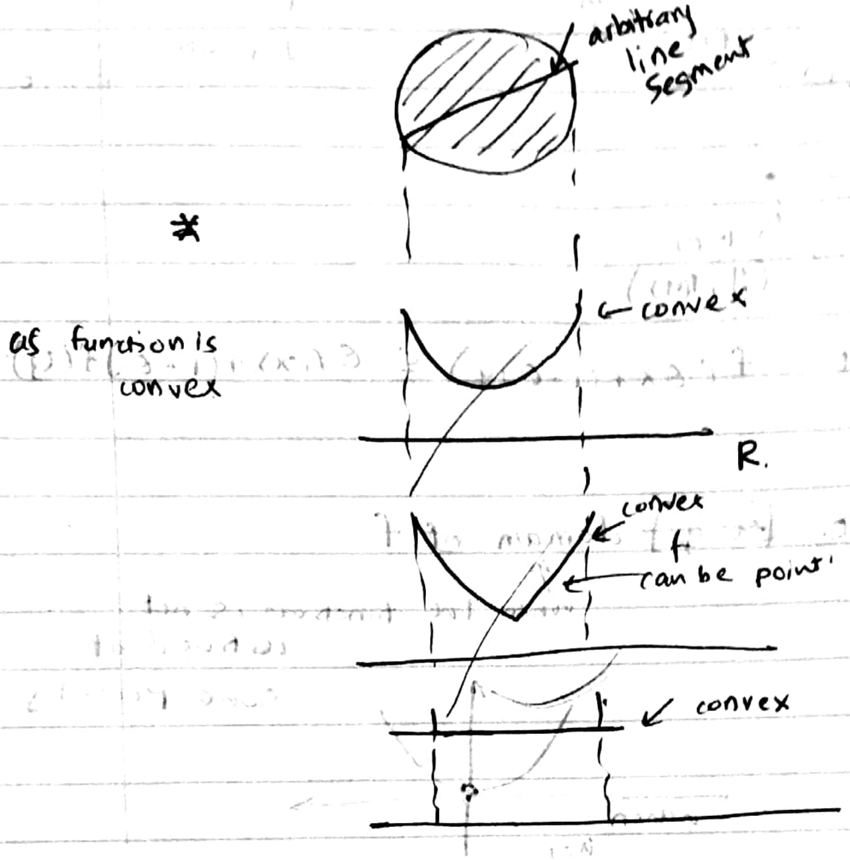
$X \in S^n_{++}$

X ~~$\Rightarrow$~~ positive definite

domain of X is possitive definite.

~~$X \in$~~ domain of f = S^n_{++}

*



$$\{ X \mid X_0 + v\alpha = X, \; \alpha \in \mathbb{R} \}$$

$$\{ X_0 + \alpha v \mid \alpha \in \mathbb{R}, (X_0 + \alpha v) \in \text{dom} f \}$$

$f(X) = X^T A X + q^T X$ is convex given ~~$\Rightarrow$~~ $A \succeq 0$

~~f~~ 1) find a line

$g(t) = f(X_0 + tv)$

(2) show $g(t)$ is convex.

$f(x) = x^T A x + q^T x$

$f(x_0 + tv) = (x_0 + tv)^T A (x_0 + tv) + q^T (x_0 + tv)$

$= (x_0^T + tv^T) A (x_0 + tv) + q^T x_0 + q^T tv$

Date: ..../.........../...........

(4)

$$g(t) = x_0^T A x_0 + t v^T A x_0 + x_0^T A t v + t v^T A t v + q^T x_0 + q^T t v$$

$$= x_0^T A x_0 + t (Av)^T x_0 + t^2 \underbrace{v^T A v}_{\text{scalar } \geq 0 \; p} + q^T x_0 + q^T t v$$

$$\cancel{g(t) = x_0^T A x_0 + ((Av) + q)^T x_0 + t^2 v^T A v + t^2 p + q^T t v}$$

$$\cancel{= x_0^T A x_0 + \; A v + [(A + q^T) v + q]} \quad \cancel{(Av + q + qt)^T}$$

$$= \cancel{x_0^T A x_0 + ((Av) + q)^T x_0 + \underbrace{t q^T v + t^2 p}_{\text{Scalar } V}}$$

$$g(t) = x_0^T A x_0 + [(Av) + q]^T x_0 + V$$

$$g(t) = (x_0 + tv)^T A (x_0 + tv) + q^T (x_0 + tv) \qquad t = \text{scalar}.$$

$$g(t) = (x_0^T + t v^T) A (x_0 + tv) + q^T x_0 + t q^T v$$

$$= (x_0^T A + t v^T A)(x_0 + tv) + q^T x_0 + t q^T v$$

$$g(t) = \underbrace{x_0^T A x_0}_{\text{Scalar}} + \underbrace{t x_0^T A v}_{\text{Scalar}} + \underbrace{t v^T A x_0}_{\text{Scalar}} + \underbrace{t^2 v^T A v}_{\alpha t^2} + \underbrace{q^T x_0}_{\text{Scalar}} + t q^T v$$

$$g(t) = \alpha t^2 + \beta t + \gamma$$

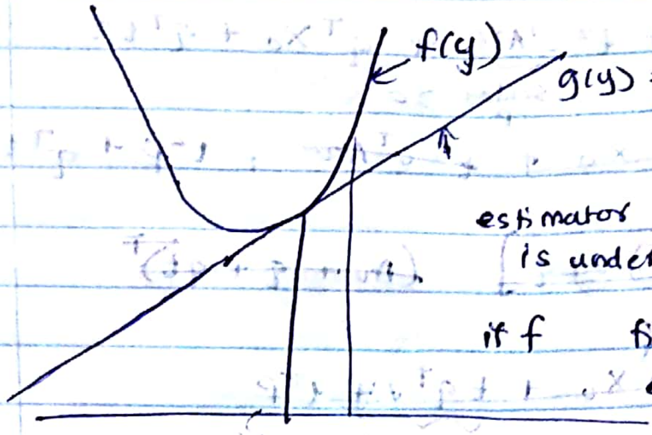
$$\alpha \geq 0$$

$g(t)$ is quadratic.

$\therefore$ $g(t)$ is convex.

$\boxed{g(t) \text{ is convex for all } x_0 \text{ and } V}$

first order convexity



$g(y) = \nabla^2 f(x,y)\,[y-x] + f(x)$

$\forall\ x, y \in$ domain f.

estimator is under function.

if f first order approximator is an under estimator for f

Second order convexity conditions

$\nabla^2 f(x) = P$

Hessian

$\dfrac{\partial f}{\partial x_i \partial x_j}$ $\quad g(t) = \alpha t^2 + \beta t + \gamma$

hessian $\Rightarrow \alpha$

~~$\nabla^2 f(x)$~~ $\quad x_1^2 + 2x_2^2$

$\nabla^2 f(x) = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

$\in \mathbb{R}^{2\times 2}$

symmetric

1) eigen values > 0

2) Hessian should be positive semidefinite.

$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$2x_1 = 2x_1$

$4x_2 = 2x_2$

$x_2 = 0$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 4\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$2x_1 = 4x_1$

vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$I V = V$

V can have any element in $\mathbb{R}$

$V \in \mathbb{R}$

Operations that preserve convexity of a convex function

when f_1 convex f_2 convex

$f_1 + f_2 \rightarrow$ convex

$f_1 - f_2 \rightarrow$ not convex

$g(t) = 2t^2 + 1$

$g_2(t) = 3t^2 + 2$

$f(Ax+b)$ is convex if f is convex.

$f(x) = \|x\|_2^2 = \sqrt{x_1^2 + x_2^2 + \dots}$

affine function. $Az+b$ ~ $g(z)$

$f(Az+b) = \| Az+b \|_2 \Rightarrow$ convex

(4) $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$

from triangle inequality

(*) exam $\rightarrow$ ~~$\|\theta x\|$~~ $\| \theta x + (1-\theta)y \| \leq \|\theta x\| + \|(1-\theta)y\|$

$\| \theta x + (1-\theta) y \| \leq \theta \|x\| + (1-\theta) \|y\|$

$\therefore \|x\|$ is a convex f^n

II Hessian method hard

$f(x) = \|x\|_2^2 = x_1^2 + x_2^2 + x_3^2 + \dots$

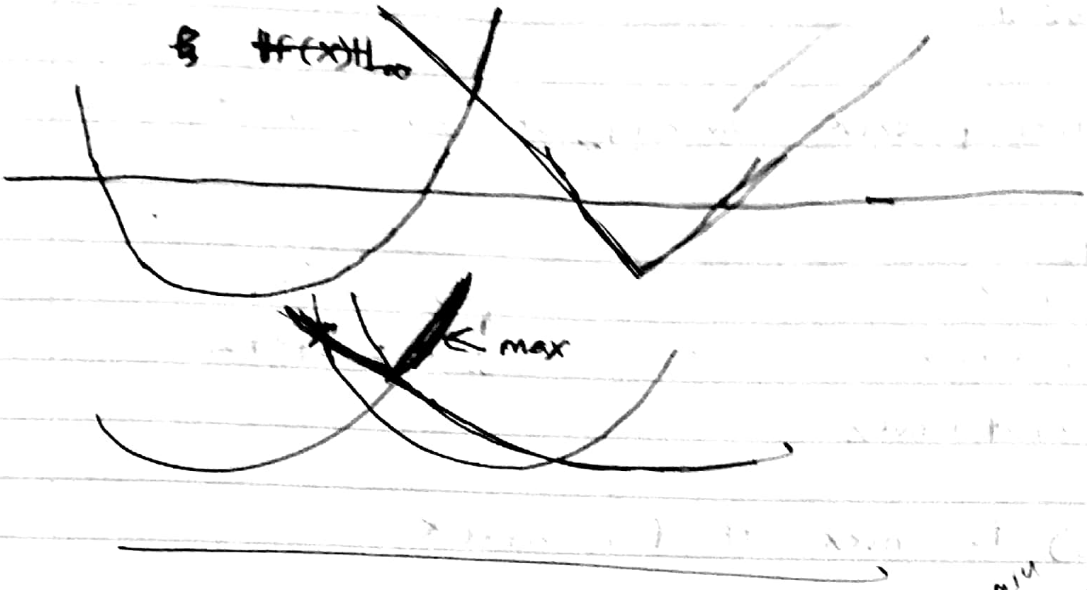
compute ~~$\nabla^2 f(x) =$~~ $h(x) = \sqrt{x_1^2 + x_2^4}$

$\nabla^2 h(x) \geq 0$

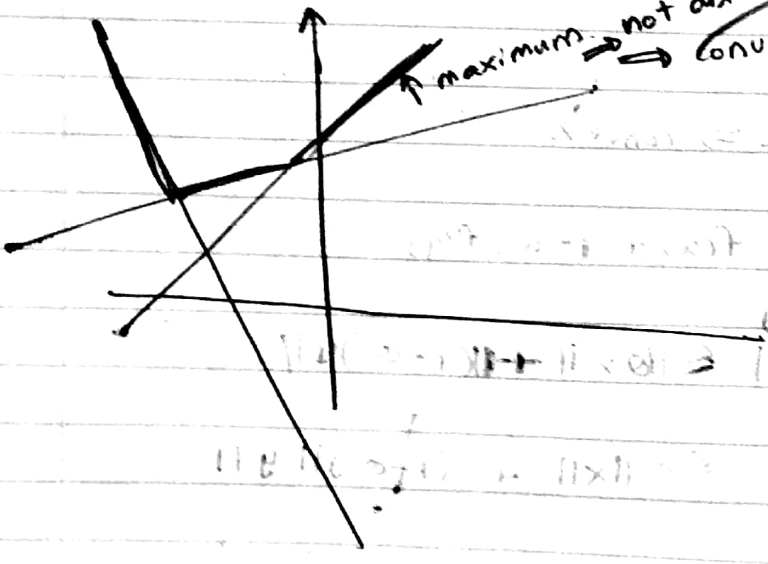
f_1, f_2, f_3 are convex

$f(x) = \max\{f_1(x), f_2(x), f_3(x)\}$

~~$\|f(x)\|_\infty$~~



$f(x) = \max_{i=(1,2,3)} (a_i^T x + b)$



maximum → not differentiable → convex

domain not continuous

sum of 2 largest components of $x \in \mathbb{R}^3$

$f(x) =$

$x \in \mathbb{R}$

$^3C_2$ = three = 3 combinations

$f(x) = \max\{(x_1+x_2), (x_1+x_3), (x_2+x_3)\}$

$\frac{3}{2\times1}$

eg $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ max $\{ (1+2), (1+3), (2+3) \} = 5$

$x = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ max $\{ (1+5), (1+3), (5+3) \} = 8$

is similar to $a_i^T x \leq b_i$ $a_i^T x$ is convex.

e.g. $f(x) \leq 5$

$f(x, y)$ is convex in x for each $y \in A$ ← no constraint that this should be convex or concave.

$f(z)$

$z = \begin{bmatrix} x \\ y \end{bmatrix}$ $g(x) =$

1 o———o 2

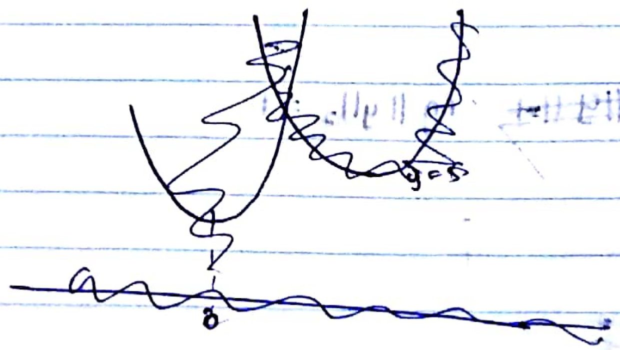
- minimum and maximum doesn't exist
  supremum, infinimum

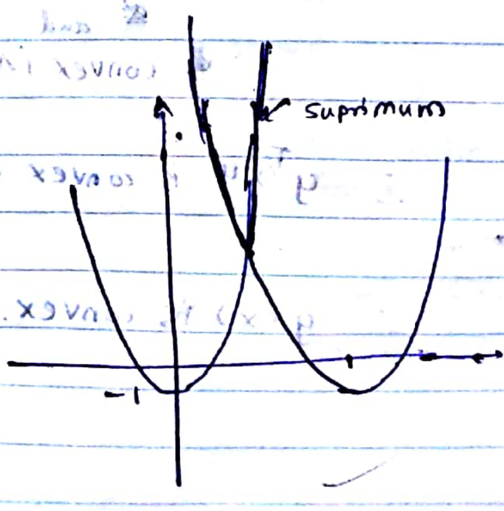
$g(x) = \underbrace{\sup_{y \in (0,5)} (x-y)^2 - 1}_{f(x,y)}$ ← quadratic. $ax^2 + bx + c$ form.

$f(x,y)$ when y is constant.

$g(x)$ is convex

$(x-0)^2 - 1$





suprimum is convex

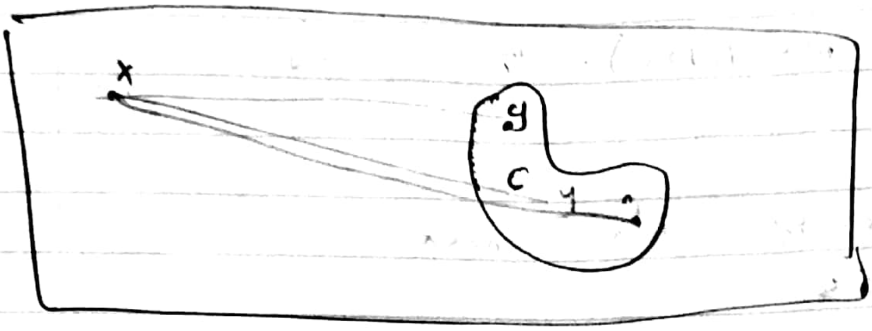
exam

$f(x,y)$ is convex in x for any y, $y \in A$ ← A doesn't have to be convex

$\Rightarrow$ then $\sup f(x,y) = g(x)$ is convex $y \in A$

* theorem

distance to farthest point in set C from x



$$f(x) = \sup_{y \in C} \overbrace{\|x-y\|}^{f(x,y)}$$

$\|x-y\|$ is convex.

$\therefore \sup \|x-y\|$ is convex.

maximize $y^T X y$ s.t. $\|y\|=1$

$$\underbrace{\lambda_{max}(X)}_{g(x)} = \sup_{\|y\|_2=1} \underbrace{y^T X y}_{f(X,Y)}$$

If $f(x,y)$ is convex for specified y (a set)

then the sup $f(x,y)$ is also convex for y (a set)

$y^T X y$ → linear in X

and

convex in X.

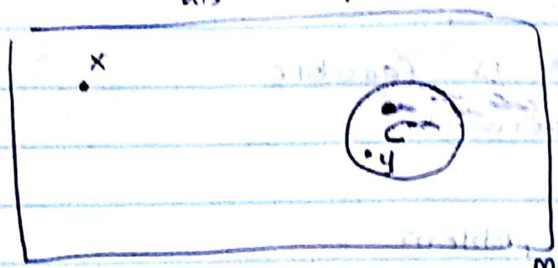
$\therefore y^T X y$ is convex in $\|y\|_2 = 1$

$\therefore g(x)$ is convex.

minimization.

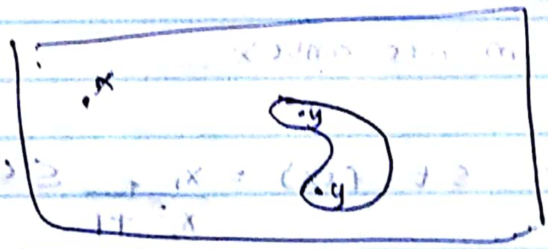
If $f(x,y)$ is convex in (x,y) and C is the convex set
then $g(x) = \inf f(x,y)$, $y \in C$ is convex

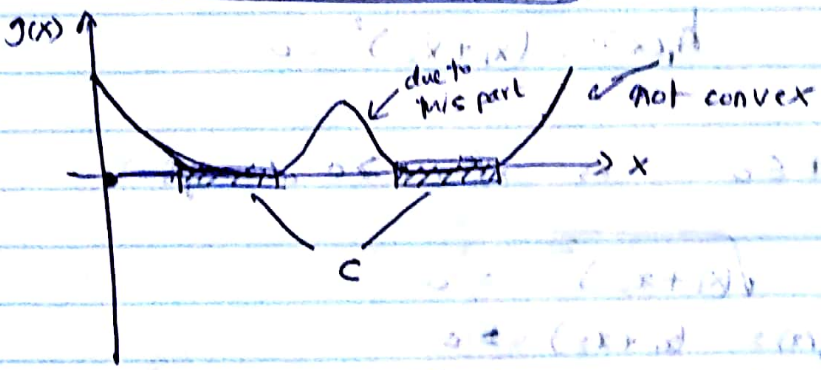
distance to c



$$g(x) = \inf_{y \in C} \|x - y\|_2 = \text{minimum distance to set}$$

$\therefore C$ has to be convex.





minimize $f_0(x)$ s.t $f_i(x) \le 0 \ \forall i$, $h_i(x) = 0 \ \forall i$ — should see whether ~~x is defined~~ f is defined ~~for all~~ for domain of f_i, h_i

(P1) feasibility problem

$$\boxed{\text{minimize} \quad 0 \quad \text{subjected to} \quad \begin{matrix} f_i(x) \le 0 & i = 1, 2, \dots, m \\ h_i(x) \le 0 & i = 1, 2, \dots, p \end{matrix}}$$

$\boxed{x}$ optimization variable

optimal value $= 0$ under condition that x is feasible

optimal value $= \infty$ if x is infeasible

$$p^* = \begin{cases} 0 & \text{if } x \text{ is feasible} \\ +\infty & \text{otherwise.} \end{cases}$$

convex optimization problem

minimize $f_0(x)$

s.t. $f_i(x) \le 0 \quad i = 1, 2, 3, \dots, m$

$a_i^T x = 0 \quad j = 1, 2, \dots, p$

$Ax = b$

$f_i \quad i = 0, 1, 2, \dots m$ are convex

minimize $x_1^2 + x_2^2$ s.t $f_1(x) = \dfrac{x_1}{x_2^2 + 1} \le 0$

$h_1(x) = (x_1 + x_2)^2 = 0$

$x_1 \le 0$ as $x_2^2 + 1 > 0$ always

$\sqrt{h_1(x)} = \sqrt{(x_1 + x_2)^2} = 0$

$g_1(x) = (x_1 + x_2) = 0$

- object is quadratic & convex
- ~~f~~ $x_1 \le 0$ is convex
- $x_1 + x_2 = 0$ ~~affined~~ convex

} convex programming problem

$x_1^2 + x_2^2 \qquad x^T A x + q^T x + r$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$q = 0$

$r = 0$

optimity conditions

read note

equivalent convex problems

2 problems are equivalent if ~~one is~~ ~~ret~~ ~~the~~ the solution of one is readily obtained by the solution of the other and vice versa

minimize $\|x\|^2 + 5$ = min $\|x\|^2$

minimize $\|x\|$ s.t $x \geq 1$, $x \in C$ (1)

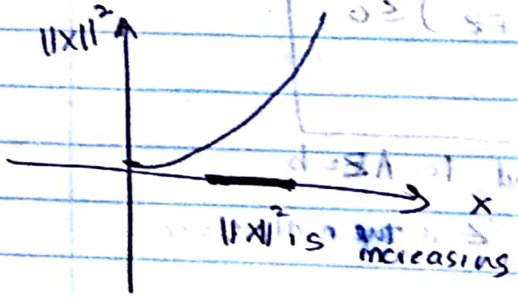
e.g

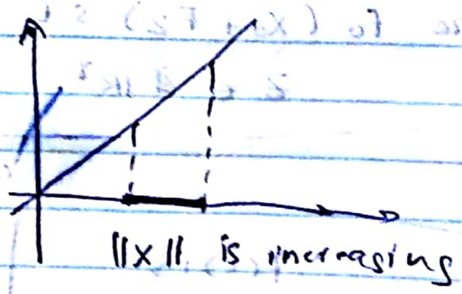
$$\boxed{\text{minimize } \|x\|^2 \quad \text{s.t} \quad x \geq 1 \quad x \in C \qquad x_1^*}$$ $\boxed{x}$ (2)

$$\boxed{\text{minimize } \log\|x\| \quad \text{s.t} \quad x \geq 1 \quad x \in C \qquad x_2^*}$$ $\boxed{x}$ (3)

equivalent because functions are increasing.

(1) and (2) are equivalent





~~$\|x_1^*\| = p^*$~~ ~~$\|x_2^*\|^2 = p^*$~~

$\|x_1^*\|^2 = p^*$ — (1)

$\log\|x_2^*\| = q^*$

$\|x_2^*\|^2 \in q^*$ — (12)

if $\|x_1^*\| < \|x_2^*\|$ q^* can be farther, pushed down

When unconstrained can scale values and obtain the minimized value

$$\boxed{\begin{aligned} &\text{* minimize } && f_0(x) \\ &\text{s.t} && f_i(x) \le 0 \\ & && Ax = b \end{aligned}} \quad x$$

$A x_0 = b$

$\therefore \{ x_0 + v \mid v \in \mathcal{N}(A) \}$

x_0 is a ~~parti~~ particular solution.

$\mathcal{N}(A)$ is a subspace

$\dim \left[\mathcal{N}(A) \right] = r$

$F = \left[f_1 \; f_2 \; f_3 \; \dots \; f_r \right]$

null space $\mathcal{N}(A) = \{ F z \mid z \in \mathbb{R}^r \}$

null vector

span of Z

$\{ x_0 + F z \mid z \in \mathbb{R}^2 \}$

$$\boxed{\begin{aligned} &\text{minimize } f_0(x_0 + Fz) \text{ s.t } f_i(x_0 + Fz) \le 0 \\ &\qquad z \in \mathbb{R}^r \end{aligned}} \quad Z$$

correspond to $AZ = b$

Z is the null space

f(x) is convex

e.g.

$$\boxed{\begin{aligned} &\text{* ~~introduce~~} \\ &\text{minimize } && f_0(A_0 x + b_0) \\ &\text{s.t } && f_i(A_i x + b_i) \le 0 \end{aligned}} \quad x \quad (1)$$

$$\boxed{\begin{aligned} &\text{minimize } && f_0(y_0) \\ &\text{s.t } && f_i(y_i) \le 0 \end{aligned}} \quad (2) \quad y_0, y_1 \dots y_m \;\; x$$

$y_0 = A_0 x + b_0$

$y_i = A_i x + b_i$

equivalent $\Rightarrow$ (1) and (2)

Introducing slack variables

$$\boxed{\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{s.t} & a_i^T x \le b_i \quad i = 1, 2, \dots, m \end{array}} \quad x$$

$$\boxed{\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{s.t} & a_i^T x + s_i = b_i & i = 1, \dots, m \\ & s_i \ge 0 & i = 1, 2, \dots, m \end{array}} \quad s_i \text{ and } x$$

non-negativity ~~introduced~~ constraint ~~to add negativity~~

constraint

introduced to add equality to eq^n

epigraph form

$$\boxed{\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{s.t} & f_i(x) \le 0 \\ & Ax = b \end{array}} \quad x$$

epigraph form

$$\boxed{\begin{array}{ll} \text{minimize} & t \quad \text{which is an upper bound} \\ \text{s.t} & f_0(x) \le t \\ & f_i(x) \le 0 \\ & Ax = b \end{array}} \quad x, t$$

$$\boxed{\begin{array}{ll} \text{minimize} & \max_i \; \underbrace{a_i^T x + b_i}_{\text{convex function}} \\ \text{s.t.} & x \in \mathcal{X} \end{array}} \quad x$$

$$\boxed{\begin{array}{l} \text{minimize } t \quad \text{s.t} \quad \max_i (a_i^T x + b) \le t \\ x \in \mathcal{X} \end{array}} \quad x, t$$

equivalent

$$\boxed{\begin{array}{lll} \text{minimize} & t & \\ \text{s.t} & a_i^T x + b_i \le t & i = 1, 2, 3, \dots, m \\ & x \in \mathcal{X} & \end{array}} \quad x, t$$

1) Linear programming (LP)
2) Quadratic programing (QP)
3) Quadratically constrained QP (QCQP)
4) Second order cone prog (SOCP)
5) Semidefinite prog (SDP)

LP

$$\text{minimize } c^T x \quad \text{s.t } Gx \leq h$$
$$Ax = b$$

$\boxed{x}$

minimize ~~lin~~ plane over polyhedron

constraints are linear, objective f^n is linear

~~AHAT~~

QP

$$\text{minimize } (x^T P x + q^T x + r)$$
$$\text{s.t} \quad Gx \leq h$$
$$Ax = b$$

$\boxed{x}$

minimize ellipsoids (quadratic) over polyhedron.

QCQP

$$\text{minimize } x^T P x + q^T x + r$$
$$x^T P_i x + q_i^T x + r_i \leq 0$$
$$Ax = 0$$

~~P_i = positive de~~

if P_i = positive definite constraints are ellipsoids quadratic

Quadratic constraints
quadratic object.

SOCP

$$\text{minimize } c^T x$$
$$\text{s.t } \|A_i x + b_i\|_2 \leq c_i^T x + d_i \quad i = 1, 2, \ldots, m$$
$$Ax = b$$

SOC constraints

SDP

semi definite cone should be involved.

$$\boxed{\begin{aligned} &\text{minimize} \quad c^T x \\ &\text{s.t} \quad x_1 F_1 + x_2 F_2 + \dots + x_n F_n + G \preceq_{S_+^n} 0 \\ &\qquad Ax = b \end{aligned}}$$

x

have $F_i \in S'^n$

$F(x) \in S^n$

symmetric

if $x \succeq_{S_+^n} 0$

$x \in S_+^n$

$-F(x) \in S_+^n$

$$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \succeq_{S_+^n} 0$$

$$\underbrace{x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{F(x)} \succeq 0$$

$F(x) \in S_+^n$

symmetric matrices

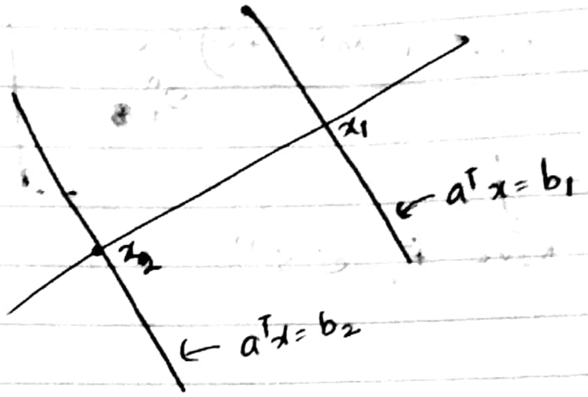
LP → SDP

SOCP → SDP

particular cases of SDP.

Homework

2.5 What is the distance between two parallel hyperplanes $\{x \in \mathbb{R}^n \mid a^T x = b_1\}$ and $\{x \in \mathbb{R}^n \mid a^T x = b_2\}$?



$\cancel{a^T x_1 = b_1}$
a_2

(assuming $b_2 < b_1$)

$x_1 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$x_2 = \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}$

$\cancel{a^T x_1 = b_1}$

$a^T x_2 = b_2$

$\cancel{\|x_1 - x_2\| = (b_1 - b_2)}$

$a^T = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

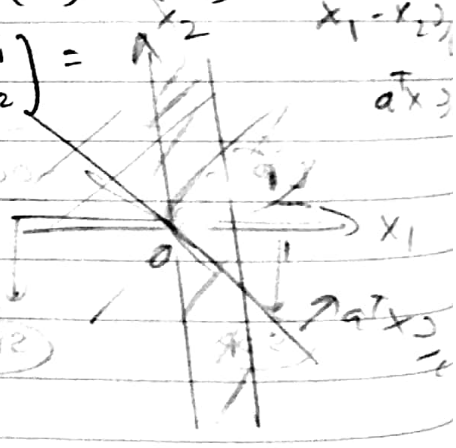
$\cancel{x_1 = (a^T)^{-1} b_1} =$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =$$

20.03.2018

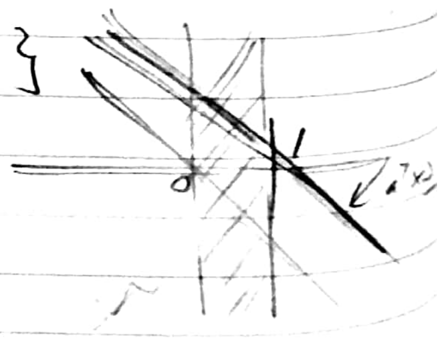
e.g. $S_1 = \{x_1 \mid x_1 \geq x_2,\ x_2 \in [0,1]\}$
$= [0, \infty)$



$S_2 = \{x_1 \mid x_1 \geq x_2,\ x_2 \in (-\infty, \infty)\}$
$= \mathbb{R}$

$S_1 = \{x_1 \mid x_1 \geq x_2$ for all $x_2 \in [1,2]\}$
$\forall$

$S_1 = [1, \infty)$



$S_1 = \{x_1 \mid x_1 \geq x_2\ \forall\ x_2 \in \mathbb{R}\}$

$= \emptyset$

$x_1 - x_2 \geq 1$

$\mathbb{R} \cup \{\infty\}$ = all possibilities
$\uparrow$
does not have ∞

p^*

minimize $f_0(x)$ ← convex

subject to $f_i(x) \le 0$ ← convex $\quad i = 1, 2, \dots, n$

$h_i(x) = 0 \quad i = 1, 2, \dots, p$

↑ affined.

· give a value x

domain $= \mathbb{R}^n$

feasible set

⇓

{ given by constraint }

⇓

minimize
$$f_0(x) + \sum_{i=1}^{n} J_1(f_i(x)) + \sum_{i=1}^{p} J_2(h_i(x))$$

J_1 (= not identity matrix)

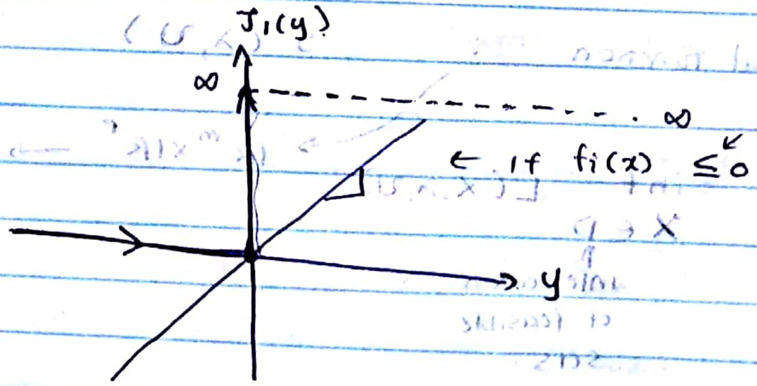
$J_1 : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$

$J_2 : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$

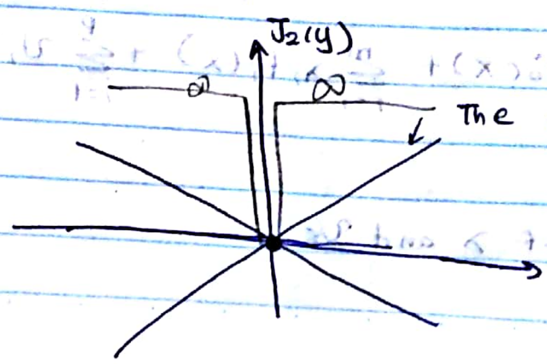
J_1 and J_2 are indicator functions.

$$J_1(y) = \begin{cases} 0 & y \le 0 \\ \infty & \text{otherwise.} \end{cases}$$

$$J_2(y) = \begin{cases} 0 & y = 0 \\ \infty & \text{otherwise} \end{cases}$$



← if $f_i(x) \le 0$ ← positive → use positive slope.



The slope can be positive, negative or 0 minimize $\underbrace{L(x,\lambda,\nu)}_{\text{Lagrangian}} = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$

Langrangian multiplier associated with ~~inequa~~ $f_i(x) \leq 0$ is λ_i.

λ_i is the cost per unit of violation

Langrangian multiplier associated with $h_i(x) = 0$ is ν_i

~~find the~~ results opti... solution p^*

maximize to λ^*

Langrangian machinery $\Rightarrow$ find ~~limits~~ lowerbounds of p^*

$p^* \geq 5$

The optimum value of the problem $\Rightarrow$ not equal to s sometimes $\Rightarrow$ may not be feasible (not ~~under~~ follow the constraints)

$L(x,\lambda,\nu)$ $\quad \lambda \nu \quad$ Lagrangian

Lagrangian dual function : $g(\lambda,\nu)$

$\mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$

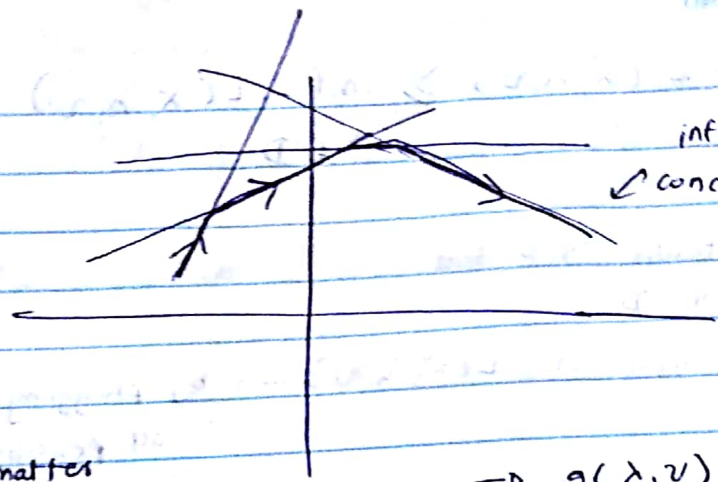
$g(\lambda,\nu) = \inf_{x \in D} L(x,\lambda,\nu)$

D: intersection of feasible sets

$= \inf_{x \in D} \left[f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right]$

L is an affined fun^n of λ and ν.

$\inf_{x \in D} (2\lambda + 5 + x,$
$-3\lambda + 5 + 2x)$



infimum of ~~the~~ Langrangian $L(x,\lambda,v)$ is concave

concave

* doesnt matter f_0, f_i, h_i is convex $\Rightarrow L(x,\lambda,v)$ is affined. $\Rightarrow g(\lambda, v)$ is concave

Langrange dual fn $\Rightarrow$ always concave.

dual problems $\Rightarrow$ can be solved always

Lower bound property

- If $\lambda \geq 0$, then $g(\lambda, v) \leq p^*$
- $p^* \rightarrow$ optimal value of primal problem
- primal problem $\rightarrow$ ~~minimum~~ minimize $f_0(x)$ s.t $f_i(x) \leq 0$, $h_i(x) = 0$ $\rightarrow f_0(x^*) = p^*$

$$L(x, \lambda, v) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} v_i h_i(x)$$

λ_i: scalar

can be a vector

range of f_i is a scalar.

if $\tilde{x}$ is feasible.

$$f_0(\tilde{x}) \geq L(\tilde{x}, \lambda, v)$$

$$L(\tilde{x}, \lambda, v) = f_0(\tilde{x}) + \underbrace{\sum_{i=1}^{m} \lambda_i f_i(\tilde{x})}_{\leq 0} + \sum_{i=1}^{n} v_i h_i(\tilde{x})$$

positive: $\lambda_i \geq 0$, $f_i(\tilde{x}) \leq 0$, $h_i(\tilde{x}) = 0$

$$f_0(\tilde{x}) \geq L(\tilde{x}, \lambda, \nu) \geq \inf_{x \in D} L(x, \lambda, \nu)$$

$\tilde{x} \rightarrow$ one particular ~~not~~ ~~but~~ point in D

~~put al~~ find minimum of $L(\tilde{x}, \lambda, \nu) \rightarrow$ by plugging all feasible $x \in D$

$$\inf L(x, \lambda, \nu) = g(\lambda, \nu)$$

$$f_0(\tilde{x}) \geq g(\lambda, \nu)$$

$\downarrow$ minimizing LHS over all feasible $\tilde{x}$

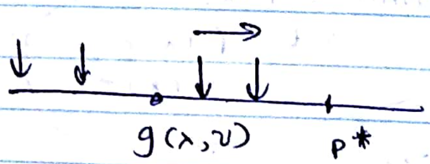
$$g(\lambda, \nu) \leq p^*$$

Lower bound property $\Rightarrow$ if $\lambda \geq 0$ then $g(\lambda, \nu) \leq p^*$

Then to make the $g(\lambda, \nu) \leq p^*$ better $\Rightarrow$ maximize $g(\lambda, \nu)$.

$\downarrow$

get as much as closer to p^*



$\Downarrow$

Dual problem. $\rightarrow$ maximize $g(\lambda, \nu)$

$\lambda \geq 0$

es minimize $x^T x$ ↙ ball → level curves are ~~spheres~~ circles.

subject to $Ax = b$ ↙ affined set → not bounded

$$L(x, v) = x^T x + \sum_{i=1}^{m} v^T (Ax - b) \qquad A \in \mathbb{R}^{m \times n}$$

$$= \underbrace{x^T x + \sum_{i=1}^{m} v_i (a_i^T x - b_i) = R(x)}_{\text{quadratic form}}$$

$$m \times n \cdot n \times 1 = m \times 1$$

↓

$$(x^T P x + q^T x + r)$$

$$= x^T I x + \sum_{i=1}^{m} v_i (a_i^T x - b_i) \qquad \nabla (q^T x) = q$$

find gradient

$$\nabla_x L(x, v) = 2x + A^T v + 0 = 0$$

$$A^T v$$

x for $\nabla_x L(x, v) \rightarrow x = -\frac{1}{2}(A^T v)$

is the point that minimized Lagrangian.

$$g(v) = \inf_{x \in D} (L(x, v)) \leftarrow \text{Minimized of Lagrangian}$$

$$g(v) = -\frac{1}{2} A^T v$$

$$\cancel{L(x,v) = (-\tfrac{1}{2} A^T v)^T (\tfrac{1}{2} A^T v) + v^T (-\tfrac{1}{2} A^T v x - b)}$$

$$L(-\tfrac{1}{2} A^T v, v) = (-\tfrac{1}{2} A^T v)^T (-\tfrac{1}{2} A^T v) + v^T (-\tfrac{1}{2} A^T v - b)$$

$$= \cancel{\tfrac{1}{4} v^T A A^T v + (\tfrac{1}{2})} \; \cancel{(Av)^T v} \; \cancel{v^T A^T v (\tfrac{1}{2} A^T v) - v^T b}$$

$$=$$

$A \in \mathbb{R}^{m \times n}$
$x \in \mathbb{R}^n$
$b \in \mathbb{R}^m$

$$L\left(-\tfrac{1}{2} A^T v, v\right) = \left(-\tfrac{1}{2} A^T v\right)^T x \left(-\tfrac{1}{2} A^T v\right) + v^T \left(A\left(-\tfrac{1}{2} A^T v\right) - b\right)$$

$$= \frac{1}{4}\left(v^T A\, A^T v\right) - \frac{1}{2}\left(v^T A A^T v\right) - v^T b$$

$\frac{1}{4} - \frac{2}{4}$

$$= -\frac{1}{4} v^T A A^T v - v^T b$$

$$v^T b = b^T v$$

v and b are scalars $\rightarrow b \in \mathbb{R}^m, v \in \mathbb{R}^m$

$$L\left(-\tfrac{1}{2} A^T v, v\right) = -\frac{1}{4} v^T A A^T v - b^T v$$

Lowerbound property

$$\underbrace{-\frac{1}{4} v^T (A A^T) v - b^T v}_{} \le p^* \qquad v \in \mathbb{R}^m$$

$g(v)$ $\downarrow$ concave

quadratic $\downarrow$ negative.

$A \rightarrow$ symmetric $\Downarrow$ always positive semidefinite

exam $\rightarrow$ $A^T A \succeq 0$

$x^T A A^T x \ge 0 \quad \forall$

$x^T A x =$

remembered $\rightarrow$ $\|Bx\|_2^2 = \begin{bmatrix} b_1 \\ \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (Bx)^T Bx$

$= x^T (B^T B) x$

$=$

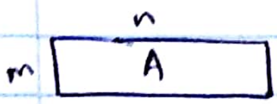
$$\|Bx\|_2^2 = (Bx)^T (Bx)$$

$x^T A A^T x = \cancel{(Ax)^T Ax} \quad (A^T x)^T (A^T x)$

$(A^T x)^T (A^T x) = \cancel{A^T x} \quad \|A^T x\|_2^2 \geq 0 \quad \forall x$

$\therefore \cancel{AA} \quad A^T A \geq 0$

A => positive semidefinite



→ full rank

show $AA^T > 0$ for $\forall x$ except $x = 0$

$x A A^T x > 0$

$(A^T x)^T (A^T x) > 0$

$\|A^T x\|_2^2 \geq 0 \quad \forall x$

$\cancel{Ax = 0}$

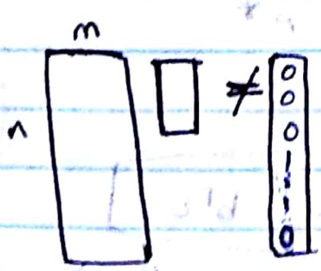
as A is full rank

~~A doesn't have $\mathcal{N}(A)$~~

need to show ~~the~~

$A^T x \neq 0$ for any x

$A^T x = 0$ has non trivial solutions ?



full rank coloumn matrix

A^T is a colour matrix

↓

full rank

↓

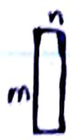
∴ no x not equal to 0 ∃ which makes

$A^T x = 0$

(no non zero x that make $A^T x = 0$)

↓

∴ reason A^T is full column rank matrix



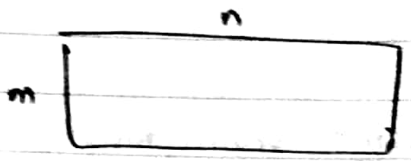
$\Leftrightarrow \mathcal{N}(B) = \{0\}$

B is full rank $n < m$

∴ $A^T x > 0$, for all x not equal to 0

A is fat matrix full rank

$m < n$

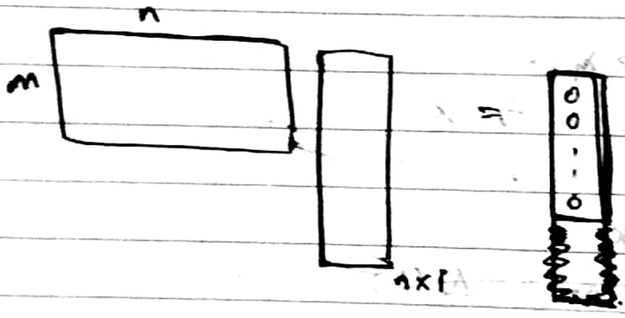


$AA^T \succeq 0$

$x^T A A^T x \geq 0$

$\cancel{(Ax)^T}$ $\quad (A^T x)^T (A^T x) \geq 0$

$\|A^T x\|_2^2 \geq 0 \quad \forall$ all x



← possible

m×1

∴ equal sign in $\|A^T x\|_2^2 \geq 0$ holds

dual problem

$$\underbrace{-\frac{1}{4} v^T (AA^T) v - b^T v}_{g(v)} \leq p^*$$

$$\boxed{\text{maximize} \quad -\frac{1}{2} v^T (AA^T) v - b^T v}$$

← no xs here

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minimize $c^T x$ subject to $Ax = b,\ \underbrace{x \geq 0}_{\text{or } -x \leq 0}$

$L(x, \lambda, v) = \cancel{c^T x + (-x)}$

$= c^T x + \lambda^T(-x) + v^T(Ax - b)$

$L(x, \lambda, v) \cancel{= \{c^T - \lambda^T + (Av)^T\} x - v^T b = 0}$

$\cancel{(c - \lambda + A)^T x}$

$= (c^T - \lambda^T + (Av)^T) x - v^T b$

$L(x, \lambda, v) = (c - \lambda + A^T v)^T x - b^T v$ — (a)

$L(x, \gamma) = 5 + (\gamma + 2)x$

$\inf_x \left[5 + (\gamma+2)x \right] = g(\gamma)$

$g(\gamma) = \begin{cases} -\infty & \gamma \neq (-2) \text{ or } \cancel{\text{other}} \\ 5 & \text{otherwise.} \end{cases}$ $\begin{cases} -\infty & \text{otherwise} \\ 5 & \gamma = -2 \end{cases}$

from (a).

$g(v) = (c - \lambda + A^T v)^T x - b^T v$

$\cancel{g(v) = \{ 0}$

$\cancel{(c - \lambda + A^T v)^T x - b^T v = 0}$

$\cancel{x = (c - \lambda + A^T v)}$

$g(v) = \begin{cases} -\infty & \text{otherwise} \\ -b^T v & \text{if } c - \lambda + A^T v = 0 \end{cases}$

at least one point in $c - \lambda + A^T v$ is non zero

c, λ are vectors

maximize $-b^T v$

Subjected to $\boxed{\begin{array}{l}\cancel{-\lambda \leq 0} \quad \lambda \geq 0 \\ c + A^T v - \lambda = 0\end{array}}$

$c + A^T v \geq 0$

$\boxed{\lambda, v}$ $\boxed{\begin{array}{ll}\text{dual problem} & -b^T v \\ & \cancel{c + A^T v} \quad c + A^T v \geq 0\end{array}}$ ✓

explicitly mention variables

$p^* \geq d^*$: weak duality → always true

optimal value of primal; optimal value of dual.

$p^* = d^*$: strong duality → not always true

Weak duality : Always holds
can be used to find non-trivial lower bound for difficult problems e.g. NP-hard.

Strong duality : Does not hold in general
+(usually) holds for convex problems
- conditions that guarantee strong duality in convex problems are called constrainst qualifications

Slater's constraint qualifications

* strong duality holds for a convex problem

minimize $f_0(x)$

Subject to $f_i(x) \le 0 \quad i = 1, 2, 3, \dots, m$

$Ax = b$

If it is strictly feasible

problem.

x should be in D

~~and~~ the set defined by constraints.

strictly feasible → $\boxed{\text{set defined by constraints should be in D}}$

$\exists\, x \in \text{int}\, D$, such that $f_i(x) < 0$ for all i

interior

and $Ax = b$

[if at least one x satisfies $f_i(x) < 0$ and $Ax = b$]

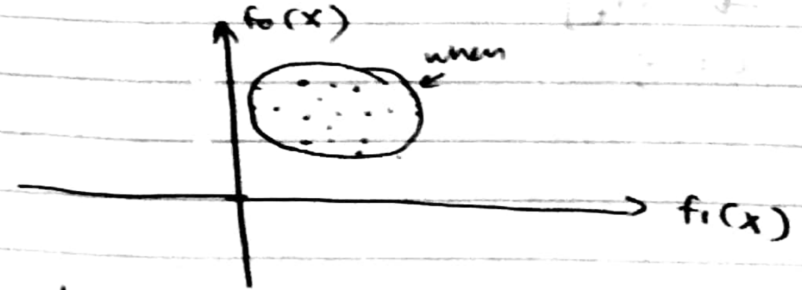
inside ball

strictly means → make x an interior point & ~~so~~ make $f_i(x) < 0$ (only less than) no equality

minimize $f_0(x)$

subject to $f_1(x) \le 0$

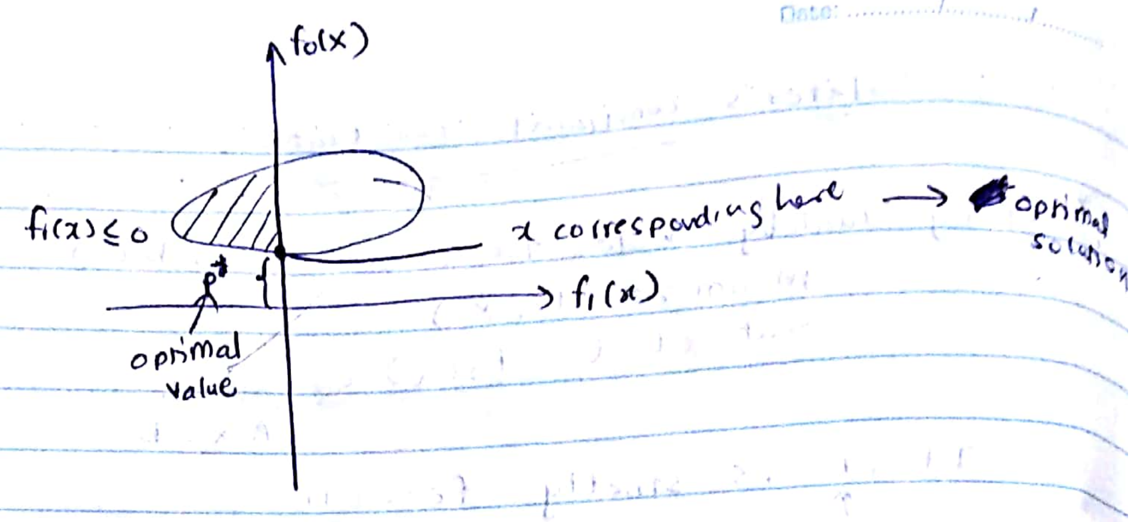
$L(x, \lambda) = f_0(x) + \cancel{x} f_1(x)$

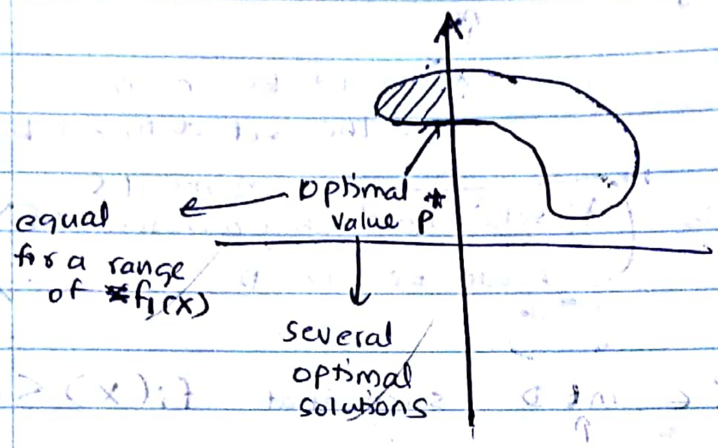


~~$f_0(x), f$~~

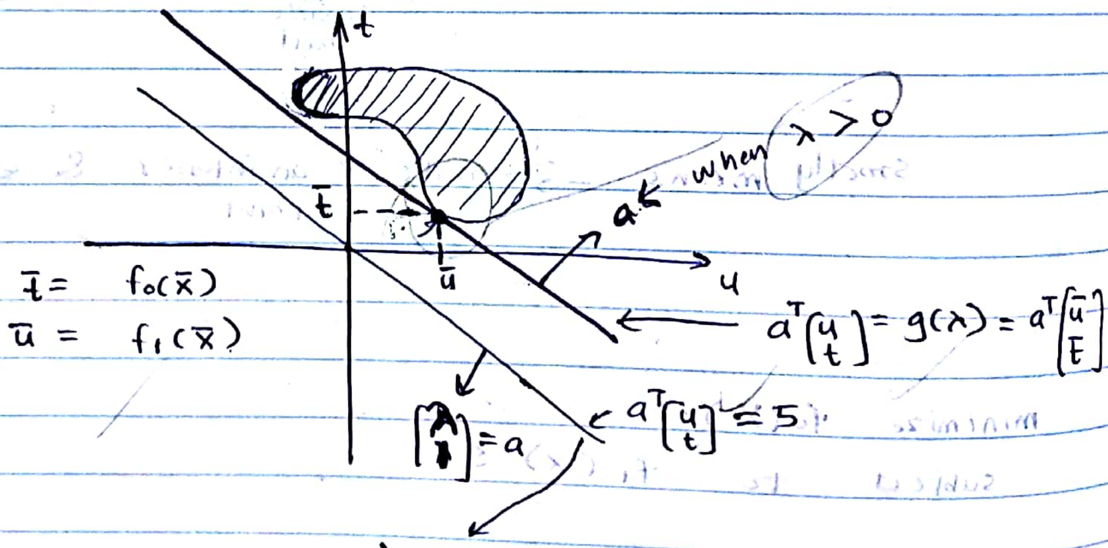
$\{ f_0(x), f_1(x) \mid x \in D \}$

draw $L(x, \lambda)$ for all $x \in \text{int}(D)$





$$g = \{ (u, t) \mid u = f_1(x),\ t = f_0(x),\ x \in D \}$$



$\lambda u + t = 5$ $\lambda \in$ scalars

minimize $t + \lambda u \Rightarrow a^T \binom{u}{t}$

subjected to $(u, t) \in g$

minimize $f_0(x) + \lambda f_1(x)$

$x \in D, \begin{bmatrix} f_0(x) \\ f_1(x) \end{bmatrix} = \begin{bmatrix} u \\ t \end{bmatrix}$ → ~~optimal~~ → $g(\lambda)$

minimized value → langrange dual function.

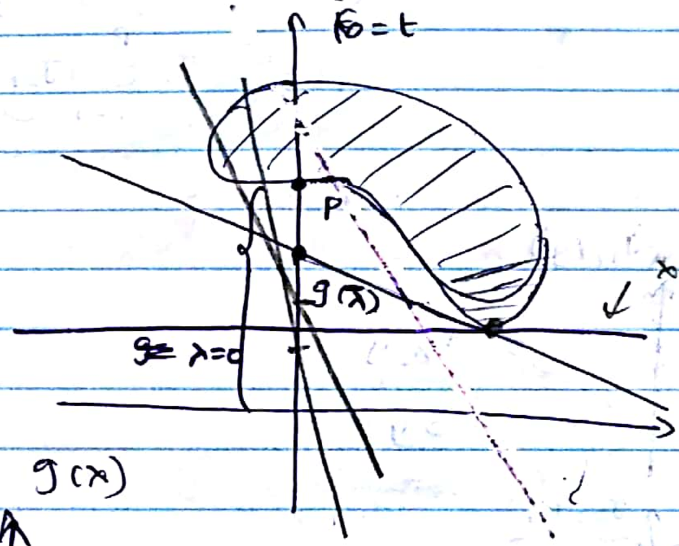
↓

assume for this case sol^n is $\bar{x}$

$a^T \begin{bmatrix} u \\ t \end{bmatrix} = \alpha$

• when you plug $u = 0$

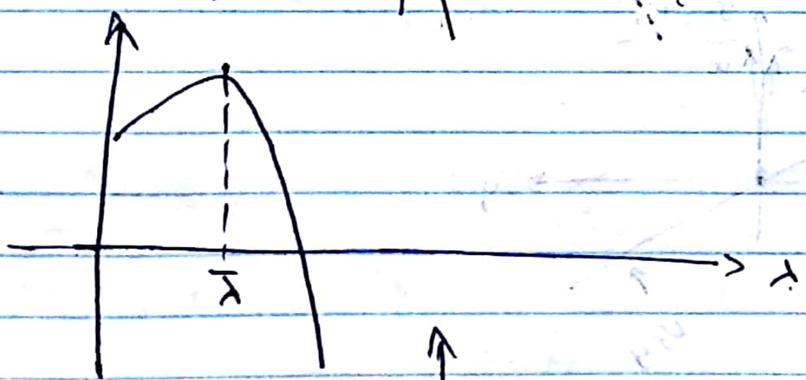
$a^T \begin{pmatrix} 0 \\ t \end{pmatrix} =$

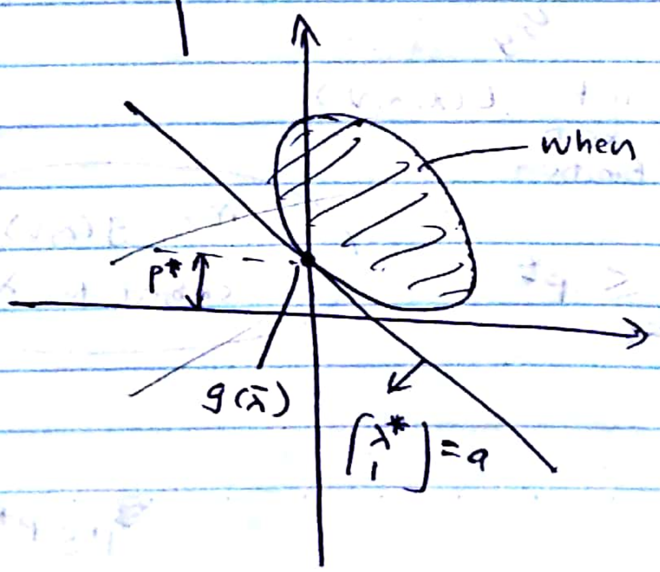


maximizing $g(\lambda)$

↓

finding the particular λ.





$p^* = g(\bar{\lambda})$

strong duality

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01.04.2015

minimize $f_0(x)$

subject to $f_i(x) \leq 0$ $\quad i = 1, 2, \ldots, m$

$h_i(x) = 0$

np-hard

Lagrangian

~~$L: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$~~

$L: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \longrightarrow \mathbb{R}$

$L(x, \lambda, \nu) =$

minimize $f_0(x)$

$$+ \sum_{i=1}^{m} J_{1i}(\underbrace{f_i(x)}_{\lambda_i f_i(x)})$$

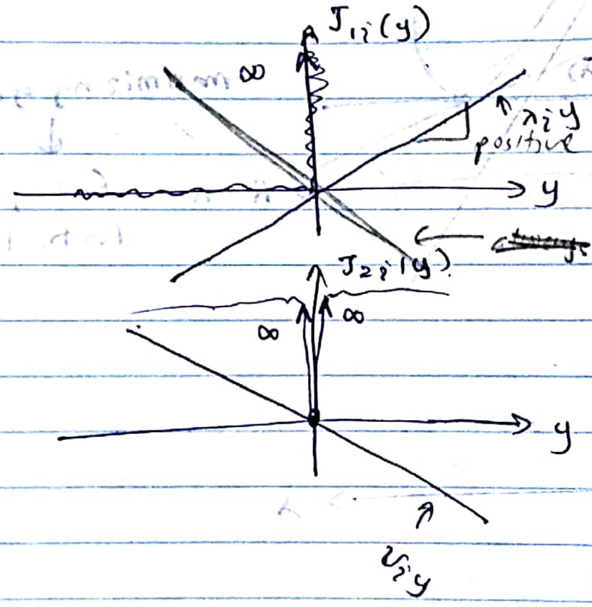
$$+ \sum_{i=1}^{p} J_{2i}(\underbrace{h_i(x)}_{\nu_i h_i(x)})$$

affined func

$Ax + C$

$f_0(x)$

$x \in \mathbb{R}^n$



~~change~~ happy because outside domain $\lambda < 0$ and inside domain $\lambda > 0$ Maximize

cost gets reduced

$d^* = p^* \longrightarrow$ convex problems

$g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu)$

Lagrangian function

always concave

$g(\lambda, \nu) \leq p^*$

lowerbound on p^*

max $g(\lambda, \nu)$

subject to $\lambda \geq 0$ $\longrightarrow$ optimal value d^*

dual problem

$d^* \leq p^*$

cannot minimize further $\longleftarrow$ lower bound

01-04-2018

convex problems → solved easily → lagrange → by decentralization (advantage: find the lower bound)

advantage ↓

divide the sum into parts

whereas primal problem can be harder. → not easy to solve primal.

Constraint qualifications → sufficient conditions for 0 duality.

minimize $c^T x$

subject to $1^T x = 1 \rightarrow 1^T x - 1 = 0$ → $x_1 + x_2 + x_3 + \dots + x_n - 1 = 0$

$x \geq 0$. $-x \leq 0$ ① for which condition need λ

② for which we need ν

$$L(x, \lambda, \nu) = c^T x + \lambda^T(-x) + \nu^T (1^T x - 1)$$

no need

only one ν is needed.

$$= \cancel{(c^T - \lambda^T + (1\nu)^T) x}$$

$$c^T x + \sum_{i=1}^{n} \lambda_i x_i + \nu(1^T x - 1)$$

$$= c^T x + \lambda^T(-x) + \nu(1^T x - 1)$$

① First understand how many equality constraints are there

$$L(x, \lambda, \nu) = (c^T - \lambda^T + \nu 1^T) x - \nu$$

$$= (c - \lambda + \nu 1)^T x - \nu$$

1 matrix

~~L(x~~

$$g(\lambda, \nu) = \begin{cases} -\nu & \text{if } c - \lambda + \nu 1 = 0 \\ -\infty & \text{if otherwise} \end{cases}$$

vector

hyperplane

dual problem

$$\boxed{\begin{array}{l} \text{maximize } -\nu \quad \text{or minimize } \nu \\ \text{subject to } \underbrace{c - \lambda + \nu 1 = 0}\ \text{and } \lambda \geq 0 \\ c + \nu 1 \geq 0 \Rightarrow c \geq -\nu 1 \end{array}}\ \boxed{\lambda, \nu}$$

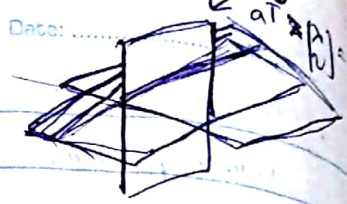
no of constraints → no of constants

$1 \longrightarrow \nu$

$n \longrightarrow \lambda_1, \lambda_2, \dots, \lambda_n$

Intersection of hyperplanes

$a^T \begin{bmatrix} \lambda \\ v \end{bmatrix}$



Date: ......

$c - \lambda + v\mathbb{1} = 0$

$\underbrace{\lambda - v\mathbb{1}}_{\text{matrix}} = c$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \lambda \quad \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda \\ v \end{bmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$

$-v \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{bmatrix} I & -\mathbb{1} \end{bmatrix} \begin{bmatrix} \lambda \\ \mathbb{1} \end{bmatrix} = c$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = -c$

$-v$

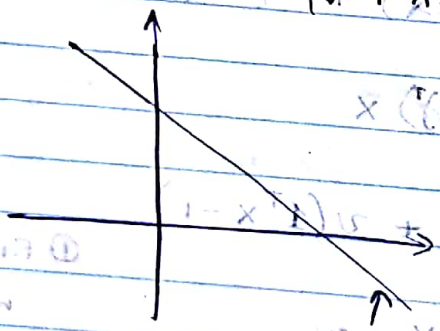
$\underbrace{\begin{bmatrix} I & \vdots & -\mathbb{1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \lambda \\ v \end{bmatrix}}_{\gamma} = -c$

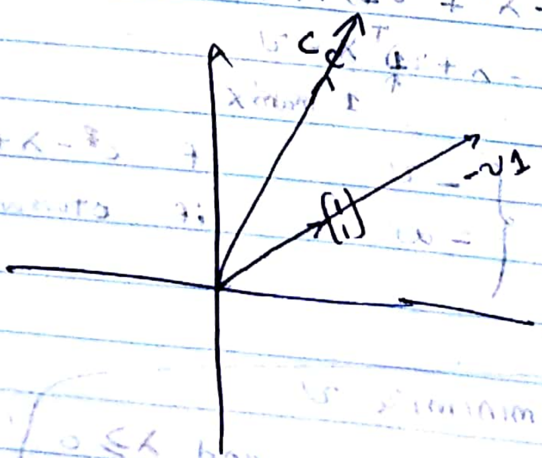
λ_1

λ_2

$-v$

$\{\gamma \mid A\gamma = -c\} = \mathcal{A}$





~~$c \geq v\mathbb{1}$~~

$c \geq -v\mathbb{1}$

$\begin{pmatrix} c_1 + v \\ c_2 + v \\ \vdots \\ c_n + v \end{pmatrix} \geq 0$

$\Downarrow$

$v = -\min_i c_i$

$\begin{pmatrix} 1 & +v \\ 2 & +v \\ -1 & +v \\ 3 & +v \end{pmatrix} \geq 0$

$v = 1 \Rightarrow \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}$

v is known find λ

$$\cancel{v^* = -\min_i c_i}$$ as $c - \lambda + v\mathbf{1} = 0$

$$\cancel{\lambda^* = c + v^*\mathbf{1}}$$

$$\lambda^* = \underset{\uparrow\ \text{known}}{c} + \left[-\min_i c_i\right]\mathbf{1}$$

~~optimal value = $+v^*$~~

~~$d^* = -(v^*)$~~

~~$= v^*$~~

$\uparrow$

~~because maximize v^*~~

but according to standards $d \rightarrow$ maximizing $-v$

$$d^* = \min_i c_i = -v^*$$

$\uparrow$ correspond to maximize $-v$

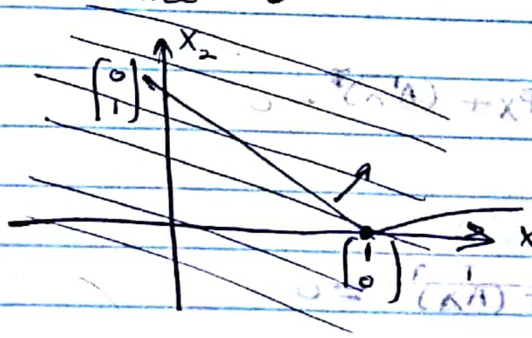
$\downarrow$

$-v^*$

$$\begin{bmatrix} 1 + v \\ 2 + v \\ 1 + v \\ 3 + v \end{bmatrix} \geq 0$$

$\uparrow$ choose v^* to minimize this

$v = -1$



minimize $c^T x$

$\mathbf{1}^T x = 1$

$\lambda \geq 0$

$$c = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$\min_i c_i = 10$$

minimize $\quad x^TPx$ $\qquad$ $P \to$ strictly positive definite

$\Downarrow$

eigenvalues > 0

$\Downarrow$

invertible

subject to $\quad Ax \le b$

$Ax - b \le 0$

$$L(x,\lambda) = x^TPx + \lambda^T(Ax-b) + 0$$

$$= x^TPx + \lambda^TAx - \lambda^Tb$$

$$\cancel{= (x^TP + \lambda^TA)x - \lambda^Tb}$$

$$=$$

$$\cancel{g(\lambda) =}$$

$$g(\lambda) = \inf_{x \in D} x^TPx + \lambda^TAx - \lambda^Tb$$

$$\nabla g(\lambda) = \inf_{x \in D} \nabla(x^TPx) + \nabla(\lambda^TAx) - \nabla(\lambda^Tb)$$

$$= 2Px + (A^T\lambda)^T - 0$$

$$\cancel{\nabla g(\lambda) = 0}$$

$$\cancel{\nabla_x L_x(\lambda) = 2Px + (A^T\lambda)^T = 0}$$

$$\cancel{(A\lambda)^T = -2P}$$

$$\Rightarrow \cancel{2Px + \lambda^TA = 0}$$

$$\cancel{2P = =}$$

$$2Px + A^T\lambda = 0$$

$$Px = -\frac{A^T\lambda}{2}$$

$$x = -\frac{P^{-1}A^T\lambda}{2} \qquad (P^{-1} \text{ exists})$$

$$\cancel{g(\lambda) = 2P\left(-\frac{P^{-1}A^T\lambda}{2}\right) + A^T\lambda}$$

$$\cancel{= -\frac{2PP^{-1}A^T\lambda}{2} + A^T\lambda}$$

$$\cancel{= -\frac{2A^T\lambda}{2} + A^T\lambda}$$

$$\cancel{= 0}$$

$P^T = P$

$\cancel{P^{-T} = P^{-1}}$

$(P^{-1})^T = (P^T)^{-1}$

$= P^{-1}$

$\cancel{L(x,\lambda)}$

$$g(\lambda) = \left(-\frac{P^{-1}A^T\lambda}{2}\right)^T P\left(-\frac{P^{-1}A^T\lambda}{2}\right) + \lambda^T A\left(-\frac{P^{-1}A^T\lambda}{2}\right) - \lambda^T b$$

$$= +\frac{\lambda^T (P^{-1}A^T)^T}{2} P\left(\frac{P^{-1}A^T\lambda}{2}\right) - \frac{\lambda^T A P^{-1} A^T \lambda}{2} - \lambda^T b$$

$$= \frac{\lambda^T A P^{-T} \overbrace{P P^{-1}}^{I} A^T \lambda}{4} - \frac{\lambda^T A P^{-1} A^T \lambda}{2} - \lambda^T b$$

$$= \frac{\lambda^T A P^{-T} A^T \lambda}{4} - \frac{\lambda^T A P^{-1} A^T \lambda}{2} - \lambda^T b$$

$$g(\lambda) = \lambda^T A\left(\frac{P^{-T}}{2} - P^{-1}\right) A^T \lambda - \lambda^T b$$

$$= \lambda^T A\left(\frac{P^{-1}}{2} - P^{-1}\right) A^T \lambda - \lambda^T b$$

$P^{-1} \Rightarrow$ is symmetric

$$= -\frac{\lambda^T A P^{-1} A^T \lambda}{4} - \lambda^T b$$

$$g(\lambda) = -\frac{\lambda^T A P^{-1} A^T \lambda}{4} - b^T\lambda$$

$(A^{-1})^T = (A^T)^{-1} = (A)^{-1}$

$A^{-T} = A^{-1}$ (A is symmetric)

inverse is also symmetric

$\lambda^T b = \cancel{b\lambda^T}\ b^T\lambda$

$$\text{maximize} \quad -\frac{\lambda^T A P^{-1} A^T \lambda}{4} - b^T \lambda$$

subject to: $\lambda \geq 0$

Slater's constraint qualification

primal problem (P) strictly feasible $\Longrightarrow$ strong duality hold for (P) (convex)

$f_0 \cap f_1 \cap f_2 \rightarrow D \cap D = \text{feasible set}$

(D: domain; D: set defined by constraints)

$h_i(x) \leq 0$

$Ax - b = 0$

Should choose a point in interior of the feasible set

$\exists \; x \in \text{int } D$

In general: (Strong duality holds) and (x, λ, ν) are optimal $\Rightarrow$ KKT condition hold.

(x: for primal problem; λ, ν: for dual problem)

KKT conditions are necessary $\rightarrow$ x to be optimal and λ

$P \Rightarrow Q$ (P: father, Q: male)

P is a sufficient for Q

Q is necessary for P

Date: ..../........../..........

for convex problems : strong duality holds and (x, λ, ν) are optimal $\Longleftrightarrow$ KKT condition holds.

KKT conditions (Karush - Kuhn - Tucker)

The following 4 conditions are called KKT conditions (for a problem with differentiable f_i, h_i)

1) primal feasibility : $f_i(x) \leq 0 \quad i = 1, 2, \ldots, m$
$h_i(x) = 0 \quad i = 1, 2, 3, \ldots, p$

2) dual feasibility : $\lambda \geq 0$

3) complimentary slackness : $\lambda_i \underbrace{f_i(x)}_{=0} = 0, \quad i = 1, 2, \ldots, m$

4) gradient of L w.r.t. x vanishes : $\nabla_x L(x, \lambda, \nu) = 0$

$$\nabla_x f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x) = 0$$

Primal problem $\Rightarrow$ minimize $c^T x$

subject to $1^T x ~~-~~ = 1$
$x \geq 0$

① primal feasibility
$\cancel{1^T x}\ -x \leq 0$
$1^T x - 1 = 0$

② dual feasibility : $\lambda \geq 0$

③ complementary slackness
$\cancel{\lambda_i f_i(x) = 0}$ $\quad \lambda_i x_i = 0$
$\cancel{f_i(x) = 0}$

④ gradient of L w.r.t X vanishes.

$$L(x,\lambda,v) = c^T x - \lambda^T x + v(1^T x - 1)$$

$$L(x,\lambda,v) = (c - \lambda + v1)^T x - v$$

$$\nabla L(x,\lambda,v) = (c - \lambda + v1) = 0$$

∴ from ②

$$c - \lambda + vI = 0$$

$$c + vI = \lambda \geq 0$$

$$c + v1 \geq 0$$

$$\begin{pmatrix} c_1 + v \\ c_2 + v \\ \vdots \\ c_n + v \end{pmatrix} \geq 0$$

Select V for minimum $c + V1$

- usually we cannot give a ~~clear~~ closed form sol^n for KKT

exam → give solution x^*, λ^* and v^* then check whether KKT holds. solution for primal problem.

$$X = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

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complementary slackness

Suppose strong duality holds and x^*, λ^*, u^* are primal and dual optimal.

(3) $\Rightarrow$ complimentary slackness $\rightarrow$

1st step = $f_0(x^*) = g(\lambda^*, v^*)$

$$= \inf_x L \; f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} v_i^* h_i(x)$$

don not have to be x^*

minimize L w.r.t x

$$\leq f_0(\bar{x}) + \sum_{i=1}^{m} \lambda_i^* f_i(\bar{x}) + \sum_{i=1}^{p} v_i^* h_i(\bar{x})$$

plug. an arbitrary $\bar{x}$

pick $\bar{x} = x^*$

$$\leq f_0(x^*) + \underbrace{\sum_{i=1}^{m} \lambda_i^* f_i(x^*)}_{\leq 0} + \underbrace{\sum_{i=1}^{p} v_i^* h_i(x^*)}_{=0 \text{ because feasible}}$$

$f_0(x^*) \leq f_0(x^*)$

replace all inequalities with equality

$$f_0(x^*) = f_0(x^*)$$

Then $\sum_{i=1}^{m} \lambda_i^* f_i(x^*) = 0$

$$\lambda_i^* f_i(x^*) = 0$$

(4) Gradient vanishes at $f_0(x^*)$

$$\inf \underbrace{(x-1)^2 + 1}_{f(x)}$$

$$1 = f(x^*)$$

• lines reserved number draft • petition form signed by US IEEE section

registration business registration university of pasadena registration

recover primal solution from the dual.

minimize lagrangian w.r.t λ^*, ν^*.
get something

exam → quadratic forms → finding x^*

* For all problems KKT doesn't hold.

SDP: semidefinite problem

minimize $c^T x$ ← semidefinite cone, matrix inequality

Subject to $x_1 F_1 + \dots + x_n F_n \preceq G$ $(x_1, x_2, \dots, x_n)$ $\underbrace{\quad}_{X}$

S_+^n

$F_i \in S^k$

$G \in S^k$

x_i is a scalar

US 2008 universal registered wire

bank or bank draws

symmetric ~~mat~~ matrix.

• Dual variable: matrix ~~symbo~~ symmetric $Z \in S^k$

$$L(x, z) = c^T x + \text{Trace}\left[Z^T \left(\underbrace{\sum_i x_i F_i - G}_{\text{Linear matrix}} \right) \right]$$

choose Z from S_+^k ↓ symmetric matrices

symmetric $Z^T = Z$

x_i = scalars

inner product of A, B

~~= Trace(A,B)~~

$= \text{Trace}(A^T B)$

$= \sum_i \sum_j A_{ij} B_{ij}$

Scalars

$$= c^T x + \text{Tr} \sum_i (x_i Z F_i) - \text{Tr}(ZG)$$

$$= \sum_{i=1}^n c_i x_i + \cancel{x_i} \sum_{i=1}^n x_i \,\text{Tr}(Z F_i) - \text{Tr}(ZG)$$

$$= \sum_{i=1}^n c_i x_i + \sum_{i=1}^n x_i \,\text{Tr}(Z F_i) - \text{Tr}(ZG)$$

$$L(x, z) = \sum_{i=1}^n \left[\cancel{\sum_{i=1}} c_i + \cancel{\sum_{i=1}} \text{Tr}(Z F_i) \right] x_i - \text{Tr}(ZG)$$

$$g(z) = \inf_x L(x,z) = \begin{cases} -Tr(ZG) & \text{if } [c_i + Tr(ZF_i)] = 0 \\ -\infty & \text{if otherwise} \end{cases}$$

$$\boxed{\begin{array}{ll} \text{maximize} & -Tr(ZG) \\ \text{subjected to} & c_i + Tr(ZF_i) = 0 \quad \forall i \\ & Z \succeq 0 \end{array}}$$

$ZG = GZ$ when dimensions match

Z has to be positive semidefinite

~~maximize X~~

~~subject to X_0 $X = r\cos\alpha$~~

maximize X +

minimize ~~maximize~~ $-X$

subject to $\begin{bmatrix} X_0 & r\cos\alpha \\ r\cos\alpha & r\sin\alpha \end{bmatrix} \succeq 0$

Z should be symmetric $Z \in \mathbb{R}^{2\times 2}$

$$L(X, Z) = -X + \text{Trace } Z^T \left(\sum_i x_i F_i - G\right)$$

X

$x_1 = X$

$x_2 = r\cos\alpha$

$x_3 = r\sin\alpha$

$$* \quad \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{F} \begin{pmatrix} X_0 \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & r\cos\alpha \\ r\cos\alpha & r\sin\alpha \end{pmatrix}}_{G} \succeq 0$$

put (−) to get criteria ≤ 0

$$= -X + \text{Trace } Z^T - (X F_1 + G)$$

~~$= -X - \text{Trace } XZF$~~

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$= -x - x \, Tr(F, Z) - Tr(G, Z)$

$= \cancel{x - x}$

$L(x, Z) = -x \left[1 + Tr(F, Z) \right] - Tr(G, Z)$

Dual function $= g(y) = \inf_x L(x, z) = - Tr(G, Z)$ if

$$g(y) = \begin{cases} -Tr(G, Z) & \text{if } 1 = -Tr(F, Z) \\ -\infty & \text{if otherwise} \end{cases}$$

$$\boxed{\begin{array}{l} \text{Maximize} \quad -Tr(G, Y) \\ \text{Subjected to} \quad 1 = -Tr(F, Z) \\ \qquad\qquad Z \succcurlyeq 0 \end{array}}$$

positive semidefinite

curly inequality

KKT conditions

① primal feasibility

$\cancel{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$ $\quad F + G \succcurlyeq 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & r\cos\alpha \\ r\cos\alpha & r\sin\alpha \end{pmatrix} \succcurlyeq 0$$

positive semi definite

② dual feasibility $- x \quad Z \succcurlyeq 0$

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④ gradient of L w.r.t X $= 0$

$$1 + Tr(F_1 Z) = 0 \qquad F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$-\,\text{Trace}\left[\left[\begin{pmatrix} 1 & \cancel{r\cos\alpha}\ 0 \\ \cancel{r\cos\alpha}\ 0 & \cancel{r\sin\alpha}\ 0 \end{pmatrix}\right] Z\right] = 1$$

$\lambda_i f_i(x) = 0$

③
$$\begin{bmatrix} x_0 & r\cos\alpha \\ r\cos\alpha & r\sin\alpha \end{bmatrix} Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix}$$

general way

$$\left[\sum_{i=1}^{n} x_i F_i - G\right] Z = 0$$

$\underbrace{\quad}_{\text{like} \to f_i(x)}$ λ

$$\text{Trace}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}\right] = 1$$

$$\text{Trace}\begin{bmatrix} z_1 & 0 \\ 0 & 0 \end{bmatrix} = -1$$

$$z_1 = -1$$

- $x_{optimal} = \dfrac{r\cos^2\alpha}{\sin\alpha}$

What are algorithms used to solve convex ~~functions~~ problems

↓

\# Gradient ~~method~~ algorithm

\# newton algorithm

\# Barrier method

gradient → can apply → unconstrained optimization

newton → unconstrained optimization + equality constraints $Ax = b$

barrier → inequality constraints + equality constraints + unconstrained.

gradient $\rightarrow f_0(x)$
newton $\rightarrow f_0(x)$, $Ax = b$
barrier $\rightarrow$ $f_0(x)$, $Ax = b$, $f_i(x) \leq 0$

- ultimately
- Sequence of $Ax = b$ are solved in all problems

Skype : chathu_pc   0719242761.

Homework

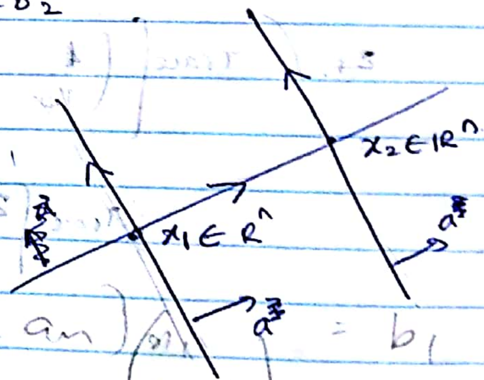
(2.5) What is the distance between two parallel hyperplanes $\{x \in \mathbb{R}^n \mid a^Tx = b_1\}$ and $\{x \in \mathbb{R}^n \mid a^Tx = b_2\}$?

Parallel means $\Rightarrow$ a^T same.

$a^Tx = b_1$ —(1) $a^Tx = b_2$ —(2)

$a^Tx_1 = b_1$ $a^Tx_2 = b_2$

~~vector between~~ $\cancel{\vec{x_1x_2} = (a^Tx_2 - a^Tx_1)}$



distance $= \dfrac{x_2 - x_1}{\|x_2 - x_1\|}$

$x_1 = \left(\dfrac{b_1}{\|a\|_2^2}\right) \cdot a$

$(a_1 \; a_2 \; \dots \; a_n)$ $= b_1$

~~unit distance along~~

$\vec{x_1} = \cancel{\|x_1\| \cdot \dfrac{a}{\|a\|}}$

$= \cancel{\dfrac{b_1}{a}}$

$(a_1x_1 + a_2x_2) = (b_1)$

x_1

a

$[a_1 \quad a_2]$

$a_{n \times 1}$, $a_{b_2} - n^{\times}$

$a_{n \times 1}$

$a^T a = a a^T$

$a^T a = a . a^T$

$a^T x_1 = b_1 a$

$\cancel{a a^T x_1 = a b_1}$

$\cancel{\|a\|_2^2 x_1 = a b_1}$

$\cancel{x_1 = \frac{a b_1}{\|a_2\|^2}}$

$\cancel{a^T x_1 a = b_1 . a}$

$\cancel{x_1 a^T a}$

$a^T . a$

$a^T x_1 a = b_1 . a$

$x_1^T a^T a = b_1 a$

$x_1^T = b . a$

Homework

What is the distance between two parallel hyperplanes $a^T x = b_1$ and $a^T x = b_2$, $x \in \mathbb{R}^n$?

① $a^T x_1 = b_1$

$\underline{x_1} = c \, \underline{a}$ ← scalar.

$a^T c \underline{a} = b_1$

$c \, a^T \underline{a} = b_1$

$c \|a\|_2^2 = b_1$

$c = \frac{b_1}{\|a\|_2^2}$

$\underline{x}_1 = \frac{b_1 . \underline{a}}{\|a\|_2^2}$ $\quad \langle u, v \rangle = \langle v, u \rangle$

similarly $\underline{x_2} = \frac{b_2 . \underline{a}}{\|a\|_2^2}$

$(x_2 - x_1) = \frac{(b_2 - b_1) . \underline{a}}{\|a\|_2^2}$

~~take absolute on both sides.~~

$\cancel{\|x_2 - x_1\| = |b_2 - b_1| a}$

$(x_2 - x_1)^T \cancel{(x_2 - x_1)} = \frac{((b_2 - b_1) a)^T}{\|a\|_2^2} \frac{(b_2 - b_1) a}{\|a\|_2^2}$

$2\times1 \quad 1\times2$

$\begin{pmatrix}1\\2\end{pmatrix} \begin{pmatrix}1 & 2\end{pmatrix}$

$\|x_2 - x_1\|_2^2 = \frac{(b_2 - b_1)^2 \; a^T a}{(\|a\|_2^2)^2}$ $\quad a^T a$

Take square root of both sides

$\|x_2 - x_1\|_2 = \frac{|b_2 - b_1|}{\|a\|_2^2}$ $\quad \begin{pmatrix}1 & 2\end{pmatrix} \begin{pmatrix}1\\2\end{pmatrix}$ $1\times2 \quad 2\times1$

Take absolute of both sides

$\|x_2 - x_1\|_2 = \frac{|b_2 - b_1| \, \|a\|}{\|a\|_2^2 \, \|a\|_2}$

1+4

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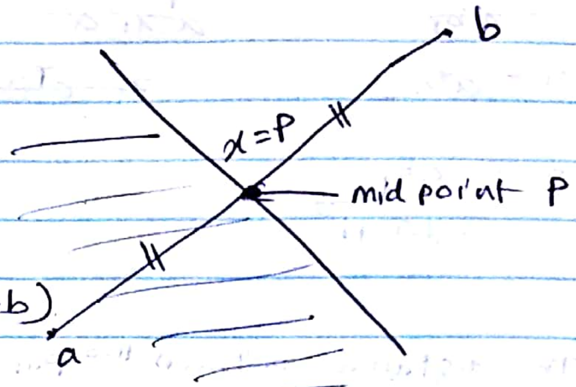
Show that the set of all points that are closer to a than b is a halfspace

(2-7) a point closer to a means

$\|x-a\|_2^2 \le \|x-b\|_2^2$



$\cancel{2(b-a)^T x}$

$(x-a)^T(x-a) \le (x-b)^T(x-b)$

$\cancel{x^Tx} - x^Ta - a^Tx + a^Ta \le \cancel{x^Tx} - x^Tb - b^Tx + b^Tb$

$-x^Ta - a^Tx + x^Tb + b^Tx \le b^Tb - a^Ta.$

$a^Tx = x^Ta$

$-a^Tx - a^Tx + b^Tx + b^Tx \le b^Tb - a^Ta.$

$2(b^T - a^T)x \le \|b\|^2 - \|a\|^2$

$(b^T - a^T)x \le \frac{\|b\|^2 - \|a\|^2}{2}$

$(b^T - a^T)x \le P$

$(b-a)^Tx \le P$

$(b-a)^T = c^T$

$P = d$

$c^Tx \le d.$

$(b-a)^Tx \le P$ represent hyperplane at x equidistant from a and b.

Thus the points that are closer to a must be below the hyperplane $c^T\underline{x} \le d$.

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2.8. Which of the following sets S are polyhedra? If possible express S in the form $S = \{x \mid Ax \preceq b, Fx = g\}$

a) $S = \{y_1 a_1 + y_2 a_2 \mid -1 \le y_1 \le 1, -1 \le y_2 \le 1\}$, where $a_1, a_2 \in \mathbb{R}^n$

$$S = \left\{ \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \;\middle|\; -1 \le y_1 \le 1, -1 \le y_2 \le 1 \right\}$$

$$S = \left\{ a^T y \;\middle|\; y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, -1 \le y_1, y_2 \le 1 \right\}$$

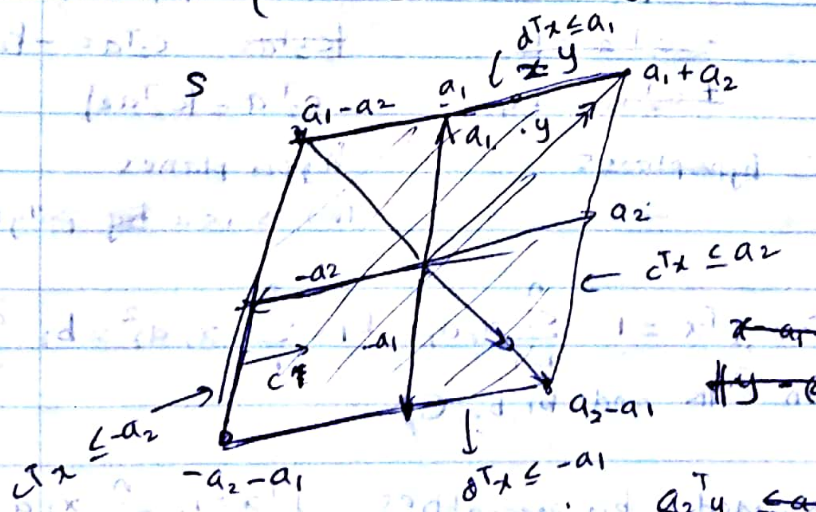
$$S = \quad (a_1 \; a_2)\begin{pmatrix} -1 \\ -1 \end{pmatrix} \le (a_1 \; a_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \cancel{-a_1 - a_2 < a^T y}$$

$$(a_1 \; a_2)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \le (a_1 \; a_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \cancel{a_1 + a_2 < a^T y}$$

$$(a_1 \; a_2)\begin{pmatrix} -1 \\ 1 \end{pmatrix} \le (a_1 \; a_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \cancel{-a_1 + a_2 < a^T y}$$

$$(a_1 \; a_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \le (a_1 \; a_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \cancel{a_1 - a_2 < a^T y}$$

$$S = \{y_1 a_1 + y_2 a_2 \mid -1 \le y_1 \le 1, -1 \le y_2 \le 1\}$$



- S is a parallogram with
- vertices $a_1 + a_2$, $a_1 - a_2$, $a_2 - a_1$, $-a_2 - a_1$

~~$x - a_1 + a_2$~~

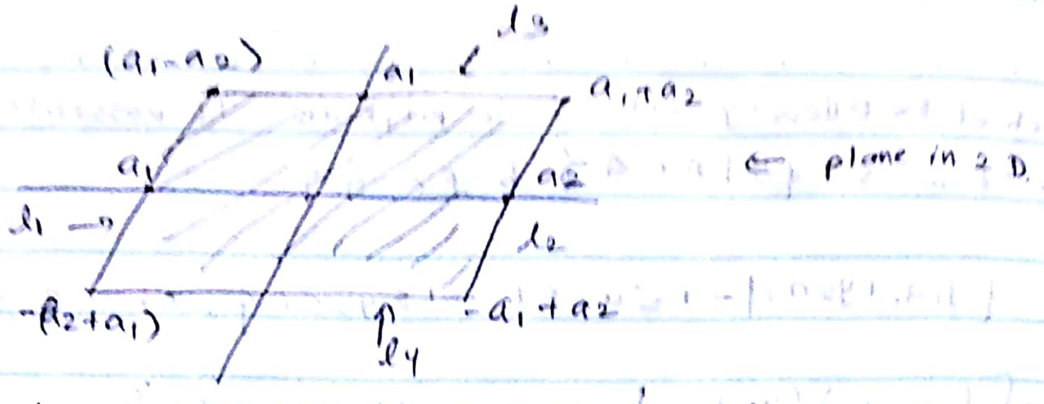
~~$\|y - (a_1 + a_2)\| \le \|b - (a_1 + a_2)\|$~~

$a_2^T y$ ~~$\le a_2$~~

S is a polyhedron with

$c^T x \le -a_2$, $d^T x \le -a_1$, as hyperplane

$c^T x \le a_2$, $d^T x \le a_1$ boundaries



Let c_1 be a vector which is in the plane of ~~a_1, a_2~~ $a_1 a_2$ and $\perp$ to a_1

then the parallogram lies between ~~$c_1^T y$~~

$$-c_1^T a_1 < y < c_1^T a_2 \longrightarrow -|c_1^T a_1| < c_1^T a < |c_1^T a_2|$$
$$l_1 < y < l_2$$

let c_2 be a vector which is in the plane of $a_1 a_2$ and $\perp$ to a_2

$$-c_2^T a_2 < y < c_2^T a_1 \qquad -|c_2^T a_2| < c_2^T a < |c_2^T a_2|$$

Thus the parallogram is bounded by

~~$c_1^T a_1$~~, ~~$-c_1^T a = b_1$~~ $\qquad c_1^T a = \cancel{c_1^T a_1} - |c_1^T a_1|$

~~$+c_1^T a = b_2$~~ $\qquad c_1^T a = |c_1^T a_2|$

~~$-c_2^T a = b_3$~~ $\qquad \cancel{c_2^T a_2} \quad c_2^T a = -|c_2^T a_2|$

~~$+c_2^T a = b_4$~~ $\qquad c_2^T a = |c_2^T a_2|$

~~hyperplanes~~ $\qquad$ hyper planes

Thus S is a polyhedron

b) $S = \{x \in \mathbb{R}^n \mid x \geq 0, \mathbf{1}^T x = 1 \quad \sum_{i=1}^{n} x_i a_i = b_1, \sum_{i=1}^{n} x_i a_i^2 = b_2 \}$

where $a_1, \ldots, a_n \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$

S is a polyhedron bounded by inequalities $\mathbf{1}^T x = 1$, $\sum_{i=1}^{n} x_i a_i = b_1$ and $\sum_{i=1}^{n} x_i a_i^2 = b_2$

$\sum_{i=1}^{n} x_i a_i = b_1$ can be written as $a^T x = b_1$

$\sum_{i=1}^{n} x_i a_i^2 = b_2$ can be written as $c^T x = b_2$

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- Thus S is bounded by $1^T x = 1$, $a^T x = b_1$, $c^T x = b_2$.

~~$x \succeq 0 \implies$ eigen values of x are ≥ 0~~

and polyhedron consist of $x \succeq 0$

Yes S is a polyhedron.

* polyhedron is consist of $x \in \mathbb{R}^n$ that satisfy $x \succeq 0$ and bounded by $1^T x = 1$, $a^T x = b_1$, $c^T x = b_2$ equalities which are hyperplanes

(5) $S = \{x \in \mathbb{R}^n \mid x \succeq 0, x^T y \leq 1$ for all y with $\|y\|_2 = 1\}$

- ~~$\|y\|_2 = 1$ is a unit ball~~

$x \succeq 0$ means x lies in

$x^T y \leq 1$

$\|x^T y\| \leq 1$

$\|x^T\|_2 \|y\|_2 \leq 1$

$\|y_2\| = 1$

$\|x^T\|_2 \leq 1$

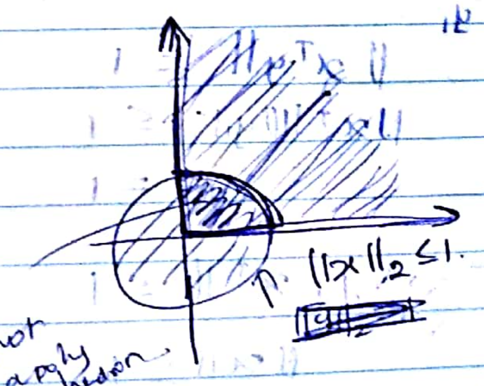
$\|x\|_2 \leq 1$.

unit ball

$x \succeq 0 \implies x$ lies in first quadrant $x \in \mathbb{R}^n_+$

positive.

non negative orthant



$x^T y \leq 1$.

$\|y_2\| = 1$

$y_1^2 + y_2^2 + \dots + y_n^2 = 1$

$x^T y \leq 1$

not a polyhedron.

because it is not bounded by ~~hyperplanes~~ half spaces

$\|y\|_2 = 1$

$\|x\|_2 =$

$x^T y \leq 1$

$\|y\|_2^2 = 1$

$\|x\|_2 \leq 1$

$x^T y \leq 1$ $\quad x^T y \leq 1$

$x x^T y y^T \leq x y^T$

$\|x\|_2^2 \|y\|_2^2 \leq x y^T$

$\|x\|_2^2 \leq x y^T$

$(y^T x)$

$y^T y$

$y^T y = 1$

$\|y_2\| = 1$

y^T

$\|y\|_2 = 1$

$\|y_2\|^2 = 1$

$y^T y = 1$

$x^T y \leq 1$

$x^T y \leq 1$

$\|y\|_2^2 = y^T y$

$x^T y \le 1$ $\quad \|y\|_2 = 1$

$\|y\|_2^2 = 1$ $\quad y \in R^n$

$y^T y = 1$ $\quad x \in \mathbb{R}^n$

$\cancel{x^T y \le 1}$ $\quad \cancel{x^T y \le 1}$

~~multiply~~ $\quad \cancel{x^T y}$

$\cancel{y^T y = 1}$

$\cancel{x^T y^T y \le y}$ $\quad \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le 1$

$x^T y \le 1$ $\quad \cancel{y_1^2 + y_2^2 = 1}$

$\cancel{(x^T y)(x^T y)^T = \|x^T y\|_2}$

$\cancel{x^T y \; y^T x = x}$ $\quad \cancel{x_1 y \quad x_1 y}$

$\cancel{x^T \|y\|^2}$ $\quad \cancel{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(x_1 \quad x_2)}$

$(x^T y)^T (x^T y) \le 1$ $\quad x_1 y_1 + x_2 y_2 \le 1$

$\|x^T y\|_2^2 \le 1$ $\quad y_1$

$\|x^T y\| \le 1$

$\|x^T\| \|y\| \le 1$

$\|y\| = 1$

$\|x^T\| \le 1$

$\|x\| \le 1$

$\cancel{\|y\|_2^2 = y^T y}$

~~d) $S = \{x \in R^n \mid x \succeq 0, x^T y \le 1$ for all y $\|y\|_2 = 1\}$~~

d) $S = \{x \in R^n \mid x \succeq 0, x^T y \le 1$ for all y with $\sum_{i=1}^{n} |y_i| = 1\}$

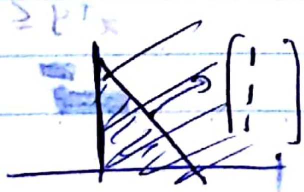
$x \succeq 0 \Rightarrow x$ lies in R^n_+

$\sum_{i=1}^{n} |y_i| = 1$ $\quad \cancel{x^T y \le 1}$

$\|x^T y\| \le$

$y_1 + y_2 + \dots + y_n = 1 \leftarrow$ hyperplane

$\mathbf{1} y = 1$



~~S is bounded~~ by $\mathbf{1} y = 1$

and ~~$\Rightarrow$~~ R^n_+

S is the intersection of R^n_+ and $\mathbf{1} y \le 1$

Thus S is a polyhedron

absolute of both sides ~~elem~~

$x^T y \le 1 \longrightarrow |x^T||y| \le 1$

Take max of both sides

$|x^T|_{max} \cdot |y|_{max} = 1$

$|x^T|_{max} \times 1 = 1$

$|x_i^T|_{max} = 1$

$0 \le x_i \le 1$

$\therefore |x_i^T| \le 1$

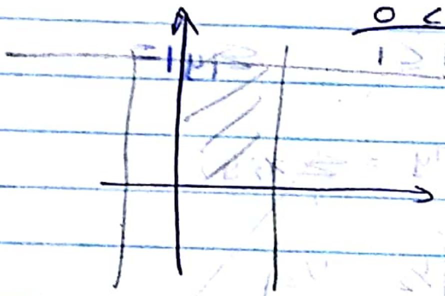
$\therefore |x_i| \le 1$

$\therefore$ ~~$|x_1| + |x_2|$~~ $\therefore$

~~$|x_1| + x_2$~~ $-1 \le x_i \le 1$

~~The intersection~~ of R_+^n and $-1 \le x_i \le 1$ is

$0 \le x_i \le 1$



~~$0 \le \mathbf{1}^T x \le 1$~~

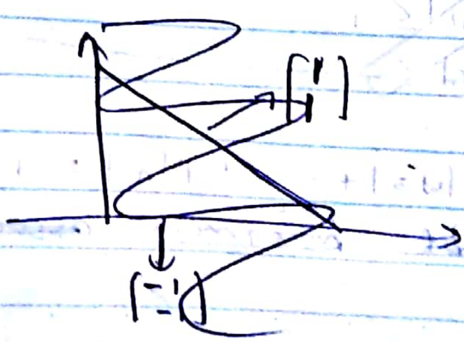
~~$\mathbf{1}^T x = [1 \; 1 \; 1 \; \dots \; 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$~~

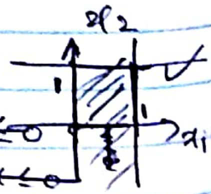
S can be described as the intersection of R_+^n with the set $\{x \mid -1 \le x \le 1\}$ which is a bounded polyhedron with halfspaces.

The ~~inequalities hyperplanes~~ ~~hyperplanes~~ inequalities that bound are.

$-x_i \le 0 \quad i = 1, 2, 3 \ldots$

$x_i \le 1 \quad i = 1, 2, 3, \ldots$





~~$-x \le 0$~~

$-\mathbf{1}^T x \le 0$

~~$\mathbf{1}^T x \le 1$~~

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$-1 \le x_1 \le 1$

$-1 \le x_2 \le 1$

$0 \le x_1 \le 1$

$0 \le x_2 \le 1$

2.9.

a) $V = \{x \in \mathbb{R}^n \mid \|x - x_0\| \le \|x - x_i\|_2, \ i = 1, 2, \ldots, K\}$ document tessarnel

Show that V is a polyhedron. Express V in the form $V = \{x \mid Ax \preceq b\}$

$\|x - x_0\| \le \|x - x_i\|_2$

$\|x - x_0\|^2 \le \|x - x_i\|_2^2$

$(x - x_0)^T(x - x_0) \le (x - x_i)^T(x - x_i)$

$\cancel{x^T x} - x^T x_0 - x_0^T x + x_0^T x_0 \le \cancel{x^T x} - x^T x_i - x_i^T x + x_i^T x_i$

$x^T(x_i - x_0) - x_0^T x + x_i^T x \le x_i^T x_i - x_0^T x_0$

$\le \|x_i\|^2 - \|x_0\|^2$

$\cancel{(x_i - x_0)^T x}$

$(x_i - x_0)^T x = x^T(x_i - x_0)$

$\cancel{x^T(x_i - x_0) + x_i^T x_0}$

$2(x_i - x_0)^T x \le \|x_i\|^2 - \|x_0\|^2$

$(x_i - x_0)^T x \le \dfrac{\|x_i\|^2 - \|x_0\|^2}{2}$

$(x_i - x_0)^T x \le b_i$ ← elementwise thing.

$$A = \begin{bmatrix} x_1 - x_0 \\ x_2 - x_0 \\ x_3 - x_0 \\ \vdots \\ x_n - x_0 \end{bmatrix} \qquad B = \begin{bmatrix} (b_1 - b_0)/2 \\ (b_2 - b_0)/2 \\ (b_3 - b_0)/2 \\ \vdots \\ (b_n - b_0)/2 \end{bmatrix}$$

$1^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_1 + x_2$

$Ax \preceq b$ ← elementwise inequality

$Ax \le b$ ← matrix wise inequality

V is a set bounded by halfspaces $\rightarrow$ V is a polyhedron

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b) $V = \{x \mid Ax \preceq b\}$ $A \in \mathbb{R}^{K \times n}$ and $b \in \mathbb{R}^K$

$x_0 \in \{x \mid Ax \preceq b\}$ with $A \in \mathbb{R}^{K \times n}$ and $b \in \mathbb{R}^K$.

Pick any $x_0 \in \{x \mid Ax \preceq b\}$ and then construct K points x_i by $\cancel{\{x \mid a_i^T x = b_i\}}$ so that $\cancel{x_i = x_0 =}$

using the fact distance from x_0 to hyperplane ~~= dis~~ is equal to the distance from x_1 to hyperplane.

The hyperplane is

$$(x_i - x_0)^T x \leq \frac{\|x_i\|^2 - \|x_0\|^2}{2} = b$$

$$\downarrow$$

$$(x_i - x_0)^T x \leq b$$

~~The~~ $\therefore$ $\cancel{(x_i - x_0)^T x_0} = \cancel{(x_i - x_0)^T x_i}$

$$\cancel{\| b - (x_i - x_0)^T x_0 \| = \| (x_i - x_0)^T x_i - b \|}$$

$$\cancel{x_i - x_0} \quad x_0 = \cancel{(x_i - x_0)} \cdot c$$

$$\cancel{2b = (x_i - x_0)^T x_0 + (x_i - x_0)^T x_i}$$

$$\cancel{2b = (x_i - x_0)^T (x_0 + x_i)}$$

$$\cancel{= (x_i^T - x_0^T)(x_0 + x_i)}$$

$$\cancel{2b = x_i^T x_0 - x_i^T x_i - x_0^T x_0 - x_0^T x_i}$$

$$\cancel{= x_0^T x_i}$$

$$\cancel{(x_i - x_0)^T x_0 = b_0}$$

$$\cancel{(x_i - x_0)^T x_1 = b_1}$$

$$\cancel{\frac{(x_i - x_0)^T x_0 + (x_i - x_0)^T x_1}{2} = \frac{b_0 + b_1}{2} = b}$$

~~$x_i = x_0 + \lambda a_i$~~

~~$(x_i - x_0) = \lambda a_i$~~ $\quad x_i - x_0 = a$

$(x_1 - x_0) = a$

$a^T x \le b$

~~$(x_i - x_0)^T (x_i - x_0) = b$~~

$x = c\,q$

~~$a^T x_0 =$~~

~~$(x_i - x_0)^T \lambda a_i = b$~~

$a^T(x_i - x_0) = b$

~~$\lambda = b$~~

Assignment 2

1. $c = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad A = \begin{pmatrix} 3 & 1 \\ 5 & 4 \\ 1 & 1 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

consider

$$\boxed{\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & A^T x \le b \end{array}}$$

Formulate (P1) ~~as a~~ equivalently as a semidefinite program (SDP). Here you have to identify how to put $A^T x \le b$ in a form of $x_1F_1 + x_2F_2 + x_3F_3 + G \le 0$, with F_i and G are symmetric

$c^T x \leftarrow$ semidefinite cone

Matrix symmetric $Z \in S^k$.

$A^T x \le b$

$$\begin{pmatrix} 3 & 1 \\ 5 & 4 \\ 1 & 1 \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \le \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 5 & 1 \\ 1 & 4 & 1 \end{pmatrix}_{2\times 3} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{3\times 1} \le \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 3x_1 + 5x_2 + x_3 \\ x_1 + 4x_2 + x_3 \end{pmatrix} \le \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{bmatrix} 3x_1 + 5x_2 + x_3 - 5 \\ x_1 + 4x_2 + x_3 - 4 \end{bmatrix} \le \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} x_1 + \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} x_2 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x_3$$

~~$+ \begin{bmatrix} 5 \\ 4 \end{bmatrix} \le 0$~~

$$+ \begin{pmatrix} -5 & 0 \\ 0 & -4 \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$P_1 \longrightarrow$ semidefinite problem

$$\text{minimize} \quad \begin{bmatrix}1\\2\\1\end{bmatrix}^T x$$

$$\text{Subject to} \quad \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix} x_1 + \begin{pmatrix}5 & 0\\0 & 4\end{pmatrix} x_2 + \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} x_3 + \begin{bmatrix}-5 & 0\\0 & -4\end{bmatrix} \leq \begin{bmatrix}0\\0\end{bmatrix}$$

$$L(x,Z) = \begin{bmatrix}1\\2\\1\end{bmatrix}^T x + \text{Trace}\left[\underset{\substack{\uparrow\\\text{Symmetric}}}{Z^T} \underbrace{\left(\sum x_i F_i + G\right)}_{\text{Linear matrix}}\right]$$

$(Z = Z^T)$ Z is symmetric

$$= \begin{bmatrix}1\\2\\1\end{bmatrix}^T x + Tr\left[\sum(x_i Z F_i)\right] + Tr(ZG)$$

$$L(x,Z) = \begin{bmatrix}1\\2\\1\end{bmatrix}^T x + Tr\left[\sum x_i Z F_i\right] + Tr(ZG)$$

$$= \sum_{i=1}^{3}\left[c_i + Tr(ZF_i)\right] x_i + Tr\, ZG$$

$$G = \begin{pmatrix}-5 & 0\\0 & -4\end{pmatrix}$$

$$g(Z) = \begin{cases} + Tr(ZG) \Rightarrow \text{if } \sum c_i + Tr(ZF_i) = 0 & \text{for } i = 1,2,3 \\ -\infty & \text{if otherwise} \end{cases}$$

$$\boxed{\begin{array}{lll} \text{maximize} & + Tr(ZG) & \\ \text{subjected to} & c_i + Tr(ZF_i) = 0 & \forall i \\ & Z \geq 0 & \end{array}}$$

$\boxed{Z,}$ $\boxed{F_i,}$

Z is a positive semidefinite.

$$G = \begin{pmatrix}-5 & 0\\0 & -4\end{pmatrix}$$

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② express the general LP

$$\boxed{\begin{array}{l} \text{minize } c^T x \\ \text{subject to } Ax \le b \end{array}} \ \boxed{x} \quad \text{as a SDP.}$$

Put ~~sub~~ $Ax \le b$ in $x_1F_1 + x_2F_2 + x_3F_3 + G \le 0$ form F_i, G are symmetric

$Ax \le b$

$x_1F_1 + \dots + x_nF_n \le G$. F_i and G are symmetric.

$$L(x, z) = c^T x + \text{Trace}\left[z^T\left(\sum_i x_i F_i - G\right)\right]$$

choose z from S^K_+

$z^T = z$

$$= c^T x + Tr \sum_i (x_i zF_i) - Tr(zG)$$

$$= \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n} x_i Tr(zF_i) - Tr(zG)$$

$$= \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n} x_i Tr(zF_i) - Tr(zG)$$

$$L(x, z) = \sum_{i=1}^{n}\left[c_i + Tr(zF_i)\right] x_i - Tr(zG)$$

$$g(z) \ge \begin{cases} -Tr(zG) & \text{if } \left[c_i + Tr(zF_i)\right] = 0 \\ -\infty & \text{if otherwise} \end{cases}$$

$$\boxed{\begin{array}{l} \text{maximize } -tr(zG) \\ \text{subjected to } c_i + Tr(zF_i) = 0 \ \forall i \\ z \succeq 0 \\ \text{Z has to be positive semidefinite} \end{array}} \ \boxed{z, F_i}$$

(3) consider the following QP (Quadratic programming)

$$\begin{cases} \text{minimize} & x^T H x + p^T x \\ \text{subject to} & Ax \geq b \end{cases} \boxed{X}$$

It is given that ~~H ⪰ 0~~ $H \succeq$ and $H \in S^n$, i.e $H \in S^n_+$

a) what is the epigraph form of the above problem (Hint: see problem (4.11) of Boyd reference text. You have to clearly mention the decision variables.)

b) show that the positive semidefinite matrix H can be written as ~~H ⪰ $\hat{H}$~~ $H = \hat{H}^T \hat{H}$ for some $\hat{H} \in \mathbb{R}^{n \times n}$

(Hint: you may consider the eigen value decomposition of H.)

a) epigraph form.

minimize t.

subject to $x^T H x + p^T x - t \leq 0$

$-Ax + b \leq 0$

t = auxillary variable introduced to form epigraph form

$$L(x, \lambda) = x^T H x + p^T x - t + \lambda^T(-Ax + b) + 0$$

$$g(\lambda) = \inf_{x \in D} [x^T H x + p^T x - t + \lambda^T(-Ax + b)]$$

$$\Delta g(\lambda) = \inf_{x \in D} \Delta(x^T H x) + \Delta[p - (A^T \lambda)]^T x - \Delta(t + \lambda^T b)$$

$$\Delta g(\lambda) = 2Hx + (p - (A^T \lambda))$$

$$\Delta g(\lambda) = 0$$

$$2Hx + (p - A^T \lambda) = 0$$

$$x = \cancel{2H\, A^T \lambda - p} \quad \frac{1}{2} H^{-1}(A^T \lambda - p) = d^*$$

$$= \frac{1}{2} H^{-1}(\lambda A^T - p)$$

$$\overline{g(\lambda)} =$$

$$\cancel{x = \tfrac{1}{2} H^{-1}(A^T \lambda - p)}$$

$$A^T\lambda = \lambda A^T$$

~~g(λ) = ...~~

$$\cancel{g(\lambda) = \frac{1}{2} H^{-1} A^T \lambda (\lambda A^T - P)}$$

$$g(\lambda) = \left(\frac{1}{2} H^{-1}(\lambda A^T - P)\right)^T H \left[\frac{1}{2} H^{-1}(\lambda A^T - P)\right] + P^T\left[\frac{1}{2} H^{-1}(\lambda A^T - P)\right] - t$$

$$+ P^T\left(\frac{1}{2} H^{-1}(\lambda A^T - P)\right] - t + \lambda^T\left(-A \frac{1}{2} H^{-1}(\lambda A^T - P) + b\right)$$

$$g(\lambda) = \frac{1}{4}(\lambda A^T - P)^T H^{-T} \underbrace{H H^{-1}}_{I} (\lambda A^T - P) + \frac{1}{2} P^T H^{-1}(\lambda A^T - P) - t -$$

$$\frac{1}{2}\lambda^T A H^{-1}(\lambda A^T - P) + \lambda^T b$$

~~H similarly~~ $H = H^{-1}$

$H^T = H^{-T} = H$? $\} \to H$ is symmetric

$$= \left(\frac{1}{2}(\lambda A^T - P)^T H^{-T} + \frac{1}{2} P^T H^{-1}\right)\cancel{(\lambda A^T - P)} - \frac{1}{2}\lambda^T A H^{-1}(\lambda A^T - P)$$

$$+ \lambda^T b - t$$

~~Maximize~~ $\frac{1}{2}$ $\cancel{\lambda A^T = \lambda^T A}$ $(\lambda A^T - P)$

$$= \left[\frac{1}{2}\lambda^T A H^{-1} - \frac{1}{2} P^T H^{-1} + \frac{1}{2} P^T H^{-1} - \frac{1}{2}\lambda^T A H^{-1}\right]$$

$$+ \lambda^T b - t$$

$= \cancel{\frac{1}{2}\lambda A^T P}$

$= \cancel{\frac{1}{2}\lambda^T A - P^T}$

$$g(\lambda) = \lambda^T b - t$$

$$\boxed{\begin{array}{ll} \text{Maximize} & \lambda^T b - t \\ \text{subject to} & \lambda \geq 0 \end{array}}\ \boxed{t, \lambda}$$

ii) $H \geq 0$ $H \subset S^n$ i.e $H \in S^n_+$

H is positive semidefinite $\longrightarrow$ all eigenvalues are non negative.

$\downarrow$

we can take square root of the eigen value

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cdot & 1 \\ - & 0 \end{pmatrix}\begin{pmatrix} 0 & \cdot \\ 1 & 0 \end{pmatrix}$$

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$H = Q \Lambda Q^T$

columns of Q are eigenvectors of H

diagonal ~~element~~ entries of diagonal matrix Λ are the eigenvalues of Y.

$$H = Q \Lambda^{1/2} \Lambda^{1/2} Q^T$$

$$H = (Q\Lambda^{1/2})(Q\Lambda^{1/2})^T \qquad (\Lambda^{1/2})^T = \Lambda^{1/2}$$

Let $\hat{H} = (Q \Lambda^{1/2})^T$

Thus $H = \hat{H}^T \hat{H}$

$Q \in \mathbb{R}^{n \times n}$

$\Lambda^{1/2} \in \mathbb{R}^{n \times n}$.

$\therefore Q\Lambda^{1/2} \in \mathbb{R}^{n \times n}$

$\hat{H} \in \mathbb{R}^{n \times n}$.

c) Use part b to show

$x^T H x + p^T x \leq t$ is equivalent to

$(t - p^T x) - (\hat{H}x)^T(\hat{H}x) \geq 0$.

in part B

$H = \hat{H}^T \hat{H}$.

$$x^T H x + p^T x \leq t$$

$$x^T \hat{H}^T \hat{H} x + p^T x \leq t$$

$$(t - p^T x) - (\hat{H}x)^T (\hat{H}x) \geq 0 \quad \text{--- } (*)$$

d) express $*$ as

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0$$

for some A, B, C defined appropriately

Hint $B = \hat{H}x$

$$\begin{bmatrix} t - p^T x & \hat{H}x \\ (\hat{H}x)^T & I \end{bmatrix}$$

$A = t - p^T x \geq 0$

$\Rightarrow H \geq 0$

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$$x^T H x + p^T x \le t$$

$$\text{epi} f = \{ x, H, t \mid H \succ 0,\ x^T H x + p^T x \le t \}$$

$$= x, H, t \mid \begin{bmatrix} t - p^T x & \hat{H} x \\ (\hat{H} x)^T & I \end{bmatrix} \succeq 0,\ H \succ 0 \}$$

(D) $A x \geq b$

$A x - b \geq 0$

~~SDP P (from above parts~~ ~~The above sections result in SDP~~

$$\text{minimize} \quad t$$

$$\text{subject to} \quad \begin{pmatrix} t - p^T x & \hat{H} x \\ (\hat{H} x)^T & I \end{pmatrix} \succeq 0$$

$$A x - b \geq 0$$

$$\begin{matrix} b & x \\ A & I \end{matrix}$$

$$F_1(\bar{x}) = \begin{pmatrix} t - p^T x & \hat{H} x \\ \hat{H} x & I \end{pmatrix}; \quad \bar{x} = \begin{pmatrix} t \\ x \end{pmatrix}$$

$$F_1(\bar{x}) \geq 0$$

~~$F_2(\bar{x}) =$~~ LMI of $Ax - b \geq 0$ is $F_2(\bar{x})$

$$F_2(\bar{x}) \geq 0$$

This result in SDP

~~This result in SDP~~ minimize t

$$\text{diag} \begin{pmatrix} F_1(\bar{x}) & 0 \\ 0 & F_2(\bar{x}) \end{pmatrix} \succeq 0$$

$$\bar{x} = \begin{bmatrix} t \\ x \end{bmatrix}$$

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(4) minimize $Tr(CX)$ — P_3

subject to $Tr(A_i X) = b_i \quad , i = 1, 2, \ldots, m$

$X \succeq 0 \longrightarrow -X \preceq 0$

where C, X and A_i's are symmetric matrices

1) What is the Lagrangian

b) Derive Langrangian dual function

c) Derive the dual problem of P_3 by explicitely expressing the related dual constraints

d) Derive and state KKT conditions for the SDP P_3

$$L(X, \lambda, \nu) = Tr(CX) + \lambda^T(-X) + \nu^T(Tr(A_i X) - b_i)$$

$$= Tr(CX) - \lambda^T X + \nu^T(Tr(A_i X)) - \nu^T b_i$$

$$= Tr(CX) - \lambda^T X + \nu^T\ Tr(A_i X) - b_i^T \nu.$$

$b_i \in \mathbb{R}$

$b_i^T \nu = \nu^T b_i$

$$L(X, \lambda, \nu) = (Tr(CX) + \nu^T\ Tr(A_i X) - \lambda^T X) - b_i^T \nu$$

$$\cancel{L(X, \lambda, \nu) =}$$

$$g(\lambda, \nu) = \begin{cases} -b_i^T \nu & \text{if } Tr(CX) + \nu^T Tr(A_i X) - \lambda^T X = 0 \\ -\infty & \text{if otherwise} \end{cases}$$

c)
$$\boxed{\begin{aligned} &\text{maximize} \quad -b_i^T \nu \quad \text{or minimize} \quad b_i^T \nu \\ &\text{subject to} \quad Tr(CX) + \nu^T Tr(A_i X) - \lambda^T X = 0 \\ &\qquad\qquad \lambda \succeq 0 \end{aligned}}$$

X

λ, ν

d) 1) primal feasibility $Tr(A_i X) = b_i \quad i = 1, 2, \ldots, m$

$-X \preceq 0$

2) dual feasibility $\lambda \succeq 0$

$$\begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_3 & c_6 & c_7 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_6 \\ x_3 & x_6 & x_5 \end{bmatrix}$$

$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$ $c_1 + c_2$

$\begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$

$c_1x_1 + c_2x_2$

$c_2x_1 + c_3x_2$

4) gradient of L w.r.t X vanishes : $\nabla_X L(X, \lambda, \nu) = 0$

$\Delta\left(Tr(CX) + \nu^T Tr(A_i X) - \lambda^T X\right) = 0$

$Tr(\Delta C X) + \nu^T Tr(\Delta A_i X) - \lambda^T \Delta X = 0$

$Tr(C) + \nu^T Tr(A_i) - \lambda^T = 0$

$Tr(C) + \nu^T Tr(A_i) = \lambda \quad (\lambda = \lambda^T)$

5) complimentary slackness $\lambda_i x_i = 0$

$$\boxed{\begin{array}{ll} \text{maximize} & -b_i^T \nu \\ \text{subject to} & Tr(C) + \nu^T Tr(A_i) \geq 0 \\ & \text{as } \lambda \geq 0 \end{array}} \quad \boxed{\nu, \lambda}$$

HW-5 ~~Boyd~~

4) Manufacturers produces 4 different products: x_1, x_2, x_3, x_4

there are three inputs to this production process

(1) Labor (in person-weeks)

(2) raw material A (in kg)

(3) raw material B (in boxes),

Each product ~~product~~ has different input requirements

when determining each week, production schedule, the manufacturer must ensure that the available amounts of revenue for labor, raw material A ~~and~~ and B cannot be exceeded.

the relevant revenue information is presented in table below

	Product x_1	x_2	x_3	x_4	Input availability
Person-weeks	1	2	1	2	25
kg of A	6	5	3	2	70
kgs of B	3	4	9	12	75
product levels	x_1	x_2	x_3	x_4	
Selling price per unit	6	7	5	2	

Formulate
~~Make~~ problems

1) ~~Fototal revenue~~ maximize the total revenue subjected to the resource availability constraints

Total revenue $= f_x$.

Resource constraints $= f_b$.

$$f_x = 6x_1 + 7x_2 + 5x_3 + 2x_4$$

~~f_c~~ f_b :

$$\underbrace{\begin{pmatrix} 1 & 2 & 1 & 2 \\ 6 & 5 & 3 & 2 \\ 3 & 4 & 9 & 12 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{x} \leq \underbrace{\begin{pmatrix} 25 \\ 70 \\ 75 \end{pmatrix}}_{b}$$

$x_i \geq 0$ ~~~~ product level non negative.

Maximize f_x

Subject to $Ax \leq b$

$x_i \geq 0$.

x — $\boxed{x_1, x_2, x_3, x_4}$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

2) maximize the minimum revenue of a product subject to the resource availability constraints.

minimum revenue of the product $= 6x_1, 7x_2, 5x_3, 2x_4$.

$$f(x_1, x_2, x_3, x_4) = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = a_i^T x \quad i = 1,2,3,4.$$

maximize $(a_i^T x)$

Subject to

$Ax \leq b$

$x \geq 0$

$\boxed{x}$

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3) max the total rev s.to the res source availibity constraint S and an additional constraint no more than half of the available resources of each input is used by any 2 products

dual problem

② ~~$\neq$~~ ~~∞~~

$$f_x = 6x_1 + 7x_2 + 5x_3 + 2x_4$$

$$\begin{bmatrix} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$x_i + x_j = 25$

$x \geq 0 \qquad Ax \leq b$

$$L(\lambda, v) = f_0 + \Sigma \lambda f_i(x) + \Sigma v h_i(x)$$
$$= c^T x + \lambda(-x) + \lambda(Ax - b) +$$

$$\begin{pmatrix} x_i + x_j \end{pmatrix} \leq \begin{pmatrix} 25/2 \\ 70/2 \\ 75/2 \end{pmatrix}$$

~~$x_i + x_j$~~

$x_1 + x_2 \quad x_1 + x$

$\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\left.\begin{matrix} (x_1 + x_2) \\ (x_1 + x_3) \\ (x_1 + x_4) \\ (x_2 + x_3) \\ (x_2 + x_4) \\ (x_3 + x_4) \end{matrix}\right\} \leq 25/2$$

$$4C_2 \quad \frac{4!}{2!\,2!} = \frac{4 \times 3 \times 2!}{2!\,2!} = 6$$

i-input

$$\text{input}/2 = \begin{pmatrix} 25/2 \\ 70/2 \\ 75/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \leq \begin{pmatrix} 25/2 \quad \text{~~input/2~~} \end{pmatrix}$$

25/2

③ ~~$x_1 + 2x_3 \leq 10$~~

$x_1 + 2x_3$

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