PHSX815: Computational Methods in Physical Sciences
Project 4
Project Paper
Title: Phase Plane Analysis for Simple Systems

Vishakha Lingadahally

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Introduction

Stability theory addresses the stability of solutions of differential equations and of trajectories of dynamical systems under small perturbations of initial conditions. The heat equation, for example, is a stable partial differential equation because small perturbations of initial data lead to small variations in temperature at a later time as a result of the maximum principle.
Many parts of the qualitative theory of differential equations and dynamical systems deal with asymptotic properties of solutions and the trajectories—what happens with the system after a long period of time. The simplest kind of behavior is exhibited by equilibrium points, or fixed points, and by periodic orbits. If a particular orbit is well understood, it is natural to ask next whether a small change in the initial condition will lead to similar behavior. Stability theory addresses the following questions: Will a nearby orbit indefinitely stay close to a given orbit? Will it converge to the given orbit? In the former case, the orbit is called stable; in the latter case, it is called asymptotically stable and the given orbit is said to be attracting.

The simplest kind of an orbit is a fixed point, or an equilibrium. If a mechanical system is in a stable equilibrium state then a small push will result in a localized motion, for example, small oscillations as in the case of a pendulum. In a system with damping, a stable equilibrium state is moreover asymptotically stable. On the other hand, for an unstable equilibrium, such as a ball resting on a top of a hill, certain small pushes will result in a motion with a large amplitude that may or may not converge to the original state.

In this project, we compute the fixed points for a system of two differential equations in x and y.

The code

In many cases there is no analytical solution to systems with nonlinear interacting dynamics. An alternative to numerical simulation is phase plane analysis. It is an important technique for studying the behavior of nonlinear systems.

We define the set of differential equations that make up the system. Steady states (or fixed points) are states in the development of a system at which the dynamics do not show any further changes. A way to find steady states is to set the differential equation expressing the change of the system per time step to 0.

Consider a system f(x,y), g(x,y) with x and y being two parameters that change with time according to the following differential equations.

$$f(x, y) = \frac{dx}{dt} = y(2 - y - x)$$
$$g(x, y) = \frac{dy}{dt} = x(-1 + y)$$

Setting the terms to 0, we obtain

$$x = 2 - y$$
$$y = 1$$

This system has two trivial steady states at (0,0) and (0,2) and a non-trivial steady state at (1,1). The non-trivial steady states can be inferred from the following plots where it is indicated as the state (or point) to which the curves seem to run up to, i.e. their 'attractor'.

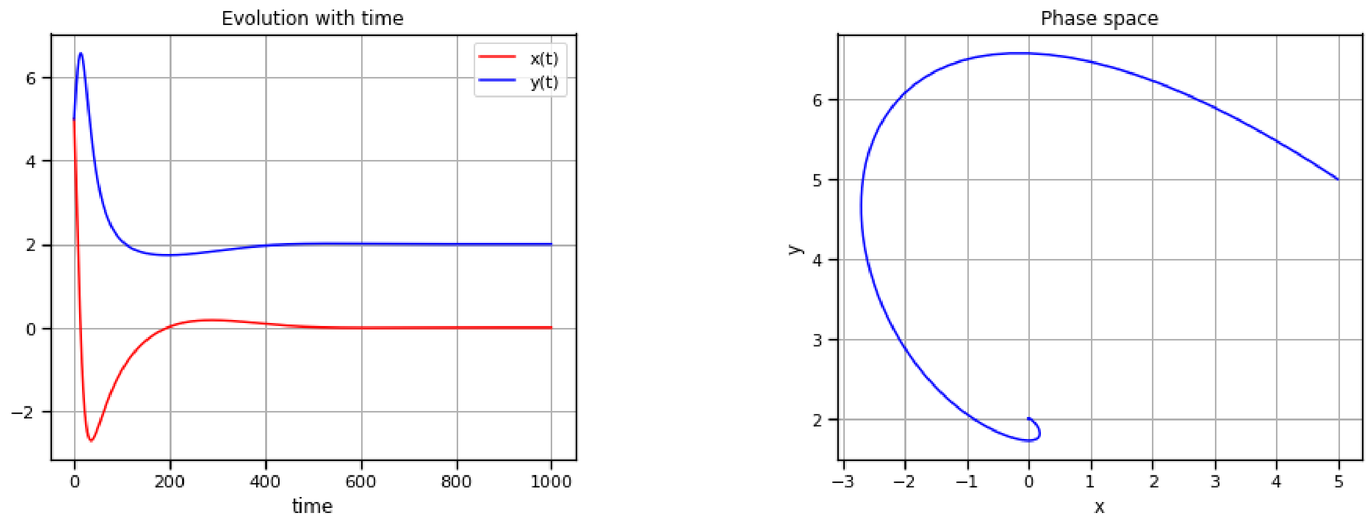

Figure 1: Plots of dynamics in time and phase space

The left plot is a temporal representation of the system's development, with time t being represented on the horizontal axis. The right plot is a phase plane (or phase space or state space) portrait of the system. It shows the size of the stock x on the horizontal axis parametrically with the size of the stock y on the vertical axis. Each point on this curve refers to one particular state of the system.

Output interpretation and the stability of steady states

The code identifies the fixed points of any given set of differential equations and intial conditions. The steady states can have different stabilities. It can either be stable or unstable. This can be seen if the system is started from a value close to a steady state. Starting it at 0.001 will induce growth. It will drive the development away from zero. Zero works as a repellor in this case. It is an unstable equilibrium. The classification of the fixed points into stable and unstable points is done as follows:

- The elements of the Jacobian matrix of our system is caclulated manually (the automation of this is a further scope of the project).
- The eigenvalues of the Jacobian are calculated.
- The real parts of the two eigenvalues obtained at the fixed points are checked if they are greater than or less than 0, simultaneously or separately, thus determining the stability of the fixed point in question.

Summary

- The fixed points for a system of two differential equation in x and y parametrized in time are computed by phase plane analysis.
- The stability of the system at the fixed points are examined.

References

- stackoverflow.com
- uni-graz.at