Assignment 1 Solutions

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1 Linear Algebra

1.1 Question 1

Given: Set of vectors $\{|k\rangle\}$ are orthonormal.

If $\{|k\rangle\}$ is a basis, then any vector $|v\rangle$ can be written as a linear combination of the basis vectors.

$\therefore |a\rangle = \sum_k a_k|k\rangle \quad |b\rangle = \sum_k b_k|k\rangle$

$\sum_k \langle a \mid k\rangle\langle k \mid b\rangle = \sum_k a_k^* b_k = \langle a \mid b\rangle$

Hence, if $\{|k\rangle\}$ is a basis, then $\langle a \mid b\rangle = \sum_k \langle a \mid k\rangle\langle k \mid b\rangle$ is a necessary condition.

If $\langle a \mid b\rangle = \sum_k \langle a \mid k\rangle\langle k \mid b\rangle$

$$\langle a| \mid I|b\rangle = \sum_k \langle a \mid k\rangle\langle k \mid b\rangle$$

$$\langle a| \left(\sum_k |k\rangle\langle k| - I\right)|b\rangle = 0 \quad \forall a, b$$

$$\therefore \sum_k |k\rangle\langle k| = I$$

Since $|k\rangle$ satsifies the completeness relation, it forms an orthonormal basis.
Hence $\langle a \mid b\rangle = \sum_k \langle a \mid k\rangle\langle k \mid b\rangle$ is a sufficient condition for $|k\rangle$ to be a basis.

1.2 Question 2

Suppose the spectral decomposition of A is given by $A = \sum_i \lambda_i|i\rangle\langle i|$ and $B = \sum_i \mu_i|i\rangle\langle i|$

Since $B = e^A$, we have $B = \sum_i e^{\lambda_i}|i\rangle\langle i|$, and hence $\mu_i = e^{\lambda_i}$

We can have infinitely many matrices A with spectral decomposition $\sum_i(\lambda_i + i2n\pi)|i\rangle\langle i| \quad \forall i \in \mathbb{Z}$, such that $e^A = \sum_i e^{\lambda_i + i2n\pi}|i\rangle\langle i| = \sum_i e^{\lambda_i}|i\rangle\langle i| = \sum_i \mu_i|i\rangle\langle i| = B$

Hence log is a non-unique function because for a given B there are infinitely many A such that $B = e^A$.

1.3 Question 3

Suppose A and B are operators with diagonal representation $A = \sum_i \lambda_i |v_i\rangle\langle v_i|$ and $B = \sum_i \mu_i |w_i\rangle\langle w_i|$

Consider the tensor product $A \otimes B$

$$(A \otimes B)(|v_i\rangle \otimes |w_j\rangle) = A|v_i\rangle \otimes B|w_j\rangle = \lambda_i |v_i\rangle \otimes \mu_j |w_j\rangle = \lambda_i \mu_j |v_i\rangle \otimes |w_j\rangle$$

Hence, the diagonal representation of $A \otimes B$ is $A \otimes B = \sum_{i,j} \lambda_i \mu_j |v_i\rangle|w_j\rangle\langle v_i|\langle w_j|$, that is the eigen values of $A \otimes B$ are $\lambda_i \mu_j$ and the corresponding eigenvectors are $|v_i\rangle|w_j\rangle$

(i) $M = \begin{pmatrix} 0 & 2 & 0 & 3 \\ 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 0 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Eigen values of $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ are $5, 1$

Eigen values of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are $1, -1$

$\therefore$ Eigen values of M are $5, 1, -1, -5$

(ii) $M = \begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 2 & 3 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

$\therefore$ Eigen values of M are $5, 1, -1, -5$

(iii) $M = \begin{pmatrix} 4 & 6 & 6 & 9 \\ 2 & 8 & 3 & 12 \\ 2 & 3 & 8 & 12 \\ 1 & 4 & 4 & 16 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \otimes \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

$\therefore$ Eigen values of M are $1, 5, 5, 25$

2 Quantum Mechanics

2.1 Question 1

$|\psi\rangle = \dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$

Unitary transformation $H = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is the Hadamard operator.

$H = \dfrac{|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|}{\sqrt{2}}$

$$H|\psi\rangle = \left(\frac{|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|}{\sqrt{2}} \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$= \frac{|0\rangle + |1\rangle - |0\rangle + |1\rangle}{2} = |1\rangle$$

$\therefore |\psi'\rangle = |1\rangle$

2.2 Question 2

Measurement of a qubit in computational basis is defined by the measurement operators $M_0 = |0\rangle\langle 0|$ and $M_1 = |1\rangle\langle 1|$

On measuring $|\psi\rangle = \dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$ in the computational basis, the probability of obtaining outcome 0 is $p(0) = |\langle 0 \,|\, \psi\rangle|^2 = \dfrac{1}{2}$ and 1 is $p(1) = |\langle 1 \,|\, \psi\rangle|^2 = \dfrac{1}{2}$

The state after measurement in both cases is:

$$\frac{M_0|\psi\rangle}{\sqrt{p(0)}} = |0\rangle$$

and

$$\frac{M_1|\psi\rangle}{\sqrt{p(1)}} = -|1\rangle$$

On measuring $|\psi'\rangle = |1\rangle$ in the computational basis, the probability of obtaining outcome 0 is $p(0) = |\langle 0 \,|\, \psi'\rangle|^2 = 0$ and 1 is $p(1) = |\langle 1 \,|\, \psi'\rangle|^2 = 1$

The state after measurement is

$$\frac{M_1|\psi'\rangle}{\sqrt{p(1)}} = |1\rangle$$