MECHANICS

Lecture notes for Phys 111

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Abstract

These notes are intended as an addition to the lectures given in class. They are NOT designed to replace the actual lectures. Some of the notes will contain less information then in the actual lecture, and some will have extra info. Not all formulas which will be needed for exams are contained in these notes. Also, these notes will NOT contain any up to date organizational or administrative information (changes in schedule, assignments, etc.) but only physics. If you notice any typos - let me know at vitaly@njit.edu. For convenience, I will keep all notes in a single file - each time you can print out only the added part. A few other things:

Graphics: Some of the graphics is deliberately unfinished, so that we have what to do in class.

Advanced topics: these will not be represented on the exams. Read them only if you are really interested in the material.

Contents

I. INTRODUCTION

A. Physics and other sciences

in class

B. Point mass

The art physics is the art of idealization. One of the central concepts in mechanics is a

"particle" or "point mass"

i.e. a body the size or structure of which are irrelevant in a given problem. Examples: electron, planet, etc.

C. Units

1. Standard units

In SI system the **basic** units are:

$$\boxed{\text{m (meter), kg (kilogram) and s (second)}}$$

Everything else in mechanics is derived. Examples of derived units (may or may not have a special name): m/s, m/s^2 (no name), kg $\cdot$ m/s^2 (Newton), kg $\cdot$ m^2/s^2 (Joule), etc.

Major variables, their typical notations and units:

variable	units	name
speed, v	m/s	-
acceleration (a or g)	m/s^2	-
force (F, f, N, T)	N=kg m/s^2	Newton
work (W), energy (K, U, E)	J=kg m^2/s^2	Joule

Standard path: all units are converted to SI. E.g., length:

$$1\,\mathrm{in} = 0.0254\,\mathrm{m}\ , \quad 1\,\mathrm{ft} = 0.3048\,\mathrm{m}\ , \quad 1\,\mathrm{mi} \simeq 1609\,\mathrm{m}$$

Examples:

$$\text{speed:}\ \ 70\,\frac{\mathrm{mi}}{\mathrm{h}} = 70\frac{1609\,\mathrm{m}}{3600\,\mathrm{s}} \simeq 31.3\,\frac{\mathrm{m}}{\mathrm{s}}$$

$$\text{area:}\ \ 3\,\mathrm{cm}^2 = 3\,\left(10^{-2}\,\mathrm{m}\right)^2 = 3\cdot 10^{-4}\,\mathrm{m}^2$$

$$\text{volume:}\ \ 1\,\mathrm{mm}^3 = 1(10^{-3}\,\mathrm{m})^3 = 10^{-9}\,\mathrm{m}^3$$

$$\text{density:}\ \ 1\frac{\mathrm{gram}}{\mathrm{cm}^3} = 1\frac{10^{-3}\mathrm{kg}}{(0.01\,\mathrm{m})^3} \approx 1000\frac{\mathrm{kg}}{\mathrm{m}^3}\ \ \text{(water)}$$

$$218\frac{\mathrm{gram}}{\mathrm{in}^3} = 218\frac{10^{-3}\mathrm{kg}}{(0.0254\,\mathrm{m})^3} \approx 13.6\cdot 10^3\frac{\mathrm{kg}}{\mathrm{m}^3}\ \ \text{(mercury)}$$

$$315\frac{\mathrm{gram}}{\mathrm{in}^3} = 315\frac{10^{-3}\mathrm{kg}}{(0.0254\,\mathrm{m})^3} \approx 19.2\cdot 10^3\frac{\mathrm{kg}}{\mathrm{m}^3}\ \ \text{(gold)}$$

Oil is spilling from a pipe at a rate of $0.2\,ft^3/min$. Express this in SI units.

$$0.2\,\frac{ft^3}{min} = 0.2\cdot\frac{(0.305\,m)^3}{60\,s} \simeq 9.4\cdot 10^{-5}\frac{m^3}{s}$$

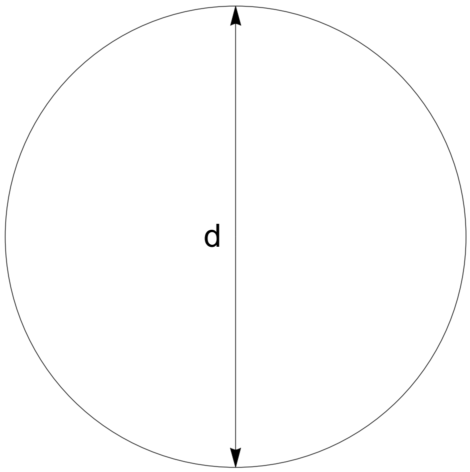
Non-standard: Convert the previous in $L/hour$:

$$1\,m^3 = 1000\,L\ ,\ 1\,s = \frac{1}{3600}hour\ \Rightarrow\ 9.4\cdot 10^{-5}\frac{m^3}{s} = 9.4\cdot 10^{-5}\frac{1000\,L}{hour/3600} \approx 340\frac{L}{hour}$$

Earth is $150\cdot 10^6\,km$ from Sun; find speed in km/s if 1 year $\simeq$ 365 days

$$2\pi\frac{150\cdot 10^6\,km}{365\cdot 24\cdot 3600\,s} \simeq 30\,km/s$$

3. Significant figures



A solid disk has a diameter of 1 cm. Find the circumference.

$$s = \pi d = 3.141592654\ldots \text{ cm. Wrong!! Why?}$$

Advanced.

The period of small oscillations of a pendulum is independent of its amplitude (Galileo). Use this to find the dependence of the period T on the length of the pendulum L, gravitational acceleration g and, possibly, mass M. Namely, look for

$$T \sim L^\alpha g^\beta M^\gamma$$

and find α, β and γ.

$$[T] = \mathrm{s}\,,\ [L] = \mathrm{m}\,,\ [g] = \frac{\mathrm{m}}{\mathrm{s}^2}\,,\ [M] = \mathrm{kg}$$

$$s = \mathrm{m}^\alpha \left(\frac{\mathrm{m}}{\mathrm{s}^2}\right)^\beta kg^\gamma = \mathrm{kg}^\gamma m^{\alpha+\beta} s^{-2\beta}$$

$$\gamma = 0\,,\ \alpha = -\beta\,,\ \beta = -1/2 \ \Rightarrow\ T \sim \sqrt{\frac{L}{g}}$$

Advanced. Less trivial example: gravitational waves. What is the speed? Can depend on $g, [m/s^2]$ on $\lambda, [m]$ and on $\rho, [kg/m^3]$

$$v \sim g^\alpha \lambda^\beta \rho^\gamma \ or\ [m/s] = [m/s^2]^\alpha [m]^\beta [kg/m^3]^\gamma$$

From dimensions,

$$\alpha = \beta = 1/2\,, \gamma = 0(!)$$

$$v \sim \sqrt{g\lambda}$$

What is neglected? Depth of the ocean, H. Thus,

$$v_{\max} \sim \sqrt{gH} \sim \sqrt{10 \cdot 4 \cdot 10^3} \sim 200\,\mathrm{m/s}$$

(the longest and fastest gravitational wave is tsunami). Note that we know very little about the precise physics, and especially the precise math of the wave, but from dimensional analysis could get a reasonable estimation.

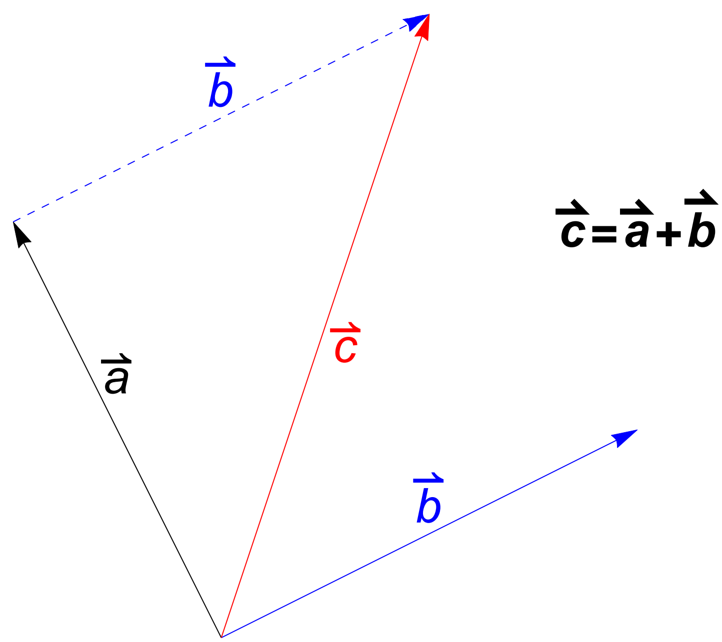
II. VECTORS

A *vector* is characterized by the following *three* properties:

- has a magnitude
- has direction (Equivalently, has several components in a selected system of coordinates).
- obeys certain addition rules ("Tail-to-Head" or, equivalently "rule of parallelogram").

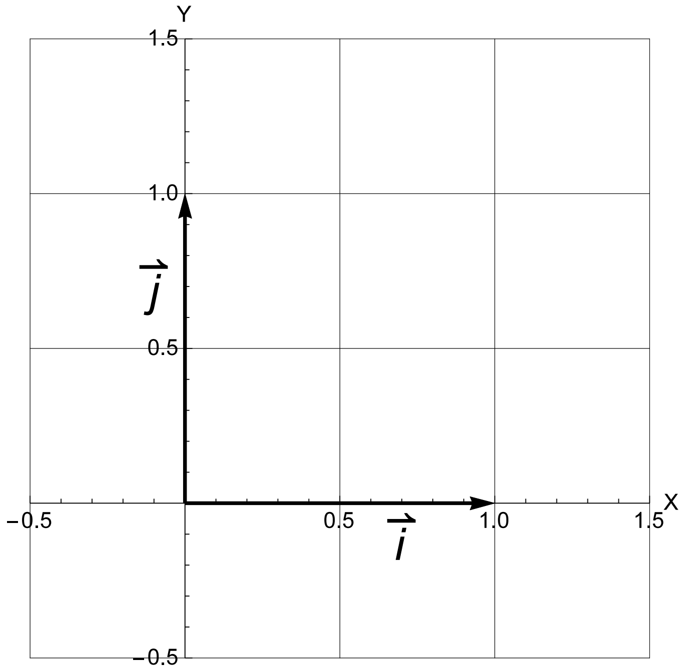
This is in contrast to a *scalar*, which has only magnitude and which is *not* changed when a system of coordinates is rotated.

How do we know which physical quantity is a vector, which is a scalar and which is neither? From experiment (of course). Examples of scalars are mass, time, kinetic energy. Examples of vectors are the displacement, velocity and force.



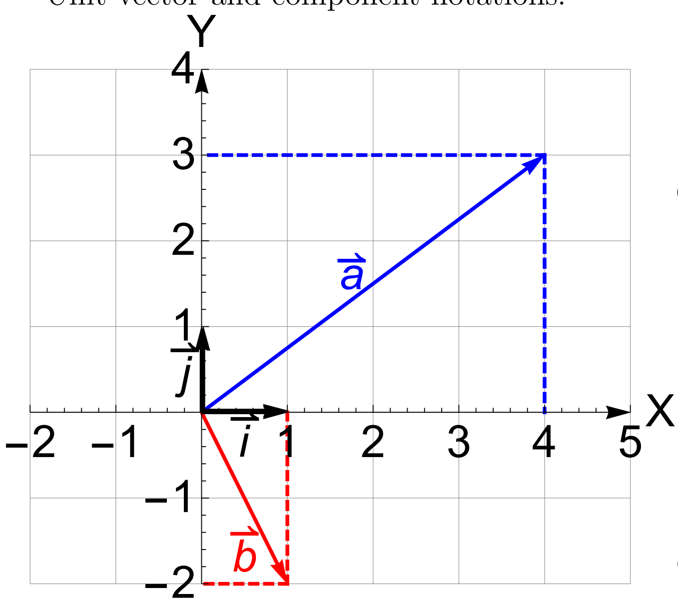
Tail-to-Head addition rule.

A. System of coordinates



Unit vectors $\vec{i}$, $\vec{j}$

Unit vector and component notations:



$$\vec{a} = 4\hat{i} + 3\hat{j} = (4, 3)$$

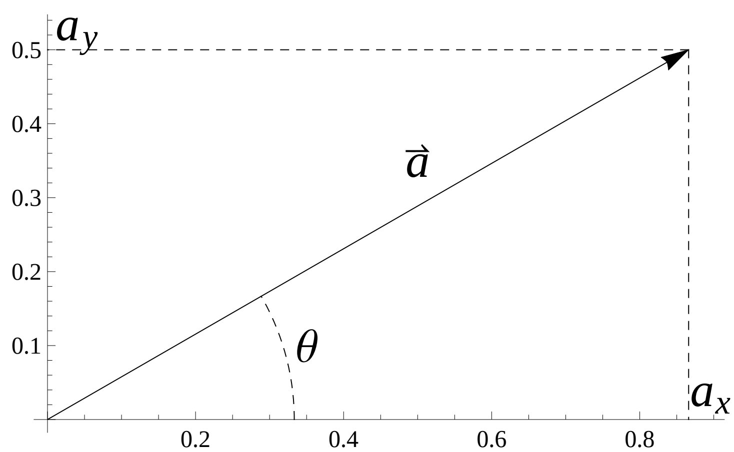
$$\vec{b} = \hat{i} - 2\hat{j} = (1, -2)$$

$$\text{Length: } a = |\vec{a}| = \sqrt{4^2 + 3^2} = 5$$

$$b = |\vec{b}| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

B. Operations with vectors

1. Single vector



Consider a vector $\vec{a}$ with components a_x and a_y (let's talk 2D for a while). The magnitude (or length) is given by the Pythagorean theorem

$$a \equiv |\vec{a}| = \sqrt{a_x^2 + a_y^2} \tag{1}$$

Note that for a different system of coordinates with axes x', y' the components $a_{x'}$ and $a_{y'}$ can be different, but the length in eq. (1) , obviously, will not change, which just means that length is a *scalar*.

Primary example: position vector (note two equivalent forms of notation)

$$\vec{r} = (x, y) = x\vec{i} + y\vec{j}$$

(sometimes, $\hat{i}$ and $\hat{j}$ is used) with $|\vec{i}| = |\vec{j}| = 1$.

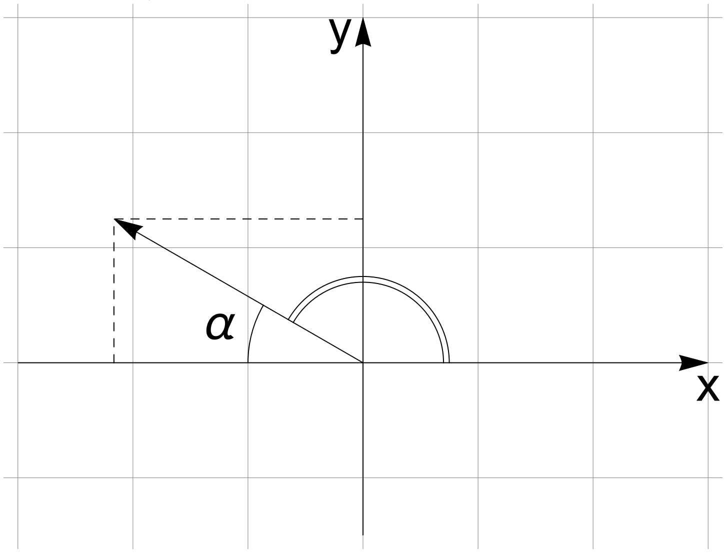
Polar coordinates:

$$r = \sqrt{x^2 + y^2}\ , \quad \theta = \arctan(y/x)$$

$$x = r\cos\theta\ , \quad y = r\sin\theta$$

Note: arctan might require adding 180^o - always check with a picture!

Example. Find the components of the vector $\vec{r}$ in the figure if its length is $r = 2.5$ units and $\alpha = 20^o$.



Solution 1 (from picture). The (x,y) components are "cut off" by the dashed lines. Thus, $x = -r\cos\alpha = -2.5\cos 20^o = -\ldots$ (note the minus!) and $y = 2.5\sin 20^o = \ldots$.

Solution 2 (from formulas). One has $x = r\cos\theta$, $y = r\sin\theta$, but θ is the angle with *positive* x-direction (double arc in the figure), or $\theta = 180^o - \alpha = 160^o$. Thus,

$$x = 2.5\cos 160^o = \cdots < 0\,,\; y = 2.5\sin 160^o > 0$$

(Check that you get the same numbers!)

Another operation allowed on a single vector is multiplication by a scalar. Note that the physical dimension ("units") of the resulting vector can be different from the original, as in $\vec{F} = m\vec{a}$.

2. Two vectors: addition

For two vectors, $\vec{a}$ and $\vec{b}$ one can define their sum $\vec{c} = \vec{a} + \vec{b}$ with components

$$c_x = a_x + b_x\ ,\quad c_y = a_y + b_y \tag{2}$$

The magnitude of $\vec{c}$ then follows from eq. (1). Note that physical dimensions of $\vec{a}$ and $\vec{b}$ must be identical.

Note: for most problems (except rotation!) it is allowed to carry a vector parallel to itself. Thus, we usually assume that every vector starts at the origin, $(0,0)$.

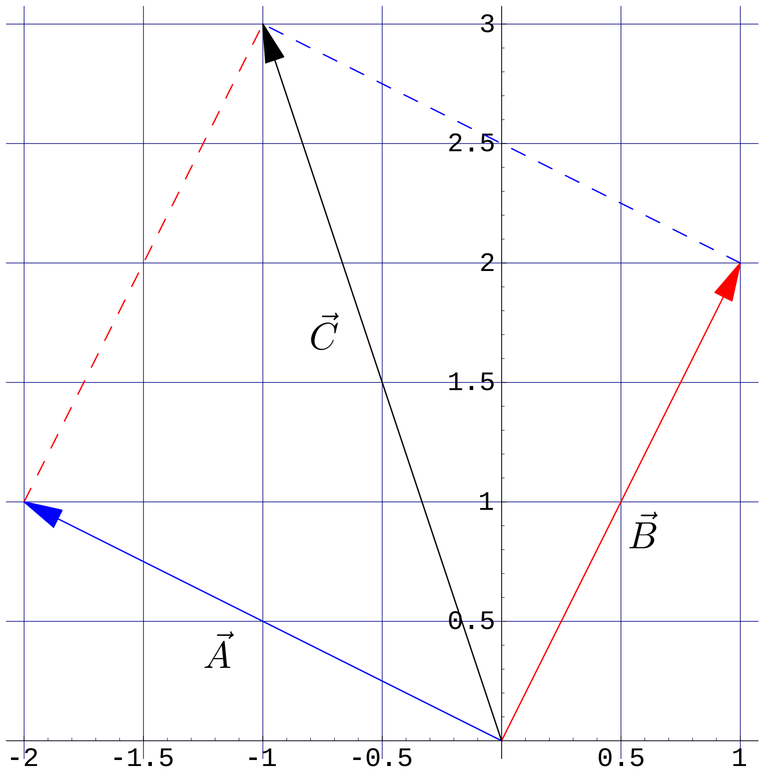


FIG. 1: Adding two vectors: $\vec{C} = \vec{A}+\vec{B}$. Note the use of rule of parallelogram (equivalently, tail-to-head addition rule). Alternatively, vectors can be added by components: $\vec{A} = (-2, 1)$, $\vec{B} = (1, 2)$ and $\vec{C} = (-2 + 1,\ 1 + 2) = (-1, 3)$.

Example Displacement $\vec{A} = -2\hat{i} + \hat{j}$ is followed by $\vec{B} = \hat{i} + 2\hat{j}$. Find magnitude and direction of the resultant displacement.

$$\vec{C} = \vec{A} + \vec{B} = -\hat{i} + 3\hat{j}$$

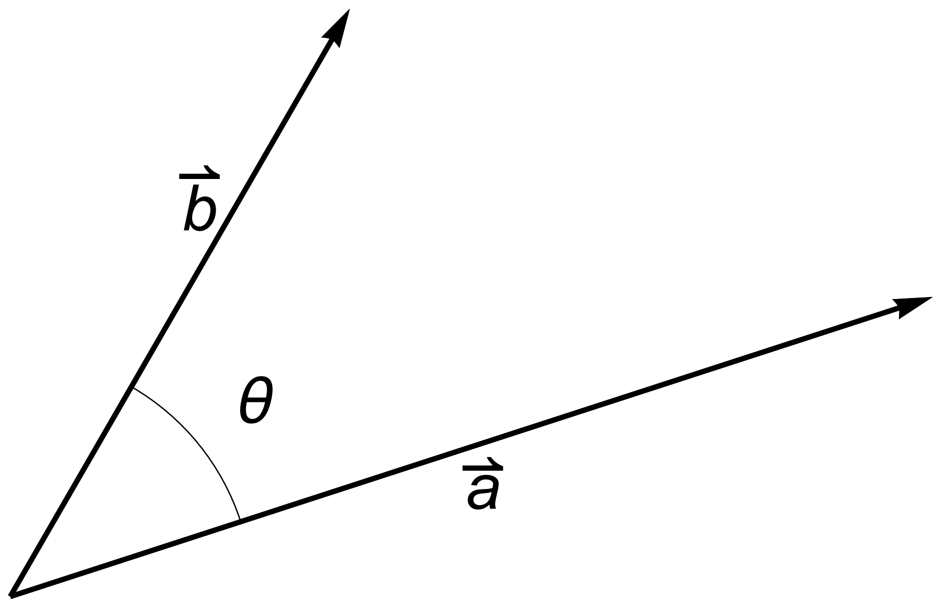
$$C = \sqrt{(-1)^2 + 3^2} = \sqrt{10}\,,$$

$$\cos\theta = \frac{C_x}{C} = \frac{-1}{\sqrt{10}} = -\ldots\,,\ \theta = \ldots > 90^o$$

Example (3D). For $\vec{A} = (1, 2, 3)$ and $\vec{B} = (-1, 1, 7)$ find $3\vec{A} + 4\vec{B}$

$$3\vec{A} + 4\vec{B} = (3 \cdot 1 + 4 \cdot (-1),\ 3 \cdot 2 + 4 \cdot 1,\ 3 \cdot 3 + 4 \cdot 7) = (-1, 10, 37)$$

3. Two vectors: dot product - need for "work"



If $\vec{a}$ and $\vec{b}$ make an angle θ with each other, their scalar (dot) product is defined as

$$\vec{a} \cdot \vec{b} = ab\cos(\theta)$$

or in components

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y \tag{3}$$

A different system of coordinates can be used, with different individual components but with the same result. For two orthogonal vectors $\vec{a} \cdot \vec{b} = 0$. *Preview.* The main application of the scalar product is the concept of work $\Delta W = \vec{F} \cdot \Delta\vec{r}$, with $\Delta\vec{r}$ being the displacement. Force which is perpendicular to displacement does not work!

Example. See Fig. 1. $\vec{A} = -2\hat{i} + \hat{j}$, $\vec{B} = \hat{i} + 2\hat{j}$

$$\vec{A} \cdot \vec{B} = (-2)1 + 1 \cdot 2 = 0$$

(thus angle is 90^o).

Example Find angle between 2 vectors $\vec{B}$ and $\vec{C}$ in Fig. 1.

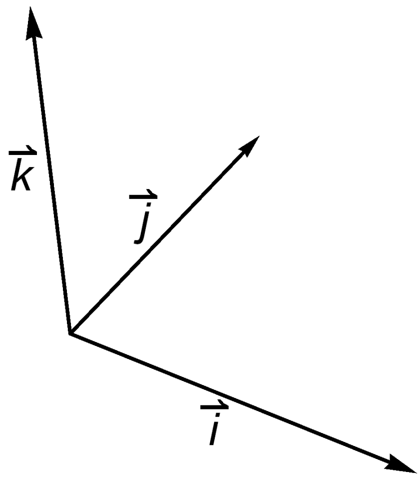
General:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{ab}$$

$$\text{In Fig. 1: } B = \sqrt{1^2 + 2^2} = \sqrt{5}\,,\ C = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\cos\theta = \frac{(-1) \cdot 1 + 3 \cdot 2}{5\sqrt{2}} = \frac{1}{\sqrt{2}}\,,\ \theta = 45^o$$

Add unit vector $\vec{k}$ in the z-direction.



$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k} = (A_x\ ,\ A_y\ ,\ A_z)\ ,\ A = \left|\vec{A}\right| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = AB\cos\theta$$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB}$$

Example. $\vec{A} = (1,2,3)$, $\vec{B} = (1,-1,0)$. Find $\vec{A}\cdot\vec{B}$ and the angle between them.

$$\vec{A}\cdot\vec{B} = 1*1 + 2*(-1) + 3*0 = -1$$

$$A = \sqrt{1^2+2^2+3^2} = \sqrt{14}\,,\ B = \sqrt{1^2+(-1)^2+0^2} = \sqrt{2}$$

$$\cos\theta = \frac{-1}{\sqrt{14}\sqrt{2}} = -\frac{1}{2\sqrt{7}}\ ,\ \theta \simeq 101^o$$

4. Two vectors: vector product - need for "torque"

At this point we must proceed to the 3D space. Important here is the correct system of coordinates, as in Fig. 2. You can rotate the system of coordinates any way you like, but you cannot reflect it in a mirror (which would switch right and left hands).

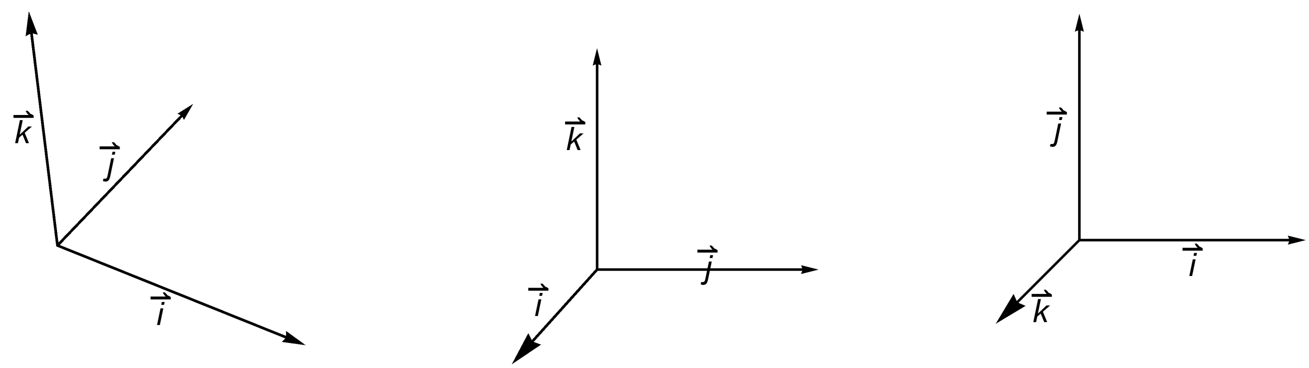


FIG. 2: The correct, "right-hand" systems of coordinates. Checkpoint - curl fingers of the RIGHT hand from x-direction ($\vec{i}$) to y-direction ($\vec{j}$), then the thumb should point into the z-direction ($\vec{k}$).

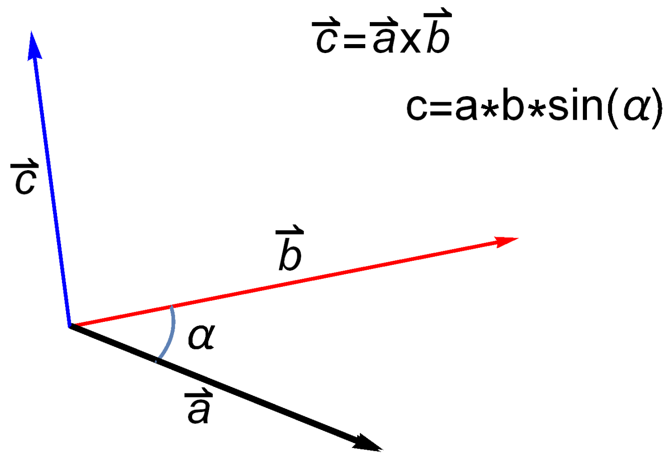


FIG. 3: Example of a cross product $\vec{c} = \vec{a} \times \vec{b}$. Direction: perpendicular to it both $\vec{a}$ and $\vec{b}$ ('right hand rule'). Magnitude - as indicated.

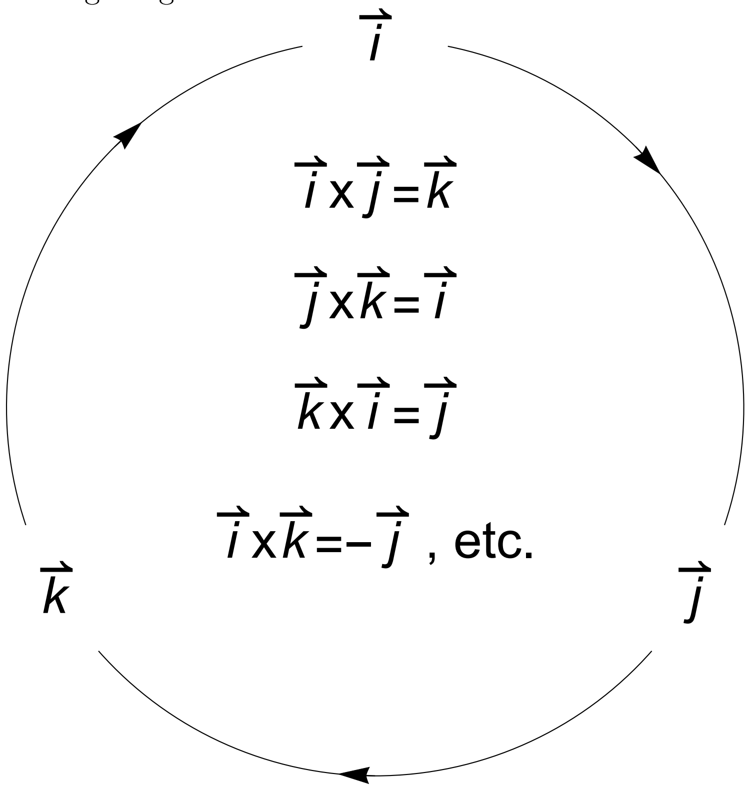
If $\vec{a}$ and $\vec{b}$ make an angle $\alpha \leq 180^o$ with each other, their vector (cross) product $\vec{c} = \vec{a} \times \vec{b}$ has a magnitude $c = ab\sin(\alpha)$. The direction is defined as perpendicular to both $\vec{a}$ and $\vec{b}$ using the following rule: curl the fingers of the right hand from $\vec{a}$ to $\vec{b}$ in the shortest direction (i.e., the angle must be smaller than 180^o). Then the thumb points in the $\vec{c}$ direction. Check with Fig. 3. Changing the order changes the sign, $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$. In particular, $\vec{a} \times \vec{a} = \vec{0}$. More generally, the cross product is zero for any two parallel vectors.

Suppose now a system of coordinates is introduced with unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ pointing in the x, y and z directions, respectively. First of all, if $\hat{\imath}$, $\hat{\jmath}, \hat{k}$ are written "in a ring", the cross product of any two of them equals the third one in clockwise direction, i.e.

$$\hat{\imath} \times \hat{\jmath} = \hat{k}\,,\; \hat{\jmath} \times \hat{k} = \hat{\imath}\,,\; \hat{k} \times \hat{\imath} = \hat{\jmath}$$

, etc.

Ring Diagram:



Example. Fig. 1:

$$\vec{A} = -2\hat{\imath} + \hat{\jmath}\ ,\ \vec{B} = \hat{\imath} + 2\hat{\jmath}$$

$$\vec{A} \times \vec{B} = (-2\hat{\imath} + \hat{\jmath}) \times (\hat{\imath} + 2\hat{\jmath}) = (-2) \cdot 2\hat{\imath} \times \hat{\jmath} + \hat{\jmath} \times \hat{\imath} =$$

$$= -4\hat{k} - \hat{k} = -5\hat{k}$$

(Note: in Fig. 1 $\hat{k}$ goes out of the page; the cross product $\vec{A} \times \vec{B}$ goes into the page, as indicated by "-".)

More generally, the cross product is now expressed as a 3-by-3 determinant

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{\imath} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\jmath} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{k} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \tag{4}$$

The two-by-two determinants can be easily expanded. In practice, there will be many zeros, so calculations are not too hard.

Preview. Vector product is most relevant to rotation.

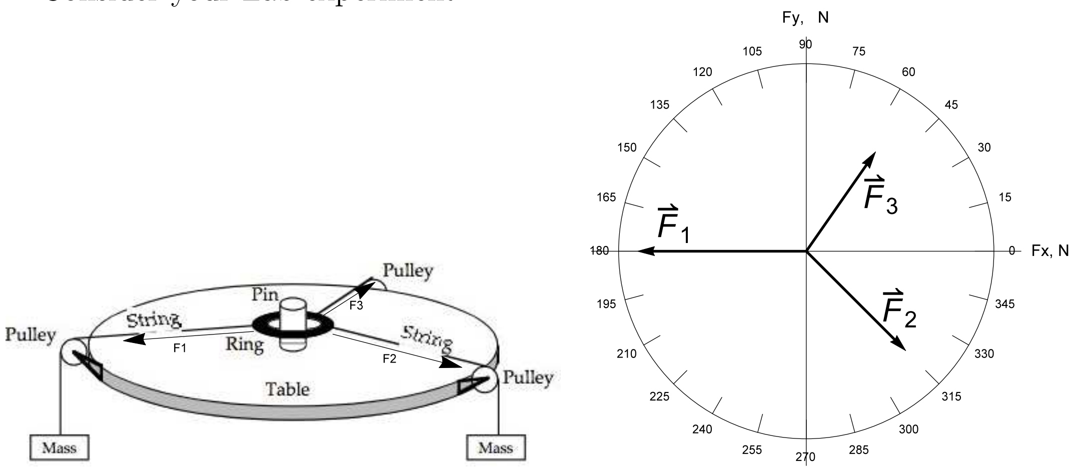
Example. See Fig. 1.

$$\vec{A} \times \vec{B} = \hat{k}((-2)2 - 1 \cdot 1) = -5\hat{k}$$

C. Preview. Forces as vectors.

Besides displacement $\vec{r}$, and velocity $\vec{v}$, forces represent another example of a vector. Note that they are measured in different units, N (*newtons*), i.e. each component of the force F_x , F_y , F_z is measured in N. How do we know that force is a vector? From experiments on static equilibrium which demonstrate that forces indeed add up following the standard vector addition rule of parallelogram (or, that they add up by components, which is the same thing).

Consider your Lab experiment



The force table (left) and its schematic representation (right)

$$\text{In equilibrium: } \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0\,,\ \Rightarrow\ \vec{F}_3 = -\left(\vec{F}_1 + \vec{F}_2\right)$$

$$\text{In the example: } F_1 = 0.9\,N\,,\ \theta_1 = 180^o\ ;\ F_2 = 0.75\,N\,,\ \theta_2 = 315^o = -45^o \Rightarrow$$

$$F_{1x} = 0.9\cos(180^o) = -0.9\,,\ F_{1y} = 0.9\sin(180^o) = 0$$

$$F_{2x} = 0.75\cos(-45^o) = 0.53\,,\ F_{2y} = 0.75\sin(-45^o) = -0.53$$

$$F_{3x} = -(F_{1x} + F_{2x}) = 0.37\,,\ F_{3y} = -(F_{1y} + F_{2y}) = 0.53$$

$$\theta_3 = \tan^{-1}\frac{F_{3y}}{F_{3x}} = \tan^{-1}\frac{0.53}{0.37} \simeq 55^o$$

(check with picture that this is about correct; the angle $55^o + 180^o = 235^o$ has the same tan). Similarly, the magnitude

$$F_3 = \sqrt{.37^2 + .53^2} = 0.646\ (N)$$

III. 1-DIMENSIONAL MOTION

Position $x(t)$

Displacement:

$$\Delta x = x(t_2) - x(t_1)$$

Distance:

$$D \geq |\Delta x| \geq 0$$

Velocity:

$$v = \frac{\Delta x}{\Delta t}\,,\ \Delta t = t_2 - t_1$$

with a small Δt (later, we distinguish between *average* velocity with a finite Δt and *instantaneous* with $\Delta t \to 0$).

Speed:

$$s = \frac{D}{\Delta t} \geq |v| \geq 0$$

A. $v = const$

See fig. 4

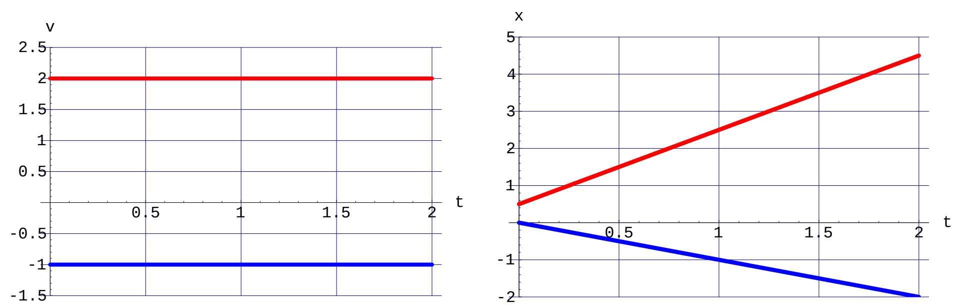


FIG. 4: Velocity (left) and position (right) plots for motion with constant velocity: Positive (red) or negative (blue). Note that area under the velocity line (positive or negative) corresponds to the change in position: E.g. (red) $2 \times 2 = 4.5 - 0.5$, or (blue) $2 \times (-1) = -2 - 0$.

Displacement

$$\Delta x = v\,\Delta t \tag{5}$$

Distance

$$D = |\Delta x| \,, \text{ for v=const only} \tag{6}$$

Speed

$$s = D/\Delta t = |v| \,, \text{ for v=const only} \tag{7}$$

Example. A motorcycle with $V_M = 60\,m/s$ is catching up with a car with $V_C = 40\,m/s$, originally $D = 200\,m$ ahead. When and where will they meet? Give the graphic solution (in class).

$$x_M = V_M t \ , \ x_C = V_C t + D \ , \text{ and } x_M = x_C \text{ - they meet}$$

$$V_M t = V_C t + D \,, \ thus, \ t = \frac{D}{V_M - V_C} = \frac{200}{60 - 40} = 10\,s$$

Above is the meeting time (note "relativity" - in a reference frame moving with V_M the motorcycle is stationary, while the car approaches it with speed of $V_M - V_C$). The meeting point - the distance from the original position of the motorcycle - is

$$x = V_M t = D\frac{V_M}{V_M - V_C} = 600\,m$$

Graphically, the solution is intersection in t, x plane.

B. Variable velocity

Example (trap!): A hiker goes from A to B with $S_1 = 2\,km/h$ and returns with $S_2 = 4\,km/h$. Find S_{av} .

$$S_1 = 2\,km/h; , \; S_2 = 4\,km/h \; . \; S_{av} - ?$$

$$D = 2AB\,, \; t_1 = AB/S_1\,, \; t_2 = AB/S_2$$

$$S_{av} = \frac{D}{t_1 + t_2} = \frac{2AB}{AB/S_1 + AB/S_2} = \frac{2S_1S_2}{S_1 + S_2} \neq 3\,km/h$$

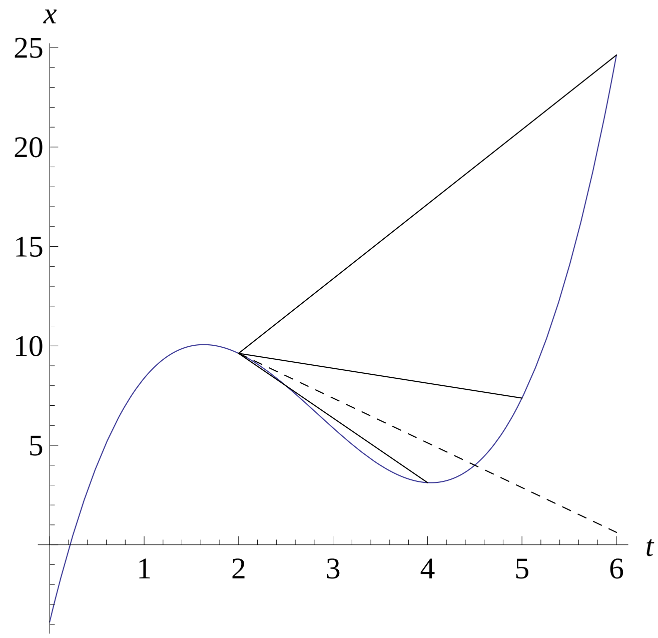


FIG. 5: A sample position vs. time plot (blue curve), and determination of the average velocities - slopes (positive or negative) of straight solid lines. Slope of dashed line (which is tangent to $x(t)$ curve) is the instantaneouus velocity at $t = 2$.

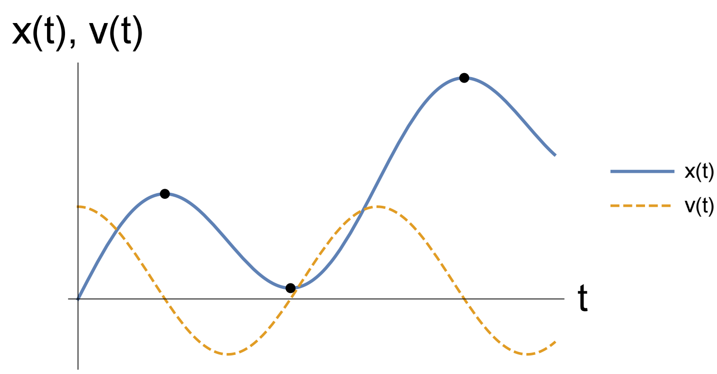


FIG. 6: Example. Instantaneous velocity as the slope of $x(t)$. Points indicate instances with $v(t) = 0$.

$$\text{Average velocity: } v_{av} = \frac{\Delta x}{\Delta t}\,, \quad \text{Instantaneous: } \quad v = \lim_{\Delta t \to 0} v_{av} = \frac{dx}{dt} \tag{8}$$

$$\text{Distance: } \quad D \geq |\Delta x|\ , \quad \text{Average speed: } \quad s_{av} = \frac{D}{\Delta t} \geq |v_{av}| \tag{9}$$

$$\text{Instantaneous speed: } s = \frac{dD}{dt} = |v| \tag{10}$$

2. Displacement from velocity

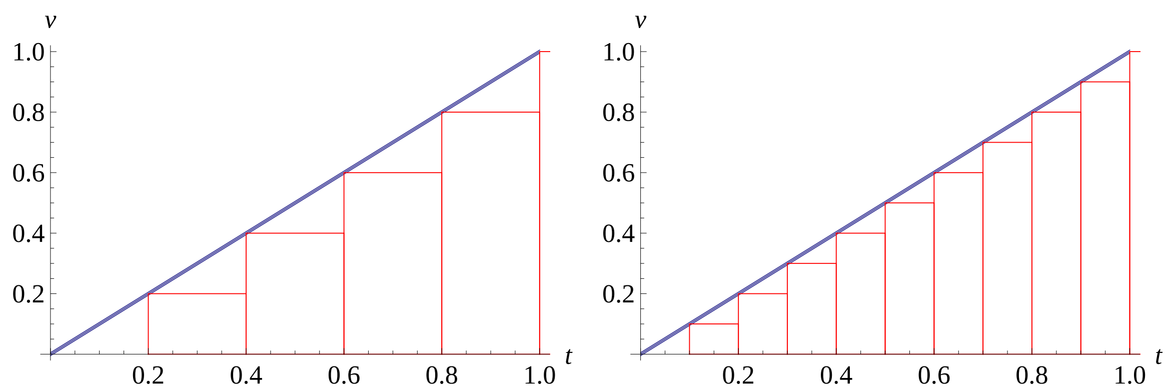


FIG. 7: Determination of displacement for a variable $v(t)$. During the ith small interval of duration Δt the velocity is replaced by a constant v_i shown by a horizontal red segment. Corresponding displacement is $\Delta x_i \approx v_i \cdot \Delta t$ (the red rectangular box). The total displacement $\Delta x = \sum \Delta x_i$ is then approximated by the area under the $v(t)$ curve.

$$\text{Displacement: } \quad \Delta x(t) = \text{"area" under the } v(t) \text{ curve } \; = \int_{t_1}^{t} v\,(t')\; dt' \tag{11}$$

$$\text{average:} \quad a_{av} = \frac{\Delta v}{\Delta t} \tag{12}$$

$$\text{instantaneous:} \;\; a = \lim_{\Delta t \to 0} a_{av} = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{13}$$

Geometric meaning: a - slope of $v(t)$. If $a(t)$ is known, Δv is the "area" under the $a(t)$ curve.

Example: Given (t in seconds, x in meters)

$$x(t) = -\frac{t^4}{4} + \frac{t^3}{3} + t^2 - t + 1$$

Find: a) v_{av} between $t_1 = 1\,s$ and $t_2 = 3\,s$; b) $v(t)$, $a(t)$

$$v_{av} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{-17/4 - 13/12}{3 - 1} = -\frac{8}{3}$$

b) from

$$\frac{d}{dt}t^n = nt^{n-1} \; : \; v(t) = -t^3 + t^2 + 2t - 1\,, \; a(t) = -3t^2 + 2t + 2$$

Example. A particle is moving according to $x = 20t^2$ (with x in meters, t in seconds. When $t = 2\,s$ (a) find a and (b) find v

$$v(t) = \frac{dx}{dt} = 40t\,, \; a = \frac{dv}{dt} = 40\,\frac{m}{s^2}$$

$$v(t = 2) = 80\,\frac{m}{s}\,, \; a = 40\,\frac{m}{s^2}$$

4. $a = const$

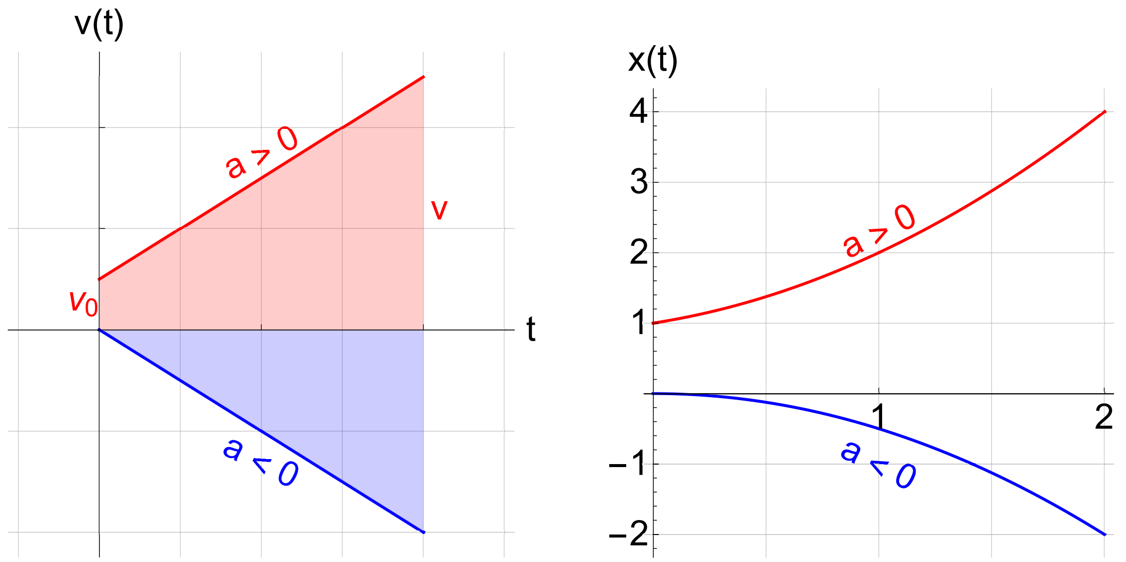


FIG. 8: Velocity (left) and position (right) plots for motion with constant acceleration: Positive (red) or negative (blue). Again, area under the velocity line (positive or negative) corresponds to the change in position. E.g. (red) $(2.5 + 0.5) \times 2/2 = 4 - 1$ or (blue) $(-2) \times 2/2 = -2 - 0$.

Notations: Start from $t = 0$, thus $\Delta t = t$; $v(0) \equiv v_0$.

$$\Delta v = at\,, \;\; v = v_0 + at \tag{14}$$

Displacement - area of the trapezoid in fig. 8 (can be negative!):

$$\Delta x = \frac{v_0 + v}{2} t = v_0 t + at^2/2 \tag{15}$$

A useful alternative: use $t = (v - v_0)\,/a$:

$$\Delta x = \frac{v_0 + v}{2}\frac{v - v_0}{a} = \frac{v^2 - v_0^2}{2a} \tag{16}$$

(A more elegant derivation follows from conservation of energy,... later)

SUMMARY: if

$$a = const\,, \tag{17}$$

$$v = v_0 + at \tag{18}$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \tag{19}$$

$$x - x_0 = \frac{v_0 + v}{2}t = \frac{v^2 - v_0^2}{2a} \tag{20}$$

Example. A motorcycle accelerates from rest for $t = 2.0\,s$ traveling $x = 20\,m$ during that time. Find acceleration a. *Solution:*. Select an equation which has no $v(t)$ (and $v_0 = 0$ since 'from rest' and $x_0 = 0$ for convenience)

$$x = \frac{1}{2}at^2 \;\Rightarrow\; a = \frac{2x}{t^2} = \frac{2\cdot 20}{2.0^2} = 10\,m/s^2$$

Example. A car accelerates from $v_0 = 5\,m/s$ to $v = 35\,m/s$ with $a = 3\,m/s^2$. (a) How far will it go? (b) How long will it take? *Solution:*. (a) time is not given, thus select the only formula which has no t:

$$x = \frac{v^2 - v_0^2}{2a} = \frac{35^2 - 5^2}{2\times 3} = \ldots$$

(b) select a formula with no x

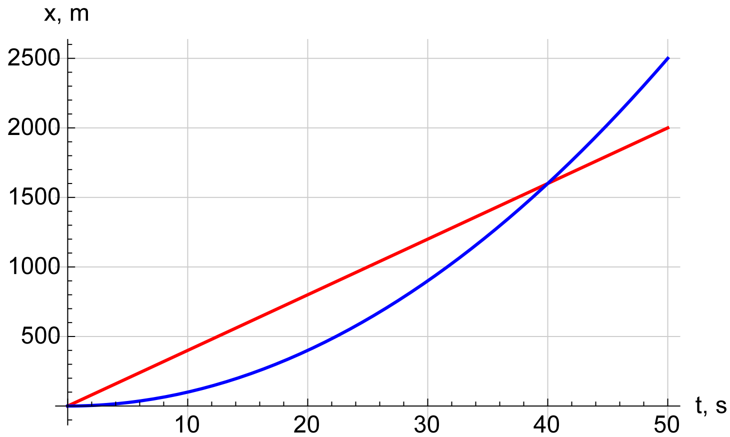
$$t = (v - v_0)\,/a = \ldots$$

Example. After the driver hits the brakes, the car skids for $10\,s$ a distance $100\,m$ before it stops. (a) Find the initial speed v_0; (b) find the deceleration. *Solution:*. (a) acceleration not given or asked for, thus use

$$x = \frac{v_0 + v}{2}t \text{ with } v = 0 \;\Rightarrow$$

$$v_0 = \frac{2x}{t} = \frac{2\times 100}{10} = 20\,m/s\,, \text{ and } a = \frac{v - v_0}{t} = -\frac{v_0}{t} = \ldots \quad \text{(b)}$$

Example: meeting problems (car $V_C = 40\,m/s$, $a_C = 0$ and motorcycle $V_M = 0$, $a_M = 2\,m/s^2$)



$$X_C = V_C t\,,\ X_M = \frac{1}{2} a_M t^2$$

$$V_C t = \frac{1}{2} a_M t^2\,,\ t = 2V_C/a_M = 40\,s\,,\ X_{meet} = t \cdot V_C = 1600\,m = \frac{1}{2} a_M t^2$$

C. Free fall

(a)

Reminder:

$$\text{if } a = const \text{ then} \tag{21}$$

$$v = v_0 + at \tag{22}$$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 \tag{23}$$

$$x - x_0 = \frac{v_0 + v}{2} t = \frac{v^2 - v_0^2}{2a} \tag{24}$$

Free fall:

$$a \ \rightarrow \ -g \, , \ x \ \rightarrow \ y \, , \ x_0 \ \rightarrow \ y_0 \ (or, \ H)$$

$$a = -g = -9.8\, m/s^2 \tag{25}$$

$$v = v_0 - gt \tag{26}$$

$$y = y_0 + v_0 t - \frac{1}{2} g t^2 \tag{27}$$

$$y - y_0 = \frac{v_0^2 - v^2}{2g} \tag{28}$$

Example: max height:

$$v = 0\,,\ y_{\max} - y_0 = v_0^2/2g \tag{29}$$

Example: the Tower of Piza ($v_0 = 0\,,\ y_0 = H \simeq 55\, m$). Find t, v upon impact.

$$0 = H + 0t - gt^2/2\,,\ t = \sqrt{\frac{2H}{g}}$$

$$0 - H = -v^2/2g\,,\ v = \sqrt{2gH}$$

What if $v_0 = 10\, m/s$? (use $g \approx 10\, m/s^2$)

$$0 = H + v_0 t - \frac{1}{2} g t^2\,,\, 0 = 55 + 10t - 5t^2$$

$$t^2 - 2t - 11 = 0,\, t = 1 \pm \sqrt{1^2 + 11} = 1 + \sqrt{12} = \ldots$$

(only positive root!)

$$0 - H = \frac{v_0^2 - v^2}{2g}\,,\, v^2 = v_0^2 + 2gH$$

$$v = \sqrt{v_0^2 + 2gH} \approx \sqrt{10^2 + 2 * 10 * 55} = \ldots$$

If $v_0 = -10\, m/s$

$$0 = 55 - 10t - 5t^2$$

$$t^2 + 2t - 11 = 0,\, t = -1 \pm \sqrt{1^2 + 11} = -1 + \sqrt{12} = \ldots$$

$v = \sqrt{v_0^2 + 2gH}$ (same: sign of v_0 does not matter!)

Other examples - ignore air friction

A shell is fired vertically up with $v_0 = 200\,m/s$. Find $h_{\max}$.

$$y - y_0 = \frac{v_0^2 - v^2}{2g}\,,\; y - y_0 = h_{\max}\,,\; v = 0 \text{ @ max } \Rightarrow$$

$$h_{\max} = \frac{v_0^2}{2g} = \frac{200^2}{2 * 9.8} \approx 2 \cdot 10^3\,m$$

A package is dropped from a helicopter moving upward at $v_0 = 20\,m/s$. If it takes $t = 15\,s$ before the package strikes the ground, (A) how high above the ground was the package when it was released if air resistance is negligible? (B) How long is the path?

$$(A)\; 0 = y_0 + v_0 t - \frac{1}{2}gt^2\,,\; y_0 = \frac{1}{2}gt^2 - v_0 t = \frac{1}{2} * 9.8 * 15^2 - 20 * 15 = \ldots$$

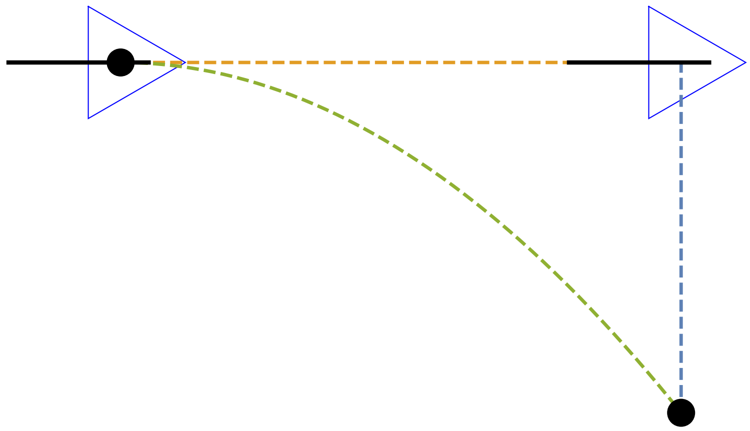
$$(B)\; D = y_0 + 2 \times h_{\max} = y_0 + 2\frac{v_0^2}{2g} = \ldots$$

Link to "hammer-vs-feather" :

`https://www.youtube.com/watch?v=KDp1tiUsZw8`

IV. PROJECTILE MOTION

A. Introduction: Object dropped from a plane



Given: $y_0 = H = 100\,m$ and $\vec{v}_0 = V\hat{i}$, with $V = 200\,m/s$ (horizontal). Find: (a) horizontal distance L to hit the ground, (b) the speed v upon impact and the angle and (c) position of the object relative to the plain. Solution:
Vertical motion (horizontal velocity does not matter!):

$$y(0) = H\,,\; v(0) = 0 \;\Rightarrow\; y(t) = H - \frac{1}{2}gt^2$$

Time to fall, $y(t) = 0$

$$t = \sqrt{2H/g}$$

(a) From x-direction

$$L = Vt = V\sqrt{2H/g} = 200\sqrt{2 \cdot 100/9.8} = \ldots$$

(b) Speed upon impact. From y-direction

$$y - y_0 = \frac{v_{0y}^2 - v_y^2}{2g} \;\Rightarrow\; v_y^2 = 2gH$$

(could use $v_y = -gt$ with calculated t). From x-direction

$$v_x = const = V \;\Rightarrow\; v^2 = v_x^2 + v_y^2 = V^2 + 2gH = 200^2 + 2 * 9.8 * 100 = \ldots$$

(the last formula is also valid for non-horizontal launch with V replaced by v_0, the full initial speed).

Angle of impact with horizontal:

$$\tan\theta = v_y/v_x = -\sqrt{2gH}/V$$

(c) since $v_x = const = V$ the object is right under the plane (!)

Another example: vertical toy cannon on a moving cart.

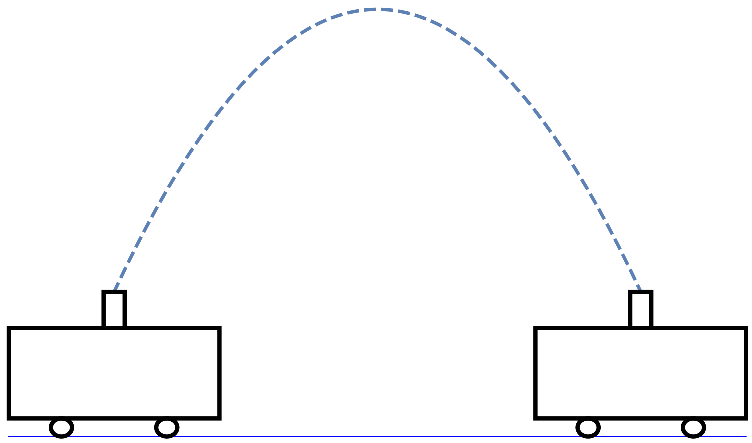


FIG. 9: The ball lands back into the cannon for any constant velocity of the cart (Galileo's relativity!). The maximum hight and the time of flight depend only on vertical velocity but not on the horizontal motion.

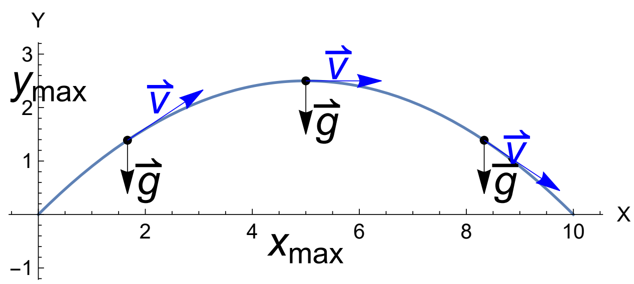
B. General

$$x\text{-axis horizontal, } y\text{-axis vertical (up): } \boxed{a_x = 0\,,\quad a_y = -g} \tag{30}$$

$$\boxed{v_x = v_{0,x} = const\,,\ v_y = v_{0,y} - gt} \text{ with } \boxed{v_{0,x} = v_0\cos\theta\,,\ v_{0,y} = v_0\sin\theta} \tag{31}$$

$$\text{Displacement: } \boxed{x = x_0 + v_{0,x}t}\,,\ \boxed{y = y_0 + v_{0,y}t - \frac{1}{2}gt^2} \tag{32}$$

$$\text{Also, } \boxed{y - y_0 = \frac{v_{0y}^2 - v_y^2}{2g}}$$



$$\text{Max elevation: } \boxed{y_{\max} - y_0 = \frac{v_{0,y}^2}{2g}}\,,\ x_{\max} = \frac{v_{0,x}v_{0,y}}{g} \tag{33}$$

Range:

$$R = 2x_{\max} = 2\frac{v_0\cos\theta\, v_0\sin\theta}{g}$$

$$\boxed{R = \frac{v_0^2}{g}\sin(2\theta)} \tag{34}$$

Note maximum for $\theta = 45^o$.

Trajectory: (use $x_0 = y_0 = 0$). Exclude time, $t = x/v_{0,x}$. Then

$$y = x\frac{v_{0,y}}{v_{0,x}} - \frac{1}{2}g\frac{x^2}{v_{0,x}^2} = x\tan\theta - \frac{g}{2v_0^2\cos^2\theta}x^2 \tag{35}$$

This is a parabola - see Fig. 10.

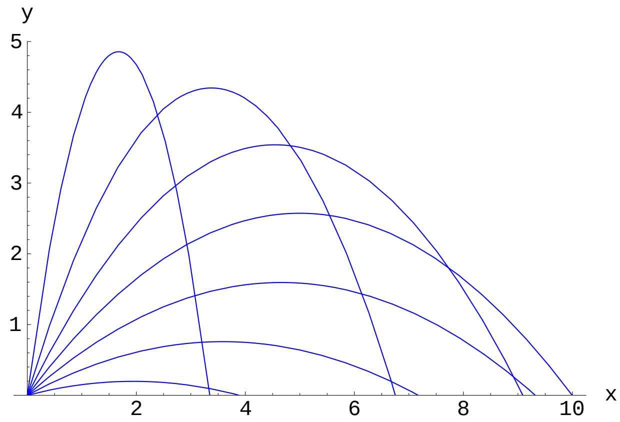


FIG. 10: Projectile motion for different values of the initial angle θ with a fixed value of initial speed v_0 (close to $10\,m/s$). Maximum range is achieved for $\theta = 45^o$.

C. Examples

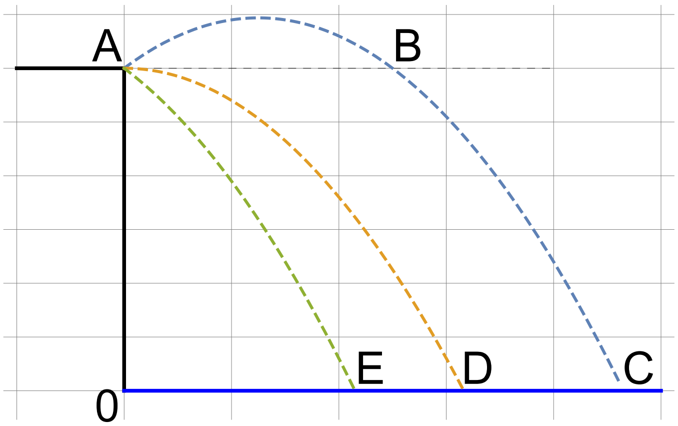
Problem. A daredevil on a motorcycle wants to jump across an L=15m - wide river starting from a horizontal cliff which is H=10m high. What should be his initial speed V?

$$y = H + 0t - \frac{1}{2}gt^2 \;\Rightarrow\; t = \sqrt{\frac{2H}{g}} = \ldots$$

$$L = Vt \;\Rightarrow\; V = \frac{L}{t} = \ldots$$

Problem. A coastguard cannon is placed on a cliff $y_0 = 60\,m$ above the sea level. Three shells are fired at angles $\theta = 0$ and $\theta = \pm 30^o$ with horizontal, each with an initial speed $v_0 = 80\,m/s$. Find the following:

1. the horizontal distance x from the cliff to the point where each projectile hits the water
2. the speed upon impact



Solution: $\underline{\theta = 0}$ exactly as for the plane, $V = v_0$: First find t, time to hit the water (from vertical motion only!):

$$y = y_0 + 0t - \frac{1}{2}gt^2\,,\; y = 0 \;\Rightarrow\; t = \sqrt{\frac{2y_0}{g}} = \ldots\,, \quad \text{horizontal motion: } x = v_0 t = \ldots$$

(b) - horizontal component of the velocity $v_x = V$, vertical component $v_y = -gt = \ldots$

$$v = \sqrt{v_x^2 + v_y^2} = \ldots \approx 87.2\frac{m}{s}$$

$$\underline{\theta = +30^o}: \quad v_x = v_0 \cos\theta \approx 69.3\frac{m}{s}\,,\; v_{0y} = v_0 \sin\theta = 40\,\frac{m}{s}$$

Note $y = 0$ at the end. Find time from vertical motion only (use here $g \approx 10\,m/s^2$)

$0 = y_0 + v_{0y}t - \frac{1}{2}gt^2 \;\Rightarrow\; 0 = 60 + 40t - 5t^2$ or $t^2 - 8t - 12 = 0$ with $t \approx 9.3\,s$ (the positive root)

Horizontal distance: $x = v_x t = \ldots$

$$\text{Speed upon impact: } v = \sqrt{v_x^2 + (v_{0,y} - gt)^2} \approx 87.2\frac{m}{s}$$

Angle θ does not affect the final speed.

$\underline{\theta = -30^o}$ - same, but $v_{0,y} = -40\frac{m}{s}$ and $0 = 60 - 40t - 5t^2$ with $t \approx 1.3\,s$.

V. 2D MOTION

A. Introduction: Derivatives of a vector

Reminder from 1D motion:

$$\boxed{v = \frac{dx}{dt}}, \ \boxed{a = \frac{dv}{dt}}$$

$$\Delta v = \int_{t_1}^{t_2} a(t)\, dt \ , \ \Delta x = \int_{t_1}^{t_2} v(t)\, dt$$

For 2D:

$$\vec{r}(t) = (x(t)\ ,\ y(t)) = x(t)\vec{i} + y(t)\vec{j}$$

$$\vec{v}(t) = \frac{d}{dt}\vec{r} = \left(\frac{dx}{dt}\ ,\ \frac{dy}{dt}\right) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$$

$$\vec{a}(t) = \frac{d^2}{dt^2}\vec{r} = \left(\frac{d^2x}{dt^2}\ ,\ \frac{d^2y}{dt^2}\right) = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j}$$

and similarly for integrals.

B. General

$$\text{Position: } \vec{r} = \vec{r}(t)$$

$$\text{Average velocity: } \vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t}$$

(see Fig. 11).

$$\text{Instantaneous velocity: } \vec{v} = \lim_{\Delta t \to 0} \vec{v}_{av} = \frac{d\vec{r}}{dt}$$

$$\text{Average acceleration: } \vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t}$$

$$\text{Instantaneous acceleration: } \vec{a} = \lim_{\Delta t \to 0} \vec{a}_{av} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

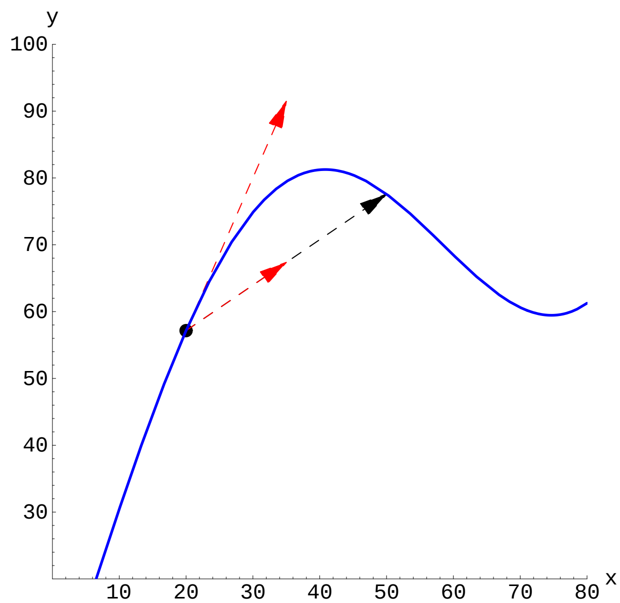


FIG. 11: Position of a particle $\vec{r}(t)$ (blue line), finite displacement $\Delta\vec{r}$ (black dashed line) and the average velocity $\vec{v} = \Delta\vec{r}/\Delta t$ (red dashed in the same direction). The instantaneous velocity at a given point is tangent to the trajectory.

C. $\vec{a} = const$

$$\Delta\vec{v} = \vec{a} \cdot \Delta t$$

or with $t_0 = 0$

$$\vec{v} = \vec{v}_0 + \vec{a} \cdot t \tag{36}$$

Displacement:

$$\vec{r} = \vec{r}_0 + \vec{v}_0 \cdot t + \frac{1}{2}\vec{a} \cdot t^2 \tag{37}$$

(The above can be proven either by integration or by writing eq. (36) in components and using known 1D results).

Example. A particle moves according to

$$\vec{r}(t) = 2t^2\hat{\imath} + (3t+4)\hat{\jmath}$$

Find $\vec{v}(t)$ and $\vec{a}(t)$ at $t = 2\,s$

Solution. Consider $x(t) = 2t^2$ and $y(t) = 3t + 4$ separately

$$v_x = \frac{dx}{dt} = 4t\,,\; a_x = \frac{dv_x}{dt} = 4$$

$$v_y = \frac{dy}{dt} = 3\,,\; a_y = \frac{dv_y}{dt} = 0$$

$$\text{at t=2}\;\; v_x = 8\,,\; a_x = 4\,,\; v_y = 3\,,\; a_y = 0$$

$$\text{or}\;\; \vec{v}(t=2) = 8\hat{\imath} + 3\hat{\jmath}\,,\; \vec{a} = 4\hat{\imath}$$

Example. Describe the x and y motion for $\vec{a} = -\hat{\imath} + 2\hat{\jmath}$ (in m/s^2) and $\vec{v}_0 = 10\hat{\imath}$ (in m/s). At $t = 0$ the particle is at the origin. Solution

$$x(t) = 10t - \frac{1}{2}t^2\,,\; y(t) = \frac{1}{2}2t^2$$

Can eliminate t via (the simpler) $y(t)$ from which we get $t = \sqrt{y}$ and

$$x = 10\sqrt{y} - y/2$$

D. $\vec{a} = \vec{g} = 0\hat{i} - 9.8\hat{j}$ **(projectile motion)**

see previous section

HW examples

A car comes to a bridge during a storm and finds the bridge washed out. The driver must get to the other side, so he decides to try leaping it with his car. The side the car is on is H=21.1 m above the river, whereas the opposite side is a mere h=2.4 m above the river. The river itself is a raging torrent L=61.0 m wide.

$$t = \sqrt{\frac{2(H-h)}{g}}\,,\; v_x = L/t = \ldots$$

$$v_y = 0 - gt\,,\; v = \sqrt{v_x^2 + v_y^2}$$

$$\text{or } h - H = \frac{0 - v_y^2}{2g}\,,\; v_y = -\sqrt{2g(H-h)}\,,\; v = \ldots$$

E. Uniform circular motion

1. Preliminaries

Radian measure of an angle:

$$\boxed{l = r\theta}$$

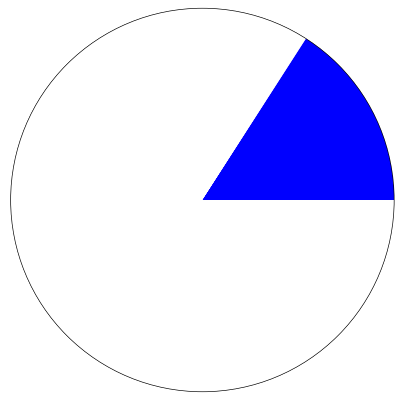


FIG. 12: Angle of $1\,rad \approx 57.3^o$. For this angle the length of the circular arc exactly equals the radius. The full angle, 360^o , is 2π radians.

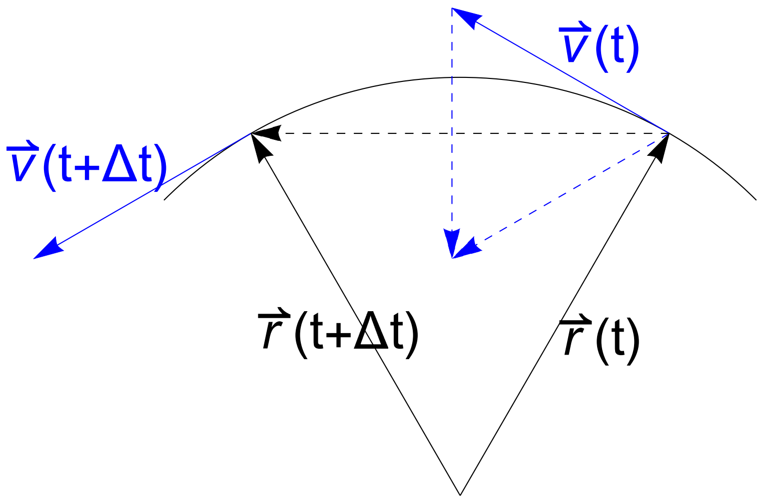
Consider motion around a circle with a constant speed v. The velocity $\vec{v}$, however, changes directions so that there is acceleration.

Period of revolution:

$$T = 2\pi r/v \tag{38}$$

with $1/T$ - "frequency of revolution". Angular velocity (in rad/s):

$$\boxed{\omega = \frac{2\pi}{T} = \frac{v}{r}} \tag{39}$$



Consider counterclockwise rotation with representative position vectors $\vec{r}(t)$ and $\vec{r}(t+\Delta t)$ (black) which are symmetric with respect to the vertical. Note that $\vec{v}$ (blue) is *always* perpendicular to $\vec{r}$. Thus, from geometry vectors $\vec{v}\,(t+\Delta t)\,, \vec{v}(t)$ and $\Delta\vec{v}$ (dashed blue) form a triangle which is similar to the one formed by $\vec{r}\,(t+\Delta t)\,, \vec{r}(t)$ and $\Delta\vec{r}$ (dashed black). Or,

$$\frac{|\Delta\vec{v}|}{v} = \frac{|\Delta\vec{r}|}{r}$$

$$a = \lim_{\Delta t\to 0}\frac{|\Delta\vec{v}|}{\Delta t} = \frac{v}{r}\lim_{\Delta t\to 0}\frac{|\Delta\vec{r}|}{\Delta t}$$

Or

$$\boxed{a_c = \frac{v^2}{r} = \omega^2 r} \tag{40}$$

We can use derivatives with the major relation

$$\frac{d}{dt}\sin(\omega t) = \omega\cos(\omega t)\ , \quad \frac{d}{dt}\cos(\omega t) = -\omega\sin(\omega t)\ , \tag{41}$$

One has

$$\vec{r}(t) = (x,\ y) = (r\cos\omega t,\ r\sin\omega t)$$
$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (-r\omega\sin\omega t,\ r\omega\cos\omega t)$$
$$\text{Note: } \vec{r}(t)\cdot\vec{v}(t) = -r^2\omega\cos(\omega t)\sin(\omega t) + r^2\omega\sin(\omega t)\cos(\omega t) = 0$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(-r\omega^2\cos\omega t,\ -r\omega^2\sin\omega t\right) = -\omega^2\vec{r} \tag{42}$$

which gives not only magnitude but also the direction of acceleration opposite to $\vec{r}$, i.e. towards the center.

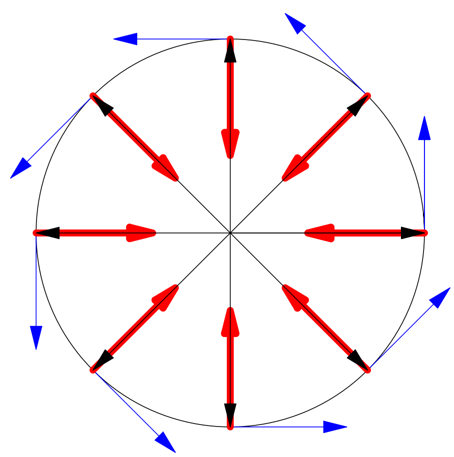


FIG. 13: Position (black), velocities (blue) and acceleration (red) vectors for a uniform circular motion in counter-clockwise direction.

VI. NEWTON'S LAWS

A. Force

1. Units

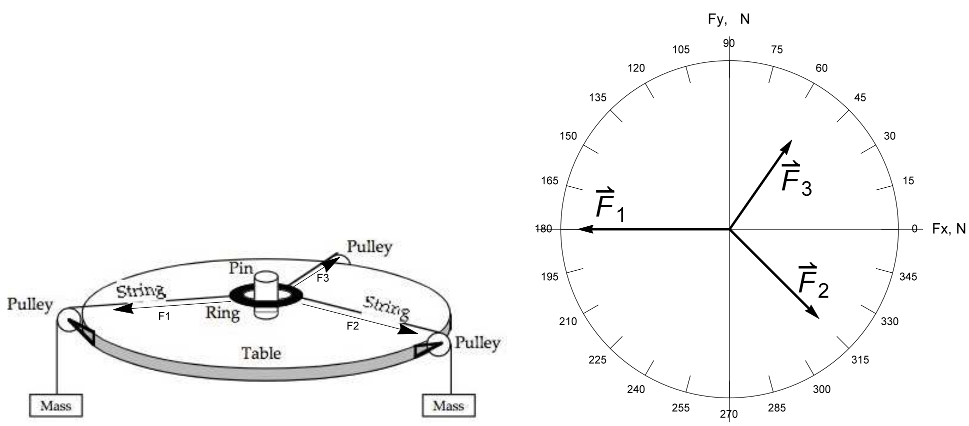
"Newton of force"

$$N = kg\frac{m}{s^2} \tag{43}$$

2. Vector nature

From experiment, action of two independent forces $\vec{F}_1$ and $\vec{F}_2$ is equivalent to action of the resultant

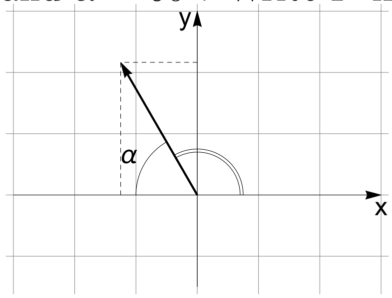
$$\vec{F} = \vec{F}_1 + \vec{F}_2 \tag{44}$$



The force table (left) and its schematic representation (right)

$$\text{In equilibrium: } \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0\,,\ \Rightarrow\ \vec{F}_3 = -\left(\vec{F}_1 + \vec{F}_2\right)$$

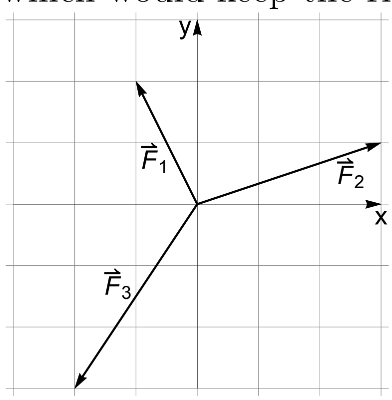
Example. In the picture below the magnitude of force $F \equiv \left|\vec{F}\right| = 2.5\,N$ and $\alpha = 60^o$. Write $\vec{F}$ in unit vector notations.



$F_x = -F\cos\alpha = F\cos(180^o - \alpha) = -1.25\,,\ F_y = F\sin\alpha = F\sin(180^o - \alpha) = 2.165\ \Rightarrow$

$$\vec{F} = -1.25\hat{\imath} + 2.165\hat{\jmath}$$

Example. Two forces $\vec{F_1} = -\hat{\imath} + 2\hat{\jmath}$ and $\vec{F_2} = 3\hat{\imath} + \hat{\jmath}$ (in Newtons) are applied to the plastic ring in the middle of the force table. Find the 3rd force which would keep the ring in equilibrium.



$$\vec{F_1} + \vec{F_2} + \vec{F_3} = 0\ \Rightarrow\ \vec{F_3} = -\left(\vec{F_1} + \vec{F_2}\right) = -\left(2\hat{\imath} + 3\hat{\jmath}\right) = -2\hat{\imath} - 3\hat{\jmath}$$

Find the angle α between $\vec{F_2}$ and the x-axis:

$$\cos\alpha = F_{2,x}/F_2 = 3/\sqrt{3^2 + 1^2} = 3/\sqrt{10}\,,\ \alpha = \ldots$$

Find the angle β between $\vec{F_1}$ and $\vec{F_2}$:

$$\cos\beta = \vec{F_1}\cdot\vec{F_2}/(F_1 * F_2) = (-1*3 + 2*1)/(\sqrt{1^2 + 2^2} * \sqrt{3^2 + 1^2}) = \ldots,\ \beta = \ldots$$

$$\text{Force of gravity: } \boxed{\vec{F}_g = m\vec{g}} \tag{45}$$

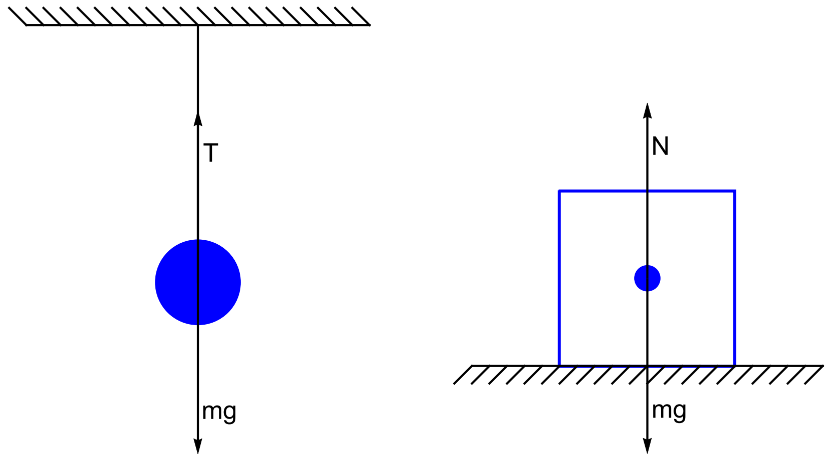
(sometimes called "weight" which is not always correct). Force $m\vec{g}$ is applied to the center of the body (center-of-mass, as we learn later.)

Tension $\vec{T}$, with magnitude T constant along a string (even if there is a massless pulley which changes the direction of $\vec{T}$).

Normal force $\vec{N}$ is perpendicular to the surface and acts on the body. Common to represent $\vec{N}$ as applied to the center of a body, (though in reality it is applied to the surface of contact).

$$\text{In equilibrium: } \vec{F}_{net} \equiv \sum_i \vec{F}_i = 0 \tag{46}$$

(all forces are applied to the same body! Note only "+" in the sum, *regardless* of the actual direction of vectors).

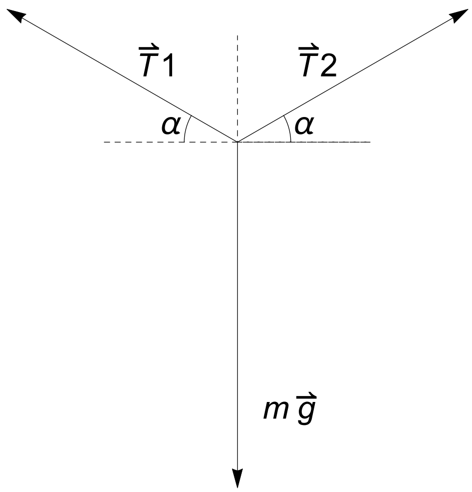


Equilibrium of a body under the action of gravity and tension (left, $m\vec{g}+\vec{T}=0$) or gravity and normal force (right, $m\vec{g}+\vec{N}=0$). In projections on a vertical axis with the positive direction - up)

$$\text{left: } T - mg = 0 \;\Rightarrow\; T = mg \;\; \text{(Note: as a rule, all symbols are positive)}$$

$$\text{right: } N - mg = 0 \;\Rightarrow\; N = mg \text{ (for } \vec{a} = 0 \text{ only!)}$$

Example: "bird in the middle of a cord"



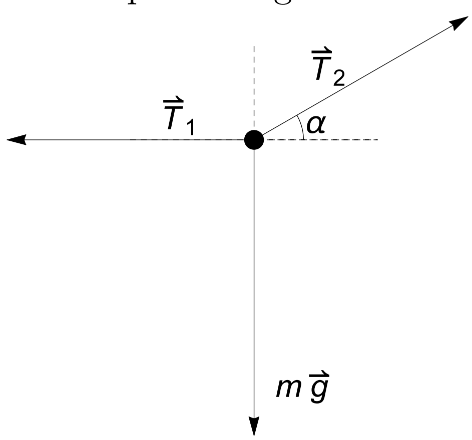
$$\text{From symmetry: } |\vec{T}_1| = |\vec{T}_2| \equiv T\ ,\ x \text{ - automatic}$$

$$\text{from y : } T\sin\alpha + T\sin\alpha - mg = 0\,,\ \Rightarrow\ T = mg/(2\sin\alpha)$$

$$\text{limits: } \alpha = 90^o\,,\ T = mg/2$$

$$\alpha \to 0\,,\ T \to \infty$$

Example: "weight between two cords"



$$\vec{T}_1 + \vec{T}_2 + m\vec{g} = 0$$

$$x:\quad -T_1 + T_2\cos\alpha + 0 = 0$$

$$y:\quad 0 + T_2\sin\alpha - mg = 0$$

$$T_2 = mg/\sin\alpha\,,\ T_1 = T_2\cos\alpha = mg\cot\alpha$$

B. The 3 Laws of motion (Newton)

1. If $\vec{F} = 0$ (no net force) then $\vec{v} = const$

2.

$$\boxed{\vec{F} = m\vec{a}} \tag{47}$$

3.

$$\boxed{\vec{F}_{12} = -\vec{F}_{21}} \tag{48}$$

Notes: the 1st Law is *not* a trivial consequence of the 2nd one for $\vec{F} = 0$, but rather it identifies inertial reference frames where a free body moves with a constant velocity.
In the 3rd Law forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are applied to *different* bodies; both forces, however, are of the same physical nature (e.g. gravitational attraction).

Example (1D): Velocity, in m/s, of an $m = 2\,kg$ particle is a polynomial function of time t (in seconds):

$$v(t) = t^4 - t^3 - t^2 + t + 4\,.\ \text{Find } F(0)$$

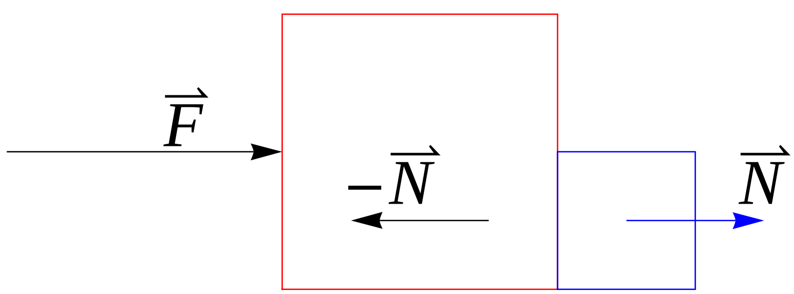
$$\text{From } \frac{d}{dt}t^n = nt^{n-1} \text{ find } a(t) = \frac{d\,v(t)}{dt} = 4t^3 - 3t^2 - 2t + 1$$

$$F(t) = ma(t) \;\Rightarrow\; F(0) = ma(0) = 2\,kg\ \times 1\,\frac{m}{s^2} = 2\,N$$

The following examples will be discusses in class:

- finding force between two accelerating blocks, finding force between cars in accelerating train
- hanging block pulling a block on a frictionless surface
- apparent weight in an elevator
- (*) Atwood machine
- block on inclined plane (no friction)

Example. Two blocks: a horizontal force $\vec{F}$ is applied to a block with mass M (red) which in turn pushes a block with mass m (blue); ignore friction. Find $\vec{N}$ and $\vec{a}$. (note that the 3rd Law is already used in the diagram).

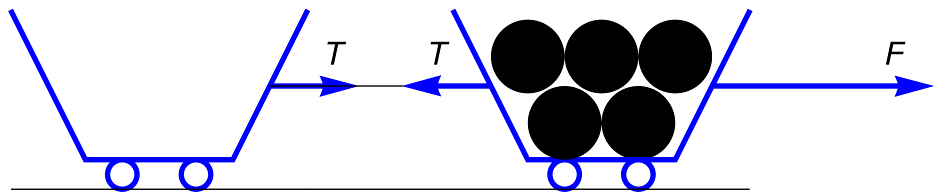


Solution: first treat the 2 blocks as a single solid body (at this stage ignore N which is an internal force). From 2nd Law for this "body" (x axis is horizontal, in direction of acceleration)

$$a = F/(M + m)$$

Now note that the blue block m is accelerated *only* due to force N. Thus, from 2nd Law for mass m alone:

$$N = ma$$



Example. A train has 2 cars: loaded with M=18,000 kg (first) and empty with m=2,000 kg (last). Find the tension T in the coupler connecting the cars if the applied force is F=40,000 N.

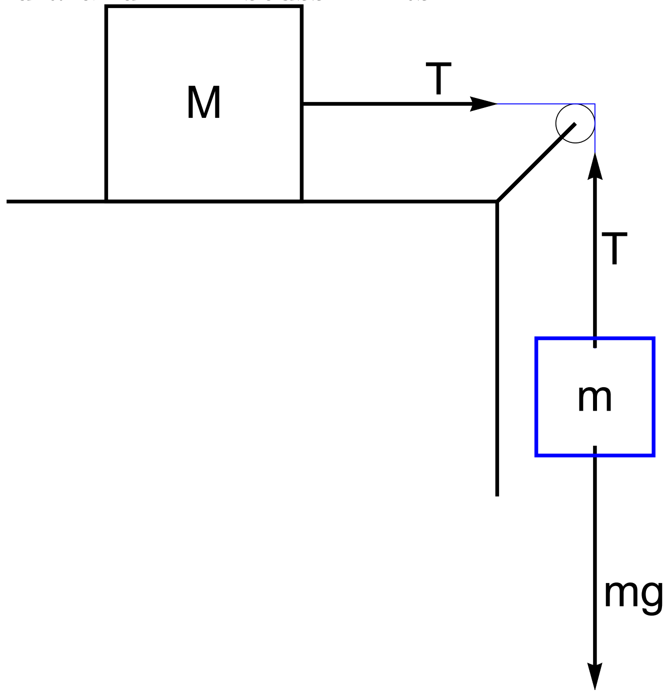
Solution. First, find a treating the train as a single body:

$$a = F/(M + m) = 4 \cdot 10^4/(2 \cdot 10^3 + 18 \cdot 10^3) = 2\, m/s^2$$

The last car alone is accelerated *only* by tension T. Thus,

$$T = ma = 2 \cdot 10^3 \times 2 = 4 \cdot 10^3\, N$$

Example: hanging block m pulling a block M on a frictionless surface. Find a and T. Discuss limits.



$$\text{mass } M\text{, axis horizontal:} \quad T = Ma$$

$$\text{mass } m\text{, axis down:} \quad mg - T = ma$$

Note: two equations for two unknowns (a and T).

$$mg = Ma + ma \ \Rightarrow \ a = g\frac{m}{M+m}$$

$$T = Ma = g\frac{mM}{M+m}$$

Limits:

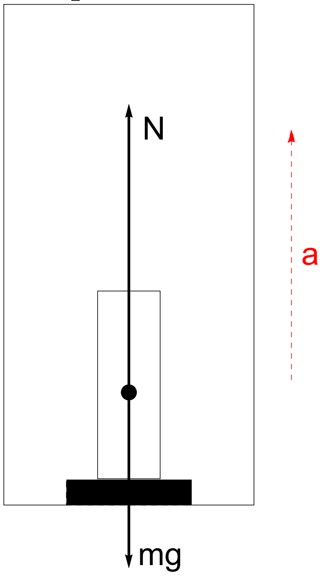
$M \to \infty$ (does not move). $a \to 0$, $T \approx mg$, as expected.

$M \to 0$ (no resistance to free fall). $a \to g$, $T \to 0$.

Quick solution. Use only *external* force mg and consider acceleration of combined mass $M + m$. (This is non-rigorous, but can be useful for verification). This immediately gives

$$mg = (M+m)a \ \Rightarrow \ a = g\frac{m}{M+m} \text{ as before}$$

Weight in an elevator: Find the "apparent weight" -reading of floor scale- of a person of mass m if the elevator accelerates up with acceleration $\vec{a}$.

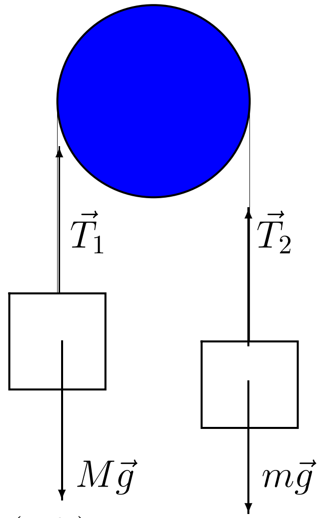


Solution. Forces on the person: $m\vec{g}$ (down, applied to center-of-mass) and $\vec{N}$ - up (reaction of the floor, applied to feet, but show as applied to center for simplicity). [A force $-\vec{N}$ -not shown- acts on the floor and determines "apparent weight"]. The 2nd Law for the person

$$-mg + N = ma \;\Rightarrow\; N = m(g+a)$$

Note: $N > mg$ for $\vec{a}$ up and $N < mg$ for $\vec{a}$ down; $N = 0$ for $\vec{a} = \vec{g}$.

Advanced. Atwood machine:



Atwood machine. Mass M (left) is almost balanced by a slightly smaller mass m. Pulley has negligible rotational inertia, so that both strings have the same tension $\left|\vec{T}_1\right| = \left|\vec{T}_2\right| = T$

Quick solution. Use external forces only and combined mass $M + m$.

$$Mg - mg = (M + m)a \;\Rightarrow\; a = g\frac{M - m}{M + m}$$

If need tension, write individual 2nd Laws:

$$Mg - T_1 = Ma \;\Rightarrow\; T_1 = M(g - a) = Mg\left(1 - \frac{M - m}{M + m}\right) = \frac{2mMg}{M + m}$$

$$T_2 - mg = ma\,,\; T_2 = m(g + a) = \ldots \text{ (same)}$$

Limits:

$$M = m\,,\; a = 0\,,\; T_1 = T_2 = Mg$$

$$m = 0\,,\; a = g\frac{M - 0}{M + 0} = g\,,\; T_1 = T_2 = 0$$

Academic solutions. Start with 2nd Law(s) for each body separately; use $T_1 = T_2 = T$:

$$Mg - T = Ma\,,\; T - mg = ma$$

Add together: T gets canceled and $(M - m)g = (M + m)a$ - same as before.

Frictionless incline - Fig. 14.

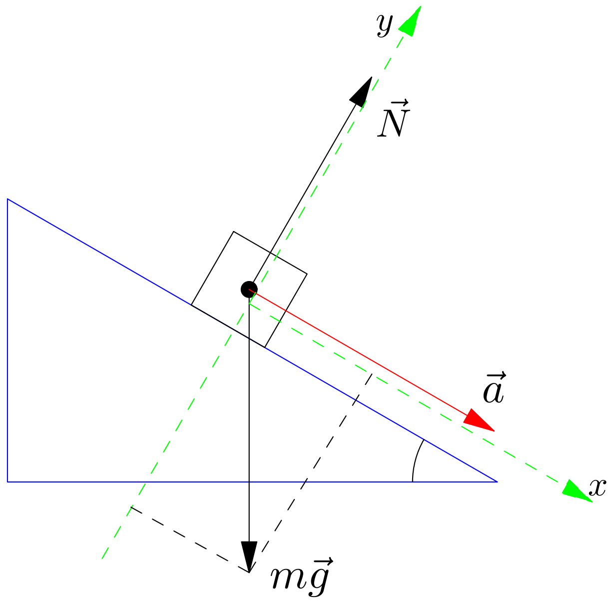


FIG. 14: A block on a frictionless inclined plane which makes an angle θ with horizontal.

- identify forces, $m\vec{g}$ and $\vec{N}$ in our case and the assumed acceleration $\vec{a}$ (magnitude still to be found).
- write the 2nd Law (vector form!) $\vec{N} + m\vec{g} = m\vec{a}$
- select a "clever" system of coordinates x, y.
- write down projections of the vector equation on the x, y axes, respectively (note, components of the force of gravity in such coordinates are always $\boxed{mg\sin\theta}$ downhill parallel to the incline and $-mg\cos\theta$ in perpendicular direction):

$$x: \quad mg\sin\theta = ma$$

$$y: \quad N - mg\cos\theta = 0$$

the x-equation will give acceleration

$$\boxed{a = g \sin \theta}$$

(which is already the solution); the y equation determines N.

- before plugging in numbers, a good idea is to check the limits. Indeed, for $\theta = 0$ (horizontal plane) $a = 0$ no acceleration, while for $\theta = \pi/2$ one has $a = g$, as should be for a free fall.

A few practical remarks to succeed in such problems.

- The original diagram should be BIG and clear. If so, you will use it as a FBD, otherwise you will have to re-draw it separately with an extra possibility of mistake.

- In the picture be realistic when dealing with "magic" angles of 30, 45, 60 and 90 degrees. Otherwise, a clear picture is more important than a true-to-life angle.

- Vectors of forces should be more distinct than anything else in the picture; *do not* draw arrows for projection of forces - they can be confused with real forces if there are many of them.

- the force of gravity in the picture should be immediately identified as $m\vec{g}$ (using an extra tautological definiton, such as $\vec{F}_g = m\vec{g}$ adds an equation and confuses the picture).

- If only one body is of interest, do not draw any forces which act on other bodies (in our case that would be, e.g. a force $-\vec{N}$ which acts on the inclined plane).

- select axes only *after* the diagram is completed and the 2'd Law is written in vector form. As a rule, in dynamic problems one axes is selected in the direction of acceleration (if this direction can be guessed).

Example. A block of mass M slides down an incline which is $L = 4\,m$ long and makes $\theta = 30^o$ with horizontal. No friction. Find the speed v at the bottom of the incline.

$$\text{from dynamics: } a = g\sin\theta \text{ (and } M \text{ does not matter)}$$

$$\text{from kinematics: } L = \frac{v^2 - v_0^2}{2a}\,,\ v_0 = 0\ \Rightarrow$$

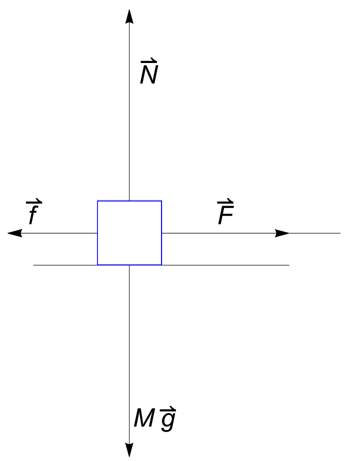
$$v^2 = 2aL = 2gL\sin\theta = 2gh\ \text{ with }\ h = L\sin\theta\,,\ \text{vertical displacement}$$

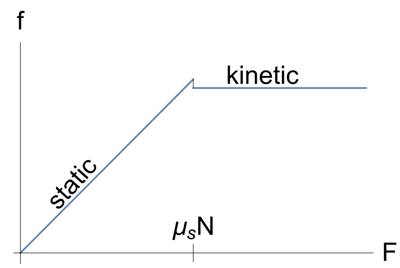
$$v = \sqrt{2gh} = \sqrt{2\cdot 9.8\cdot 4\cdot \sin 30^o} = \ldots$$

(the same will be later re-derived from conservation of energy)

VII. NEWTON'S LAWS: APPLICATIONS TO FRICTION AND TO CIRCULAR MOTION

A. Force of friction





Left: $\vec{F}$ - external force, $\vec{f}$ - friction force, kinetic if the block is moving ($F \geq f$), static if not moving ($f = F$).
Right: The f vs. F dependence. Note: the *maximum* static friction is usually slightly bigger than kinetic.

Force of friction on a moving body:

$$f = \mu N \tag{49}$$

Direction - against velocity; μ (or μ_k) - kinetic friction coefficient.

Static friction:

$$f_s \leq \mu_s N \tag{50}$$

with μ_s - static friction coefficient.

1. Example: block on inclined plane

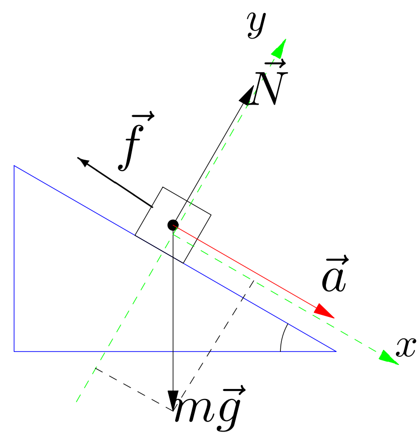


FIG. 15: A block sliding down an inclined plane with friction. The force of friction $\vec{f}$ is opposite to the direction of motion and equals $\mu_k N$.

- identify forces, $\vec{f}$, $m\vec{g}$ and $\vec{N}$ and the acceleration $\vec{a}$ (magnitude still to be found).
- write the 2nd Law (vector form!): $\vec{f} + \vec{N} + m\vec{g} = m\vec{a}$
- select a "clever" system of coordinates and write down projections of the vector equation on the x, y axes:

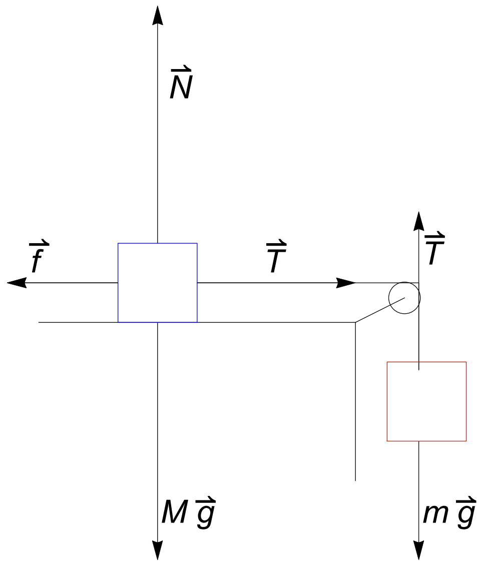
$$x: \quad -f + mg\sin\theta = ma$$

$$y: \quad N - mg\cos\theta = 0 \;\Rightarrow\; N = mg\cos\theta$$

- relate friction to normal force: $f = \mu_k N = \mu_k mg\cos\theta$. This goes into the above equation for the x-axis: $-\mu_k mg\cos\theta + mg\sin\theta = ma \;\Rightarrow$

$$a = g\,(\sin\theta - \mu_k\cos\theta) \;>\; 0$$

- keeps moving: $\boxed{\mu_k \leq \tan\theta}$, starts moving: $\boxed{\mu_s < \tan\theta}$



Quick solution: First, recall without friction. External force mg, thus

$$mg = (M+m)a \;\Rightarrow\; a = g\frac{m}{m+M}$$

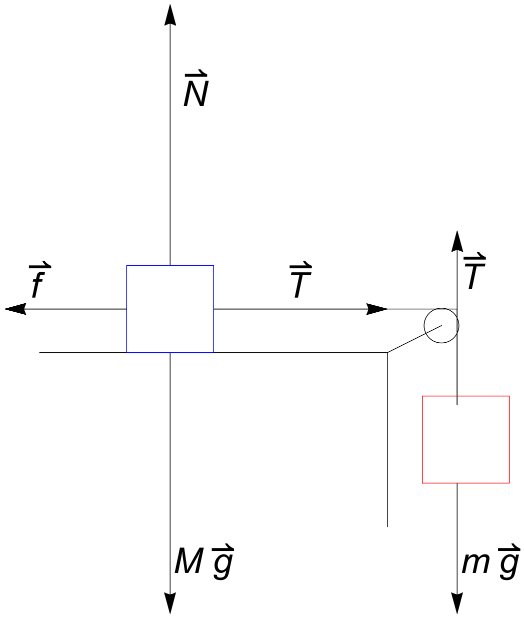
$$\text{for mass } M \text{ on the table: } T = Ma \;\Rightarrow\; T = g\frac{Mm}{m+M}$$

With friction: Assume motion. Use $\boxed{f = \mu_k mg\cos\theta}$ for a general incline and $\boxed{f = \mu_k mg}$ for a horizontal surface, $\theta = 0$.

$$\text{external forces } mg \text{ and } f\text{, net } mg - f = mg - \mu_k Mg \;\Rightarrow$$

$$mg - \mu_k Mg = (M+m)a \;\Rightarrow\; a = \frac{mg - \mu_k Mg}{M+m} = g\frac{m - \mu_k M}{M+m} > 0$$

(if $\mu_k M > m$ motion is impossible - friction too strong, see below).



If not moving:

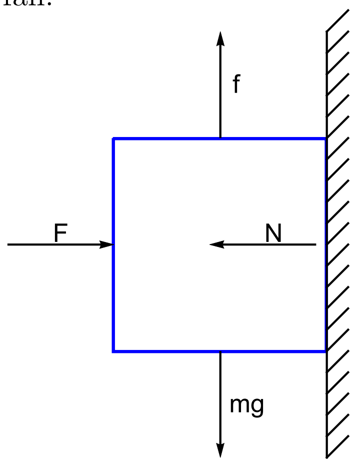
Hanging mass: $mg - T = 0 \Rightarrow T = mg$

Mass on the table: $T - f = 0 \Rightarrow f = T = mg$

Restriction: $f \leq \mu_s M g \Rightarrow \mu_s M \geq m$

Recall that friction force f is related to normal force N by $f = \mu_k N$ (kinetic) or $f \le \mu_s N$ (static). Usually, N is determined by Mg (horizontal surface) or by $Mg\cos\theta$ (inclined). But not always...

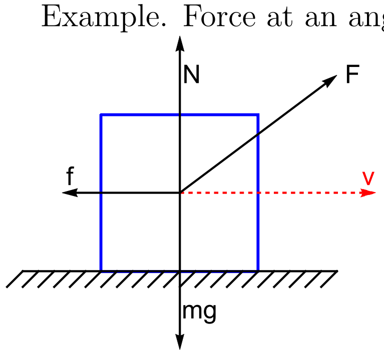
Example. Block pressed to a rough vertical wall. Find the minimal force F for it not to fall.



$$\text{from horizontal: } N = F\,, \quad \text{from vertical: } f = mg$$

$$f \le \mu_s N \;\Rightarrow\; mg \le \mu_s F\,, \text{ or } F \ge mg/\mu_s$$

Example. Force at an angle - reduced friction.



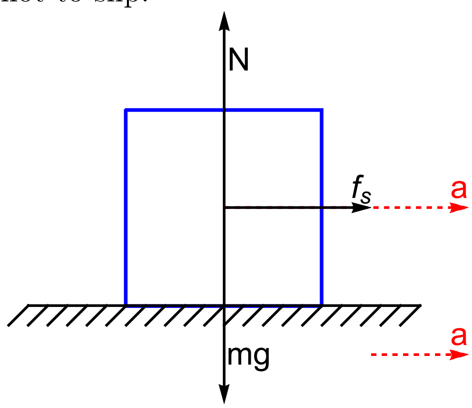
$$x: \quad -f + F\cos\theta = 0\,, \quad y: \quad N - mg + F\sin\theta = 0\,, \quad f = \mu N$$

$$N = mg - F\sin\theta \;\Rightarrow\; -\mu(mg - F\sin\theta) + F\cos\theta = 0\,,\ F = \frac{\mu mg}{\cos\theta + \mu\sin\theta}$$

$$F = \min \ \text{if} \ \cos\theta + \mu\sin\theta = \max$$

$$0 = \frac{d}{d\theta}(\cos\theta + \mu\sin\theta) = -\sin\theta + \mu\cos\theta \;\Rightarrow\; \theta = \tan^{-1}\mu$$

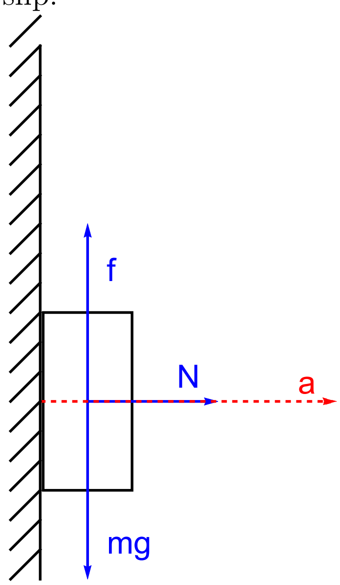
Example. Coin on accelerating rough horizontal surface. Find maximum a for the coin not to slip.



$$\text{from horizontal: } f_s = ma\,, \quad \text{from vertical: } N = mg$$

$$f \leq \mu_s N \;\Rightarrow\; ma \leq \mu_s mg\,, \text{ or } a \leq \mu_s g$$

Example. The same as above for a vertical wall. Find the *minimal* a for the coin not to slip.



$$\text{from horizontal: } N = ma\,, \quad \text{from vertical: } f = mg$$

$$f \leq \mu_s N \;\Rightarrow\; mg \leq \mu_s ma\,, \text{ or } a \geq g/\mu_s$$

Note: if μ_s is small, a needs to be VERY large.

B. Centripetal force

Newton's laws are applicable to *any* motion, centripetal including. Thus, for centripetal force of any physical origin

$$F_c = ma_c \equiv m\frac{v^2}{r} = m\omega^2 r \tag{51}$$

Centripetal force is always directed towards center, perpendicular to the velocity - see Fig. 16. Examples: tension of a string, gravitational force, friction.

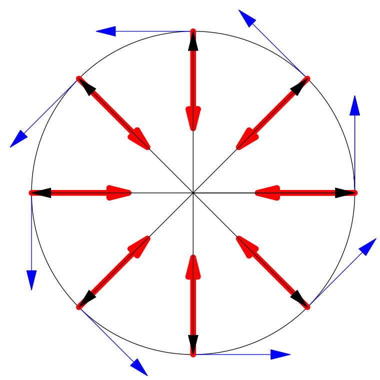


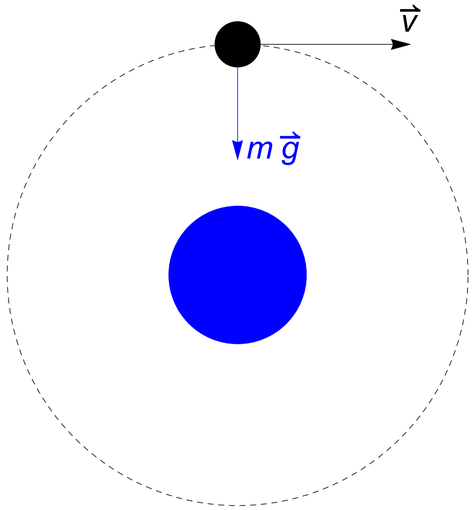
FIG. 16: Position (black), velocities (blue) and centripetal force (red) vectors for a uniform circular motion in counter-clockwise direction. The direction of force coincides with centripetal acceleration (towards the center). The value of centripetal force at each point is determined by the vector sum of actual physical forces, e.g. normal force and gravity in case of a Ferris Wheel, or tension plus gravity in case of a conic pendulum.

Simple example: The centripetal force is due to tension.
A particle with mass $m = 4.0\,kg$ is attached to a string with length $l = 1.0\,m$ and is moving at a constant speed around a horizontal circle. It takes $t = 3.0\,s$ to complete one revolution. Find the magnitude of the tension force F in the string. (Ignore gravity)
Solution: Since $F = ma_c$, need acceleration. Use

$$a_c = \omega^2 r \;\text{ with } r = l = 1.0\,m \text{ and } \omega = 2\pi/t = \frac{2\pi}{3.0} = \ldots$$

$$F = ma_c = m\omega^2 r = \ldots$$

1. Example of Satellite. "Centripetal force" is force of gravity

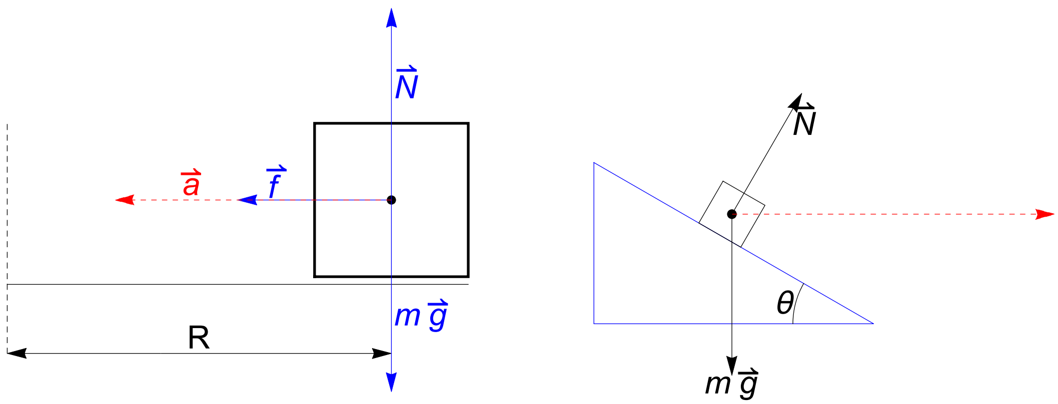


$$\vec{F}_g = m\vec{a}_c \text{ and } \vec{F}_g = m\vec{g} \;\Rightarrow\; \vec{a}_c = \vec{g}$$

$$a_c = \frac{v^2}{r} \;\Rightarrow\; \frac{v^2}{r} = g \;\Rightarrow\; \boxed{v = \sqrt{gr}}$$

$$\text{for Earth, low orbit } r \simeq 6400\,km\text{: } v = \left(9.8\frac{m}{s^2} \times 6.4 \times 10^3 \times 10^3 m\right)^{1/2} \simeq 8 \times 10^3 \frac{m}{s} = 8\frac{km}{s}$$

$$\text{Period of revolution: } \frac{2\pi r}{v} \approx \frac{40,000\,km}{8\,km/s} = 5000\,s \text{ (for Earth)}$$



Left: car making a left turn of radius R (view from back); find the max speed for a given friction coefficient μ_s. Right: car making a turn on an inclined road (no friction force); find the speed v.

$$\text{horizontal road with friction, 2nd Law}: \ \vec{N} + \vec{f} + m\vec{g} = m\vec{a}$$

$$x\text{-axis - towards the center (dashed):} \ f = ma = mv^2/R$$

$$y\text{-axis, up:} \ N - mg = 0$$

and

$$f \le \mu_s N = \mu_s mg \quad \Rightarrow$$

$$v^2/R \le \mu_s g\,, \ v_{\max} = \sqrt{\mu_s g R}$$

$$\text{incline}: \ \vec{N} + m\vec{g} = m\vec{a}$$

$$x: \quad N \sin\theta = mv^2/R$$

$$y: \quad N \cos\theta - mg = 0$$

$$mg \tan\theta = mv^2/R$$

$$\tan\theta = \frac{v^2}{gR}$$

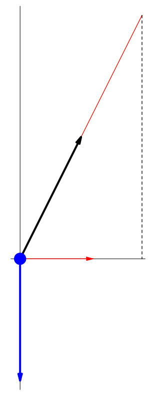


FIG. 17: A conic pendulum rotating around a vertical axis (dashed line). Two forces gravity (blue) and tension (black) create a centripetal acceleration (red) directed towards the center of rotation. The angle with vertical is θ, length of the s tring is L and radius of revolution is $r = L \sin\theta$.

3. Advanced: Conic pendulum

See Fig. 17, forces will be labeled in class.

One has the 2nd Law:

$$\vec{T} + m\vec{g} = m\vec{a}_c$$

In projections:

$$x: \quad T\sin\theta = ma_c = m\omega^2 r$$

$$y: \quad T\cos\theta - mg = 0$$

Thus,

$$\omega = \sqrt{\frac{g}{L\cos\theta}} \simeq \sqrt{\frac{g}{L}}$$

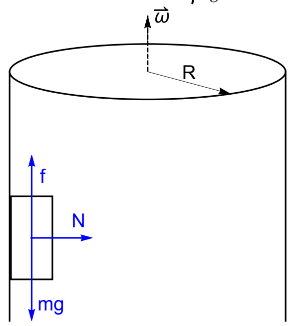
(the approximation is valid for $\theta \ll 1$). The period of revolution

$$\frac{2\pi}{\omega} \simeq 2\pi\sqrt{\frac{L}{g}}$$

Note that mass and angle (if small) do not matter.

4. "Barrel-of-fun"

In the "barrel-of-fun" attraction a person stays in a spinning room with his back against the wall. Suddenly, the floor falls out but the person does not (!) Find the minimal ω if $R = 5\,m$ and $\mu_s = 0.3$.



$$m\vec{g} + \vec{N} + \vec{f} = m\vec{a}_c$$

$\vec{a}$ in the diagram - to the right (towards center). Thus select x-axis -right, y-axis - up

$$x: \quad N = ma_c\,, \quad y: \quad f - mg = 0$$

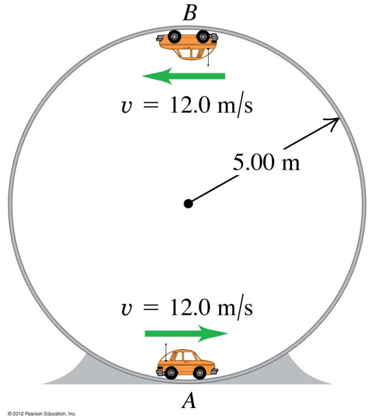
and $f \leq \mu_s * N$. Thus,

$$mg \leq \mu_s m a_c\,,\; g \leq \mu_s a_c = \mu_s \omega^2 R$$

$$\omega^2 \geq \frac{g}{\mu_s R}\,,\; \omega_{\min} = \sqrt{\frac{g}{\mu_s R}} = \sqrt{\frac{9.8}{0.3 * 5}} \approx 2.6\,\frac{rad}{s}$$

5. "Loop-of-death"

Find N (""weight") of the toy car ($m = 0.3\,kg$) at the top and the bottom parts of the circle ($R = 5\,m$) if $v = 12\,m/s$. Find $v_{\min}$ for the car not to fall.



$$\vec{N} + m\vec{g} = m\vec{a}_c$$

BOT.: $N = N_1$, up; a_c - up

TOP: $N = N_2$, down; a_c - down

In each case x-axis - along $\vec{a}_c$:

$$\text{BOT.: } N_1 - mg = ma_c\,,\; N_1 = mg + ma_c = mg + mv^2/R = ...$$

$$\text{TOP: } N_2 + mg = ma_c\,,\; N_2 = ma_c - mg = mv^2/R - mg = ... > 0$$

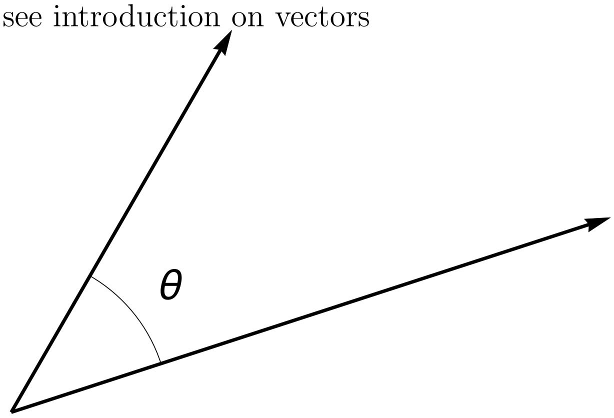
Smallest speed: $N_2 = 0$

$$mv_{\min}^2/R - mg = 0\,,\; v_{\min}^2/R = g\,,\; v_{\min} = \sqrt{Rg} = \sqrt{5 * 9.8} \simeq 7\,\frac{m}{s}$$

VIII. WORK

A. Scalar (dot) product in 3D

see introduction on vectors



$$\vec{a}\cdot\vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}\,,\ b = \sqrt{b_x^2 + b_y^2 + b_z^2}$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab}$$

Example: $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k} = (1,2,3)$, $\vec{b} = 2\hat{i} - 4\hat{j} + \hat{k} = (2,-4,1)$. Find $\vec{a}\cdot\vec{b}$ and find θ.

$\vec{a}\cdot\vec{b} = 1\times 2 + 2\times(-4) + 3\times 1 = -3\,,\ a = \sqrt{1^2+2^2+3^2} = \sqrt{14}\,,\ b = \sqrt{2^2+4^2+1^2} = \sqrt{21}$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab} = \frac{-3}{\left(\sqrt{14}\right)\times\left(\sqrt{21}\right)}\,,\ \theta \approx 100^o$$

Note: if $\cos\theta < 0$, then $\theta > 90^o$.

B. Units

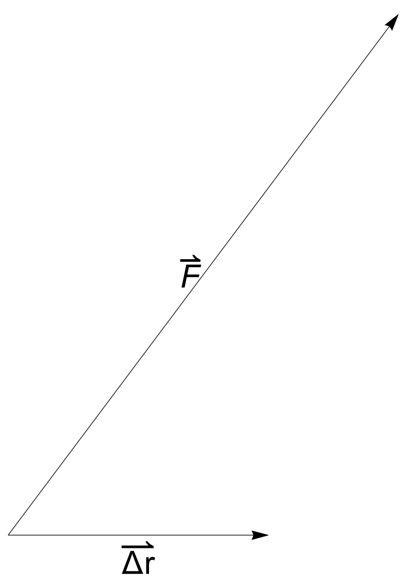
Joule (J):

$$1\,J = 1\,N \cdot m = kg\frac{m^2}{s^2} \tag{52}$$

C. Definitions

Constant force:

$$W = \vec{F} \cdot \Delta\vec{r} \equiv F\Delta r\cos\theta \equiv F_x\Delta x + F_y\Delta y + F_z\Delta z \tag{53}$$



Example: work by force of gravity ; x-"east", y-"North", z-up

$$F_x = 0\,,\ F_y = 0\,,\ F_z = -mg$$

$$\boxed{W_g = -mg\Delta z} \tag{54}$$

(and Δx , Δy do not matter!)

Let $m = 3\,kg$, and the particle is moved from $\vec{r}_1 = 2\hat{\imath} - 4\hat{\jmath} + \hat{k}$ to $\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ (in *meters*). Find W_g

$$W_g = -3\,kg \cdot 9.8\frac{m}{s^2} \cdot (3-1)\,m \simeq -60\,J$$

Example:

$$\vec{F} = \vec{i} + 2\vec{j}\,,\ \vec{r}_1 = 3\vec{i} + 4\vec{j}\,,\ \vec{r}_2 = 6\vec{i} - 4\vec{j}$$

$$\Delta\vec{r} = 3\vec{i} - 8\vec{j}\,,\ W = 3 \cdot 1 + (-8) \cdot 2 = -13\,(J)$$

Example:

$$\vec{F} = 2\vec{j}\,,\ \vec{r}_1 = 3\vec{i} + 4\vec{j} + 5\vec{k}\,,\ \vec{r}_2 = 6\vec{i} - 4\vec{j} + \vec{k}$$

$$W = (-4 - 4) \cdot 2 = -16\,(J)$$

(x and z components of displacement do not matter!)

Example: An $M = 80\,kg$ skydiver falls $200\,m$ with a constant speed of $100\,m/s$. Find the work done by viscous air friction.

Since $v = const$

$$F_y - Mg = 0\,,\ \text{thus } F_y = Mg$$

(otherwise speed does not matter!)

$$\Delta y = -200\,m\,,\ W = Mg\,\Delta y = 80 \cdot 9.8 \cdot (-200) = -\ldots$$

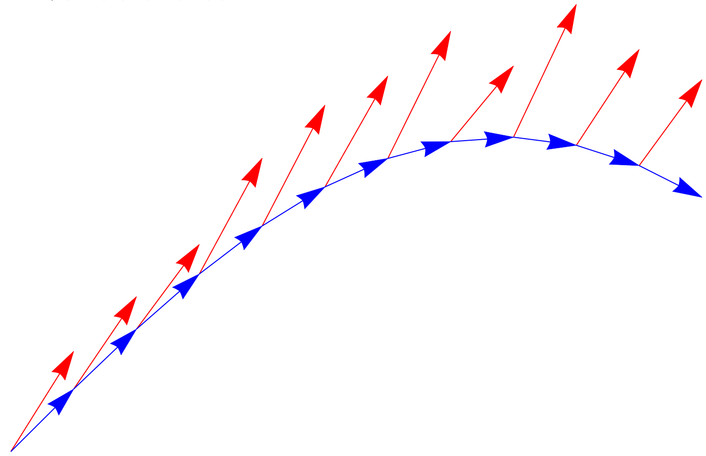
[Alternatively: the work done by gravity $W_g = -Mg\Delta y = 80 \cdot 9.8 \cdot 200 > 0$ (since goes down), and $W = -W_g < 0$]

Example: An $10\,kg$ projectile is displaced $200\,m$ horizontally and $50\,m$ vertically with respect to its initial position. Find W_g .

Horizontal motion does not matter!

$$W_g = -Mgy = -10 \cdot 9.8 \cdot 50 \simeq -5\,kJ$$

Variable force:



Let us break the path from $\vec{r}_1$ to $\vec{r}_2$ in small segments $\Delta\vec{r}_i$ (blue), each with a force $\vec{F}_i \simeq const$ (red). Then

$$W_i \simeq \vec{F}_i \cdot \Delta\vec{r}_i$$

and total work

$$W = \sum_i W_i \ \rightarrow \ \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \tag{55}$$

Note: forces which are perpendicular to displacement do not work, e.g. the centripetal force, normal force.

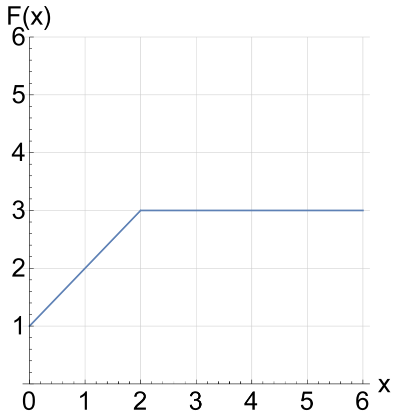
D. 1D motion and examples

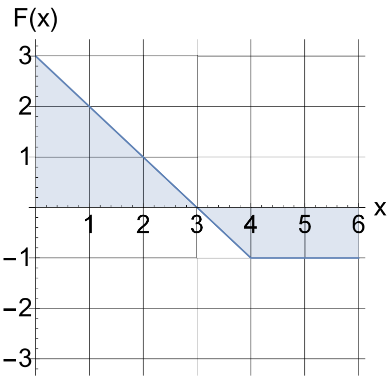
For motion in x-direction only

$$W = \int_{x_1}^{x_2} F_x \, dx \tag{56}$$

If F_x is given by a graph, work is the area under the curve (can be negative!).

Example: find work from graphs for $0 \leq x \leq 6$



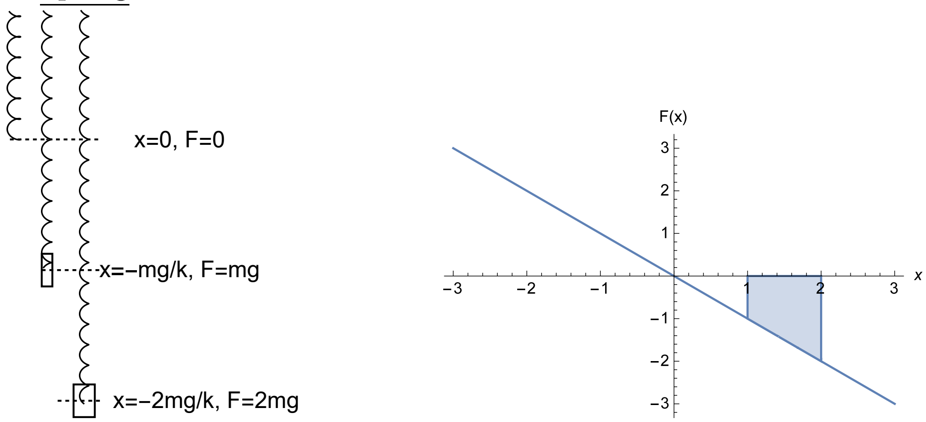


$$\text{left}: \; W = 16\, J\,,$$

$$\text{right}: \; W = \frac{1}{2} 3 \cdot 3 + \frac{1}{2}[(6-3)+(6-4)](-1) = 2\, J$$

(note: part of area can be negative)

Spring:



Left: the "experiment". Elongation of the spring is proportional to the stretching force. Right: the Hook's law. The shaded area is the (negative) work done by the spring when stretching from x_1 to $x_2 > x_1$.

$$F = -kx \tag{57}$$

k - "spring constant" (also known as "Hook's law"). Work done by the spring

$$W_{sp} = \frac{1}{2}(F_2 + F_1)(x_2 - x_1) = \frac{1}{2}(-kx_2 - kx_1)(x_2 - x_1) = -\frac{k}{2}(x_2 + x_1)(x_2 - x_1) =$$

$$= -\frac{1}{2}kx_2^2 + \frac{1}{2}kx_1^2$$

$$W_{sp} = +\frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 \tag{58}$$

(If the spring is stretched from rest, the term with x_1^2 will be absent and the work done by the spring will be negative for any $x_2 \neq 0$.)

Example. When a $3.6\,kg$ mass hangs from a spring it is extended by $x_1 = 12\,cm$. An extra force is applied to extend the spring by additional $\Delta x = 8\,cm$. What is the work done by the spring between x_1 and x_2?

$$\text{first, find } k\text{: } kx_1 = mg \;\Rightarrow\; k = \frac{mg}{x_1} = \frac{3.6 \cdot 9.8}{0.12} = \ldots \simeq 300\,\frac{N}{m}$$

$$x_2 = x_1 + \Delta x = 20\,cm \;\Rightarrow\; W = \frac{k}{2}(x_1^2 - x_2^2) = \frac{300}{2}(0.12^2 - 0.2^2) = -3.8\,J$$

IX. KINETIC ENERGY

A. Definition and units

$$\boxed{K = \frac{1}{2}mv^2} \tag{59}$$

or if many particles, the sum of individual energies.

Units: J (same as work).

Note: $v^2 = v_x^2 + v_y^2 + v_z^2$, thus $K = \frac{1}{2}m\left(v_x^2 + v_y^2 + \ldots\right)$.

B. Relation to work

1. Constant force

$$W = F_x \Delta x + F_y \Delta y = m\left[a_x \Delta x + a_y \Delta y\right]$$

According to kinematics

$$\Delta x = \frac{v_x^2 - v_{0x}^2}{2a_x}\ , \quad \Delta y = \frac{v_y^2 - v_{0y}^2}{2a_y}$$

and

$$a_x \Delta x = \frac{v_x^2 - v_{0x}^2}{2}\ , \quad a_y \Delta y = \frac{v_y^2 - v_{0y}^2}{2}$$

Thus,

$$W = \Delta K \tag{60}$$

which is the "work-energy" theorem.

Examples: gravity and friction (in class).

(gravity). Max hight for vertical initial v:

$$-mgh = 0 - \frac{1}{2}mv^2\ ,\ h = \frac{v^2}{2g}$$

(friction). The "policeman problem". Given: L, μ, find v_o

Friction: $f = \mu mg$.

$$W = -fL = -\mu mgL \ < 0 \ (!) \tag{61}$$

$$\Delta K = 0 - \frac{1}{2}mv_0^2 \tag{62}$$

$$-mv_0^2/2 = -\mu mgL \tag{63}$$

$$v = \sqrt{2\mu gL} \tag{64}$$

(friction+another force). A heavy crate is pushed from A to B (with $AB = 3\,m$) by a force $P = 200\,N$ at 60^o to horizontal. Find the work W_f done by friction if $K_A = 300\,J$ and $K_B = 100\,J$.

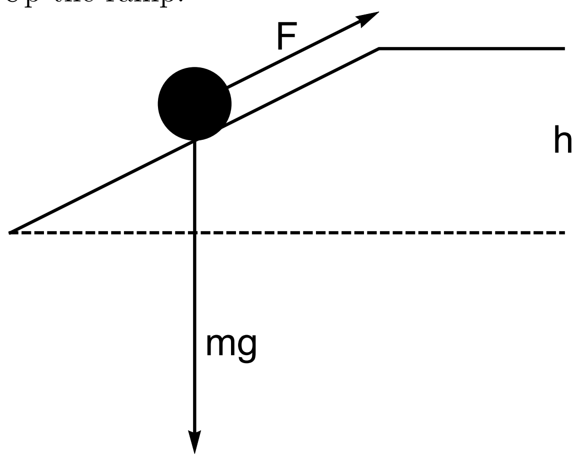
$$K_B - K_A = W_f + W_P\,,\ W_P = P \cdot AB \cdot \cos 60^o \tag{65}$$

$$W_f = K_B - K_A - W_P = 100 - 300 - 3 \cdot 200 \cdot \frac{1}{2} = -500\,J$$

(Note: negative!)

Ramp (no friction)

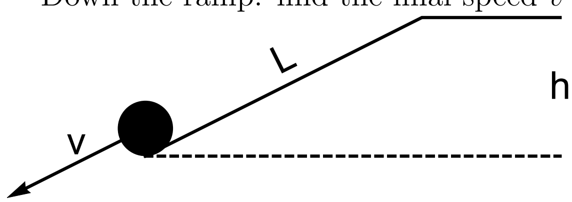
Up the ramp:

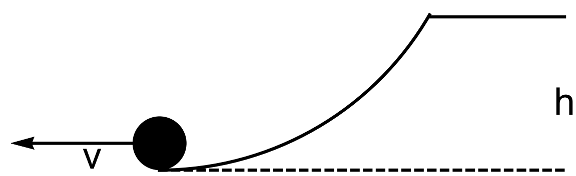


$$F = mg\sin\theta = mg\frac{h}{L} < mg\,(!)$$

$$\text{work } W = FL = mgh \;-\; \text{same}$$

Down the ramp: find the final speed v





$$\text{Kinetic energy: } K - 0 = W_g = mgh \Rightarrow K = mgh$$

$$\text{speed: from } K = \frac{1}{2}mv^2\,,\; v = \sqrt{2gh}$$

Note: shape of the slope does not matter; kinetic energy is proportional to mass, final v is mass-independent

Again, break the displacement path into a large number N of small segments Δr_i . For each

$$W_i = \Delta K_i \equiv K_i - K_{i-1}$$

Thus,

$$W = \sum_i^N W_i = (K_1 - K_0) + (K_2 - K_1) + \ldots + (K_{i-1} - K_{i-2}) + (K_i - K_{i-1}) + \ldots + (K_N - K_0) \equiv \Delta K$$

or

$$\boxed{W = \Delta K} \tag{66}$$

which is the "work-energy" theorem in a general form (also, can be applied to a system of particles).

Example: spring. A mass $M = 3\,kg$ is attached to a spring with $k = 10\,N/m$. The spring is at equilibrium (neither stretched nor compressed). The mass is given an initial speed of $v_0 = 2\,m/s$. Find the maximum absolute deviations from equilibrium.

$$\Delta K = \frac{1}{2} M \left(v^2 - v_0^2\right) = -\frac{1}{2} M v_0^2 \ \text{ since } v = 0 \text{ at max deviation}$$

$$W_{sp} = -\frac{k}{2}\left(x_{\max}^2 - x_0^2\right) = -\frac{k}{2} x_{\max}^2 \text{ since } x_0 = 0$$

$$\Delta K = W_{sp} \ \Rightarrow \ -\frac{1}{2} M v_0^2 = -\frac{k}{2} x_{\max}^2 \ \Rightarrow$$

$$x_{\max}^2 = v_0^2 \frac{M}{k}\ ,\ x_{\max} = \pm v_0 \sqrt{\frac{M}{k}} = \pm 2\sqrt{\frac{3}{10}} \simeq \pm 1.1\,m$$

C. Power

$$P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}$$

Units: *Watts*. $1\,\mathcal{W} = 1\,J/s$

Re-derivation of work-energy theorem:

$$\frac{d}{dt}K = \frac{d}{dt}\frac{m}{2}v^2 = \frac{m}{2}\frac{d}{dt}\left(\vec{v} \cdot \vec{v}\right) = m\frac{d\vec{v}}{dt} \cdot \vec{v} = \vec{F} \cdot \vec{v} = \frac{dW}{dt}$$

Example: An $M = 500\,kg$ horse is running up an $\alpha = 30^o$ slope with $v = 4\,m/s$. Find P:

$$P = mg\sin\alpha \cdot v \simeq 500 \cdot 9.8\frac{1}{2}4 \approx 10^4\,\mathcal{W}$$

$1\,hp \simeq 746\,\mathcal{W}$.

Example. For a fast bike the air resistance is $\sim v^2$. How much more power is needed to double v?

$$F(v) = \alpha \times v^2 \;\; \text{(unknown } \alpha\text{)}$$

$$P(v) = F(v) \times v = \alpha \times v^3 \;\Rightarrow\; P(2v) = 8 \times P(v)$$

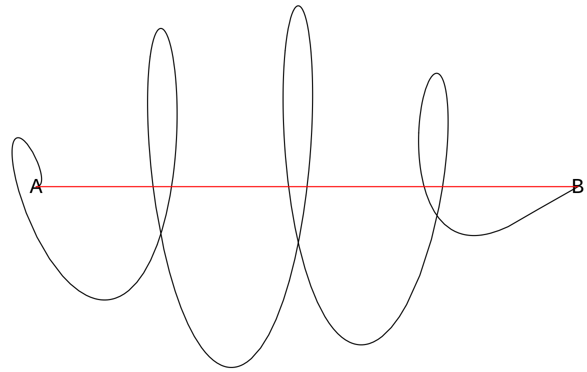


FIG. 18: Generally, the work of a force between points A and B depends on the actual path. However, for some "magic" (conservative) forces the work is path-independent. For such forces one can introduce potential energy U and determine work along *any* path as $W = U_A - U_B = -\Delta U$.

Dr. Vitaly A. Shneidman, Phys 111, 7th Lecture

X. POTENTIAL ENERGY

A. Some remarkable forces with path-independent work

See Fig. 18 and caption.

Examples:

- Constant force

$$W = \sum \vec{F} \cdot \Delta \vec{r}_i = \vec{F} \cdot \sum \Delta \vec{r}_i = \vec{F} \cdot \Delta r = \vec{F} \cdot \vec{r}_B - \vec{F} \cdot \vec{r}_A \tag{67}$$

with potential energy

$$U\left(\vec{r}\right) = -\vec{F} \cdot \vec{r} \tag{68}$$

Example force of gravity with $F_x = 0\,, F_y = -mg$ and

$$\boxed{U_g = mgh} \tag{69}$$

- Elastic (spring) force

$$W \simeq -\sum k \frac{x_i + x_{i+1}}{2}\left(x_{i+1} - x_i\right) = -\frac{k}{2} \sum x_{i+1}^2 - x_i^2 = -\frac{k}{2}\left(x^2 - x_0^2\right) \tag{70}$$

with potential energy

$$\boxed{U_{sp}(x) = \frac{1}{2}kx^2} \tag{71}$$

- Other: full force of gravity $F = -mg$ with r-dependent g, and any force which does not depend on velocity but depends only on distance from a center.

- Non-conservative: kinetic friction (depends on velocity, since points against $\vec{v}$)

B. Relation to force

$$U(x) = -\int F(x)\, dx \tag{72}$$

$$F = -dU/dx \tag{73}$$

XI. CONSERVATION OF ENERGY

Start with

$$\Delta K = W$$

If only conservative forces

$$W = -\Delta U \tag{74}$$

thus

$$\boxed{K + U = const \equiv E} \tag{75}$$

Examples:

- Maximum hight.

$$E = mgh + \frac{1}{2}mv^2$$

$$0 + \frac{1}{2}mv_0^2 = mgh_{\max} + 0$$

$$h_{\max} = \frac{v_0^2}{2g}$$

- Galileo's tower. Find speed upon impact

$$mgh + 0 = 0 + \frac{1}{2}mv^2$$

$$v^2 = 2gh$$

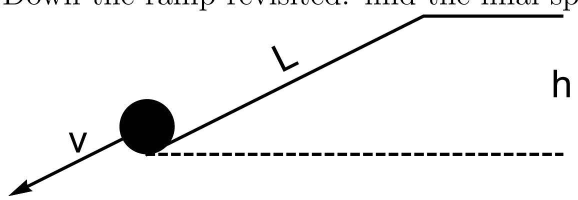
- Coastguard cannon (from recitation on projectiles). Find speed upon impact

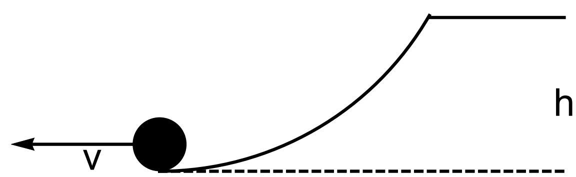
$$mgH + \frac{1}{2}mv_0^2 = 0 + \frac{1}{2}mv^2$$

$$v^2 = v_0^2 + 2gH$$

(note that the angle does not matter!).

- Down the ramp revisited: find the final speed v



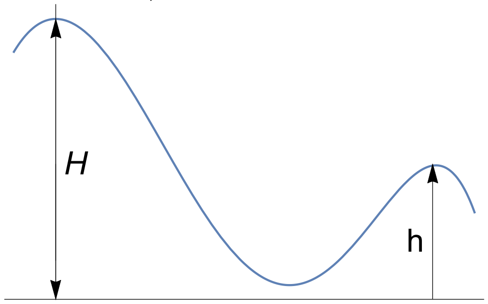


$$\text{initial mech. energy: } 0 + mgh\,,\ \text{final mech.energy: } \frac{1}{2}mv^2 + 0$$

$$\Rightarrow\ \frac{1}{2}mv^2 = mgh\,,\ v = \sqrt{2gh}$$

Note: shape of the slope or angle or mass do not matter

- A skier with no initial velocity slides down from a hill which is $H = 18\,m$ high and then, without losing speed, up a smaller hill which is $h = 10\,m$ high. What is his speed in m/s at the top of the smaller hill? Ignore friction.

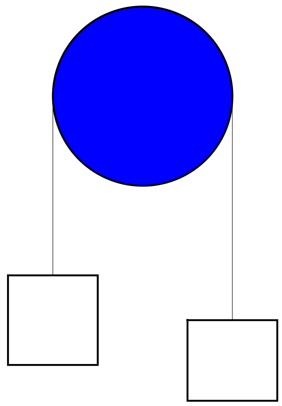


$$\text{(full initial energy) } MgH + 0 \;=\; Mgh + \frac{1}{2}Mv^2 \text{ (full final)}$$

$$v^2 = 2g(H-h) > 0\,,\; v = \ldots$$

- In the Atwood machine the heavier body on left has mass $M = 1.01\,kg$, while the lighter body on right has mass $m = 1\,kg$. The system is initially at rest, there is no friction and the mass of the pulley is negligible.

  Find the speed after M lowers by $h = 50\,cm$; use *only* energy considerations.



$$\text{(M on top)}:\ E = Mgh + 0 + 0 \ = \ mgh + \frac{1}{2}(M+m)v^2 \ \text{(m on top)}$$

$$v^2 = 2gh\frac{M-m}{M+m}\,,\ v = \ldots$$

What if we need a? For smaller mass, for example

$$h = \frac{v^2 - v_0^2}{2a} = \frac{v^2}{2a},\ \Rightarrow\ a = \frac{v^2}{2h} = g\frac{M-m}{M+m}$$

- A spring with given k, m is stretched by X meters and released. Find $v_{\max}$.

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

$$\frac{1}{2}kX^2 + 0 = 0 + \frac{1}{2}mv_{\max}^2\,,\ v_{\max} = X\sqrt{k/m}$$

- A block approaches the spring with speed v_0 . Find the maximum compression

$$\text{(full final energy)}:\ \frac{1}{2}kx_{\max}^2 + 0 \ = \ 0 + \frac{1}{2}mv_0^2\ \text{(full initial energy)}$$

$$|x_{\max}| = v_0\sqrt{m/k}$$

- find speed v at some given location x

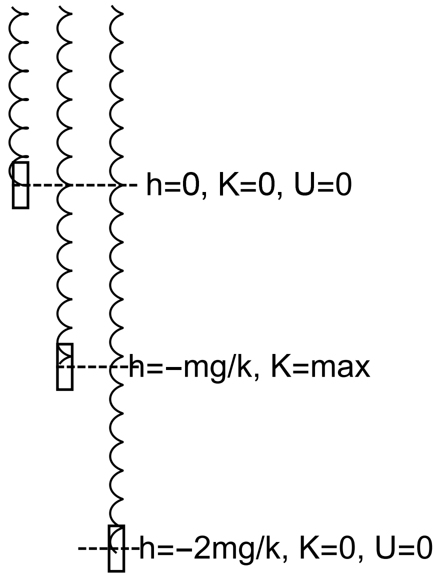
  first find kinetic energy at x

$$K(x) + U(x) = E = 0 + \frac{1}{2}mv_0^2$$

$$K(x) = \frac{1}{2}mv_0^2 - U(x) = \frac{1}{2}mv_0^2 - \frac{1}{2}kx^2 \; > \; 0$$

$$v(x) = \sqrt{2K(x)/m}\,,\; |x| \leq |x_{max}|$$

Example. A block of $m = 10\,kg$ is placed on a vertical spring with $k = 40\,N/m$, originally unstretched . Find the max compression distance and $v_{\max}$.



$$\text{From work-energy theorem: } \Delta K = 0 - 0 = 0 \;\Rightarrow, W = W_g + W_s = 0$$

$$W_g = -mgh\,,\; W_s = -\frac{1}{2}kh^2 \;\Rightarrow h = -2mg/k = \ldots$$

$$\text{From energy conservation: } U(h) = U_g(h) + U_s(h) = mgh + \frac{1}{2}kh^2$$

$$E = K + U = K(0) + U(0) = 0,\; K_{fin} = 0 \;\Rightarrow\; U_{fin} = 0 \;\Rightarrow\; U_{fin} = 0$$

$$mgh + \frac{1}{2}kh^2 = 0\,,\; h = -\frac{2mg}{k}$$

Maximum speed is achieved when the block passes through equilibrium at $h = -mg/k$ (middle picture).

$$E = K + U = 0 \;\Rightarrow\; K_{\max} = -U_g\left(-\frac{mg}{k}\right) - U_s\left(-\frac{mg}{k}\right) = mg * \frac{mg}{k} - \frac{1}{2}k\left(-\frac{mg}{k}\right)^2 = \frac{1}{2}\frac{(mg)^2}{k}$$

$$\frac{1}{2}mv_{\max}^2 = K_{\max} \;\Rightarrow\; v_{\max}^2 = (mg)^2/(mk)$$

Alternnatively, could consider deviation from new equilibrium with *only* spring energy:

$$h_{eq} = -\frac{mg}{k}\,,\; x = h - h_{eq}\,,\; U = \frac{1}{2}kx^2\,,\; x_{\max} = -h_{eq}\,,\; v_{\max} = \pm\sqrt{\frac{k}{m}}x_{\max}$$

A. Conservative plus non-conservative forces

For conservative forces introduce potential energy U, and then define the total *mechanical* energy

$$E_{mech} = K + U$$

One has

$$W = -\Delta U + W_{non-cons} \tag{76}$$

Then, from work-energy theorem

$$\boxed{\Delta E_{mech} = \Delta\left(K + U\right) = W_{non-cons}} \tag{77}$$

Examples:

- Galileo's Tower revisited (with friction/air resistance). Given: $H = 55\,m$, $M = 1\,kg$ and $W_f = -100\,J$ is lost to friction (i.e. the mechanical energy is lost, the thermal energy of the mass M and the air is increased by $+100\,J$). Find the speed upon impact.

$$\left(0 + \frac{1}{2}Mv^2\right) - (mgH + 0) = W_f < 0$$

$$\frac{1}{2}Mv^2 = Mgh + W_f\,,\ v^2 = 2gH - \frac{2|W_f|}{M} = 2 \times g \times 55 - 2 \times \frac{100}{1}\,,\ v \sim 30\frac{m}{s}$$

- Sliding crate (down). Given L, m, θ, μ find the speed v at the bottom.

$$h = L\sin\theta\,,\ f = \mu mg\cos\theta\,,\ W_f = -fL$$

$$(0 + \frac{1}{2}mv^2) - (mgh + 0) = W_f\ \Rightarrow\ v^2 = 2gh - \frac{2}{m}|W_f|$$

$$v^2 = 2gL\sin\theta - \frac{2}{m}\mu Lmg\cos\theta = 2gL(\sin\theta - \mu\cos\theta) \geq 0\ \Rightarrow\ \mu < \tan\theta \text{ (or will not slide)}$$

- Sliding crate (up). Given v find L (a) no friction; (b) with friction

$$\text{(a): } \frac{1}{2}mv^2 + 0 = mgh\,,\ h = v^2/(2g)\,,\ L = h/\sin\theta$$

$$\text{(b): } W_f = -fL\ \Rightarrow\ (mgh + 0) - (0 + \frac{1}{2}mv^2) = -fL = -\mu mg\cos\theta \times \frac{h}{\sin\theta}$$

$$mgh(1 + \mu\cot\theta) = \frac{1}{2}mv^2\,,\ h = \frac{v^2}{2g(1 + \mu\cot\theta)}\ \Rightarrow\ L = \frac{h}{\sin\theta} = \frac{v^2}{2g(\sin\theta + \mu\cos\theta)}$$

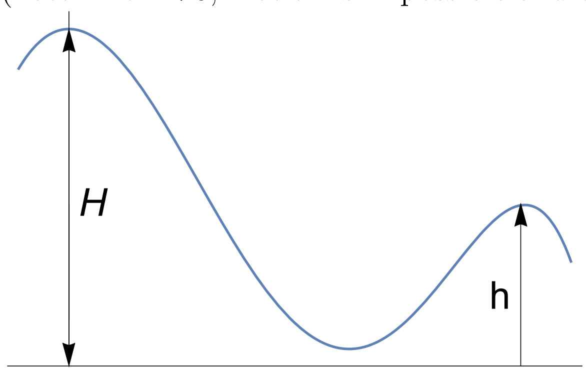
- The same skier as before, but assume a small average friction force of $40\,N$ and the combined length of the slopes $L = 200\,m$. The mass of the skier is $M = 80\,kg$; find the speed at the top of the smaller hill.

$$W_f = -fL < 0$$

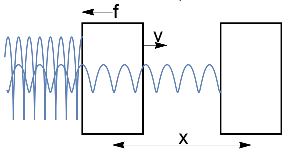
(full final energy - full initial) $(Mgh + \frac{1}{2}Mv^2) - (MgH + 0) \; = \; W_f$(work by non-cons. force)

$$v^2 = 2g(H-h) + \frac{2W_f}{M} \geq 0\,,\; v = \ldots$$

(note: if $v^2 < 0$, motion is impossible and the skier will stop before reaching the 2nd top.)



Example (*Advanced*) .



A block $m = 2\,kg$ is attached to an unstretched spring with spring constant $k = 20\,N/m$ and is given an initial speed $v = 10\,m/s$ along a rough horizontal surface with $\mu = 0.5$. At which distance x will the block stop during the first swing to the right?

$$E_i = \frac{1}{2}mv^2 + 0\,,\; E_f = 0 + \frac{1}{2}kx^2$$

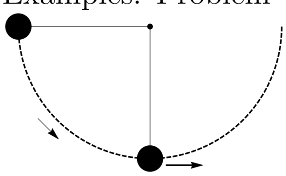
$$E_f - E_i = W_{n.-c.} = -fx \text{ (work done by friction)}$$

$$f = \mu mg \;\Rightarrow\; \frac{1}{2}kx^2 - \frac{1}{2}mv^2 = -\mu mgx \text{ (a quadratic equation for x)}$$

$$x^2 + 2\mu g\frac{m}{k} - \frac{m}{k}v^2 = 0$$

$$x = -\mu g\frac{m}{k} + \sqrt{(\mu g\frac{m}{k})^2 + \frac{m}{k}v^2}$$

Examples: Problem which are too hard without energy



A pendulum of length L swings starting from a horizontal position (i.e. $\theta_0 = \pi/2$). Find the speed V at the lowest point. Generalize for arbitrary θ_0.

$$E = mgh + \frac{1}{2}mv^2$$

$$\text{init. } \theta_0 = \frac{\pi}{2}\text{: } h = L\,,\; v = 0\,;\; \text{fin. } \theta = 0\text{: } h = 0\,,\; v = V$$

$$mgL = \frac{1}{2}mV^2, V^2 = 2gL$$

Arbitrary θ_0:

$$\text{init. } \theta = \theta_0\text{: } h = L(1 - \cos\theta)\,,\; v = 0\,;\; \text{fin. } \theta = 0\text{: } h = 0\,,\; v = V$$

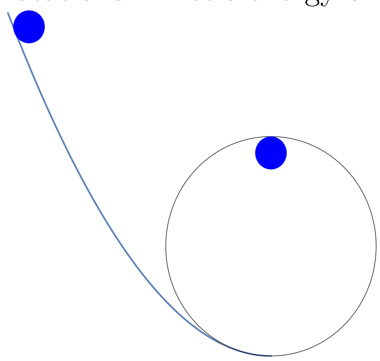
$$mgL(1 - \cos\theta) = \frac{1}{2}mV^2\,,\; V^2 = 2gL(1 - \cos(\theta))$$

What if need T? Lowest point

$$T - mg = mV^2/r\,,\; r = L \;\Rightarrow\; T = mg + m\frac{V^2}{L}$$

.

In the "loop-the-loop" demo a small ball is sent along a looping track of radius R. Find the minimal initial hight H so that the ball makes the loop. (Ignore at this stage the rotational kinetic energy of the spinning ball).



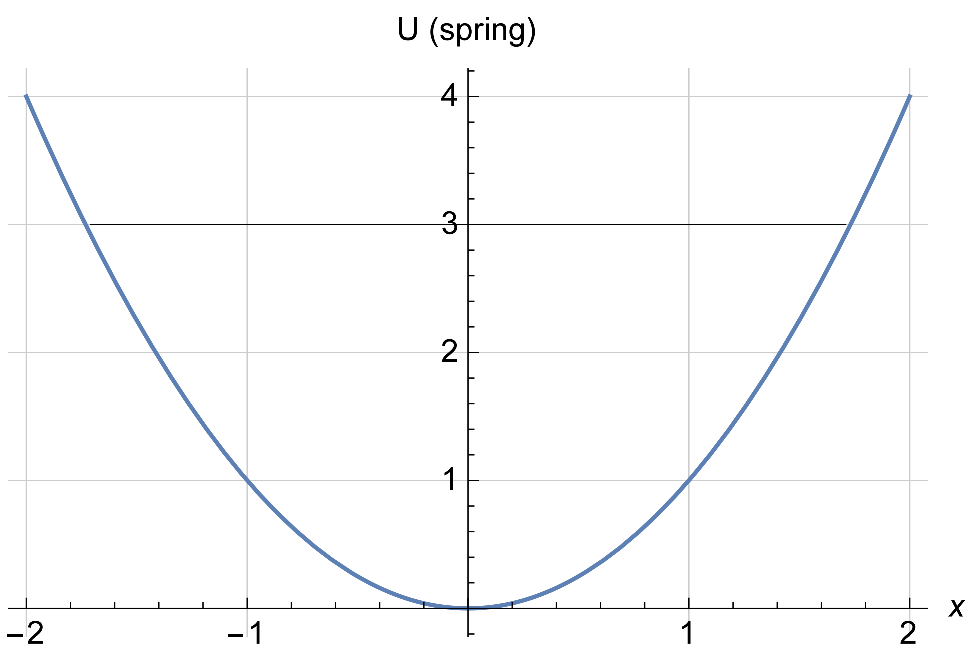
$$E = mhg + \frac{1}{2}mv^2$$

$$\text{init.: } h = H,\ v = 0\,;\ \text{top of the loop: } h = 2R, v = V$$

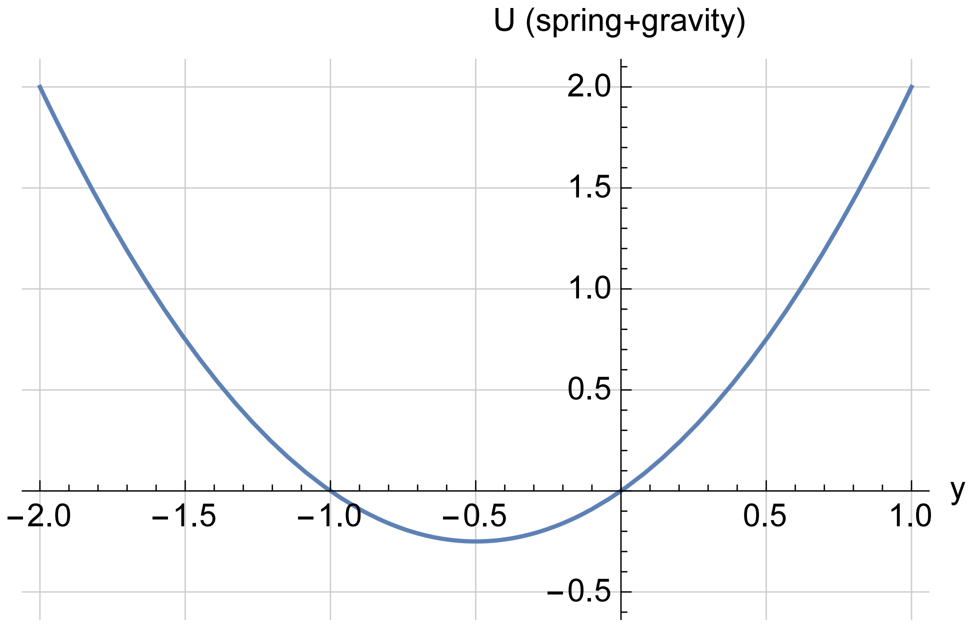
$$mgH = 2mgR + \frac{1}{2}mV^2 \text{ and } \frac{V^2}{R} \geq g \text{ not to fall}$$

$$gH_{\min} = 2gR + \frac{1}{2}gR\,,\ H_{\min} = 2.5R$$

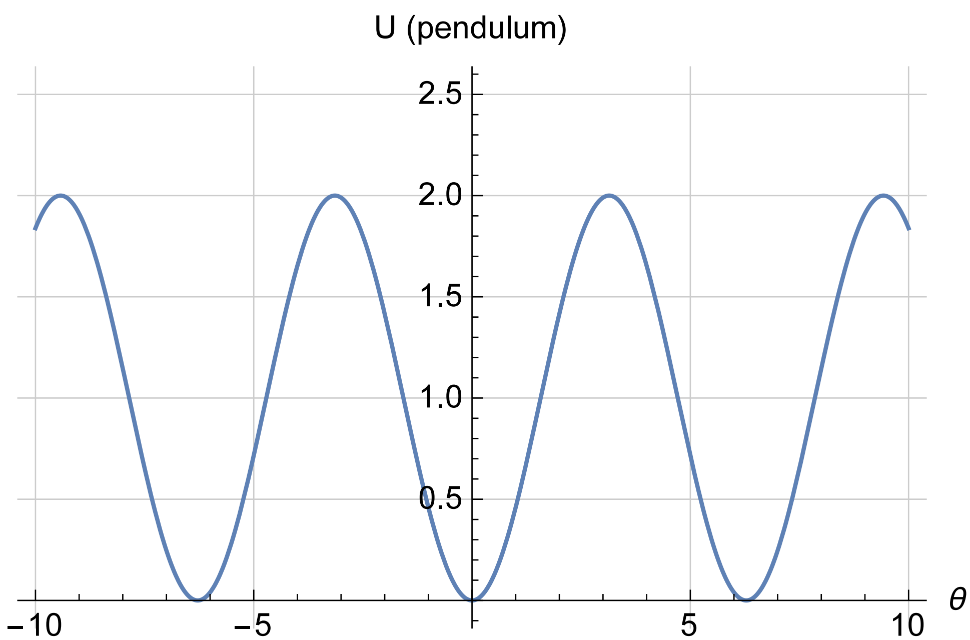
B. Advanced: Typical potential energy curves



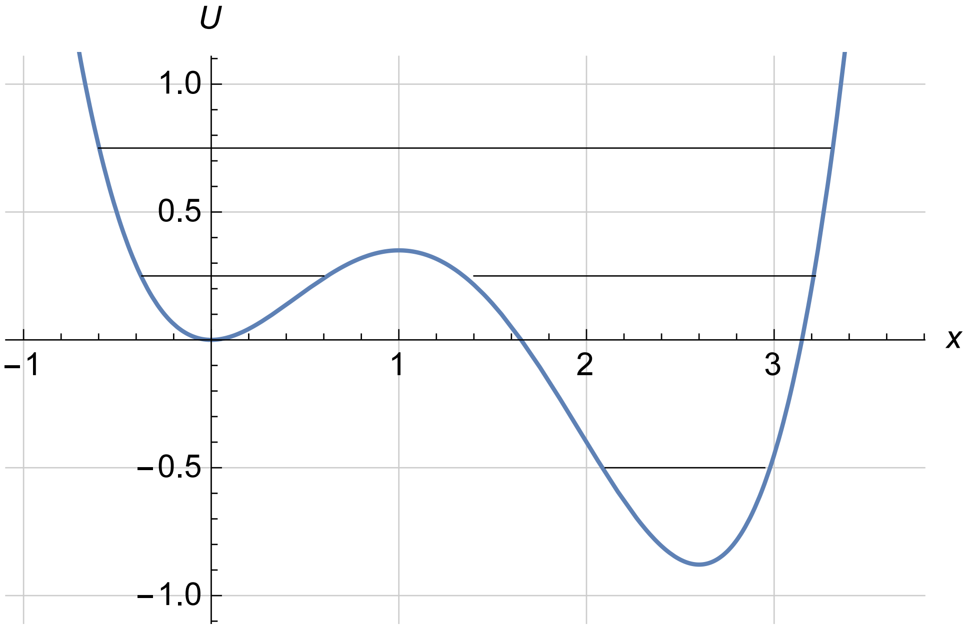
$$U = \frac{1}{2}kx^2 \tag{78}$$



$$U = \frac{1}{2}ky^2 + mgy \tag{79}$$



$$U = mgL(1 - \cos\theta) \tag{80}$$



$$K = E - U \geq 0\,,\ v = \pm\sqrt{2K/m} \tag{81}$$

C. (Advanced) "Forces of inertia"

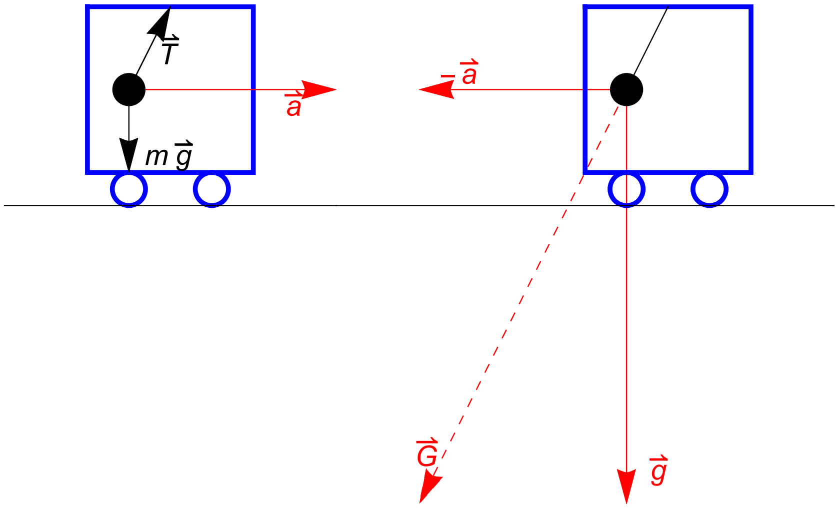
$$\vec{F} = m\vec{a}$$

Define

$$\vec{F}_i = -m\vec{a}$$

$$\vec{F} + \vec{F}_i = 0$$

"statics".



Deflection of a pendulum in an accelerating car. Left: from the point of inertial reference frame. Right: from the point an accelerating reference frame associated with the car; note a new "field of gravity" $-\vec{a}$ which combines with regular $\vec{g}$.

Regular solution (ϕ - angle with vertical):

$$\vec{T} + m\vec{g} = m\vec{a}$$

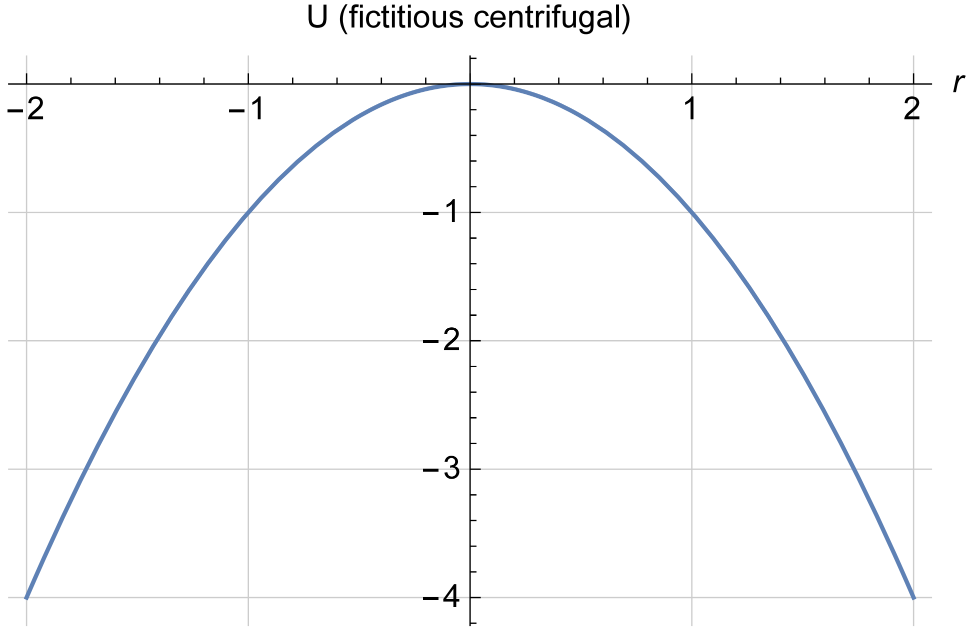
$$x: \; T\sin\phi = ma\,, \quad y: \; T\cos\phi - mg = 0$$

Thus,

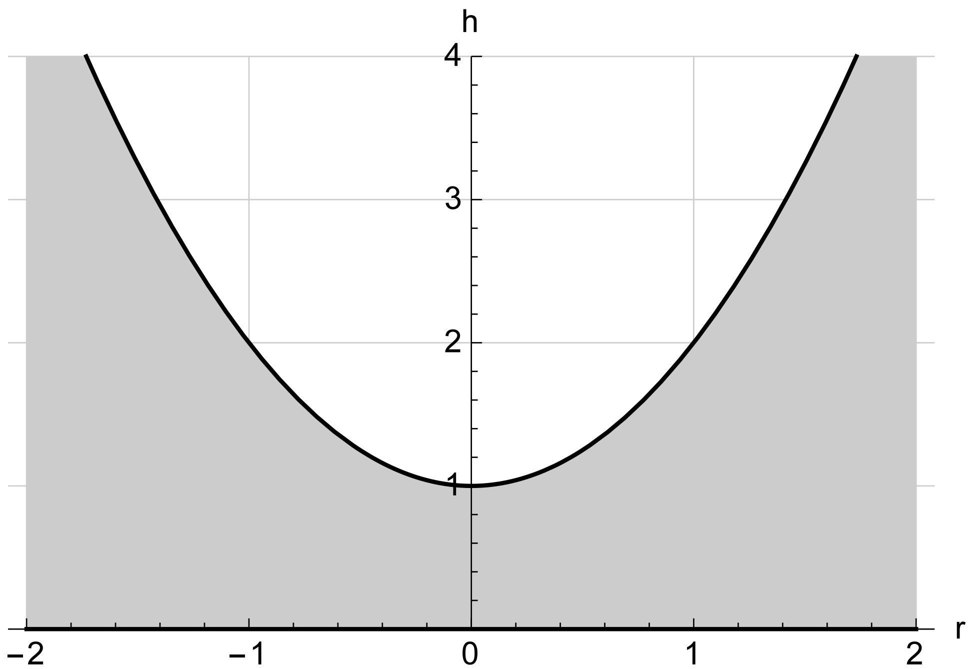
$$\tan\phi = \frac{a}{g}$$

Solution in non-inertial reference frame: no acceleration; pendulum hangs in the direction of new "full gravity" $\vec{G} = \vec{g} + (-\vec{a})$. Thus, $\tan\phi = a/g$, the same.

D. Advanced: Fictitious "centrifugal energy"



$$U = -\frac{1}{2}m\omega^2 r^2 \tag{82}$$



$$U = U_g + U_{cent} = mgh - \frac{1}{2}m\omega^2 r^2 = const \tag{83}$$

$$h(r) = h(0) + \frac{\omega^2}{g}r^2 \tag{84}$$

E. Advanced: Mathematical meaning of energy conservation

Start from 2nd Law with $F = F(x)$ (not v or t !)

$$m\ddot{x} = F(x) \mid \times \dot{x} \tag{85}$$

$$\dot{x}\ddot{x} = \frac{d}{dt}(\dot{x})^2/2$$
$$\dot{x}F(x) = \frac{d}{dt}\int F(x)\,dx = -\frac{d}{dt}U(x)$$

Thus

$$\frac{d}{dt}(K+U) = 0 \tag{86}$$
$$K + U = const = E \tag{87}$$