MTL732 Assignment

Vishal
2018MT60797

December 9, 2022

Problem Statement

Let the returns on n companies be $K_1, K_2, K_3, \ldots, K_n$, and let the corresponding weights be $w_1, w_2, w_3, \ldots, w_n$, and the corresponding mean be $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$, respectively. Also, let the variance-covariance matrix be $C = [C_{ij}]$. Suppose short selling is not allowed. A feasible portfolio is defined by:

$$\sum_{i=1}^{n} w_i = 1 \tag{1}$$

$$w_i \geq 0, \quad i = 1, 2, \ldots, n \tag{2}$$

The return of the portfolio K_v is given by:

$$K_v = \boldsymbol{w}^\top \boldsymbol{K}, \quad \text{where } \boldsymbol{K} = (K_1, K_2, \ldots, K_n), \boldsymbol{w} = (w_1, w_2, \ldots, w_n).$$

We define $-K_v$ as the loss. Thus, the investor maximizes the return such that:

$$\text{loss } \leq d$$

The problem simplifies to finding:

$$\max_{\boldsymbol{w}} (\boldsymbol{w}^\top \boldsymbol{K})$$

such that:

$$-w^\top \boldsymbol{K} \leq d, \quad \sum_{i=1}^{n} w_i = 1, \quad w_i \geq 0, \quad i = 1, 2, \ldots, n.$$

This is a stochastic optimization problem:

$$\max_{\boldsymbol{w}} \mathbb{E}[\boldsymbol{w}^\top \boldsymbol{K}] \quad \text{such that } P(-\boldsymbol{w}^\top \boldsymbol{K} \leq d) \geq \alpha,$$

$$\sum_{i=1}^{n} w_i = 1, \quad w_i \geq 0, \quad i = 1, 2, \ldots, n.$$

The above is called **chance constraints**:

$$P\left(w \mid -\boldsymbol{w}^\top \boldsymbol{K} \leq d\right) \geq \alpha.$$

Solution

Now, let $\boldsymbol{K}$ be a multivariate normal distribution. Suppose:

$$\boldsymbol{K} \sim N(\boldsymbol{\mu}, \boldsymbol{\sigma}),$$

whose probability density function (pdf) is:

$$\frac{1}{\sqrt{(2\pi)^n |\boldsymbol{C}|}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{C}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right).$$

Then:

$$-\boldsymbol{w}^\top \boldsymbol{K} \sim N(-\boldsymbol{w}^\top \boldsymbol{\mu}, \boldsymbol{w}^\top \boldsymbol{C}\boldsymbol{w}) \tag{3}$$

Assuming $|\boldsymbol{C}| \neq 0$ (as $\boldsymbol{C}$ is a positive definite matrix), we have:

$$P\left(\frac{-\boldsymbol{w}^\top \boldsymbol{K} + \boldsymbol{w}^\top \boldsymbol{\mu}}{\sqrt{\boldsymbol{w}^\top \boldsymbol{C}\boldsymbol{w}}} \leq \frac{d + \boldsymbol{w}^\top \boldsymbol{\mu}}{\sqrt{\boldsymbol{w}^\top \boldsymbol{C}\boldsymbol{w}}}\right) \geq \alpha \tag{4}$$

$$P\left(z \leq \frac{d + \boldsymbol{w}^\top \boldsymbol{\mu}}{\sqrt{\boldsymbol{w}^\top \boldsymbol{C}\boldsymbol{w}}}\right) \geq \alpha \tag{5}$$

$$\Phi\left(\frac{d + \boldsymbol{w}^\top \boldsymbol{\mu}}{\sqrt{\boldsymbol{w}^\top \boldsymbol{C}\boldsymbol{w}}}\right) \geq \alpha \tag{6}$$

$$\frac{d + \boldsymbol{w}^\top \boldsymbol{\mu}}{\sqrt{\boldsymbol{w}^\top \boldsymbol{C}\boldsymbol{w}}} \geq \Phi^{-1}(\alpha) \tag{7}$$

$$d \geq -\boldsymbol{w}^\top \boldsymbol{\mu} + \Phi^{-1}(\alpha)\sqrt{\boldsymbol{w}^\top \boldsymbol{C}\boldsymbol{w}} \tag{8}$$

As we have:

$$\sqrt{\boldsymbol{w}^\top \boldsymbol{C}\boldsymbol{w}} = \|\boldsymbol{C}^{1/2}\boldsymbol{w}\|_2, \tag{9}$$

and:

$$d \geq -\boldsymbol{w}^\top \boldsymbol{\mu} + \Phi^{-1}(\alpha)\|\boldsymbol{C}^{1/2}\boldsymbol{w}\|_2 \tag{10}$$

So, the optimization problem is:

$$\max_{\boldsymbol{w}} (\boldsymbol{w}^\top \boldsymbol{\mu})$$

subject to:

$$\sum_{i=1}^{n} w_i = 1, \quad w_i \geq 0, \quad i = 1, 2, \ldots, n,$$

$$-\boldsymbol{w}^\top \boldsymbol{\mu} + \Phi^{-1}(\alpha)\|\boldsymbol{C}^{1/2}\boldsymbol{w}\|_2 \leq d.$$

Clearly, this is a convex optimization problem, and the function in total is convex for $\alpha \in [0.5, 1)$ as $\Phi^{-1}(\alpha)$ is positive in this range, and both $-\boldsymbol{w}^\top \boldsymbol{\mu}$ and $\|\boldsymbol{C}^{1/2}\boldsymbol{w}\|_2$ are convex functions. Their positive linear combination (with positive coefficients) will also be a convex function.