Polynomials

25

POLYNOMIAL

A function $p(x)$ of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where a_0, a_1, a_2, ..., a_n are real numbers, $a_n \neq 0$ and n is a non-negative integer is called a *polynomial* in x over reals.

The real number a_0, a_1, ..., a_n are called the *coefficients of the polynomial*.

If a_0, a_1, a_2, ..., a^n are all integers, we call it a *polynomial over integers*.

If they are rational numbers, we call it a *polynomial over rationals*.

Illustration 1:

(a) $4x^2 + 7x - 8$ is a polynomial over integers.

(b) $\frac{7}{4}x^3 + \frac{2}{3}x^2 - \frac{8}{7}x + 5$ is a polynomial over rationals.

(c) $4x^2 - \sqrt{3}\,x + \sqrt{5}$ is a polynomial over reals.

Monomial

A polynomial having only one term is called a monomial. For example, 7, $2x$, $8x^3$ are monomials.

Binomial

A polynomial having two terms is called a binomial. For example, $2x + 3$, $7x^2 - 4x$, $x^2 + 8$ are binomials.

Trinomial

A polynomial having three terms is called a trinomial. For example, $7x^2 - 3x + 8$ is a trinomial.

Degree of a Polynomial

The exponent in the term with the highest power is called the degree of the polynomial.

For example, in the polynomial $8x^6 - 4x^5 + 7x^3 - 8x^2 + 3$, the term with the highest power is x^6. Hence, the degree of the polynomial is 6.

A polynomial of degree 1 is called a *linear polynomial*.

It is of the form $ax + b$, $a \neq 0$.

A polynomial of degree 2 is called a *quadratic polynomial*.

It is of the form $ax^2 + bx + c$, $a \neq 0$.

Division of a Polynomial by a Polynomial

Let, $p(x)$ and $f(x)$ are two polynomials and $f(x) \neq 0$. Then, if we can find polynomials $q(x)$ and $r(x)$, such that

$$p(x) = f(x) \cdot q(x) + r(x),$$

where degree $r(x)$ < degree $f(x)$, then we say that $p(x)$ divided by $f(x)$, gives $q(x)$ as *quotient* and $r(x)$ as *remainder*.

If the remainder $r(x)$ is zero, we say that *divisor* $f(x)$ is a factor of $p(x)$ and we have

$$p(x) = f(x) \cdot q(x).$$

Illustration 2: Divide $f(x) = 5x^3 - 70x^2 + 153x - 342$ by $g(x) = x^2 - 10x + 16$. Find the quotient and the remainder.

Solution:

$$\begin{array}{r|l}
 & 5x - 20 \\
\hline
x^2 - 10x + 16 & 5x^3 - 70x^2 + 153x - 342 \\
 & 5x^3 - 50x^2 + 80x \\
 & - \quad + \qquad\quad - \\
\hline
 & -20x^2 + 73x - 342 \\
 & -20x^2 + 200x - 320 \\
 & + \qquad\quad - \quad + \\
\hline
 & -127x - 22
\end{array}$$

$\therefore$ Quotient $= 5x - 20$ and

Remainder $= -127x - 22$.

Illustration 3: Determine if $(x - 1)$ is a factor of $p(x) = x^3 - 3x^2 + 4x + 2$.

$$
\begin{array}{r|l}
 & x^2 - 2x + 2 \\
\hline
x - 1 & x^3 - 3x^2 + 4x + 2 \\
 & x^3 - x^2 \\
 & - \quad + \\
\hline
 & -2x^2 + 4x \\
 & -2x^2 + 2x \\
 & \quad - \quad + \\
\hline
 & \qquad 2x + 2 \\
 & \qquad 2x - 2 \\
 & \qquad - \quad + \\
\hline
 & \qquad\quad 4
\end{array}
$$

Since the remainder is not zero, $(x - 1)$ is not a factor of $p(x)$.

SOME BASIC THEOREMS

Factor Theorem

Let, $p(x)$ be a polynomial of degree $n > 0$. If $p(a) = 0$ for a real number a, then $(x - a)$ is a factor of $p(x)$.
Conversely, if $(x - a)$ is a factor of $p(x)$, then $p(a) = 0$.

Illustration 4: Use factor theorem to determine if $(x - 1)$ is a factor of $x^8 - x^7 + x^6 - x^5 + x^4 - x + 1$.

Solution: Let, $p(x) = x^8 - x^7 + x^6 - x^5 + x^4 - x + 1$.

Then, $p(1) = (1)^8 - (1)^7 + (1)^6 - (1)^5 + (1)^4 - 1 + 1 = 1 \neq 0$.

Hence, $(x - 1)$ is not a factor of $p(x)$.

Remainder Theorem

Let, $p(x)$ be any polynomial of degree ≥ 1 and a any number.
If $p(x)$ is divided by $x - a$, the remainder is $p(a)$.

Illustration 5: Let, $p(x) = x^5 + 5x^4 - 3x + 7$ be divided by $(x - 1)$. Find the remainder.

Solution: Remainder $= p(1) = (1)^5 + 5(1)^4 - 3(1) + 7 = 10$.

Some Useful Results and Formulae

1. $(A + B)^2 = A^2 + B^2 + 2AB$
2. $(A - B)^2 = A^2 + B^2 - 2AB = (A + B)^2 - 4AB$
3. $(A + B)(A - B) = A^2 - B^2$
4. $(A + B)^2 + (A - B)^2 = 2(A^2 + B^2)$
5. $(A + B)^2 - (A - B)^2 = 4AB$
6. $(A + B)^3 = A^3 + B^3 + 3AB(A + B)$
7. $(A - B)^3 = A^3 - B^3 - 3AB(A - B)$
8. $A^2 + B^2 = (A + B)^2 - 2AB$
9. $A^3 + B^3 = (A + B)(A^2 + B^2 - AB)$
10. $A^3 - B^3 = (A - B)(A^2 + B^2 + AB)$
11. $(A + B + C)^2 = A^2 + B^2 + C^2 + 2(AB + BC + CA)$
12. $(A^3 + B^3 + C^3 - 3ABC$
    $= (A + B + C)(A^2 + B^2 + C^2 - AB - CA - BC)$
13. $A + B + C = 0 \Rightarrow A^3 + B^3 + C^3 = 3ABC$.
14. $A^n - B^n$ is divisible by $(A - B)$ for all values of n.
15. $A^n - B^n$ is divisible by $(A + B)$ only for even values of n.
16. $A^n + B^n$ is never divisible by $(A - B)$.
17. $A^n + B^n$ is divisible by $(A + B)$ only when n is odd.

A Useful Shortcut Method

When a polynomial $f(x)$ is divided by $x - a$ and $x - b$, the respective remainders are A and B. Then, if the same polynomial is divided by $(x - a)(x - b)$, the remainder will be

$$\frac{A - B}{a - b}x + \frac{Ba - Ab}{a - b}.$$

M25.indd 2 9/21/2015 12:20:39 PM

Illustration 6: When a polynomial $f(x)$ is divided by $(x-1)$ and $(x-2)$, the respective remainders are 15 and 9. What is the remainder when it is divided by

$$(x-1)(x-2)?$$

Solution: Remainder $= \frac{A-B}{a-b}x + \frac{Ba-Ab}{a-b}$

$= \frac{15-9}{1-2}x + \frac{9(1)-15(2)}{1-2}$

$= (-x+21)$.

Exercise-I

1. If $(x-2)$ is a factor of the polynomial $x^3 - 2ax^2 + ax - 1$, then find the value of a.

(a) $\frac{5}{6}$ (b) $\frac{7}{6}$
(c) $\frac{11}{6}$ (d) None of these

2. If $x + a$ is a factor of the polynomial $x^3 + ax^2 - 2x + a + 4$, then find the value of a.

(a) $-\frac{4}{3}$ (b) $+\frac{2}{3}$
(c) $+\frac{4}{3}$ (d) None of these

3. Find the value of k if $f(x) = x^3 - kx^2 + 11x - 6$ and $(x-1)$ is a factor of $f(x)$.

(a) 6 (b) 4
(c) 8 (d) None of these

4. If $5x^2 - 4x - 1$ is divided by $x - 1$, then the remainder is:

(a) 0 (b) 2
(c) 1 (d) None of these

5. Find the values of m and n in the polynomials $2x^3 + mx^2 + nx - 14$, such that $(x-1)$ and $(x+2)$ are its factors.

(a) $m = 4$, $n = 5$ (b) $m = 9$, $n = 3$
(c) $m = 6$, $n = 7$ (d) None of these

6. What value should a possess so that $x + 1$ may be a factor of the polynomial.

$f(x) = 2x^3 - ax^2 - (2a - 3)x + 2$?

(a) 2 (b) -2
(c) 3 (d) None of these

7. Divide the polynomial $4y^3 - 3y^2 + 2y - 4$ by $y + 2$ and find the quotient and remainder.

(a) $4y^2 - 11y + 24, -52$
(b) $6y^2 - 13y + 36, -64$
(c) $4y^2 + 13y - 24, +52$
(d) None of these

8. Resolve into factors: $16(x-y)^2 - 9(x+y)^2$.

(a) $(x-5y)(5x-y)$ (b) $(x+7y)(7x+y)$
(c) $(x-7y)(7x-y)$ (d) None of these

9. Resolve into factors: $4x^2 + 12xy + 9y^2 - 8x - 12y$.

(a) $(3x+2y)(4x+2y-3)$
(b) $(2x+3y)(2x+3y-4)$
(c) $(2x-3y)(2x+3y+4)$
(d) None of these

10. Resolve into factors: $16x^2 - 72xy + 81y^2 - 12x + 27y$.

(a) $(6x-7y)(6x-7y-5)$
(b) $(4x-9y)(4x-9y-3)$
(c) $(4x+9y)(4x+9y+3)$
(d) None of these

11. Resolve into factors: $(a+b)^2 - 14c(a+b) + 49c^2$.

(a) $(a-b-9c)^3$ (b) $(a+b-7c)^2$
(c) $(a+b+9c)^2$ (d) None of these

12. Resolve into factors: $81x^2y^2 + 108xyz + 36z^2$.

(a) $(6xy+9z)^2$ (b) $(9xy-7z)^2$
(c) $(9xy+6z)^2$ (d) None of these

13. Factorize: $(a-b+c)^2 + (b-c+a)^2 + 2(a-b+c)(b+c-a)$.

(a) $4a^2$ (b) $6a^2$
(c) $8a^2$ (d) None of these

14. Resolve into factors: $9(3x+5y)^2 - 12(3x+5y)(2x+3y) + 4(2x+3y)^2$.

(a) $(7x+9y)^2$ (b) $(5x+9y)^2$
(c) $(5x-9y)^2$ (d) None of these

15. Factorize: $(2x+3y)^2 + 2(2x+3y)(2x-3y) + (2x-3y)^2$.

(a) $16x^2$ (b) $18x^2$
(c) $12x^2$ (d) None of these

16. Factorize: $45a^3b + 5ab^3 - 30a^2b^2$.

(a) $5ab(5a-b)^2$ (b) $7ab(5a-b)^2$
(c) $5ab(3a-b)^2$ (d) None of these

17. Find the factors of $(a - b)^3 + (b - c)^3 + (c - a)^3$.
(a) $3(a + b)(b + c)(c + a)$
(b) $5(a - b)(b - c)(c - a)$
(c) $3(a - b)(b - c)(c - a)$
(d) None of these

18. Factorize: $a^2 + \frac{1}{a^2} + 3 - 2a - \frac{2}{a}$.
(a) $\left(a + \frac{1}{a} - 1\right)\left(a - \frac{1}{a} + 1\right)$
(b) $\left(a + \frac{1}{a} - 1\right)\left(a + \frac{1}{a} + 1\right)$
(c) $\left(a + \frac{1}{a} + 1\right)\left(a + \frac{1}{a} + 1\right)$
(d) $\left(a + \frac{1}{a} - 1\right)\left(a + \frac{1}{a} - 1\right)$.

19. If $x + \frac{1}{x} = 2$, then find the value of $x^4 + \frac{1}{x^4}$.
(a) 2 (b) 4
(c) 6 (d) 8

20. If $\frac{x}{y} + \frac{y}{x} = 6$, then find the value of $\frac{x^3}{y^3} + \frac{y^3}{x^3}$.
(a) 176 (b) 198
(c) 184 (d) None of these

21. If $x + y + 2 = 0$, what will be the value of $\frac{x^2 + y^2 + z^2}{x^2 - yz}$?
(a) 4 (b) 6
(c) 2 (d) 8

22. If $\left(x^3 + \frac{1}{x^3}\right) = 52$, then the value of $x + \frac{1}{x}$ is:
(a) 4 (b) 3
(c) 6 (d) 13

23. If $x = 3$ and $y = 4$, then find the value of $256x^4 + 160x^2y^2 + 25y^4$.
(a) 114967 (b) 50176
(c) 103976 (d) 914976

24. If $x + \frac{1}{x} = 2$, then $x^3 + \frac{1}{x^3}$ is equal to:
(a) 64 (b) 14
(c) 8 (d) 2

25. If $\sqrt{x} + \frac{1}{\sqrt{x}} = 5$, what will be the value of $x^2 + \frac{1}{x^2}$.
(a) 927 (b) 727
(c) 527 (d) 627

26. If $x + \frac{1}{x} = 3$, then the value of $x^6 + \frac{1}{x^6}$ is:
(a) 927 (b) 414
(c) 364 (d) 322

27. Factors of $a^2 + \frac{1}{4} + a$ will be:
(a) $\left(a + \frac{1}{2}\right)\left(a - \frac{1}{2}\right)$ (b) $\left(a + \frac{1}{2}\right)^2$
(c) $\left(a + \frac{1}{2}\right)^3$ (d) $\left(a + \frac{1}{2}\right) \cdot a$

28. If $a + b + c = 0$, then the value of $\left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}\right)$ is:
(a) 1 (b) 0
(c) −1 (d) 3

29. If $x + y + z = 9$ and $xy + yz + zx = 23$, then the value of $x^3 + y^3 + z^3 - 3xyz$ is:
(a) 108 (b) 207
(c) 669 (d) 729

30. If $x = \sqrt{3}$, then the value of $x^4 + 2 + \frac{1}{4x}$ will be:
(a) $\frac{9}{100}$ (b) $\frac{81}{100}$
(c) $\frac{101}{9}$ (d) $\frac{100}{9}$

31. If $x + \frac{1}{y} = 1$ and $y + \frac{1}{z} = 1$, find the value of $z + \frac{1}{x}$.
(a) 2 (b) 1
(c) 0 (d) 3

32. Resolve into factors:
$(a + b)^2 - 2(a^2 - b^2) + (a - b)^2$
(a) $6b^2$ (b) $2b^2$
(c) $4b^2$ (d) None of these

33. When $(x^3 - 2x^2 + px - q)$ is divided by $x^2 - 2x - 3$ the remainder is $(x - 6)$. The values of p and q are:
(a) $p = -2, q = -6$ (b) $p = 2, q = -6$
(c) $p = -2, q = 6$ (d) $p = 2, q = 6$

34. Let, $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$, where, $a_0, a_1, a_2, \ldots, a_n$ are constants. If $f(x)$ is divided by $ax - b$, then the remainder is:

(a) $f\left(\frac{b}{a}\right)$ (b) $f\left(\frac{-b}{a}\right)$

(c) $f\left(\frac{a}{b}\right)$ (d) $f\left(\frac{-a}{b}\right)$

35. If $(x^{3/2} - xy^{1/2} + x^{1/2}y - y^{3/2})$ is divided by $(x^{1/2} - y^{1/2})$, then the quotient is:

(a) $x + y$ (b) $x - y$

(c) $x^{1/2} + y^{1/2}$ (d) $x^2 - y^2$

36. When $4x^3 - ax^2 + bx - 4$ is divided by $x - 2$ and $x + 1$, the respective remainders are 20 and −13. Find the values of a and b.

(a) $a = 3, b = 2$ (b) $a = 5, b = 4$

(c) $a = 7, b = 6$ (d) $a = 9, b = 8$

37. When a polynomial $f(x)$ is divided by $x - 3$ and $x + 6$, the respective remainders are 7 and 22. What is the remainder when $f(x)$ is divided by $(x - 3)(x + 6)$?

(a) $\frac{-5}{3}x+12$ (b) $\frac{-7}{3}x+14$

(c) $\frac{-5}{3}x+16$ (d) $\frac{-7}{3}x+12$

38. If $(x - 1)$ is a factor of $Ax^3 + Bx^2 - 36x + 22$ and $2^B = 64^A$, find A and B.

(a) $A = 4, B = 16$ (b) $A = 6, B = 24$

(c) $A = 2, B = 12$ (d) $A = 8, B = 16$

Exercise-2
(Based on Memory)

1. If $x = 11$, then the value of $x^5 - 12x^4 + 12x^3 - 12x^2 + 12x - 1$ is:

(a) 5 (b) 10

(c) 15 (d) 20

[SSC, 2014]

2. If $p = 99$, then the value of $p(p^2 + 3p + 3)$ is:

(a) 10000000 (b) 999000

(c) 999999 (d) 990000

[SSC, 2014]

ANSWER KEYS

Exercise-1

1. (b)	**2.** (a)	**3.** (a)	**4.** (a)	**5.** (b)	**6.** (c)	**7.** (a)	**8.** (c)	**9.** (b)	**10.** (b)	**11.** (b)	**12.** (c)	**13.** (a)
14. (b)	**15.** (a)	**16.** (c)	**17.** (c)	**18.** (d)	**19.** (a)	**20.** (b)	**21.** (c)	**22.** (a)	**23.** (b)	**24.** (d)	**25.** (c)	**26.** (d)
27. (b)	**28.** (d)	**29.** (a)	**30.** (d)	**31.** (b)	**32.** (c)	**33.** (c)	**34.** (a)	**35.** (a)	**36.** (a)	**37.** (a)	**38.** (c)	

Exercise-2

1. (b)	**2.** (c)

EXPLANATORY ANSWERS

EXERCISE-1

1. (b) Let, $p(x) = x^3 - 2ax^2 + ax - 1$

Since, $x - 2$ is a factor of $p(x)$, we must have $p(2) = 0$

$\therefore\ (2)^3 - 2a(2)^2 + 2a - 1 = 0$

$\Rightarrow\ 8 - 8a + 2a - 1 = 0$

$\Rightarrow\ -6a = -7 \ \Rightarrow\ a = \dfrac{7}{6}.$

2. (a) Let, $p(x) = x^3 + ax^2 - 2x + a + 4$

Since, $x + a$, i.e., $x - (-a)$ is a factor of $p(x)$, we must have $p(-a) = 0$

$\Rightarrow\ (-a)^3 + a(-a)^2 - 2(-a) + a + 4 = 0$

$\Rightarrow\ -a^3 + a^3 + 2a + a + 4 = 0$

$\Rightarrow\ 3a + 4 = 0 \ \Rightarrow\ a = -\dfrac{4}{3}.$

3. (a) $\because\ (x - 1)$ is a factor of $f(x)$.

$\therefore$ By factor theorem, $f(1) = 0$

$\Rightarrow\ (1)^3 - k(1)^2 + 11(1) - 6 = 0$

$\Rightarrow\ 1 - k + 11 - 6 = 0$

$\Rightarrow\ -k + 6 = 0 \ \Rightarrow\ k = 6.$

4. (a) $f(x) = 5x^2 - 4x - 1$

$\therefore\ f(1) = 5(1)^2 - 4(1) - 1 = 0.$

5. (b) Let, $f(x) = 2x^3 + mx^2 + nx - 14.$

Since, $x - 1$ is a factor of $f(x)$.

$\therefore\ f(1) = 0$ [By factor theorem]

$\Rightarrow\ 2(1)^3 + m(1)^2 + n(1) - 14 = 0$

$\Rightarrow\ 2 + m + n - 14 = 0 \ \Rightarrow\ m + n = 12$ …(1)

Since, $x + 2$, i.e., $x - (-2)$ is factor of $f(x)$.

$\therefore\ f(-2) = 0$ [By factor theorem]

$\Rightarrow\ 2(-2)^3 + m(-2)^2 + n(-2) - 14 = 0$

$\Rightarrow\ -16 + 4m - 2n - 24 = 0 \ \Rightarrow\ 4m - 2n - 30 = 0$

$\Rightarrow\ 2m - n = 15$ …(2)

Adding (1) and (2), we get $3m = 27 \ \Rightarrow\ m = 9$

Put $m = 9$ in (1), we get $9 + n = 12 \ \Rightarrow\ n = 3.$

6. (c) $f(x) = 2x^3 - ax^2 - (2a - 3)x + 2?$

If, $x + 1$, i.e., $x - (-1)$ is a factor of $f(x)$, then $f(-1) = 0$

[By factor theorem]

$\Rightarrow\ 2(-1)^3 - a(-1)^2 - (2a - 3)(-1) + 2 = 0$

$\Rightarrow\ -2 - a + 2a - 3 + 2 = 0$

$\Rightarrow\ a - 3 = 0 \ \Rightarrow\ a = 3.$

7. (a)

$$
\begin{array}{r|l}
 & 4y^2 - 11y + 24 \\
\hline
y+2 & 4y^3 - 3y^2 + 2y - 4 \\
 & 4y^3 + 8y^2 \\
 & - \quad - \\
\hline
 & -11y^2 + 2y - 4 \\
 & -11y^2 - 22y \\
 & + \quad + \\
\hline
 & 24y - 4 \\
 & 24y + 48 \\
 & - \quad - \\
\hline
 & -52
\end{array}
$$

$\therefore$ Quotient $= 4y^2 - 11y + 24$

Remainder $= -52.$

8. (c) $16(x - y)^2 - 9(x + y)^2$

$= [4(x - y)]^2 - [3(x + y)]^2$

$= [4(x - y) - 3(x + y)][4(x - y) + 3(x + y)]$

$= (4x - 4y - 3x - 3y)\,(4x - 4y + 3x + 3y)$

$= (x - 7y)(7x - y).$

9. (b) $4x^2 + 12xy + 9y^2 - 8x - 12y$

$= [(2x)^2 + 2(2x)(3y) + (3y)^2] - 4(2x + 3y)$

$= (2x + 3y)^2 - 4(2x + 3y)$

$= (2x + 3y)(2x + 3y - 4).$

10. (b) $16x^2 - 72xy + 81y^2 - 12x + 27y$

$= (4x)^2 - 2(4x)(9y) + (9y)^2] - 3(4x - 9y)$

$= (4x - 9y)^2 - 3(4x - 9y)$

$= (4x - 9y)(4x - 9y - 3).$

11. (b) $(a + b)^2 - 14c(a + b) + 49c^2$

$= (a + b)^2 - 2(a + b)\cdot(7c) + (7c)^2$

$= (a + b - 7c)^2.$

12. (c) $81x^2y^2 + 108xyz + 36z^2$

$= (9xy)^2 + 2(9xy)(6z) + (6z)^2$

$= (9xy + 6z)^2$

13. **(a)** $(a - b + c)^2 + (b - c + a)^2 + 2(a - b + c)(b + c - a)$

$= (a - b + c)^2 + 2(a - b + c)(b + c - a) + (b - c + a)^2$ [rearranging]

$= [(a - b + c) + (b - c + a)]^2 = (2a)^2 = 4a^2.$

14. **(b)** $9(3x + 5y)^2 - 12(3x + 5y)(2x + 3y) + 4(2x + 3y)^2$

$= [3(3x+5y)]^2 - 2[3(3x+5y)][2(2x+3y)] + [2(2x+3y)]^2$

$= [3(3x + 5y) - 2(2x + 3y)]^2$

$= (9x + 15y - 4x - 6y)^2 = (5x + 9y)^2.$

15. **(a)** $(2x + 3y)^2 + 2(2x + 3y)(2x - 3y) + (2x - 3y)^2$

$= [(2x + 3y) + (2x - 3y)]^2 = (4x)^2 = 16x^2.$

16. **(c)** $45a^3b + 5ab^3 - 30a^2b^2$

$= 5ab[9a^2 + b^2 - 6ab]$

$= 5ab[9a^2 - 6ab + b^2]$

$= 5ab[(3a)^2 - 2(3a)(b) + (b)^2]$

$= 5ab[3a - b]^2.$

17. **(c)** Suppose, $a - b = x$, $b - c = y$, $c - a = z$

$\therefore \quad (a - b) + (b - c) + (c - a) = x + y + z$

$\Rightarrow \quad 0 = x + y + z$

$\therefore \quad x + y = -z$ ...(1)

$\therefore \quad (x + y)^3 = (-z)^3$

or, $x^3 + y^3 + 3xy(x + y) = -z^3$

or, $x^3 + y^3 + z^3 + 3xy(-z) = -z^3$

[On substituting $x + y = -z$ from Equation (1)]

or, $x^3 + y^3 - 3xyz = -z^3$

or, $x^3 + y^3 + z^3 = 3xyz$

$\therefore \quad (a - b)^3 + (b - c)^3 + (c - a)^3$

$= 3(a - b)(b - c)(c - a)$

18. **(d)** $a^2 + \frac{1}{a^2} + 3 - 2a - \frac{2}{a}$

$= \left(a^2 + \frac{1}{a^2} + 2\right) - 2a - \frac{2}{a} + 1$

$= \left(a + \frac{1}{a}\right)^2 - 2\left(a + \frac{1}{a}\right) + 1$

$= x^2 - 2x + 1 \quad \left[\text{suppose } a + \frac{1}{a} = x\right]$

$= (x - 1)^2$

$= \left(a + \frac{1}{a} - 1\right)^2.$

19. **(a)** $x + \frac{1}{x} = 2 \;\Rightarrow\; \left(x + \frac{1}{x}\right)^2 = (2)^2$

$\therefore \quad x^2 + \frac{1}{x^2} + 2x \cdot \frac{1}{x} = 4 \;\Rightarrow\; x^2 + \frac{1}{x^2} + 2 = 4$

$\Rightarrow \quad x^2 + \frac{1}{x^2} = 2$

$\therefore \quad \left(x^2 + \frac{1}{x^2}\right)^2 = (2)^2 \;\Rightarrow\; x^4 + \frac{1}{x^4} + 2x^2 \cdot \frac{1}{x^2} = 4$

$\Rightarrow \quad x^4 + \frac{1}{x^4} + 2 = 4$

$\therefore \quad x^4 + \frac{1}{x^4} = 2.$

20. **(b)** $\frac{x}{y} + \frac{y}{x} = 6 \;\Rightarrow\; \left(\frac{x}{y} + \frac{y}{x}\right)^3 = (6)^3$

$\therefore \quad \frac{x^3}{y^3} + \frac{y^3}{x^3} + 3\left(\frac{x}{y} + \frac{y}{x}\right) = 216$

$\therefore \quad \frac{x^3}{y^3} + \frac{y^3}{x^3} + 3 \times 6 = 216$

$\therefore \quad \frac{x^3}{y} + \frac{y^3}{x^3} = 216 - 18 = 198.$

21. **(c)** $\because \quad x + y + z = 0 \;\Rightarrow\; (x + y + z)^2 = 0$

$\therefore \quad x^2 + y^2 + z^2 + 2(xy + yz + zx) = 0$

$\therefore \quad x^2 + y^2 + z^2 = -2(xy + yz + zx)$

$= -2[x(y + z) + yz]$

$= -2(x \times -x + yz)$

$(\because \quad x + y + z = 0)$

$= 2(x^2 - yz)$

$\therefore \quad \frac{x^2 + y^2 + z^2}{x^2 - yz} = 2.$

22. **(a)** $\left(x + \frac{1}{x}\right)^3 = \left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$

$\therefore \quad \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) = x^3 + \frac{1}{x^3} = 52$

$\Rightarrow \quad y^3 - 3y = 52$ where, $y = x + \frac{1}{x}$

i.e., $y^3 - 3y - 52 = 0$

Clearly $y = 4$, satisfies $y^3 - 3y - 52 = 0$

$\therefore \quad x + \frac{1}{x} = 4.$

23. (b) $256x^4 + 160x^2y^2 + 25y^4$

$= (16x^2)^2 + 2.16x^2 \times 5y^2 + (5y^2)^2$

$= (16x^2 + 5y^2)^2$

On substituting $x = 3$ and $y = 4$

$(16x^2 + 5y^2)^2 = (16 \times 3^2 + 5 \times 4^2)^2$

$= (16 \times 9 + 5 \times 16)^2$

$= (144 + 80)^2 = (224)^2$

$= 50176.$

24. (d) $x+\frac{1}{x}=2 \Rightarrow \left(x+\frac{1}{x}\right)^3 = 23$

$\Rightarrow x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)=8$

$\Rightarrow x^3+\frac{1}{x^3}+3\times 2=8$

$\Rightarrow x^3+\frac{1}{x^3}=2.$

25. (c) $\sqrt{x}+\frac{1}{\sqrt{x}}=5 \Rightarrow \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2=(5)^2$

$\therefore x+2\cdot\sqrt{x}\cdot\frac{1}{\sqrt{x}}+\frac{1}{x}=25$

$\therefore 2+x+\frac{1}{x}=25 \Rightarrow x+\frac{1}{x}=23$

$\therefore \left(x+\frac{1}{x}\right)^2=(23)^2 \Rightarrow x^2+\frac{1}{x^2}+2=529$

$\Rightarrow x^2+\frac{1}{x^2}=527.$

26. (d) $\left(x+\frac{1}{x}\right)^2=3^2 \Rightarrow x^2+\frac{1}{x^2}=7$

$\Rightarrow \left(x^2+\frac{1}{x^2}\right)^3=7^3$

$\therefore x^6+\frac{1}{x^6}+3\left(x^2+\frac{1}{x^2}\right)=343$

$\Rightarrow x^6+\frac{1}{x^6}+3\times 7=343$

$\therefore x^6+\frac{1}{x^6}=343-21=322.$

27. (b) $a^2+\frac{1}{4}+a=a^2+\left(\frac{1}{2}\right)^2+2\cdot a\left(\frac{1}{2}\right)$

$=\left(a+\frac{1}{2}\right)^2.$

28. (d) $a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$

$\therefore \frac{a^3}{abc}+\frac{b^3}{abc}+\frac{c^3}{abc}=3$ or, $\frac{a^2}{bc}+\frac{b^2}{ac}+\frac{c^2}{ab}=3.$

29. (a) $x^3 + y^3 + z^3 - 3xyz$

$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$= (x + y + z)[(x + y + z)^2 - 3(xy + yz + zx)]$

$= 9[(9)^2 - 3(23)] = 9[81 - 69]$

$= 9 \times 12 = 108.$

30. (d) $x^4+2+\frac{1}{x^4}=(x^2)^2+2\cdot x^2\cdot\frac{1}{x^2}+\left(\frac{1}{x^2}\right)^2$

$=\left(x^2+\frac{1}{x^2}\right)^2$

$\therefore$ On substituting $x=\sqrt{3}$

$=\left[(\sqrt{3})^2+\frac{1}{(\sqrt{3})^2}\right]^2$

$=\left(3+\frac{1}{3}\right)^2=\left(\frac{10}{3}\right)^2$

$=\frac{100}{9}.$

31. (b) $x+\frac{1}{y}=1 \Rightarrow x=1-\frac{1}{y}=\frac{y-1}{y}$

$\Rightarrow \frac{1}{x}=\frac{y}{y-1}$

and, $y+\frac{1}{z}=1 \Rightarrow \frac{1}{z}=1-y \Rightarrow z=\frac{1}{1-y}$

$\therefore z+\frac{1}{x}=\frac{1}{1-y}+\frac{y}{y-1}=\frac{1}{1-y}-\frac{y}{1-y}$

$=\frac{1-y}{1-y}=1.$

32. (c) $(a + b)^2 - 2(a^2 - b^2) + (a - b)^2$

$= (a + b)^2 - 2(a + b)(a - b) + (a - b)^2$

$= \{(a + b) - (a - b)\}^2 = (2b)^2 = 4b^2.$

33. **(c)** On actual division, remainder is $(p + 3)x - q$.

$\therefore \quad (p + 3)x - q = x - 6 \quad \Rightarrow \quad p + 3 = 1$ and $q = 6$

$\Rightarrow \quad p = -2,\ q = 6.$

34. **(a)** $ax - b = 0 \Rightarrow \quad x = \dfrac{b}{a}$

So, $\quad$ remainder $= f\left(\dfrac{b}{a}\right).$

35. **(a)** $x^{3/2} - xy^{1/2} + x^{1/2}y - y^{3/2}$

$= x(x^{1/2} - y^{1/2}) + y(x^{1/2} - y^{1/2})$

$= (x + y)(x^{1/2} - y^{1/2})$

$$\therefore \quad \frac{x^{3/2} - xy^{1/2} + x^{1/2}y - y^{3/2}}{x^{1/2} - y^{1/2}} = (x + y).$$

36. **(a)** Let, $f(x) = 4x^3 - ax^2 + bx - 4$. When the expression $f(x)$ is divided by $x - 2$, the remainder is

$f(2) = 4(2)^3 - a(2)^2 + b(2) - 4 = 20$ (given)

$2b - 4a + 28 = 20 \quad \Rightarrow \quad 2a - b = 4$ (1)

Similarly, when the expression $f(x)$ is divided by $x - (-1)$, the remainder is

$f(-1) = 4 \times (-1)^3 - a(-1) + b(-1) - 4 = -13$ (given)

$\Rightarrow \quad -4 - a - b - 4 = -13$

$\Rightarrow \quad a + b = 5$ …(2)

Solving (1) and (2), we get

$a = 3,\ b = 2.$

37. **(a)** The function $f(x)$ is not known

Here, $a = 3,\ b = -6$

$A = 7,\ B = 22$

Required remainder

$$= \frac{A - B}{a - b}x + \frac{Ba - Ab}{a - b}$$
$$= \frac{7 - 22}{3 - (-6)}x + \frac{22 \times 3 - 7 \times (-6)}{3 - (-6)}$$
$$= -\frac{5}{3}x + 12.$$

38. **(c)** Since $x - 1$ is a factor of $Ax^3 + Bx^2 - 36x + 22$

$\therefore \quad A(1)^3 + B(1)^2 - 36(1) + 22 = 0 \quad \Rightarrow \quad A + B = 14$

and, $2B = (2^6)A \quad \Rightarrow \quad B = 6A$

$\therefore \quad A = 2,\ B = 12.$

Exercise-2
(Based on Memory)

1. **(b)** $x = 11$ (Given)

$\therefore \quad x^5 - 12x^4 + 12x^3 - 12x^2 + 12x - 1$

$= x^5 - (11+1)x^4 + (11+1)x^3 - (11+1)x^2 + (11+1)x - 1$

$= x^5 - 11x^4 - x^4 + 11x^3 + x^3 - 11x^2 - x^2 + 11x + x - 1$

When $x = 11$,

$= 11^5 - 11^5 - 11^4 + 11^4 + 11^3 - 11^3 - 11^2 + 11^2 + 11 - 1 = 10$

2. **(c)** $p = 99$ (Given)

$\therefore \quad p(p^2 + 3p + 3) = p^3 + 3p^2 + 3p$

$= p^3 + 3p^2 + 3p + 1 - 1$

$= (p + 1)^3 - 1 = (99 + 1)^3 - 1$

$= (100)^3 - 1 = 999999$