TRIGONOMETRIC RATIOS & IDENTITIES

1. INTRODUCTION TO TRIGONOMETRY :

The word 'trigonometry' is derived from the Greek words 'trigon' and 'metron' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analysing a musical tone and in many other areas.

(a) **Measurement of angles** : There are three systems of measurement of angles.

(i) **Sexagesimal or English System** : Here 1 right angle = 90 (degrees)

1 = 60' (minutes)

1' = 60" (seconds)

(ii) **Centesimal or French System** : Here 1 right angle = 100^g (grades)

1^g = 100' (minutes)

1' = 100" (seconds)

(iii) **Circular system** : Here an angle is measured in radians. One radian corresponds to the angle subtended by an arc of length 'r' at the centre of the circle of radius r. It is a constant quantity and does not depend upon the radius of the circle.

(b) Relation between the three systems : $\frac{D}{90}=\frac{G}{100}=\frac{R}{\pi/2}$

(c) If θ is the angle subtended at the centre of a circle of radius 'r',

by an arc of length 'ℓ' then $\frac{\ell}{r}=\theta$.

Note that here ℓ, r are in the same units and θ is always in radians.

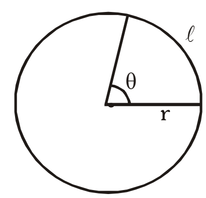


Illustration 1 : If the arcs of same length in two circles subtend angles of 60 and 75 at their centres. Find the ratio of their radii.

Solution : Let r_1 and r_2 be the radii of the given circles and let their arcs of same length s subtend angles of 60 and 75 at their centres.

Now, $60 = \left(60\times\frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$ and $75° = \left(75\times\frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$

$\therefore \quad \frac{\pi}{3}=\frac{s}{r_1}$ and $\frac{5\pi}{12}=\frac{s}{r_2}$

$\Rightarrow \quad \frac{\pi}{3}r_1 = s$ and $\frac{5\pi}{12}r_2 = s \quad \Rightarrow \quad \frac{\pi}{3}r_1 = \frac{5\pi}{12}r_2 \quad \Rightarrow 4r_1 = 5r_2 \quad \Rightarrow \quad r_1 : r_2 = 5 : 4$ **Ans.**

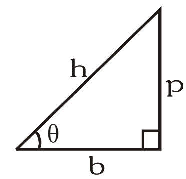
Do yourself - 1 :

(i) Express in the three systems of angular measurement, the magnitude of the angle of a regular decagon.

(ii) The radius of a circle is 30 cm. Find the length of an arc of this circle if the length of the chord of the arc is 30 cm.

2. T-RATIOS (or Trigonometric functions) :

In a right angle triangle $\sin\theta = \frac{p}{h}$; $\cos\theta = \frac{b}{h}$; $\tan\theta = \frac{p}{b}$; $\operatorname{cosec}\theta = \frac{h}{p}$; $\sec\theta = \frac{h}{b}$ and $\cot\theta = \frac{b}{p}$



'p' is perpendicular ; 'b' is base and 'h' is hypotenuse.

Note : The quantity by which the cosine falls short of unity i.e. $1 - \cos\theta$, is called the versed sine θ of θ and also by which the sine falls short of unity i.e. $1 - \sin\theta$ is called the coversed sine of θ.

3. BASIC TRIGONOMETRIC IDENTITIES :

(1) $\sin\theta . \operatorname{cosec}\theta = 1$

(2) $\cos\theta . \sec\theta = 1$

(3) $\tan\theta . \cot\theta = 1$

(4) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ & $\cot\theta = \frac{\cos\theta}{\sin\theta}$

(5) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

(6) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

(7) $\sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta}$

(8) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$ or $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \operatorname{cosec}^2\theta - 1$

(9) $\operatorname{cosec}\theta + \cot\theta = \frac{1}{\operatorname{cosec}\theta - \cot\theta}$

(10) Expressing trigonometrical ratio in terms of each other :

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\operatorname{cosec}\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$
$\tan\theta$	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\cot\theta$	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\operatorname{cosec}^2\theta-1}$
$\sec\theta$	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\frac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\operatorname{cosec}\theta$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\operatorname{cosec}\theta$

Illustration 2 : If $\sin\theta + \sin^2\theta = 1$, then prove that $\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta - 1 = 0$

Solution : Given that $\sin\theta = 1 - \sin^2\theta = \cos^2\theta$

L.H.S. $= \cos^6\theta(\cos^2\theta + 1)^3 - 1 = \sin^3\theta(1 + \sin\theta)^3 - 1 = (\sin\theta + \sin^2\theta)^3 - 1 = 1 - 1 = 0$

Illustration 3 : $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$ is equal to

(A) 0 (B) 1 (C) –2 (D) none of these

Solution : $2[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)] - 3[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta] + 1$

$= 2[1 - 3\sin^2\theta\cos^2\theta] - 3[1 - 2\sin^2\theta\cos^2\theta] + 1$

$= 2 - 6\sin^2\theta\cos^2\theta - 3 + 6\sin^2\theta\cos^2\theta + 1 = 0$ **Ans.(A)**

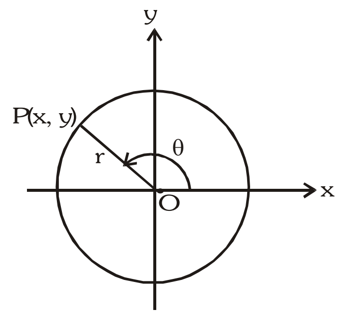
Do yourself - 2 :

(i) If $\cot\theta = \frac{4}{3}$, then find the value of sinθ, cosθ and cosecθ in first quadrant.

(ii) If sinθ + cosecθ = 2, then find the value of $\sin^8\theta + \text{cosec}^8\theta$

4. NEW DEFINITION OF T-RATIOS :

By using rectangular coordinates the definitions of trigonometric functions can be extended to angles of any size in the following way (see diagram). A point P is taken with coordinates (x, y). The radius vector OP has length r and the angle θ is taken as the directed angle measured anticlockwise from the x-axis. The three main trigonometric functions are then defined in terms of r and the coordinates x and y.



sinθ = y/r,

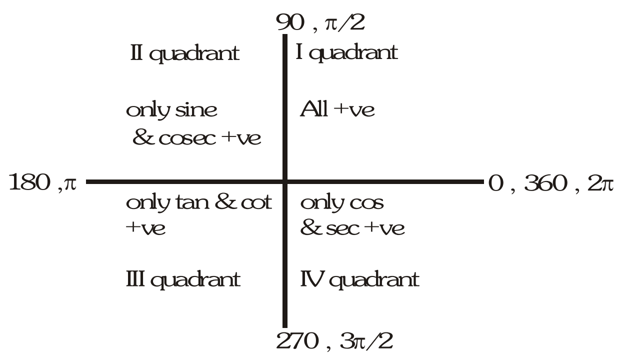
cosθ = x/r

tanθ = y/x,

(The other function are reciprocals of these)

This can give negative values of the trigonometric functions.

5. SIGNS OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS :



6. TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES :

(a) sin (2nπ + θ) = sin θ, cos (2nπ + θ) = cos θ, where n ∈ I

(b)

sin (–θ) = – sin θ	cos (–θ) = cos θ
sin(90 – θ) = cosθ	cos(90 – θ) = sinθ
sin(90 + θ) = cosθ	cos(90 + θ) = –sinθ
sin(180 – θ) = sinθ	cos(180 – θ) = –cosθ
sin(180 + θ) = –sinθ	cos(180 + θ) = –cosθ
sin(270 – θ) = –cosθ	cos(270 – θ) = –sinθ
sin(270 + θ) = –cosθ	cos(270 + θ) = sinθ
sin (360 – θ) = –sinθ	cos(360 – θ) = cosθ
sin (360 + θ) = sinθ	cos(360 + θ) = cosθ

7. VALUES OF T-RATIOS OF SOME STANDARD ANGLES :

Angles / T-ratio	0	30	45	60	90	180	270
	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	-1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	-1	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	N.D.	0	N.D.
$\cot\theta$	N.D.	$\sqrt{3}$	1	$1/\sqrt{3}$	0	N.D.	0
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	N.D.	-1	N.D.
$\text{cosec}\theta$	N.D.	2	$\sqrt{2}$	$2/\sqrt{3}$	1	N.D.	-1

N.D. $\rightarrow$ Not Defined

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(b) $\sin(2n+1)\frac{\pi}{2} = (-1)^n$; $\cos(2n+1)\frac{\pi}{2} = 0$ where $n \in I$

Illustration 4 : If $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$ then θ is equal to -

(A) 30 (B) 150 (C) 210 (D) none of these

Solution : Let us first find out θ lying between 0 and 360 .

Since $\sin\theta = -\frac{1}{2} \Rightarrow \theta = 210$ or 330 and $\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30$ or 210

Hence , $\theta = 210$ or $\frac{7\pi}{6}$ is the value satisfying both. **Ans. (C)**

Do yourself - 3 :

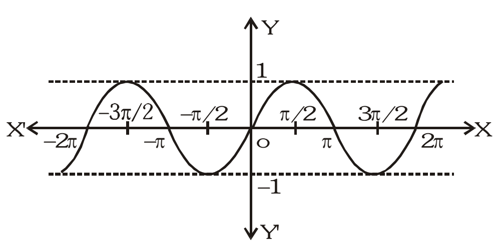
(i) If $\cos\theta = -\frac{1}{2}$ and $\pi < \theta < \frac{3\pi}{2}$, then find the value of $4\tan^2\theta - 3\text{cosec}^2\theta$.

(ii) Prove that : (a) $\cos 570 \sin 510 + \sin(-330)\cos(-390) = 0$

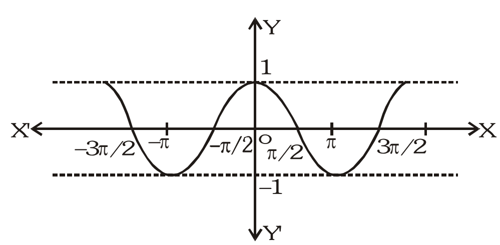
(b) $\tan\frac{11\pi}{3} - 2\sin\frac{9\pi}{3} - \frac{3}{4}\text{cosec}^2\frac{\pi}{4} + 4\cos^2\frac{17\pi}{6} = \frac{3-2\sqrt{3}}{2}$

8. GRAPH OF TRIGONOMETRIC FUNCTIONS :

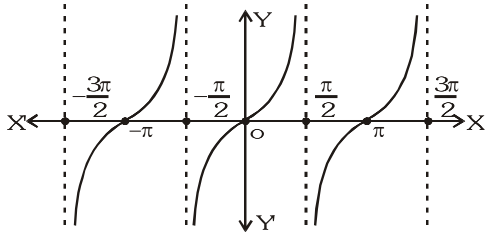
(i) $y = \sin x$



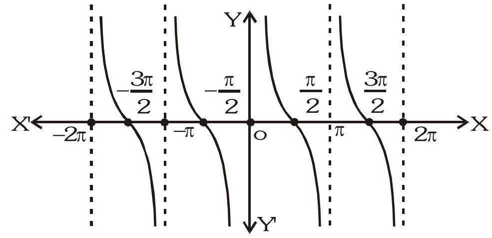
(ii) $y = \cos x$



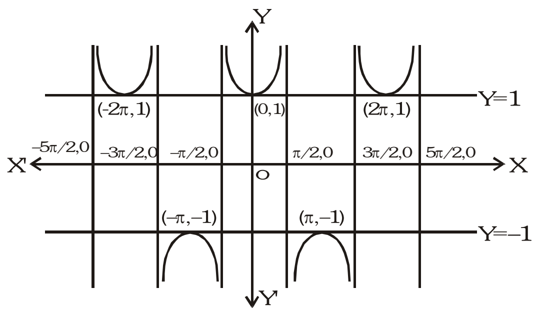
(iii) y = tanx



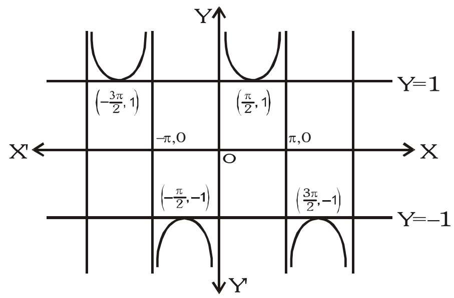
(iv) y = cotx



(v) y = secx



(vi) y = cosecx



9. DOMAINS, RANGES AND PERIODICITY OF TRIGONOMETRIC FUNCTIONS :

T-Ratio	Domain	Range	Period
sin x	R	[–1,1]	2π
cos x	R	[–1,1]	2π
tan x	R–{(2n+1)π/2 ; n∈I}	R	π
cot x	R–{nπ : n ∈ I}	R	π
sec x	R– {(2n+1) π/2 : n ∈ I}	(–∞,–1] ∪[1,∞)	2π
cosec x	R– {nπ : n ∈ I}	(–∞,–1] ∪[1,∞)	2π

10. TRIGONOMETRIC RATIOS OF THE SUM & DIFFERENCE OF TWO ANGLES :

(i) sin (A + B) = sin A cos B + cos A sin B.

(ii) sin (A – B) = sin A cos B – cos A sin B.

(iii) cos (A + B) = cos A cos B – sin A sin B

(iv) cos (A – B) = cos A cos B + sin A sin B

(v) tan (A + B) = $\frac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) tan (A – B) = $\frac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) cot (A + B) = $\frac{\cot B \cot A - 1}{\cot B + \cot A}$

(viii) cot (A – B) = $\frac{\cot B \cot A + 1}{\cot B - \cot A}$

Some more results :

(i) $\sin^2 A - \sin^2 B = \sin(A + B).\sin(A - B) = \cos^2 B - \cos^2 A$.

(ii) $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A - B)$.

Illustration 5 : Prove that $\sqrt{3}$ cosec20 – sec20 = 4.

Solution : L.H.S. = $\frac{\sqrt{3}}{\sin 20°} - \frac{1}{\cos 20°} = \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°.\cos 20°}$

$$= \frac{4\left(\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right)}{2\sin 20° \cos 20°} = \frac{4(\sin 60.\cos 20° - \cos 60°.\sin 20°)}{\sin 40°}$$

$$= 4.\frac{\sin(60° - 20°)}{\sin 40°} = 4.\frac{\sin 40°}{\sin 40°} = 4 = \text{R.H.S.}$$

Illustration 6 : Prove that tan70 = cot70 + 2cot40 .

Solution : L.H.S. = $\tan 70° = \tan(20° + 50°) = \dfrac{\tan 20° + \tan 50°}{1 - \tan 20° \tan 50°}$

or tan70 – tan20 tan50 tan70 = tan20 + tan50

or tan70 = tan70 tan50 tan20 + tan20 + tan50 = 2 tan 50 + tan20

= cot70 + 2cot40 = R.H.S.

Do yourself - 4 :

(i) If $\sin A = \dfrac{3}{5}$ and $\cos B = \dfrac{9}{41}$, $0 < A \;\&\; B < \dfrac{\pi}{2}$, then find the value of the following :

(a) sin(A + B) (b) sin(A – B) (c) cos(A + B) (d) cos(A – B)

(ii) If x + y = 45 , then prove that :

(a) (1 + tanx)(1 + tany) = 2 (b) (cotx – 1)(coty – 1) = 2

11. FORMULAE TO TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE :

(i) 2 sin A cos B = sin (A+ B) + sin (A – B).

(ii) 2 cos A sin B = sin (A + B) – sin (A – B).

(iii) 2 cos A cos B = cos (A + B) + cos (A – B)

(iv) 2 sin A sin B = cos (A – B) – cos (A + B)

Illustration 7 : If sin2A = λ sin2B, then prove that $\dfrac{\tan(A+B)}{\tan(A-B)} = \dfrac{\lambda+1}{\lambda-1}$

Solution : Given sin2A = λ sin2B

$$\Rightarrow \frac{\sin 2A}{\sin 2B} = \frac{\lambda}{1}$$

Applying componendo & dividendo,

$$\frac{\sin 2A + \sin 2B}{\sin 2B - \sin 2A} = \frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow \frac{2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)}{2\cos\left(\frac{2B+2A}{2}\right)\sin\left(\frac{2B-2A}{2}\right)} = \frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\sin\{-(A-B)\}} = \frac{\lambda+1}{1-\lambda} \qquad \Rightarrow \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\times -\sin(A-B)} = \frac{\lambda+1}{-(\lambda-1)}$$

$$\Rightarrow \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\sin(A-B)} = \frac{\lambda+1}{\lambda-1} \qquad \Rightarrow \tan(A+B)\cot(A-B) = \frac{\lambda+1}{\lambda-1}$$

$$\Rightarrow \frac{\tan(A+B)}{\tan(A-B)} = \frac{\lambda+1}{\lambda-1}$$

12. FORMULAE TO TRANSFORM SUM OR DIFFERENCE INTO PRODUCT :

(i) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(ii) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(iii) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(iv) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

Illustration 8 : $\frac{\sin 5\theta + \sin 2\theta - \sin\theta}{\cos 5\theta + 2\cos 3\theta + 2\cos^2\theta + \cos\theta}$ is equal to -

(A) $\tan\theta$ (B) $\cos\theta$ (C) $\cot\theta$ (D) none of these

Solution : L.H.S.= $\frac{2\sin 2\theta\cos 3\theta + \sin 2\theta}{2\cos 3\theta.\cos 2\theta + 2\cos 3\theta + 2\cos^2\theta} = \frac{\sin 2\theta[2\cos 3\theta + 1]}{2\left[\cos 3\theta(\cos 2\theta + 1) + (\cos^2\theta)\right]}$

$= \frac{\sin 2\theta[2\cos 3\theta + 1]}{2\left[\cos 3\theta(2\cos^2\theta) + \cos^2\theta\right]} = \frac{\sin 2\theta(2\cos 3\theta + 1)}{2\cos^2\theta(2\cos 3\theta + 1)} = \tan\theta$ **Ans. (A)**

Illustration 9 : Show that sin12 .sin48 .sin54 = 1/8

Solution : L.H.S. = $\frac{1}{2}[\cos 36° - \cos 60°]\sin 54° = \frac{1}{2}\left[\cos 36°\sin 54° - \frac{1}{2}\sin 54°\right]$

$= \frac{1}{4}[2\cos 36°\sin 54° - \sin 54°] = \frac{1}{4}[\sin 90° + \sin 18° - \sin 54°]$

$= \frac{1}{4}[1 - (\sin 54° - \sin 18°)] = \frac{1}{4}[1 - 2\sin 18°\cos 36°]$

$= \frac{1}{4}\left[1 - \frac{2\sin 18°}{\cos 18°}\cos 18°\cos 36°\right] = \frac{1}{4}\left[1 - \frac{\sin 36°\cos 36°}{\cos 18°}\right]$

$= \frac{1}{4}\left[1 - \frac{2\sin 36°\cos 36°}{2\cos 18°}\right] = \frac{1}{4}\left[1 - \frac{\sin 72°}{2\sin 72°}\right] = \frac{1}{4}\left[1 - \frac{1}{2}\right] = \frac{1}{8} = \text{R.H.S.}$

Do yourself - 5 :

(i) Simplify $\frac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°}$

(ii) Prove that

(a) (sin3A + sinA)sinA + (cos3A – cosA)cosA = 0

(b) cos20 cos40 cos60 cos80 = $\frac{1}{16}$

(c) $\frac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta} = \tan 2\theta$

13. TRIGONOMETRIC RATIOS OF SUM OF MORE THAN TWO ANGLES :

(i) $\sin(A+B+C) = \sin A\cos B\cos C + \sin B\cos A\cos C + \sin C\cos A\cos B - \sin A\sin B\sin C$

$= \Sigma\sin A\cos B\cos C - \Pi\sin A$

$= \cos A\cos B\cos C\,[\tan A + \tan B + \tan C - \tan A\tan B\tan C]$

(ii) $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

$= \Pi\cos A - \Sigma\sin A\sin B\cos C$

$= \cos A\cos B\cos C\,[1 - \tan A\tan B - \tan B\tan C - \tan C\tan A]$

(iii) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A} = \dfrac{S_1 - S_3}{1 - S_2}$

Do yourself - 6 :

Prove the above identities

14. TRIGONOMETRIC RATIOS OF MULTIPLE ANGLES :

(a) Trigonometrical ratios of an angle 2θ in terms of the angle θ :

(i) $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

(ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

(iii) $1 + \cos 2\theta = 2\cos^2\theta$ (iv) $1 - \cos 2\theta = 2\sin^2\theta$

(v) $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta} = \dfrac{\sin 2\theta}{1+\cos 2\theta}$ (vi) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Illustration 10 : Prove that : $\dfrac{2\cos 2A + 1}{2\cos 2A - 1} = \tan(60^\circ + A)\tan(60^\circ - A)$.

Solution : R.H.S. = $\tan(60 + A)\tan(60 - A)$

$$= \left(\frac{\tan 60^\circ + \tan A}{1 - \tan 60^\circ \tan A}\right)\left(\frac{\tan 60^\circ - \tan A}{1 + \tan 60^\circ \tan A}\right) = \left(\frac{\sqrt{3} + \tan A}{1 - \sqrt{3}\tan A}\right)\left(\frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}\right)$$

$$= \frac{3 - \tan^2 A}{1 - 3\tan^2 A} = \frac{3 - \dfrac{\sin^2 A}{\cos^2 A}}{1 - 3\dfrac{\sin^2 A}{\cos^2 A}} = \frac{3\cos^2 A - \sin^2 A}{\cos^2 A - 3\sin^2 A} = \frac{2\cos^2 A + \cos^2 A - 2\sin^2 A + \sin^2 A}{2\cos^2 A - 2\sin^2 A - \sin^2 A - \cos^2 A}$$

$$= \frac{2(\cos^2 A - \sin^2 A) + \cos^2 A + \sin^2 A}{2(\cos^2 A - \sin^2 A) - (\sin^2 A + \cos^2 A)} = \frac{2\cos 2A + 1}{2\cos 2A - 1} = \text{L.H.S.}$$

Do yourself - 7 :

(i) Prove that :

(a) $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$ (b) $\dfrac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta} = \cot\theta$

(c) $\dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$

(b) **Trigonometrical ratios of an angle 3θ in terms of the angle θ :**

(i) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. (ii) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(iii) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

Illustration 11 : Prove that : $\tan A + \tan(60 + A) + \tan(120 + A) = 3\tan 3A$

Solution : L.H.S. $= \tan A + \tan(60 + A) + \tan(120 + A)$

$= \tan A + \tan(60 + A) + \tan\{180 - (60 - A)\}$

$= \tan A + \tan(60 + A) - \tan(60 - A)$ $[\because \tan(180 - \theta) = -\tan\theta]$

$$= \tan A + \frac{\tan 60^\circ + \tan A}{1 - \tan 60^\circ \tan A} - \frac{\tan 60^\circ - \tan A}{1 + \tan 60^\circ \tan A} = \tan A + \frac{\sqrt{3} + \tan A}{1 - \sqrt{3}\tan A} - \frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}$$

$$= \tan A + \frac{\sqrt{3} + \tan A + 3\tan A + \sqrt{3}\tan^2 A - \sqrt{3} + \tan A + 3\tan A - \sqrt{3}\tan^2 A}{(1 - \sqrt{3}\tan A)(1 + \sqrt{3}\tan A)}$$

$$= \tan A + \frac{8\tan A}{1 - 3\tan^2 A} = \frac{\tan A - 3\tan^3 A + 8\tan A}{1 - 3\tan^2 A}$$

$$= \frac{9\tan A - 3\tan^3 A}{1 - 3\tan^2 A} = 3\left(\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}\right) = 3\tan 3A = \text{R.H.S.}$$

Do yourself - 8 :

(i) Prove that :

(a) $\cot\theta \cot(60 - \theta)\cot(60 + \theta) = \cot 3\theta$ (b) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(c) $\sin 4\theta = 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$

15. TRIGONOMETRIC RATIOS OF SUB MULTIPLE ANGLES :

Since the trigonometric relations are true for all values of angle θ, they will be true if instead of θ be substitute $\dfrac{\theta}{2}$

(i) $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} = \dfrac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$

(ii) $\cos\theta = \cos^2\dfrac{\theta}{2} - \sin^2\dfrac{\theta}{2} = 2\cos^2\dfrac{\theta}{2} - 1 = 1 - 2\sin^2\dfrac{\theta}{2} = \dfrac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$

(iii) $1 + \cos\theta = 2\cos^2\dfrac{\theta}{2}$

(iv) $1 - \cos\theta = 2\sin^2\dfrac{\theta}{2}$

(v) $\tan\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1 + \cos\theta}$

(vi) $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(vii) $\sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$

(viii) $\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$

(ix) $\tan\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

(x) $2\sin\dfrac{\theta}{2} = \pm\sqrt{1+\sin\theta} \pm \sqrt{1-\sin\theta}$

(xi) $2\cos\dfrac{\theta}{2} = \pm\sqrt{1+\sin\theta} \mp \sqrt{1-\sin\theta}$

(xii) $\tan\dfrac{\theta}{2} = \dfrac{\pm\sqrt{1+\tan^2\theta}-1}{\tan\theta}$

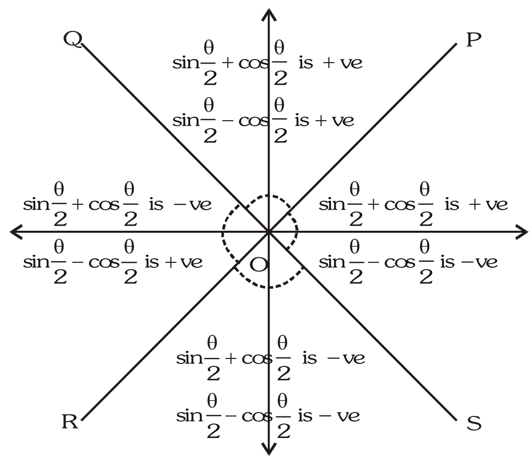


Illustration 12: $\sin 67\frac{1}{2} + \cos 67\frac{1}{2}$ is equal to

(A) $\dfrac{1}{2}\sqrt{4+2\sqrt{2}}$ (B) $\dfrac{1}{2}\sqrt{4-2\sqrt{2}}$ (C) $\dfrac{1}{4}\left(\sqrt{4+2\sqrt{2}}\right)$ (D) $\dfrac{1}{4}\left(\sqrt{4-2\sqrt{2}}\right)$

Solution : $\sin 67\frac{1}{2} + \cos 67\frac{1}{2} = \sqrt{1+\sin 135^\circ} = \sqrt{1+\dfrac{1}{\sqrt{2}}}$ (using $\cos A + \sin A = \sqrt{1+\sin 2A}$)

$= \dfrac{1}{2}\sqrt{4+2\sqrt{2}}$

Ans.(A)

Do yourself - 9 :

(i) Find the value of

(a) $\sin\dfrac{\pi}{8}$ (b) $\cos\dfrac{\pi}{8}$ (c) $\tan\dfrac{\pi}{8}$

16. TRIGONOMETRIC RATIOS OF SOME STANDARD ANGLES :

(i) $\sin 18^\circ = \sin\dfrac{\pi}{10} = \dfrac{\sqrt{5}-1}{4} = \cos 72^\circ = \cos\dfrac{2\pi}{5}$

(ii) $\cos 36^\circ = \cos\dfrac{\pi}{5} = \dfrac{\sqrt{5}+1}{4} = \sin 54^\circ = \sin\dfrac{3\pi}{10}$

(iii) $\sin 72^\circ = \sin\dfrac{2\pi}{5} = \dfrac{\sqrt{10+2\sqrt{5}}}{4} = \cos 18^\circ = \cos\dfrac{\pi}{10}$

(iv) $\sin 36^\circ = \sin\dfrac{\pi}{5} = \dfrac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54^\circ = \cos\dfrac{3\pi}{10}$

(v) $\sin 15^\circ = \sin\dfrac{\pi}{12} = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ = \cos\dfrac{5\pi}{12}$

(vi) $\cos 15^\circ = \cos\dfrac{\pi}{12} = \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ = \sin\dfrac{5\pi}{12}$

(vii) $\tan 15^\circ = \tan\dfrac{\pi}{12} = 2-\sqrt{3} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = \cot 75^\circ = \cot\dfrac{5\pi}{12}$

(viii) $\tan 75^\circ = \tan\dfrac{5\pi}{12} = 2+\sqrt{3} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = \cot 15^\circ = \cot\dfrac{\pi}{12}$

(ix) $\tan(22.5^\circ) = \tan\dfrac{\pi}{8} = \sqrt{2}-1 = \cot(67.5^\circ) = \cot\dfrac{3\pi}{8}$

(x) $\tan(67.5^\circ) = \tan\dfrac{3\pi}{8} = \sqrt{2}+1 = \cot(22.5^\circ) = \cot\dfrac{\pi}{8}$

Illustration 13 : Evaluate sin78 – sin66 – sin42 + sin6 .

Solution : The expression = (sin78 – sin42) – (sin66 – sin6) = 2cos(60) sin(18) – 2cos36 . sin30

$$= \sin 18 - \cos 36 = \left(\frac{\sqrt{5}-1}{4}\right) - \left(\frac{\sqrt{5}+1}{4}\right) = -\frac{1}{2}$$

Do yourself - 10 :

(i) Find the value of

(a) $\sin\frac{\pi}{10} + \sin\frac{13\pi}{10}$ (b) $\cos^2 48° - \sin^2 12°$

17. CONDITIONAL TRIGONOMETRIC IDENTITIES :

If A + B + C = 180 , then

(i) tan A + tan B + tan C = tan A tan B tan C

(ii) cot A cot B + cot B cot C + cot C cot A = 1

(iii) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(iv) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(v) sin 2A + sin 2B + sin 2C = 4 sinA sinB sinC

(vi) cos 2A + cos 2B + cos 2C =–1–4 cosA cosB cosC

(vii) sin A + sin B + sin C = 4 $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(viii) cos A + cos B + cos C = 1 + 4 $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Illustration 14 : In any triangle ABC, sin A – cos B = cos C, then angle B is

(A) $\pi/2$ (B) $\pi/3$ (C) $\pi/4$ (D) $\pi/6$

Solution : We have , sin A – cos B = cos C

sin A = cos B + cos C

$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\cos\left(\frac{\pi-A}{2}\right)\cos\left(\frac{B-C}{2}\right) \qquad \because \ A + B + C = \pi$$

$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow \quad \cos\frac{A}{2} = \cos\frac{B-C}{2} \quad \text{or} \quad A = B - C \quad ; \quad \text{But} \quad A + B + C = \pi$$

Therefore 2B = π $\Rightarrow$ B = $\pi/2$ **Ans.(A)**

Illustration 15 : If A + B + C = $\frac{3\pi}{2}$, then cos 2A + cos 2B + cos2C is equal to-

(A) 1 – 4cosA cosB cosC (B) 4 sinA sin B sinC

(C) 1 + 2cosA cosB cosC (D) 1 – 4 sinA sinB sinC

Solution : cos 2A + cos 2B + cos 2C = 2 cos (A + B) cos (A – B) + cos 2C

$$= 2\cos\left(\frac{3\pi}{2} - C\right)\cos(A-B) + \cos 2C \qquad \because \ A + B + C = \frac{3\pi}{2}$$

$$= -2\sin C\cos(A-B) + 1 - 2\sin^2 C = 1 - 2\sin C\,[\cos(A-B) + \sin C\,)$$

$$= 1 - 2\sin C\left[\cos(A-B) + \sin\left(\frac{3\pi}{2} - (A+B)\right)\right]$$

$$= 1 - 2\sin C\,[\cos(A-B) - \cos(A+B)] = 1 - 4\sin A\sin B\sin C$$ **Ans.(D)**

Do yourself - 11 :

(i) If ABCD is a cyclic quadrilateral, then find the value of sinA + sinB − sinC − sinD

(ii) If $A + B + C = \frac{\pi}{2}$, then find the value of tanA tanB + tanBtanC + tanC tanA

18. MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS :

(i) $a\cos\theta + b\sin\theta$ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$ i.e. the maximum and minimum values are $\sqrt{a^2+b^2}$, $-\sqrt{a^2+b^2}$ respectively.

(ii) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$ where a, b > 0

(iii) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le a\cos(\alpha+\theta) + b\cos(\beta+\theta) \le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where α and β are known angles.

(iv) If $\alpha, \beta, \in \left(0, \frac{\pi}{2}\right)$ and $\alpha + \beta = \sigma$ (constant) then

(i) Maximum value of the expression $\cos\alpha\cos\beta$, $\cos\alpha + \cos\beta$, $\sin\alpha\sin\beta$ or $\sin\alpha + \sin\beta$ occurs when $\alpha = \beta = \sigma/2$

(ii) Minimum value of $\sec\alpha + \sec\beta$, $\tan\alpha + \tan\beta$, $\operatorname{cosec}\alpha + \operatorname{cosec}\beta$ occurs when $\alpha = \beta = \sigma/2$

(v) If A, B, C are the angles of a triangle then maximum value of sin A + sin B + sin C and sin A sin B sin C occurs when A= B =C = 60

(vi) In case a quadratic in $\sin\theta$ & $\cos\theta$ is given then the maximum or minimum values can be obtained by making perfect square.

Illustration 16 : Prove that : $-4 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 10$, for all values of θ.

Solution : We have, $5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) = 5\cos\theta + 3\cos\theta\cos\frac{\pi}{3} - 3\sin\theta\sin\frac{\pi}{3} = \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$

Since, $-\sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le \sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2}$

$\Rightarrow \quad -7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le 7$

$\Rightarrow \quad -7 \le 5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) \le 7$ for all θ.

$\Rightarrow \quad -7+3 \le 5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) + 3 \le 7+3$ for all θ.

$\Rightarrow \quad -4 \le 5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) + 3 \le 10$ for all θ.

Illustration 17 : Find the maximum value of $1 + \sin\left(\frac{\pi}{4}+\theta\right) + 2\cos\left(\frac{\pi}{4}-\theta\right)$ -

(A) 1 (B) 2 (C) 3 (D) 4

Solution : We have $1 + \sin\left(\frac{\pi}{4}+\theta\right) + 2\cos\left(\frac{\pi}{4}-\theta\right)$

$= 1 + \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta) + \sqrt{2}(\cos\theta + \sin\theta) = 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)(\cos\theta + \sin\theta)$

$= 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) . \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$

$\therefore$ maximum value $= 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) . \sqrt{2} = 4$ **Ans. (D)**

Do yourself - 12 :

(i) Find maximum and minimum value of $5\cos\theta + 3\sin\left(\theta+\frac{\pi}{6}\right)$ for all real values of θ.

(ii) Find the minimum value of $\cos\theta + \cos2\theta$ for all real values of θ.

(iii) Find maximum and minimum value of $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$.

19. IMPORTANT RESULTS :

(i) $\sin\theta \sin(60° - \theta)\sin(60° + \theta) = \frac{1}{4}\sin3\theta$ (ii) $\cos\theta.\cos(60° - \theta)\cos(60° + \theta) = \frac{1}{4}\cos3\theta$

(iii) $\tan\theta \tan(60° - \theta)\tan(60° + \theta) = \tan3\theta$ (iv) $\cot\theta \cot(60° - \theta)\cot(60° + \theta) = \cot3\theta$

(v) (a) $\sin^2\theta + \sin^2(60° + \theta) + \sin^2(60° - \theta) = \frac{3}{2}$ (b) $\cos^2\theta + \cos^2(60° + \theta) + \cos^2(60° - \theta) = \frac{3}{2}$

(vi) (a) If $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, then $A + B + C = n\pi, n \in I$

(b) If $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$, then $A + B + C = (2n + 1)\frac{\pi}{2}, n \in I$

(vii) $\cos\theta\cos2\theta\cos4\theta .... \cos(2^{n-1}\theta) = \frac{\sin(2^n\theta)}{2^n\sin\theta}$

(viii) (a) $\cot A - \tan A = 2\cot2A$
(b) $\cot A + \tan A = 2\operatorname{cosec}2A$

(ix) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + ... \sin(\alpha + \overline{n-1}\,\beta) = \frac{\sin\left\{\alpha+\left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

(x) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha + 2\beta) + ... + \cos(\alpha + \overline{n-1}\,\beta) = \frac{\cos\left\{\alpha+\left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

Illustration 18 : Prove that $\tan A + 2\tan2A + 4\tan4A + 8\cot8A = \cot A$.

Solution : $8\cot8A = \cot A - \tan A - 2\tan2A - 4\tan4A$

$= 2\cot2A - 2\tan2A - 4\tan4A$ (using viii (a) in above results)

$= 4\cot4A - 4\tan4A$ (using viii (a) in above results)

$= 8\cot8A$.

Aliter Method : L.H.S. $= \tan A + 2\tan2A + 4\tan4A + 8\left(\frac{1-\tan^2 4A}{2\tan4A}\right)$

$= \tan A + 2\tan2A + \left(\frac{4\tan^2 4A + 4 - 4\tan^2 4A}{\tan4A}\right)$

$= \tan A + 2\tan2A + 4\cot4A = \tan A + 2\tan2A + 4\left(\frac{1-\tan^2 2A}{2\tan2A}\right)$

$= \tan A + \left[\frac{2\tan^2 2A + 2 - 2\tan^2 2A}{\tan2A}\right] = \tan A + 2\cot2A$

$= \tan A + 2\left(\frac{1-\tan^2 A}{2\tan A}\right) = \frac{\tan^2 A + 1 - \tan^2 A}{\tan A} = \cot A = \text{R.H.S.}$

Illustration 19 : Evaluate $\sum_{r=1}^{n-1}\cos^2\left(\frac{r\pi}{n}\right)$; $n \ge 2$

Solution :

$$\text{Sum} = \frac{1}{2}\sum_{r=1}^{n-1}\left(1+\cos\frac{2r\pi}{n}\right) = \frac{1}{2}(n-1)+\frac{1}{2}\left\{\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\ldots\ldots+\cos\frac{(2n-2)\pi}{n}\right\}$$

$$= \frac{1}{2}(n-1)+\frac{1}{2}\left\{\frac{\sin(n-1)\frac{2\pi}{2n}}{\sin\frac{2\pi}{n.2}}.\cos\left\{\frac{2\left(\frac{2\pi}{n}\right)+(n-2)\frac{2\pi}{n}}{2}\right\}\right\}$$

$$\left\{\text{Using}, \cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots\ldots.+\cos(\alpha+(n-1)\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.\cos\left\{\frac{2\alpha+(n-1)\beta}{2}\right\}\right\}$$

$$= \frac{1}{2}(n-1)+\frac{1}{2}\left\{\frac{\sin\frac{(n-1)\pi}{n}.\cos\pi}{\sin\left(\frac{\pi}{n}\right)}\right\} = \frac{1}{2}(n-1)-\frac{1}{2} = \frac{n}{2}-1$$

$$\therefore \quad \sum_{r=1}^{n-1}\cos^2\left(\frac{r\pi}{n}\right) = \frac{n-2}{2}$$ **Ans.**

Illustration 20 : Prove that : $(1 + \sec2\theta)(1 + \sec2^2\theta)(1 + \sec2^3\theta)\ldots\ldots(1 + \sec2^n\theta) = \tan2^n\theta.\cot\theta$.

Solution :

$$\text{L.H.S.} = \left(1+\frac{1}{\cos2\theta}\right)\left(1+\frac{1}{\cos2^2\theta}\right)\left(1+\frac{1}{\cos2^3\theta}\right)\ldots.\left(1+\frac{1}{\cos2^n\theta}\right)$$

$$= \left(\frac{1+\cos2\theta}{\cos2\theta}\right)\left(\frac{1+\cos2^2\theta}{\cos2^2\theta}\right)\left(\frac{1+\cos2^3\theta}{\cos2^3\theta}\right)\ldots.\left(\frac{1+\cos2^n\theta}{\cos2^n\theta}\right)$$

$$= \frac{2\cos^2\theta.2\cos^2 2\theta.2\cos^2 2^2\theta\ldots.2\cos^2 2^{n-1}\theta}{\cos2\theta.\cos2^2\theta.\cos2^3\theta\ldots.\cos2^n\theta}$$

$$= \cos\theta(2\cos\theta)(2\cos2\theta)(2\cos2^2\theta)\ldots(2\cos2^{n-1}\theta).\frac{1}{\cos2^n\theta}$$

$$= \frac{\cos\theta}{\sin\theta}(2\sin\theta\ \cos\theta)(2\cos2\theta)(2\cos2^2\theta)\ldots(2\cos2^{n-1}\theta).\frac{1}{\cos2^n\theta}$$

$$= \frac{\cos\theta}{\sin\theta}(2\sin2\theta.\ \cos2\theta)(2\cos2^2\theta)\ldots(2\cos2^{n-1}\theta).\frac{1}{\cos2^n\theta}$$

$$= \frac{\cos\theta}{\sin\theta}(2\sin2^{n-1}\theta.\cos2^{n-1}\theta).\frac{1}{\cos2^n\theta} = \frac{\cos\theta}{\sin\theta}.\sin2^n\theta.\frac{1}{\cos2^n\theta} = \tan2^n\theta.\cot\theta = \text{R.H.S.}$$

Do yourself - 13 :

(i) Evaluate $\sin\frac{\pi}{n}+\sin\frac{3\pi}{n}+\sin\frac{5\pi}{n}+\ldots\ldots\ldots$ to n terms

(ii) If $(2^n + 1)\theta = \pi$, then find the value of $2^n\cos\theta\cos2\theta\cos2^2\theta\ldots\ldots\ldots\cos2^{n-1}\theta$.

__Miscellaneous Illustration :__

Illustration 21 : Prove that

$\tan\alpha + 2\tan2\alpha + 2^2\tan^2\alpha + \ldots\ldots + 2^{n-1}\tan2^{n-1}\alpha + 2^n\cot2^n\alpha = \cot\alpha$

Solution : We know $\tan\theta = \cot\theta - 2\cot2\theta$ .....(i)

Putting $\theta = \alpha, 2\alpha, 2^2\alpha$, ...in (i), we get

$\tan\alpha = (\cot\alpha - 2\cot2\alpha)$

$2(\tan2\alpha) = 2(\cot2\alpha - 2\cot2^2\alpha)$

$2^2(\tan2^2\alpha) = 2^2(\cot2^2\alpha - 2\cot2^3\alpha)$

....

$2^{n-1}(\tan2^{n-1}\alpha) = 2^{n-1}(\cot2^{n-1}\alpha - 2\cot2^n\alpha)$

Adding,

$\tan\alpha + 2\tan 2\alpha + 2^2\tan^2\alpha + \ldots\ldots + 2^{n-1}\tan 2^{n-1}\alpha = \cot\alpha - 2^n\cot 2^n\alpha$

$\therefore\quad \tan\alpha + 2\tan 2\alpha + 2^2\tan^2\alpha + \ldots\ldots + 2^{n-1}\tan 2^{n-1}\alpha + 2^n\cot 2^n\alpha = \cot\alpha$

Illustration 22 : If A,B,C and D are angles of a quadrilateral and $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2} = \frac{1}{4}$, prove that $A = B = C = D = \pi/2$.

Solution :

$$\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)\left(2\sin\frac{C}{2}\sin\frac{D}{2}\right) = 1$$

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right\}\left\{\cos\left(\frac{C-D}{2}\right) - \cos\left(\frac{C+D}{2}\right)\right\} = 1$$

Since, $A + B = 2\pi - (C + D)$, the above equation becomes,

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right\}\left\{\cos\left(\frac{C-D}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right\} = 1$$

$$\Rightarrow \cos^2\left(\frac{A+B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\left\{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{C-D}{2}\right)\right\} + 1 - \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{C-D}{2}\right) = 0$$

This is a quadratic equation in $\cos\left(\frac{A+B}{2}\right)$ which has real roots.

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{C-D}{2}\right)\right\}^2 - 4\left\{1 - \cos\left(\frac{A-B}{2}\right).\cos\left(\frac{C-D}{2}\right)\right\} \geq 0$$

$$\left(\cos\frac{A-B}{2} + \cos\frac{C-D}{2}\right)^2 \geq 4$$

$\Rightarrow \cos\frac{A-B}{2} + \cos\frac{C-D}{2} \geq 2$, Now both $\cos\frac{A-B}{2}$ and $\cos\frac{C-D}{2} \leq 1$

$$\Rightarrow \cos\frac{A-B}{2} = 1 \ \& \ \cos\frac{C-D}{2} = 1$$

$$\Rightarrow \frac{A-B}{2} = 0 = \frac{C-D}{2}$$

$\Rightarrow$ A = B, C = D.

Similarly A = C, B = D $\Rightarrow$ A = B = C = D = $\pi/2$

ANSWERS FOR DO YOURSELF

1 :	(i)	144 , 160^g, $\left(\frac{4\pi}{5}\right)^c$		(ii)	10π cm	
2 :	(i)	$\frac{3}{5}, \frac{4}{5}, \frac{5}{3}$		(ii)	2	
3 :	(i)	8				
4 :	(i)	(a) $\frac{187}{205}$	(b) $\frac{-133}{205}$	(c) $\frac{-84}{205}$		(d) $\frac{156}{205}$
5 :	(i)	$\frac{1}{\sqrt{3}}$				
9:	(i)	(a) $\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$		(b) $\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$		(c) $\sqrt{2}-1$
10 :	(i)	(a) $-\frac{1}{2}$		(b) $\frac{\sqrt{5}+1}{8}$		
11 :	(i)	0	(ii) 1			
12 :	(i)	7 & –7	(ii) $-\frac{9}{8}$	(iii)	$4+\sqrt{10}$ & $4-\sqrt{10}$	
13 :	(i)	0	(ii) 1			