UNIT # 04
LIMIT
EXERCISE - 01 CHECK YOUR GRASP

2. $\lim\limits_{n\to\infty}\left(\frac{1}{1.3}+\frac{1}{3.5}+.....+\text{to n terms}\right)$

$=\lim\limits_{n\to\infty}\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+.....+\frac{1}{2n-1}-\frac{1}{2n+1}\right)$

$=\lim\limits_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{2n+1}\right)=\frac{1}{2}$

5. $\lim\limits_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$

$=\lim\limits_{x\to\infty}\left(\frac{x+\sqrt{x+\sqrt{x}}-x}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}\right)$ (By Rationalising)

$=\lim\limits_{x\to\infty}\left(\frac{\sqrt{1+\frac{\sqrt{x}}{x}}}{\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}+1}\right)=\frac{1}{2}$

9. Let $\lim\limits_{x\to a}f(x)=L$ & $\lim\limits_{x\to a}g(x)=M$

$\because$ L + M = 2 & L − M = 1

$\Rightarrow$ $L=\frac{3}{2}$ & $M=\frac{1}{2}$

So $\lim\limits_{x\to a}f(x)\,g(x)=L.M=\frac{3}{4}$

11. $\lim\limits_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$

$=\lim\limits_{x\to\alpha}\left(\frac{1-\cos a(x-\alpha)(x-\beta)}{(x-\alpha)^2(x-\beta)^2a^2}\right)(x-\beta)^2\ a^2$

$=\frac{1}{2}\ a^2\ (\alpha-\beta)^2$

15. $\lim\limits_{x\to1}(1-x)\tan\left(\frac{\pi x}{2}\right)=\lim\limits_{h\to0}(1-1+h)\tan\left(\frac{\pi}{2}-\frac{\pi h}{2}\right)$

$=\lim\limits_{h\to0}h\cot\frac{\pi}{2}h=\lim\limits_{h\to0}\frac{h}{\tan\frac{\pi h}{2}}=\frac{2}{\pi}$

16. $\lim\limits_{x\to\frac{\pi}{2}}\frac{(1-\tan x/2)}{(1+\tan x/2)}\frac{(1-\sin x)}{(\pi-2x)^3}$

$=\lim\limits_{x\to\frac{\pi}{2}}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\frac{(1-\sin x)}{(\pi-2x)^3}$

$=\lim\limits_{h\to0}\frac{\tan\left(\frac{\pi}{4}-\frac{\pi}{4}+\frac{h}{2}\right)(1-\cos h)}{(2h)^3}$

$=\lim\limits_{h\to0}\frac{2\tan\frac{h}{2}\sin^2\frac{h}{2}}{8h^3}=\frac{1}{32}$

17. $\lim\limits_{x\to0}\frac{(\tan(\{x\}-1))\sin\{x\}}{\{x\}(\{x\}-1)}$

$LHL=\lim\limits_{h\to0^-}\frac{(\tan((1-h)-1))\sin(1-h)}{(1-h)((1-h)-1)}$

$[\because\ \{x\}=\{0-h\}=1-h]$

$=\lim\limits_{h\to0^-}\frac{-\tan h\sin(1-h)}{(1-h)(-h)}=\sin1$

$RHL=\lim\limits_{h\to0^+}\frac{\tan(h-1)\sin h}{h(h-1)}=\tan1$

$\because$ LHL $\neq$ RHL

$\therefore$ limit does not exist

18. $\lim\limits_{x\to\infty}\sin\sqrt{x+1}-\sin\sqrt{x}$

$=\lim\limits_{x\to\infty}2\cos\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right)$

$=\lim\limits_{x\to\infty}\frac{2\cos\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right)\times\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right)}$

$=\lim\limits_{x\to\infty}2\cos\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)\times\frac{1}{2\left(\sqrt{x+1}+\sqrt{x}\right)}=0$

22. $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x}=e^2\quad\Rightarrow\quad e^{\lim\limits_{x\to\infty}2x\left(1+\frac{a}{x}+\frac{b}{x^2}-1\right)}=e^2$

$\Rightarrow e^{2a}=e^2$

$\Rightarrow a=1\ \&\ b\in R$

26. $\lim\limits_{x\to0}\frac{(1+3x)^{1/3}-1-x}{(1+x)^{101}-1-101x}$

$=\lim\limits_{x\to0}\frac{\frac{3}{3}(1+3x)^{-2/3}-1}{(101)(1+x)^{100}-101}$ (By L' Hospital rule)

$=\lim\limits_{x\to0}\frac{-\frac{2}{3}(1+3x)^{-5/3}.3}{(101)(100)(1+x)^{99}}=-\frac{1}{5050}$

27. $\lim\limits_{x\to0}\frac{e^{x^3}-\tan x+\sin x-1}{x^n}$

$=\lim\limits_{x\to0}\frac{e^{x^3}-1}{x^n}+\frac{-\tan x(1-\cos x)}{x^n}$

Now for existence of limit n should be 3

1. $\lim_{x\to-\infty} [(x^5 + 7x^4 + 2)^c - x]$

$$= \lim_{x\to-\infty}\left(x^{5c}\left(1+\frac{7}{x}+\frac{2}{x^5}\right)^c - x\right)$$

for limit to exist c = 1/5

Now to find limit put c = 1/5

$$\Rightarrow \lim_{x\to-\infty}\left(x\left(1+\frac{7}{x}+\frac{2}{x^5}\right)^{1/5} - x\right) = \frac{7}{5} \text{ (by expansion)}$$

3. $\lim_{n\to\infty}\left(\left(\frac{n}{n+1}\right)^{\alpha}+\sin\frac{1}{n}\right)^n$ [1^∞ form]

$$= e^{\lim_{n\to\infty} n\left(\left(\frac{n}{n+1}\right)^{\alpha}+\sin\left(\frac{1}{n}\right)-1\right)} = e^{\lim_{n\to\infty}\left(n\left(\frac{n}{n+1}\right)^{\alpha}+\frac{\sin(1/n)}{(1/n)}-n\right)}$$

$$= e^{\lim_{n\to\infty}\left(n\left(1-\frac{1}{n+1}\right)^{\alpha}+1-n\right)} = e^{\lim_{n\to\infty}\left(n\left(1-\frac{\alpha}{n+1}+......\right)+1-n\right)} = e^{1-\alpha}$$

5. Let $\left(\sqrt{(1-\cos x)+\sqrt{(1-\cos x)+\sqrt{(1-\cos x)+......\infty}}}\right) = y$

$y = \sqrt{(1-\cos x)+y}$

$\Rightarrow y^2 - y + (\cos x - 1) = 0$

$$y = \frac{1+\sqrt{5-4\cos x}}{2}$$

Now $\lim_{x\to 0}\frac{\sqrt{5-4\cos x}-1}{2x^2}$

$$= \lim_{x\to 0}\frac{4(1-\cos x)}{2x^2\left(\sqrt{5-4\cos x}+1\right)} = \frac{1}{2}$$

7. $\lim_{n\to\infty}\left(\left(1-\frac{0}{n}\right)^n+\left(1-\frac{1}{n}\right)^n+\left(1-\frac{2}{n}\right)^n+.....\left(1-\frac{n-1}{n}\right)^n\right)$

$$= e^{\lim_{n\to\infty}\left[n\left(\frac{-0}{n}\right)\right]}+e^{\lim_{n\to\infty} n\left(\frac{-1}{n}\right)}+e^{\lim_{n\to\infty} n\left(\frac{-2}{n}\right)}+.....+e^{\lim_{n\to\infty} n\left(\frac{n-1}{n}\right)}$$

$= e^0 + e^{-1} + e^{-2} + ...e^{n-1}$

$$= \frac{1}{1-e^{-1}} = \frac{e}{e-1}$$

9. $\lim_{n\to\infty}\cos\left(\pi\sqrt{n^2+n}\right) = \lim_{n\to\infty}\cos\left(n\pi\left(1+\frac{1}{n}\right)^{1/2}\right)$

$$= \lim_{n\to\infty}\cos n\pi\left(1+\frac{1}{2n}+\frac{1}{2}\cdot\frac{1}{2}\left(\frac{1}{2}-1\right)\frac{1}{n^2}+....\right)$$

$$= \lim_{n\to\infty}\cos\left(n\pi+\frac{\pi}{2}+0+.......\right) = -\sin n\pi = 0$$

10. $\lim_{x\to 0^+}\frac{1}{x\sqrt{x}}\left(a\tan^{-1}\frac{\sqrt{x}}{a}-b\tan^{-1}\frac{\sqrt{x}}{b}\right)$

$$= \lim_{x\to 0^+}\frac{a\left(\frac{\sqrt{x}}{a}-\frac{(\sqrt{x})^3}{3a^3}+\frac{(\sqrt{x})^5}{5a^5}+...\right)-b\left(\frac{\sqrt{x}}{b}-\frac{(\sqrt{x})^3}{3b^3}+\frac{(\sqrt{x})^5}{5b^5}+...\right)}{x\sqrt{x}}$$

$$= \frac{1}{3}\left(\frac{1}{b^2}-\frac{1}{a^2}\right) = \frac{a^2-b^2}{3a^2b^2}$$

15. Put $x = \frac{\pi}{2} - h$

$$\ell = \lim_{h\to 0}\frac{a^{\tan h}-a^{\sin h}}{\tan h-\sin h}$$

$$= \lim_{h\to 0} a^{\sin h}\left(\frac{a^{\tan h-\sin h}-1}{\tan h-\sin h}\right)$$

$= \log_e a$

$$m = \lim_{x\to-\infty}\frac{x^2+ax-x^2+ax}{\sqrt{x^2+ax}+\sqrt{x^2-ax}}$$

$$= \lim_{x\to-\infty}\frac{2ax}{|x|\left(\sqrt{1+\frac{a}{x}}+\sqrt{1-\frac{a}{x}}\right)}$$

$$= \frac{2ax}{-x(2)} = -a$$

True & False :

3. $\lim\limits_{x\to 0} x^a \dfrac{\sin^b x}{\sin^c x} = \lim\limits_{x\to 0} \dfrac{\sin^{b-c} x}{x^{-a}}$ exist

$\Rightarrow$ b – c = –a $\Rightarrow$ a + b = c

Match the Column :

1. (A) $L = \lim\limits_{h\to 0} \dfrac{\cos\left(\tan^{-1}\left(\tan\left(\dfrac{\pi}{2}+h\right)\right)\right)}{\dfrac{\pi}{2}+h-\dfrac{\pi}{2}}$

$= \dfrac{\cos\left(-\dfrac{\pi}{2}+h\right)}{h} = \dfrac{\sinh}{h} = 1$

Now cos [2π (1)] = 1

(B) $k = \lim\limits_{n\to\infty} \prod\limits_{r=2}^{n} \dfrac{r^3-1}{r^3+1} = \lim\limits_{n\to\infty} \prod\limits_{r=2}^{n} \dfrac{(r-1)}{(r+1)} \dfrac{(r^2+r+1)}{(r^2-r+1)}$

$= \lim\limits_{n\to\infty} \dfrac{2}{n(n+1)} \times \dfrac{n^2+n+1}{3} = \dfrac{2}{3}$

so $\text{cosec}\theta = \dfrac{2}{3} \Rightarrow$ No. of solution is zero

(C) $\lim\limits_{x\to\infty} \left(\dfrac{x+c}{x-c}\right)^x = 4$

$\Rightarrow e^{\lim\limits_{x\to\infty} x\left(\frac{x+c-x+c}{x-c}\right)} = 4 \Rightarrow e^{\lim\limits_{x\to\infty} x\left(\frac{2c}{x-c}\right)} = 4$

$\Rightarrow e^{2c} = 4 \Rightarrow e^c = 2$ (only positive value)

$\Rightarrow \dfrac{-e^c}{2} = -1$

(D) $\lim\limits_{x\to -\infty} \dfrac{(3x^4+2x^2)\sin(1/x)-x^3+5}{(-x)^3+(-x)^2-x+1} = k$

$\Rightarrow \lim\limits_{x\to -\infty} \dfrac{\left(3+\dfrac{2}{x^2}\right)\dfrac{\sin(1/x)}{(1/x)}-1+\dfrac{5}{x^3}}{-1+\dfrac{1}{x}-\dfrac{1}{x^2}+\dfrac{1}{x^3}} = k$

$\Rightarrow \dfrac{3-1}{-1} = k \Rightarrow \dfrac{k}{2} = -1$

Assertion & Reason :

3. For all x, $x-1 < [x] \le x$, where [.] denotes greatest integer function.

$\Rightarrow x^n - 1 < [x^n] \le x^n \Rightarrow \dfrac{1}{x^n} \le \dfrac{1}{[x^n]} < \dfrac{1}{x^n-1}$

Multiplying the inequation by $x^n + nx^{n-1}+1$ and taking the limit as $x \to \infty$,
we get,

$$\lim_{x\to\infty} \frac{x^n+nx^{n-1}+1}{x^n} \le \lim_{x\to\infty} \frac{x^n+nx^{n-1}+1}{[x^n]} < \lim_{x\to\infty} \frac{x^n+nx^{n-1}+1}{x^n-1}$$

Evaluating the limits on the left and right side of the inequality, we obtain

$$\lim_{x\to\infty} \frac{x^n+nx^{n-1}+1}{x^n} = \lim_{x\to\infty} \frac{x^n+nx^{n-1}+1}{x^n-1} = 1$$

And hence by sandwich theorem,

$$\lim_{x\to\infty} \frac{x^n+nx^{n-1}+1}{[x^n]} = 1$$

$\Rightarrow$ Statement 1 is false.

4. $\tan^2 x > 1 \Rightarrow \tan x > \dfrac{1}{\sqrt{3}}$ or $\tan x < \dfrac{-1}{\sqrt{3}}$

$\Rightarrow \dfrac{-\pi}{2} < x < \dfrac{-\pi}{6}$ or $\dfrac{\pi}{6} < x < \dfrac{\pi}{2}$

$$f(x) = \begin{cases} 0, & \dfrac{-\pi}{2} < x < -\dfrac{\pi}{6} \\ \dfrac{x}{2}, & x = -\dfrac{\pi}{6} \\ x, & \dfrac{-\pi}{6} < x < \dfrac{\pi}{6} \\ \dfrac{x}{2}, & x = \dfrac{\pi}{6} \\ 0, & \dfrac{\pi}{6} < x < \dfrac{\pi}{2} \end{cases}$$

$\because$ obviously statement-I is false and statement-II is true.

Comprehension # 2 :

$$T(x) = \frac{1}{2}\left(\sin\left(\frac{x}{2}\right)\tan\left(\frac{x}{2}\right)\cos\left(\frac{\pi}{2}-\frac{x}{2}\right)\right) \times 2$$

$$= \sin^2\frac{x}{2} \tan\frac{x}{2}$$

$$s(x) = \frac{1}{2}(1)^2 (x - \sin x)$$

1. $\lim\limits_{x\to 0} \dfrac{\sin^2 x/2 \tan x/2}{x^3} = \dfrac{1}{8}$

2. $\lim\limits_{x\to 0} \dfrac{1}{2} \dfrac{(x-\sin x)}{x} = 0$

3. $\lim\limits_{x\to 0} \dfrac{\sin^2(x/2)\tan(x/2)}{\dfrac{1}{2}(x-\sin x)}$

$= \lim\limits_{x\to 0} \dfrac{1}{4}\left(\dfrac{x^3}{x-\sin x}\right) = \dfrac{3}{2}$

3. $\lim\limits_{x\to1}\dfrac{\left[\sum\limits_{K=1}^{100}x^K-100\right]}{x-1}$

$= \lim\limits_{x\to1}\left[\dfrac{x+x^2+x^3+\ldots\ldots\ldots+x^{100}-100}{x-1}\right]$

$= \lim\limits_{x\to1}\left[\dfrac{1+2x+3x^2+\ldots\ldots.+100x^{99}}{1}\right]$

(using L' Hospital)

$= 5050.$

4. (b) We put $x = -t$

$\lim\limits_{t\to\infty}\left(\sqrt{t^2+t+1}+at-b\right) = 0$

[For $\infty - \infty$, a must be negative.]

$\lim\limits_{t\to\infty}\left(\dfrac{t^2+t+1-(at-b)^2}{\sqrt{t^2+t+1}-(at-b)}\right) = 0$

$\lim\limits_{t\to0}\left(\dfrac{t^2(1-a^2)+t(1+2ab)+1-b^2}{\sqrt{t^2+t+1}-(at-b)}\right)$

For limit coefficient of t^2 in numerator should be zero

$1 - a^2 = 0$ & $1 + 2ab = 0$

$\Rightarrow \quad a^2 = \pm 1 \ \Rightarrow \ a = -1 \ \Rightarrow b = \dfrac{1}{2}$

(a=1 is rejected)

7. Let $f(x) = \left(\dfrac{1}{\sqrt{n^2}}+\dfrac{1}{\sqrt{n^2}}+\dfrac{1}{\sqrt{n^2}}+\ldots+\dfrac{1}{\sqrt{n^2}}\right) = \dfrac{2n+1}{\sqrt{n^2}}$

... (2n+1) terms

$\lim\limits_{n\to\infty}f(x) = 2$

$g(x) = \left(\dfrac{1}{\sqrt{n^2+2n}}+\dfrac{1}{\sqrt{n^2+2n}}+\ldots.+\dfrac{1}{\sqrt{n^2+2n}}\right)$

...(2n+1) terms

$\lim\limits_{n\to\infty}g(x) = 2$

$\therefore \ \lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2}}+\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\ldots.+\dfrac{1}{\sqrt{n^2+2n}}\right) = 2$

9. (Using L' Hospital rule)

$\lim\limits_{x\to\frac{3\pi}{4}}\dfrac{1+(\tan x)^{\frac{1}{3}}}{1-2\cos^2 x} = \lim\limits_{x\to\frac{3\pi}{4}}\dfrac{\frac{1}{3}\tan x^{\frac{-2}{3}}\sec^2 x}{4\cos x\,\sin x} = -\dfrac{1}{3}.$

10. $\lim\limits_{\theta\to\frac{\pi}{4}}\dfrac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2} = \lim\limits_{\theta\to\frac{\pi}{4}}\dfrac{\sqrt{2}-\sqrt{2}\left[\cos\left(\theta-\frac{\pi}{4}\right)\right]}{(4\theta-\pi)^2}$

$= \lim\limits_{\theta\to\frac{\pi}{4}}\dfrac{\sqrt{2}\left[1-\cos\left(\theta-\frac{\pi}{4}\right)\right]}{16\left(\theta-\frac{\pi}{4}\right)^2} = \dfrac{1}{16\sqrt{2}}.$

12. $\lim\limits_{x\to1}\dfrac{\ell n\left(\frac{1+x}{2}\right)\times3.(4^{x-1}-x)}{\left[(7+x)^{\frac{1}{3}}-(1+3x)^{\frac{1}{2}}\right]\sin(x-1)}$

$= \lim\limits_{h\to0}\dfrac{\ell n\left(\frac{2+h}{2}\right)\times3\left(4^h-(1+h)\right)}{\left[(8+h)^{\frac{1}{3}}-(4+3h)^{1/2}\right].\sin h}$

$= \lim\limits_{h\to0}\dfrac{\dfrac{\ell n\left(1+\frac{h}{2}\right)}{2(h/2)}\,.\,3\,.\,\left(4^h-(1+h)\right)}{2\left\{\left[1+\frac{h}{8}\times\frac{1}{3}+\ldots\right]-\left[1+\frac{3h}{4}\times\frac{1}{2}+\ldots\right]\right\}\dfrac{\sin h}{h}}$

$= \lim\limits_{h\to0}\dfrac{\frac{3}{4}\left[\frac{4^h-1}{h}-1\right]}{\frac{1}{h}\left[\frac{h}{24}-\frac{3h}{8}\right]} = \dfrac{-9}{4}\ell n\dfrac{4}{e}.$

15. $\lim\limits_{x\to2}\dfrac{(\cos\alpha)^x+(\sin\alpha)^x-1}{x-2}$

$= \lim\limits_{h\to0}\dfrac{\cos^2\alpha \ \ \cos^h\alpha+\sin^2\alpha \ \ \sin^h\alpha-1}{h}$

$= \lim\limits_{h\to0}\dfrac{\cos^h\alpha-\sin^2\alpha\cos^h\alpha+\sin^2\alpha\sin^h\alpha-1}{h}$

$= \lim\limits_{h\to0}\dfrac{\cos^h\alpha-1}{h} + \sin^2\alpha\left(\dfrac{\sin^h\alpha-1-\cos^h\alpha+1}{h}\right)$

$= \ell n\cos\alpha + \sin^2\alpha\,(\ell n\sin\alpha - \ell n\cos\alpha)$

$= \cos^2\alpha\,\ell n\cos\alpha + \sin^2\alpha\,\ell n\sin\alpha.$

18. $\lim\limits_{x\to\pi/4}\dfrac{\sqrt{1-\sqrt{\sin2x}}}{\pi-4x}$

LHL $= \lim\limits_{h\to0}\dfrac{\sqrt{1-\sqrt{\sin\left(\frac{\pi}{2}-2h\right)}}}{\pi-\pi+4h}$

$= \lim\limits_{h\to0}\dfrac{\sqrt{1-\sqrt{\cos2h}}}{4h}\ \dfrac{\sqrt{1+\sqrt{\cos2h}}}{\sqrt{1+\sqrt{\cos2h}}}$

$= \lim\limits_{h\to0}\dfrac{\sqrt{1-\cos2h}}{4h}\times\dfrac{1}{\sqrt{1+\sqrt{\cos2h}}}$

$= \dfrac{\sqrt{2}\,|\sin h|}{4h\times\sqrt{2}} = \dfrac{1}{4}$

Similarly RHL $= \dfrac{-1}{4}$

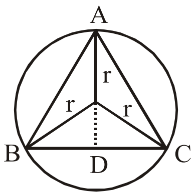
Hence LHL $\neq$ RHL $\quad\therefore\quad$ limit does not exist.

1. $BD = \sqrt{r^2-(h-r)^2}$, $AD = h$

$= \sqrt{2hr-h^2}$

$AB = \sqrt{h^2+2hr-h^2}$

$= \sqrt{2hr}$



Now $\lim\limits_{h\to0} \dfrac{\Delta}{P^3} = \lim\limits_{h\to0} \dfrac{\frac{1}{2}\,.\,h\,.\,2\sqrt{2hr-h^2}}{(2\sqrt{2hr}+2\sqrt{2hr-h^2})^3}$

$$= \lim_{h\to0} \frac{h^{\frac{3}{2}}\sqrt{2r-h}}{h^{\frac{3}{2}}\left(2\sqrt{2r}+2\sqrt{2r-h}\right)^3}$$

$$= \frac{\sqrt{2r}}{\left(2\sqrt{2r}+2\sqrt{2r}\right)^3} = \frac{1}{128\,r}\,.$$

4. $\lim\limits_{x\to\infty} x^2 \sin\left(\ell n\sqrt{\cos\dfrac{\pi}{x}}\right)$

$$= \lim_{x\to\infty} \frac{x^2\sin\left(\ell n\sqrt{\cos(\pi/x)}\right)}{\left(\ell n\sqrt{\cos(\pi/x)}\right)} \times \left(\ell n\sqrt{\cos\left(\frac{\pi}{x}\right)}\right)$$

$$= \lim_{x\to\infty} \frac{x^2}{2}\ell n\left(1+\cos\left(\frac{\pi}{x}\right)-1\right)$$

$$= \lim_{x\to\infty} \frac{x^2}{2}\frac{\ell n\left(1+\left(\cos\left(\frac{\pi}{x}\right)-1\right)\right)}{\left(\cos\left(\frac{\pi}{x}\right)-1\right)} \times \left(\cos\left(\frac{\pi}{x}\right)-1\right)$$

$$= \lim_{x\to\infty} \frac{x^2}{2}\left(\cos\left(\frac{\pi}{x}\right)-1\right)$$

Now put $x = 1/y$ $\Rightarrow \lim\limits_{y\to0} \dfrac{\cos(\pi y)-1}{2y^2} = \dfrac{-\pi^2}{4}$

5. $\lim\limits_{x\to a} \dfrac{1}{(a^2-x^2)^2}\left[\dfrac{a^2+x^2}{ax} - 2\sin\left(\dfrac{a\pi}{2}\right)\sin\left(\dfrac{x\pi}{2}\right)\right]$

$$= \lim_{h\to0} \frac{1}{(a^2-a^2-h^2-2ah)^2}\left[\frac{a^2+a^2+h^2+2ah}{a^2+ah} - 2\sin\left(\frac{a\pi}{2}\right)\sin\left((a+h)\frac{\pi}{2}\right)\right]$$

$$= \lim_{h\to0} \frac{1}{h^2(h+2a)^2}\left[\frac{2a^2+h^2+2ah}{a^2+ah} - \left(\cos\frac{h\pi}{2} - \cos\left(a\pi+\frac{h\pi}{2}\right)\right)\right]$$

$$= \lim_{h\to0} \frac{1}{h^2(h+2a)^2}\left[\frac{2a^2+h^2+2ah}{a^2+ah} - 2\cos\frac{\pi h}{2}\right]$$

[$\because$ a is odd integer]

$$= \lim_{h\to0} \frac{1}{h^2(h+2a)^2}\left[\frac{h^2}{a^2+ah} + 2 - 2\cos\frac{\pi h}{2}\right]$$

$$= \lim_{h\to0} \frac{1}{4a^2}\left[2\left(\frac{1-\cos(\pi/2)h}{h^2}\right) + \frac{1}{a^2}\right] = \frac{\pi^2a^2+4}{16a^4}$$

7. Let $x_0 = 2\cos\theta$

$x_1 = \sqrt{2+2\cos\theta} = 2\cos\ \theta/2$

$x_2 = \sqrt{2+2\cos\theta/2} = 2\cos\ \theta/4$

$x_n = 2\cos\dfrac{\theta}{2^n}$

Now $\lim\limits_{n\to\infty} 2^{n+1}\sqrt{2-2\cos\dfrac{\theta}{2^n}} = \lim\limits_{n\to\infty} \dfrac{2\sin\dfrac{\theta}{2^{n+1}}}{\dfrac{1}{2^{n+1}}}$

$= 2\theta = 2.\dfrac{\pi}{6} = \pi/3$

11. $\lim\limits_{x\to0} \dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$

$$= \lim_{x\to0} \frac{ae^x+ce^{-x}-b\cos x}{x^2} = 2$$

$$= \lim_{x\to0} \frac{a\left(1+x+\frac{x^2}{2!}+\ldots\right)+c\left(1-x+\frac{x^2}{2!}+\ldots\right)-b+b-b\cos x}{x^2}$$

$= 2$

$\Rightarrow a + c - b = 0,\ a - c = 0\ \&\ \dfrac{a+c}{2}+\dfrac{b}{2} = 2$

$\Rightarrow a = c = 1,\ b = 2$

13. $\lim\limits_{n\to\infty}\left(1-\tan^2\dfrac{\theta}{2}\right)\left(1-\tan^2\dfrac{\theta}{2^2}\right)\ldots.\left(1-\tan^2\dfrac{\theta}{2^n}\right)$

$$= \lim_{n\to\infty}\left(\frac{\cos^2\theta/2-\sin^2\theta/2}{\cos^2\theta/2}\right)\left(\frac{\cos^2\theta/2^2-\sin^2\theta/2^2}{\cos^2\theta/2^2}\right)\ldots\left(\frac{\cos^2\frac{\theta}{2^n}-\sin^2\frac{\theta}{2^n}}{\cos^2\frac{\theta}{2^n}}\right)$$

$$= \lim_{n\to\infty} \frac{\cos\theta}{\cos^2\frac{\theta}{2}} \times \frac{\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2^2}} \times \frac{\cos\frac{\theta}{4}}{\cos^2\frac{\theta}{2^3}} \ldots. \frac{\cos\frac{\theta}{2^{\,n-1}}}{\cos^2\frac{\theta}{2^n}}$$

$$= \lim_{n\to\infty} \frac{\cos\theta}{\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots..\cos\frac{\theta}{2^n}}\,.\,\frac{1}{\cos\frac{\theta}{2^n}}$$

$$= \lim_{n\to\infty} \frac{\cos\theta 2^n\sin\left(\frac{\theta}{2^n}\right)}{\sin\left(2^n\frac{\theta}{2^n}\right)} = \lim_{n\to\infty} \frac{\cos\theta}{\sin\theta}\frac{\sin\left(\frac{\theta}{2^n}\right)}{\frac{1}{2^n}} = \frac{\theta}{\tan\theta}$$

1. $y = \lim\limits_{x\to 1} \dfrac{\sqrt{f(x)}-1}{\sqrt{x}-1}$

$\Rightarrow y = \lim\limits_{x\to 1} \dfrac{\left(\sqrt{f(x)}-1\right)}{\left(\sqrt{x}-1\right)}.\dfrac{\left(\sqrt{f(x)}+1\right)}{\left(\sqrt{x}+1\right)}.\dfrac{\left(\sqrt{x}+1\right)}{\left(\sqrt{f(x)}+1\right)}$

$\Rightarrow y = \lim\limits_{x\to 1} \dfrac{f(x)-1}{x-1}.\dfrac{\sqrt{x}+1}{\sqrt{f(x)}+1}$

$\Rightarrow y = \lim\limits_{x\to 1} \dfrac{f(x)-f(1)}{x-1}.\lim\limits_{x\to 1}\dfrac{\sqrt{x}+1}{\sqrt{f(x)}+1}$

$\Rightarrow y = f'(1).\dfrac{2}{\sqrt{f(1)}+1} \Rightarrow y = 2.\dfrac{2}{2} = 2$

Aliter : Applying L-Hospital's rule.

3. We have,

$\lim\limits_{x\to\infty} f(x) \quad = \lim\limits_{x\to\infty}\left(\dfrac{x^2+5x+3}{x^2+x+2}\right)^x$

$= \lim\limits_{x\to\infty}\left(1+\dfrac{4x+1}{x^2+x+2}\right)^x = e^{\lim\limits_{x\to\infty}\frac{x(4x+1)}{x^2+x+2}} = e^4$

4. $\lim\limits_{x\to\infty}\dfrac{\log x^n - [x]}{[x]} = \lim\limits_{x\to\infty}\dfrac{\log x^n}{[x]} - \lim\limits_{x\to\infty}\dfrac{[x]}{[x]} = 0 - 1 = -1$

5. $\lim\limits_{x\to 0}\dfrac{\log(3+x)-\log(3-x)}{x} = k$

By L-Hospital's rule,

$\lim\limits_{x\to 0}\dfrac{\dfrac{1}{3+x}+\dfrac{1}{3-x}}{1} = k \quad \Rightarrow \dfrac{2}{3} = k$

6. We have, $\lim\limits_{x\to a}\dfrac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)} = 4$

$\Rightarrow \lim\limits_{x\to a}\dfrac{f(a)g'(x)-0-g(a)f'(x)+0}{g'(x)-f'(x)} = 4$

$\Rightarrow \dfrac{f(a)g'(a)-g(a)f'(a)}{g'(a)-f'(a)} = 4$

$\Rightarrow \dfrac{k\{g'(a)-f'(a)\}}{\{g'(a)-f'(a)\}} = 4 \qquad [\because f(a) = g(a) = k]$

$\Rightarrow k = 4$

7. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)(1-\sin x)}{(\pi-2x)^3}$

Let $x = \dfrac{\pi}{2} + y : y \to 0 \Rightarrow \lim\limits_{y\to 0}\dfrac{\tan\left(-\frac{y}{2}\right)(1-\cos y)}{(-2y)^3}$

$= \lim\limits_{y\to 0}\dfrac{-\tan\frac{y}{2}.2\sin^2\frac{y}{2}}{(-8)y^3} = \lim\limits_{y\to 0}\dfrac{1}{32}\dfrac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)}.\left[\dfrac{\sin\frac{y}{2}}{\frac{y}{2}}\right]^2 = \dfrac{1}{32}$

8. Since, $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}+\dfrac{b}{x^2}\right)^{2x} = e^2$

$\therefore \lim\limits_{x\to\infty}\left[\left(1+\dfrac{ax+b}{x^2}\right)^{\frac{x^2}{ax+b}}\right]^{\frac{2(ax+b)}{x}} = e^2$

$\Rightarrow \lim\limits_{x\to\infty} e^{\frac{2(ax+b)}{x}} = e^2$

$\Rightarrow \lim\limits_{x\to\infty}\dfrac{2(ax+b)}{x} = 2 \Rightarrow 2a = 2 \Rightarrow a = 1$

Thus a = 1 and $b \in R$

12. $\lim\limits_{x\to 5}\dfrac{(f(x))^2-9}{\sqrt{|x-5|}} = 0$

$\therefore$ Question must be in $\dfrac{0}{0}$ form

$\therefore$ $(f(5))^2 - 9 = 0$

$\Rightarrow$ $f(5) = 3$

13. $\lim\limits_{x\to a}\dfrac{x^2 f(a)-a^2 f(x)}{x-a} \qquad \left[\dfrac{0}{0}\text{ form}\right]$

Use L'Hospital rule

$= \lim\limits_{x\to a}\dfrac{2xf(a)-a^2f'(x)}{1}$

$= 2af(a) - a^2f'(a)$

14. $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{4x^2}.(4x^2).\dfrac{(3+1)}{\dfrac{x\tan 4x}{4x}(4x)}$

$\dfrac{1}{2}.4 = 2$

6. $$\frac{a-a\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}-\frac{x^2}{4}}{x^4}$$

$$=\frac{a-a\left(1-\frac{x^2}{2a^2}-\frac{1}{8}\frac{x^4}{a^4}\right)-\frac{x^2}{4}}{x^4}$$

$a = 2$, (coefficient of $x^2 = 0$)

$\therefore L = \frac{1}{64}$.

7. $$\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right)=4$$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{(1-a)x^2+x(1-a-b)+1-b}{x+1}\right)=4$$

$$\Rightarrow \lim_{x\to\infty}\frac{\left((1-a).x+1-a-b+\left(\frac{1-b}{x}\right)\right)}{1+\frac{1}{x}}=4$$

for limit to exist finitely

$1 - a = 0$ and $1 - a - b = 4$

$\Rightarrow a = 1$ and $b = -4$.

8. $$\left(\left(1+\frac{a}{3}\right)-1\right)x^2+\left(\left(1+\frac{a}{2}\right)-1\right)x+\left(1+\frac{a}{6}-1\right)=0$$

$$a\left(\frac{x^2}{3}+\frac{x}{2}+\frac{1}{6}\right)=0 \Rightarrow 2x^2+3x+1=0$$

$$\Rightarrow x=-\frac{1}{2},-1$$

$\Rightarrow \lim_{a\to0^+}\alpha(a)$ and $\lim_{a\to0^+}\beta(a)$ are $-\frac{1}{2}$ and -1