AI1110
Assignment 5

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Question

The order statistics of the random variables x_i are n random variables y_k defined as follows: For a specific outcome ζ, the random variables x_i; take the values $x_i(\zeta)$. Ordering these numbers. we obtain the sequence

$$x_{r_1}(\zeta) \leq \ldots \leq x_{r_k} \leq \ldots \leq x_{r_n} \tag{1}$$

and we define the random variable y_k such that

$$y_1(\zeta) = x_{r_1}(\zeta) \leq \ldots \leq y_k(\zeta) = x_{r_k}(\zeta) \leq \ldots \leq y_n(\zeta) = x_{r_n}(\zeta) \tag{2}$$

We note that for a specific i, the values $x_i(\zeta)$ of x_i occupy different locations in the above ordering as ζ changes. We maintain that the density $f_k(y)$ of the kth statistic y_k is given by :

$$f_k(y) = \frac{n!}{(k-1)!(n-k)!} F_x^{k-1}(y) \left[1 - F_x(y)\right]^{n-k} f_x(y) \tag{3}$$

where $F_x(x)$ is the distribution of the i.i.d. random variables x_i and $f_x(x)$ is their density.

Solution

Proof:

Here ; $f_k(y)dy$ is p.d.f. & $F_x(x)$ is c.d.f. The event $B = \{\Pr(y < y_k \leq y + dy)\}$ occurs if and only if exactly $(k-1)$ of the random variables x_i are less than y and one is in the interval $(y, y + dy)$. The events be A_1, A_2, A_3.

1. $A_1 = \{x \leq y\}$
2. $A_2 = \{y < x \leq y + dy\}$
3. $A_3 = \{x > y + dy\}$

And partitions be :

1. $\Pr(A_1) = F_x(y)$
2. $\Pr(A_2) = f_x(y)dy$
3. $\Pr(A_3) = 1 - F_x(y)$

Solution

The event B happens if and only if A_1 occurs $k-1$ times, A_2 occurs once, and A_3 occurs $n-k$ times. With $k_1 = k-1$, $k_2 = 1$, $k_3 = n-k$.

$$\Pr(B) = \frac{n!}{(k-1)!.1!.(n-k)!} Pr^{k-1}(A_1) Pr(A_2) Pr^{n-k}(A_3) \tag{4}$$

As,

$$f_1(y) = n\left[1 - F_x(y)\right]^{n-1} f_x(y) \tag{5}$$

$$f_n(y) = nF_x^n - 1(y) f_x(y) \tag{6}$$

$$f_k(y) = \frac{n!}{(k-1)!(n-k)!} F_x^{k-1}(y)\left[1 - F_x(y)\right]^{n-k} f_x(y) \tag{7}$$