Assignment

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CS21BTECH11010*

Contents

***Abstract*—This manual provides solutions to the Assignment of Random Numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/q1.1.c
wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/coeffs.h
```

Download the above files and execute the following commands

```
$ gcc q1.1.c −lm
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/q1.2.py
```

Download the above files and execute the following commands to produce Fig.1.2

```
$ python3 q1.2.py
```

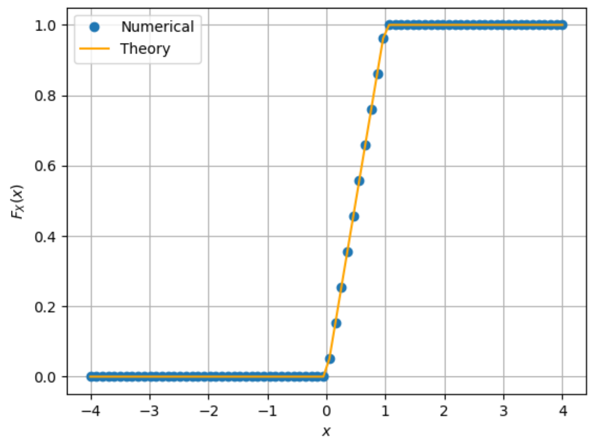

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given U is a uniform Random Variable

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{1.2}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.3}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.4}$$

Write a C program to find the mean and variance of U.
Solution: Download the following files and execute the C program.

```
wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/q1.4.c
```

```
wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/coeffs.h
```

Download the above files and execute the following commands

```
$ gcc q1.4.c −lm
$ ./a.out
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.5}$$

Solution:

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

$$\implies \text{var}[U] = E\left[U^2\right] - E[U]^2 \tag{1.7}$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.8}$$

$$E[U] = \int_0^1 x \tag{1.9}$$

$$\implies \boxed{E[U] = \frac{1}{2}} \tag{1.10}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.11}$$

$$E\left[U^2\right] = \int_0^1 x^2 dF_U(x) \tag{1.12}$$

$$\implies E\left[U^2\right] = \frac{1}{3} \tag{1.13}$$

$$\implies \boxed{\text{var}[U] = \frac{1}{12} = 0.0833} \tag{1.14}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and execute the C program.

```
wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/q2.1.c
wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/coeffs.h
```

Download the above files and execute the following commands

```
$ gcc q2.1.c −lm
$ ./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2 using the code below

```
wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/q2.2.py
```

Download the above files and execute the following commands to produce Fig.2.2

```
$ python3 q2.2.py
```

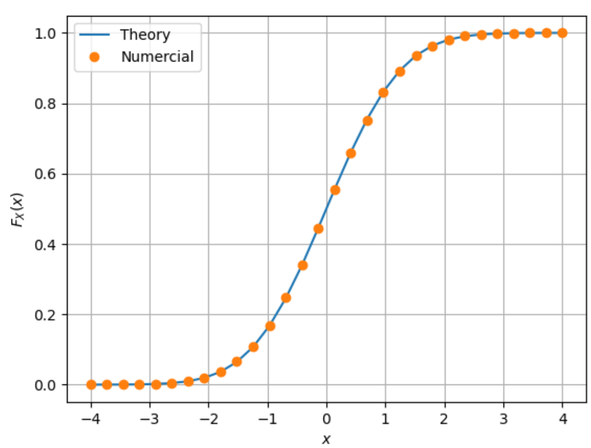

Fig. 2.2: The CDF of X

Some of the properties of CDF

a) $\lim_{x\to\infty} F_X(x) = 1$
b) $F_X(x)$ is non decreasing function.
c) Symmetric about one point.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/q2.3.py
```

Download the above files and execute the following commands to produce Fig.2.3

```
$ python3 q2.3.py
```

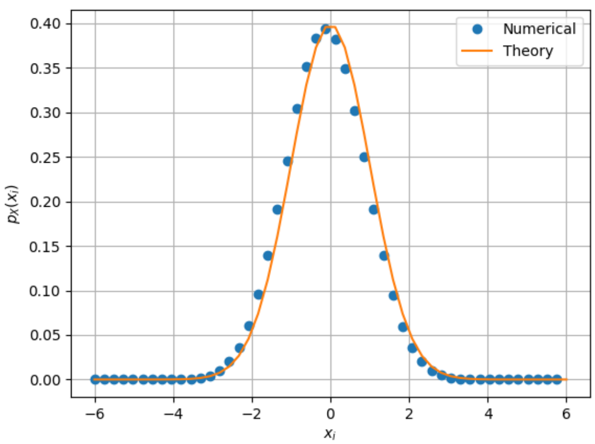

Fig. 2.3: The PDF of X

Some of the properties of the PDF:

a) Symmetric about $x = \mu$ in this case
b) Decreasing function for $x > \mu$ and increasing for $x < \mu$ and attains maximum at $x = \mu$
c) Area under the curve is unity.

2.4 Find the mean and variance of X by writing a C program.
Solution: Download the following files and execute the C program.

```
wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/q2.4.c
wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/coeffs.h
```

Download the above files and execute the following commands

```
$ gcc q2.4.c −lm
$ ./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.

Solution:

1) CDF is given by

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.4}$$

$$= \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \tag{2.5}$$

$$= 1 \tag{2.6}$$

2) Mean is given by

$$\int_{-\infty}^{\infty} x \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.7}$$

$$= 0 \qquad + \tag{2.8}$$

3) Variance is given by

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) = 0 \tag{2.9}$$

$$E\left[X^2\right] \int_{-\infty}^{\infty} x \times x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.10}$$

$$Var(U) = x \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\Bigg|_{-\infty}^{\infty} \tag{2.11}$$

$$+ \int_{-\infty}^{\infty} 1 \times \int \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.12}$$

$$= x \times 0 + \int_{-\infty}^{\infty} 1 \times \int \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.13}$$

$$= \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \tag{2.14}$$

$$Var(X) = \sqrt{2} \tag{2.15}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF.
Solution: Download the following files and execute the C program.

```
wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/q3.1.c
wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/coeffs.h
```

Download the above files and execute the following commands

```
$ gcc q3.1.c −lm
$ ./a.out
```

The CDF of V is plotted in Fig. 3.1 using the code below

```
wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/q3.1.py
```

Download the above files and execute the following commands to produce Fig.3.1

```
$ python3 q3.1.py
```

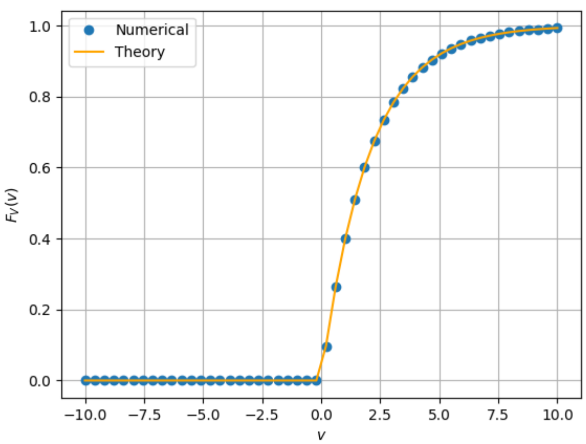

Fig. 3.1: The CDF of X

3.2 Find a theoretical expression for $F_V(x)$.
Solution: If Y = g(X), we know that $F_Y(y) = F_X(g^{-1}(y))$, here

$$V = -2\ln(1-U) \tag{3.2}$$
$$1-U = e^{\frac{-V}{2}} \tag{3.3}$$
$$U = 1-e^{\frac{-V}{2}} \tag{3.4}$$
$$F_V(x) = F_U(1-e^{\frac{-x}{2}}) \tag{3.5}$$

$$\implies F_V(x) = \begin{cases} 0 & x<0 \\ 1-e^{\frac{-x}{2}} & x \geq 0 \end{cases} \tag{3.6}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: The c code for generating T is at :

```
wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/q4.1.c
```

And then execute the code :

```
gcc q4.1.c −lm
./a.out
```

This generates a file named ”td.dat”

4.2 Find the CDF of T.
Solution: The Python code for the plot is :

```
$ wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/q4.2.py
```

and can be run using

```
$ python3 q4.2.py
```

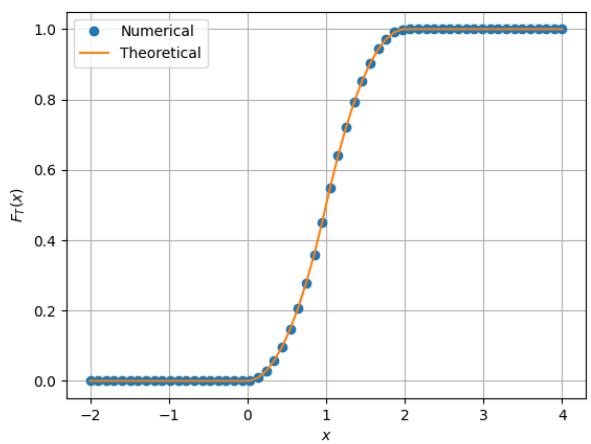

Fig. 4.2: The CDF of T

4.3 Find the PDF of T.
Solution: The Python code for the figure is available at :

```
$ wget https://github.com/VishalBurra27/
    randomvar/blob/main/code/q4.3.py
```

and run using

```
$ python3 q4.3.py
```

4.4 Find the theoretical PDF and CDF of T.
Solution: We write,

$$F_T(t) = \Pr(U_1 + U_2 \leq t) \tag{4.2}$$
$$= \Pr(U_1 \leq t - U_2) \tag{4.3}$$
$$= \int_0^1 F_{U_1}(t-x)p_{U_2}(x)dx \tag{4.4}$$

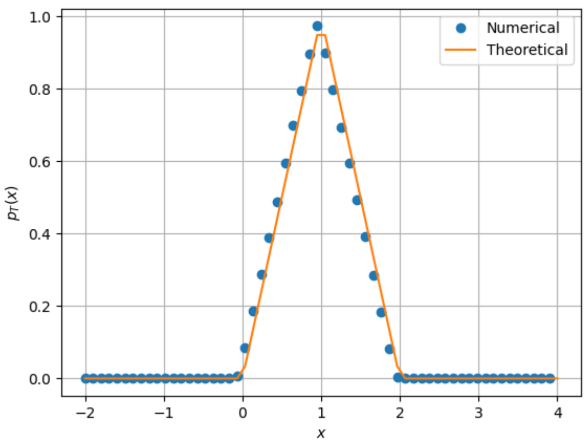

Fig. 4.3: The PDF of T

where U_1 and U_2 are uniform i.i.d. random variables in [0, 1]. Then, $0 \le U_1 + U_2 \le 2$. We have three cases:

a) $t < 0$: Using equation (1.2) $F_T(t) = 0$.

b) $0 \le t < 1$: We have,

$$F_T(t) = \int_0^t (t - x)dx = \frac{t^2}{2} \tag{4.5}$$

c) $1 \le t < 2$: Here, we get

$$F_T(t) = \int_0^{t-1} dx + \int_{t-1}^1 (t - x)dx \tag{4.6}$$

$$= t - 1 + t(2 - t) - \frac{1 - (t - 1)^2}{2} \tag{4.7}$$

$$= -\frac{t^2}{2} + 2t - 1 \tag{4.8}$$

d) $t \ge 2$: Here, $F_T(t) = 1$.

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t < 1 \\ -\frac{t^2}{2} + 2t - 1 & 1 \le t < 2 \\ 1 & t \ge 2 \end{cases} \tag{4.9}$$

Using

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{4.10}$$

,

$$p_T(t) = \begin{cases} t & 0 \le t < 1 \\ 2 - t & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

4.5 Verify your results through a plot.

Solution: This has been done in the plots (4.2) and (4.3).