Cart-Pendulum Model

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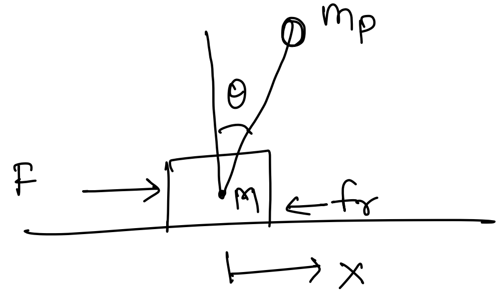

f_r = friction coefficient
X = cart displacement
X_m, Y_m = pendulum mass (m) displacement.
F = applied force on pendulum.
θ = pendulum angle with vertical axis (Y)
M = mass of cart
m = mass of pendulum
I = Inertia of pendulum
L, l = Length of pendulum.

$X_m = X + l\sin\theta \quad y_m = l\cos\theta$

$\dot{x}_m = \dot{X} + l\cos\theta.\dot{\theta} \quad \dot{y}_m = -l\sin\theta.\dot{\theta}$

Kinetic energy of pendulum:

$$= \frac{1}{2} m\left[(\dot{X} + l\cos\theta.\dot{\theta})^2 + (-l\sin\theta.\dot{\theta})^2\right] + \frac{1}{2} I \dot{\theta}^2$$

Kinetic energy of cart:

$$= \frac{1}{2} M \dot{x}^2$$

Total kinetic energy (T)

$$= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left(\dot{x}^2 + l^2 \dot{\theta}^2 \cos^2\theta + 2l \dot{x} \dot{\theta} \cos\theta + l^2 \dot{\theta}^2 \sin^2\theta \right) + \frac{1}{2} I \dot{\theta}^2$$

$$= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left(\dot{x}^2 + l^2 \dot{\theta}^2 + 2l \dot{x} \dot{\theta} \cos\theta \right) + \frac{1}{2} I \dot{\theta}^2$$

$$= \frac{1}{2} (M+m) \dot{x}^2 + \frac{1}{2} (ml^2 + I) \dot{\theta}^2 + ml \dot{x} \dot{\theta} \cos\theta$$

Potential Energy (V)

$$= m.g.h = m.g.l.\cos\theta.$$

Lagrangian (L) = T−V

$$L = \frac{1}{2} (M+m) \dot{x}^2 + \frac{1}{2} (ml^2 + I) \dot{\theta}^2 + ml \dot{x} \dot{\theta} \cos\theta - m.g.l \cos\theta.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F - f_r \dot{x}$$

$$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + ml\dot{\theta}\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta.$$

$$\frac{\partial L}{\partial x} = 0$$

$$\boxed{(M+m)\ddot{x} = F - f_r\dot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta.} \quad \text{—}(1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = (ml^2+I)\dot{\theta} + ml\dot{x}\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = (ml^2+I)\ddot{\theta} + ml\ddot{x}\cos\theta - ml\dot{x}\dot{\theta}\sin\theta.$$

$$\frac{\partial L}{\partial \theta} = -ml\dot{x}\dot{\theta}\sin\theta + mgl\sin\theta$$

$$(ml^2+I)\ddot{\theta} = -ml\ddot{x}\cos\theta + \cancel{ml\dot{x}\dot{\theta}\sin\theta} \cancel{- ml\dot{x}\dot{\theta}\sin\theta} + mgl\sin\theta$$

$$\boxed{(ml^2+I)\ddot{\theta} = -ml\ddot{x}\cos\theta + mgl\sin\theta} \quad \text{—}(2)$$

$(ml^2 + I) \Rightarrow \lambda$

$$\ddot{\theta} = -\frac{ml\cos\theta}{\lambda}\ddot{x} + \frac{mgl\sin\theta}{\lambda}$$

Put this $\ddot{\theta}$ in (1)

$$\left[\lambda(M+m) - m^2 l^2 \cos^2\theta\right]\ddot{x} = F\lambda - f_r\dot{x}\lambda$$
$$+ ml\dot{\theta}^2\sin\theta.\lambda + m^2 l^2 g\cos\theta\sin\theta$$

$$\boxed{\ddot{x} = \frac{-f_r\lambda\dot{x} + ml\dot{\theta}^2\lambda\sin\theta. - m^2l^2 g\cos\theta\sin\theta + F\lambda.}{(\lambda(M+m) - m^2l^2\cos^2\theta)}}$$

at equillibrium $\dot{\theta} = 0$, $\sin\theta = \theta$, $\cos\theta = 1$

$$\ddot{x} = \frac{-f_r.\lambda.\dot{x} - m^2l^2.g.\theta + F\lambda}{\lambda(M+m) - m^2l^2}$$

$$mMl^2 + \cancel{m^2l^2} + I(M+m) - \cancel{m^2l^2}$$

$$= I(M+m) + mMl^2$$

$$\boxed{A_{22} = \frac{-f_r\lambda}{I(M+m) + mMl^2} \qquad A_{23} = \frac{-m^2l^2g}{I(M+m) + M.m.l^2} \qquad B_{21} = \frac{\lambda}{I(M+m) + mMl^2}}$$

$$\ddot{x} = \frac{F - f_r\dot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta}{(M+m)}$$

Put this value of $\ddot{x}$ in eq^n (2)

$$\lambda\ddot{\theta} = -ml\cos\theta\left(\frac{F - f_r\dot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta}{(M+m)}\right) + mgl\sin\theta$$

$$\left(\frac{\cancel{(M+m)}\lambda - m^2l^2\cos^2\theta}{\cancel{(M+m)}}\right)\ddot{\theta} = \frac{-ml\cos\theta F + f_r ml\cos\theta\dot{x} - m^2l^2\dot{\theta}^2\sin\theta.\cos\theta + (M+m)m g.l.\sin\theta}{\cancel{(M+m)}}$$

$$\boxed{\ddot{\theta} = \frac{f_r ml\cos\theta\dot{x} - m^2l^2\dot{\theta}^2\sin\theta\cos\theta + (M+m)mgl\sin\theta - ml\cos\theta.F}{(M+m)\lambda - m^2l^2\cos^2\theta}}$$

$\dot{\theta}^2 = 0 \quad \sin\theta = \theta \quad \cos(0) = 1$

$$\boxed{A_{42} = \frac{f_r ml}{I(M+m) + mMl^2} \qquad A_{43} = \frac{(M+m).m.g.l}{I(M+m) + mMl^2} \qquad B_{41} = \frac{-ml}{I(M+m) + mMl^2}}$$