Set 4 - Constrained expansion beyond the bounds of the Logistic model

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In this lab, we use mathematical models to study biological population dynamics. These models include the Allee effect, which helps us understand population growth rates under different initial population sizes, and the Gompertz equation, which describes the growth of tumors.

I. THE GOMPERTZ LAW

A. Model

The Gompertz equation is given by:

$$\dot{x} = -ax\ln(bx) \tag{1}$$

After rescaling with $T = at$ and $X = x/b^{-1}$, the Gompertz equation becomes:

$$\dot{X} = -X\ln(X) \tag{2}$$

After integrating the rescaled Gompertz equation, we get:

$$X = \exp\left(\ln(X_0)e^{-T}\right) \tag{3}$$

We solve the Gompertz equation numerically using Euler's method and compare it with the analytical solution. Euler's method is given by:

$$X_{i+1} = X_i + \Delta T \cdot (-X_i \ln(X_i)) \tag{4}$$

B. Results

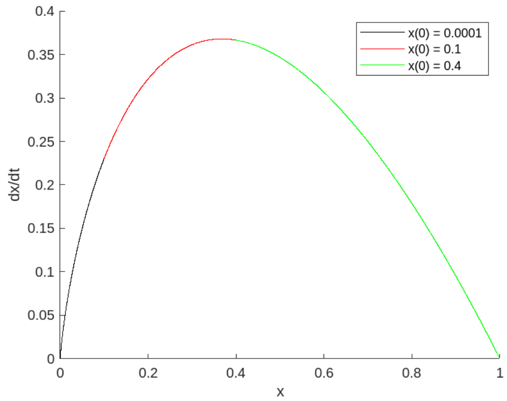

FIG. 1: Plot of $\dot{X}$ vs X for initial values 0.0001 (Closer to 0), 0.4 (Greater than e^{-1}), and 0.1 (An Intermediate Value) with $\Delta X = 0.0001$.

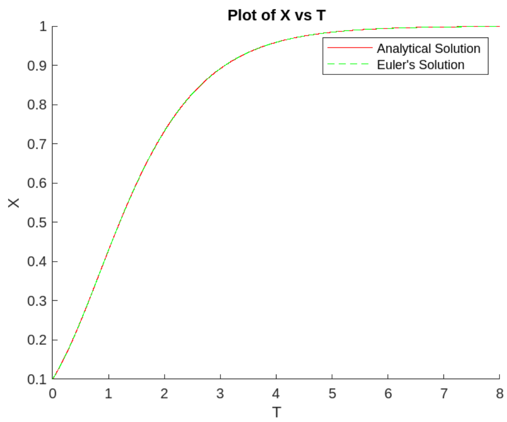

FIG. 2: Plot of X vs T using Euler's method and the analytical solution with $X(0) = 0.1$ and $\Delta T = 0.001$.

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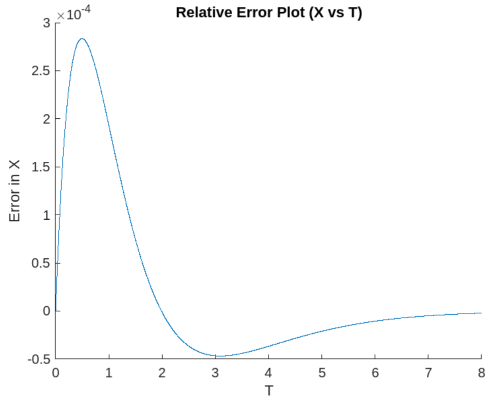

FIG. 3: Relative Error Plot of Euler's and Analytical Solution.

II. THE ALLEE EFFECT

A. Model

A key idea in population ecology, the Allee Effect emphasizes the complex relationship between population dynamics and individual behavior. It illustrates how cooperative activities and social interactions influence growth rates, showing a critical threshold that populations cannot continue to grow over.

The model equation for the Allee effect is given by:

$$\dot{x} = x\left(r - a(x-b)^2\right) \tag{5}$$

where a, b, and r are positive constants. The model is effective only when $r < ab^2$.

At equilibrium, the equation has two solutions:

$$x = 0 \quad \text{(extinction equilibrium)} \tag{6}$$

$$x = b \pm \sqrt{\frac{a}{r}} \tag{7}$$

B. Results

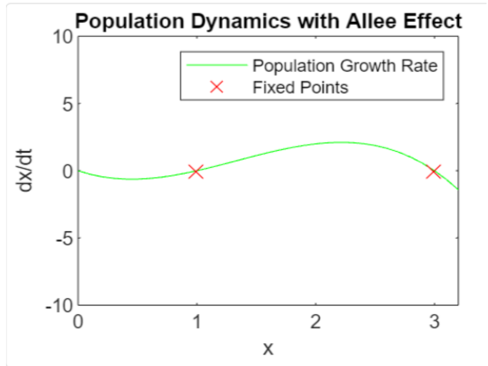

FIG. 4: Plot of $\dot{x}$ vs x where $a = r = 1$ and $b = 2$. The fixed points occur at $x = 1$ and $x = 3$.

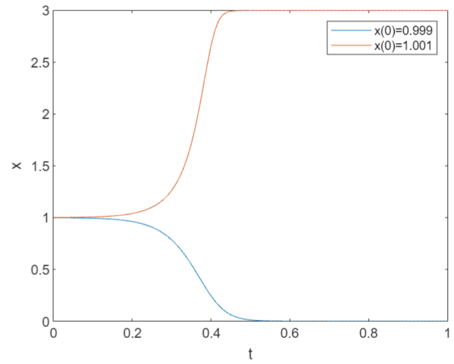

FIG. 5: Plot of Euler's Method for initial values $x(0) = 0.99999$ and $x(0) = 1.00001$.

Here as x tends to infinity:

1. Starting at $x(0) = 0.99999$, the population size converges to the extinction equilibrium $x = 0$, illustrating the difficulty populations face in maintaining low densities owing to slower growth rates.
2. In contrast, the population tends toward the stable equilibrium $x = 3$ for $x(0) = 1.00001$. This shows how the Allee effect, which encourages growth and persistence, promotes population stability at greater densities.