Random variable $\rightarrow$ $X \Rightarrow \{H, T\}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; \downarrow \;\; \downarrow$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; \tfrac{1}{2} \;\; \tfrac{1}{2}$

X	H	T
Prob	0.5	0.5

$\rightarrow$ probability distribution table

Expectation

- The expectation of a variable X is a bit like the mean, but for probability distributions.
- Generalization of the weighted average.
- The expected value is arithmetic mean of independently selected outcomes of a random variable.

__Expected Value__:

$$E(X) = \sum_{i=1}^{k} x_i p_i$$

Value of X:	x_1	x_2	...	x_k
Probability:	p_1	p_2	...	p_k

$$E(X) = x_1p_1 + x_2p_2 + \ldots + x_kp_k$$

$P(X=H) = P(H)$

random variable

event

prob. distribution

X	x_1	x_2	x_3	x_4	x_5
P	p_1	p_2	p_3	p_4	p_5

$P = \frac{n(E)}{n(S)}$

$\frac{\Sigma xf}{f}$ → mean for frequency dist.

weighted sum

Expectation $= \boxed{x_1 p_1} + x_2 p_2 + x_3 p_3 + x_4 p_4 + x_5 p_5$

$$\boxed{E(x) = \sum_{i=1}^{5} x_i p_i}$$

$P(\$) = 0.1$ $\qquad$ $P(\text{lemons}) = 0.2$

$P(\text{cherry}) = 0.2$ $\qquad$ $P(\text{other}) = 0.5$

---

① $P(\text{all } \$) = 0.1 \times 0.1 \times 0.1$

$$= 0.001$$

$$= \frac{1}{1000}$$

$	$	$

$0.1 \times 0.1 \times 0.1$

$P(\$) = 0.1 \quad P(\text{cherry}) = 0.2$

(11) P(two dollars & 1 cherry)

$\$$	$\$$	cherry

$\frac{6}{1000}$

or

$\$$	cherry	$\$$

or

cherry	$\$$	$\$$

$$P = 0.1 \times 0.1 \times 0.2 + 0.1 \times 0.2 \times 0.1 + 0.2 \times 0.1 \times 0.1$$

$$= 0.002 + 0.002 + 0.002$$

$$= 0.006$$

$P(\text{lemon}) = 0.2$

(iii) all lemons

$P = 0.2 \times 0.2 \times 0.2 = 0.008$

(iv) all cherries

$P = 0.2 \times 0.2 \times 0.2 = 0.008$

$P(\text{not winning}) = 1 - (0.001 + 0.006 + 0.008 + 0.008)$

$= 0.977$

Charge = \$1

combinations	None	lemons	cherries	\$ & ch.	\$
Prob.	0.977	0.008	0.008	0.006	0.001
Gains	-\$1	\$4	\$9	\$14	\$19

all \$ = \$20 all cherries = \$10

all lemons = \$5 \$ & cherries = \$15

	None	lemons	cherries	\$ & cherries	\$
Gain($\underline{x}$)	-1	4	9	14	19
Prob.	0.977	0.008	0.008	0.006	0.001

$$E(x) = \sum x_i p_i$$

$$= (-1 \times 0.977) + (4 \times 0.008) + (9 \times 0.008)$$

$$+ (14 \times 0.006) + (19 \times 0.001)$$

$$= -\$0.77$$

$$\frac{\sum (x-\mu)^2}{n}$$

$$\underline{\sum (x-\mu)^2 p}$$

Variance

- The expectation gives the typical or average value of a variable but it doesn't tell you anything about how the values are spread out.
- Hence, variance is calculated.
- It is same as variance we studied in statistics, except it is for probability distributions.

Go through each value x and work out what $(x-\mu)^2$ is. Then multiply it by the probability of getting x...

$$E(X-\mu)^2 = \underline{\sum (x-\mu)^2 P(X=x)}$$

...and then add these results together.

Gains	-1	4	9	14	19
Prob	0.977	0.008	0.008	0.006	0.001

$$Var \Rightarrow \sum (x - E(x))^2 \, P(x) \qquad E(x) = -0.77$$

$$Var = (-1 + 0.77)^2 \times 0.977 + (4+0.77)^2 \times 0.008 + (9+0.77)^2 \times 0.008 + (14+0.77)^2 \times 0.006 + (19+0.77)^2 \times 0.001$$

$$= 2.6971$$

Standard Deviation

- It serves a similar function to the standard deviation of a set of values. It's a way of measuring how far away from the center you can expect your values to be.

$$\sigma = \sqrt{Var(X)}$$

$$\sigma = \sqrt{2.6971} = 1.64$$

$(1-2.95)^2 \times 0.1$

Question to Play!

- Find E(x) → 2.95
- Find Variance → 1.24
- Find Standard Deviation

↳ 1.116

Here's the probability distribution of a random variable X:

x	1	2	3	4	5
P(X = x)	0.1	0.25	0.35	0.2	0.1

Game charge = \$2 , Prize is now 5 times of original price.

All dollars = \$100   All lemons = \$25

All cherries = \$50   \$ & cherries = \$75

(Y)

Gain	-2	23	48	73	98
Prob	0.977	0.008	0.008	0.006	0.001

$$E(Y) = -0.85$$

$$Var(Y) = (-2+0.85)^2 \times 0.977 + (23+0.85)^2 \times 0.008$$
$$+ (48+0.85)^2 \times 0.008 + (73+0.85)^2 \times 0.006$$
$$+ (98+0.85)^2 \times 0.001 = 67.4275$$

MCT

$\downarrow$

$+, -, \times, \div$

$\uparrow$

expectation

MS

$\downarrow$

$\times, \div$

$\uparrow$

variance

Let X & Y be R.V.

gains

win = charge + gains

gain = win − charge

$$\underset{\text{win}}{\text{original}} = X + 1$$

$$Y = 5(\text{original win}) - \overset{\text{new}}{\text{charge}} \Rightarrow 5(X+1) - 2$$

$$Y = 5X + 3$$

$$Y = 5x + 3$$

$$E(Y) = 5E(x) + 3 \Rightarrow E(5x+3)$$

$$= 5(-0.77) + 3 = -0.85$$

$$E(\underline{ax+b}) = aE(x) + b$$

$$\mathrm{var}(y) = \mathrm{var}(5x+3) = 5^2 \times \mathrm{var}(x)$$

$$= 25 \times 2.6971$$

$$= 67.4275$$

$$\mathrm{var}(ax+b) = a^2\,\mathrm{var}(x)$$

Linear Transformations

- Eliminate the need for you to have to calculate the expectation and variance of a probability distribution every time the values change.

$$E(aX + b) = aE(X) + b$$

$$Var(aX + b) = a^2Var(X)$$

Question to Play!

- Find E(y)
- Find Variance(y)
- E(X) is -0.77 and E(Y) = -0.85. What is 5 x E(X)? What is 5 x E(X) + 3? How does this relate to E(Y)?
- Var(X) = 2.6971 and Var(Y) = 67.4275. What is 5 x Var(X)? What is 5^2 x Var(X)? How does this relate to Var(Y)?

What's the expectation and variance of the new probability distribution? How do these values compare to the previous payout distribution's expectation of -0.77 and variance of 2.6971?

y	-2	23	48	73	98
P(Y = y)	0.977	0.008	0.008	0.006	0.001

Properties

If $X_1, X_2, \ldots, X_n$ are independent observations of X then:

$E(X_1 + X_2 + \ldots + X_n) = nE(X)$
$Var(X_1 + X_2 + \ldots X_n) = nVar(X)$

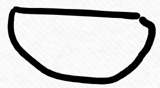
If X and Y are independent random variables, then:

$E(X + Y) = E(X) + E(Y)$
$E(X - Y) = E(X) - E(Y)$
$Var(X + Y) = Var(X) + Var(Y)$
$Var(X - Y) = Var(X) + Var(Y)$

The expectation and variance of linear transforms of X and Y are given by

$E(aX + bY) = aE(X) + bE(Y)$
$E(aX - bY) = aE(X) - bE(Y)$
$Var(aX + bY) = a2Var(X) + b2Var(Y)$
$Var(aX - bY) = a2Var(X) + b2Var(Y)$

$E(x) = 16\,oz$



$E(y) = 4\,oz$

$$18 \le \overbrace{(X+Y)}^{4\ (2+2)} \le 22$$

$$\overbrace{15 \le X \le 17}^{2} \qquad \underbrace{3 \le y \le 5}_{2}$$

$$\underbrace{10 \le X - Y \le 14}_{\substack{4 \\ (2+2)}}$$

$E(x) = 16$

$25 \Rightarrow 16 \times 25 = \underline{400}$

Question to Play!

$E(y) = 20.75$

$20 \Rightarrow 20 \times 20.75 = 415$

A restaurant offers two menus, one for weekdays and the other for weekends. Each menu offers four set prices, and the probability distributions for the amount someone pays is as follows:

Weekday:

x	**10**	**15**	**20**	**25**
P(X = x)	0.2	0.5	0.2	0.1

Weekend:

y	**15**	**20**	**25**	**30**
P(Y = y)	0.15	0.6	0.2	0.05

Who would you expect to pay the restaurant most: a group of 20 eating at the weekend, or a group of 25 eating on a weekday?

$E(x) = 30.5$ $E(y) = 14.6$

$E(X-Y) = 30.5 - 14.6 = 15.9$

$Var(X) = 72.25$

$Var(Y) = 6.64$

Question to Play!

$Var(X-Y) = 72.25 + 6.64 = 78.89$

Sam likes to eat out at two restaurants. Restaurant A is generally more expensive than restaurant B, but the food quality is generally much better.

Below you'll find two probability distributions detailing how much Sam tends to spend at each restaurant. As a general rule, what would you say is the difference in price between the two restaurants? What's the variance of this?

Restaurant A:

x	**20**	**30**	**40**	**45**
P(X = x)	0.3	0.4	0.2	0.1

Restaurant B:

y	**10**	**15**	**18**
P(Y = y)	0.2	0.6	0.2