CS5691: Pattern Recognition and Machine Learning
Programming Assignment – 1

Motivation:

a) To understand the difference between EigenValue Decomposition (EVD) and Singular Value Decomposition (SVD)
b) To implement SVD using EVD and matrix operations
c) To compare and contrast between the methods in image reconstruction

Mathematical Background:
EigenValue Decomposition (EVD):

Let **A** be a square ***n* × *n*** matrix with ***n*** linearly independent eigenvectors q_i (where $i = 1, \ldots, n$). Then **A** can be factorized as

$$\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^{-1}$$

where **Q** is the square ***n* × *n*** matrix whose ***ith*** column is the eigenvector q_i of **A**, and **Λ** is the diagonal matrix whose diagonal elements are the corresponding eigenvalues, $\Lambda_{ii} = \lambda_i$.

To compute the k-rank approximation of **A,** the eigenvalues of **A** are sorted (based on magnitude) and all other eigenvalues, other than the top k eigenvalues, are set to zero in **Λ**. Let us call this matrix **Λk.** It should be noted that if the eigenvalues are complex, we should make sure that the diagonal elements in **Λk** occur in complex conjugate pairs. Then the k-rank approximation of **A** is computed as,

$$\mathbf{A}_k = \mathbf{Q}\Lambda_k\mathbf{Q}^{-1}$$

Singular Value Decomposition (SVD):

The singular value decomposition of an ***m* x *n*** real matrix **M** is a factorization of the form

$$\mathbf{M} = \mathbf{U}\Sigma\mathbf{V}^T$$

where **U (*m* x *m*)** and **V (*n* x *n*)** are orthogonal matrices and **Σ** is an ***m* x *n*** rectangular diagonal matrix with non-negative real numbers on the diagonals. These are called the singular values of **M**. SVD can be applied to any ***m* x *n*** matrix whereas EVD can be applied only to diagonalizable matrices. Given an SVD, the following two relations hold:

$$\mathbf{M}\mathbf{M}^T = \mathbf{U}(\Sigma\Sigma^T)\mathbf{U}^T$$
$$\mathbf{M}^T\mathbf{M} = \mathbf{V}(\Sigma^T\Sigma)\mathbf{V}^T$$

The columns of **V** are the eigenvectors of M^TM and the columns of **U** are the eigenvectors of MM^T. The non-zero elements of **Σ** (non-zero singular values) are the square roots of the non-zero eigenvalues of M^TM or MM^T. The columns of the matrices **U** and **V** can always be arranged in such a way that the singular values in **Σ** appear in descending order. To compute the k-rank approximation of **M,** we truncate the matrices **U** and **V** such that only the first k columns are included. Also, **Σ** is truncated such that only the first k singular values are present along the main diagonal. Let the truncated matrices be **Uk, Σk, and Vk.** Then,

$$\mathbf{M}_k = \mathbf{U}_k\Sigma_k\mathbf{V}_k^T$$

The shape of the given grayscale image is **256 x 256 x 1**. Error matrix is defined as the difference between the original and reconstructed images. We plot the frobenius norm of the error matrix against the order of approximation (k - number of eigen/singular values retained) for both EVD and SVD.

Experimental Results:

Results for EVD:

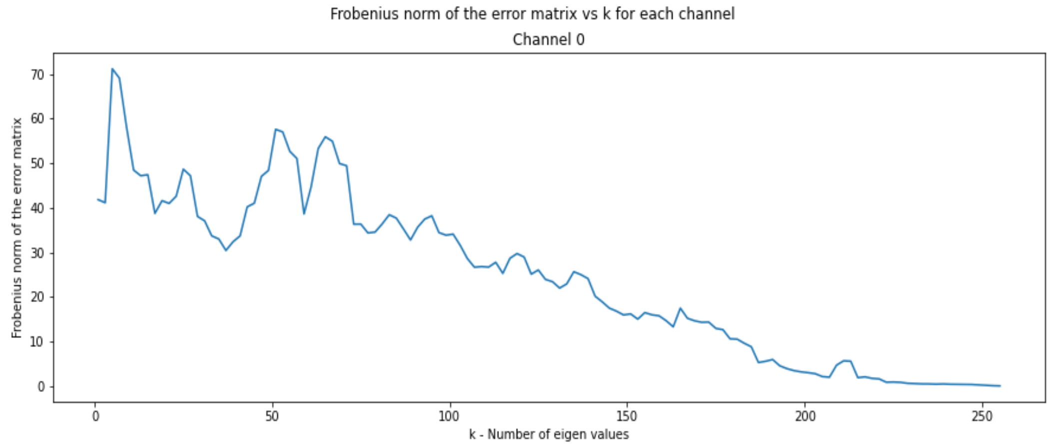

Figure 1: Plot of the Frobenius norm of the error matrix (image - reconstruction) against the order of approximation (k - the number of eigenvalues retained)

The best trade-off between reconstruction quality (low error in reconstruction) and compression (lower order of approximation) is obtained when **k = 180**. The Frobenius norm of the reconstruction error matrix obtained using EVD is **10.489**.

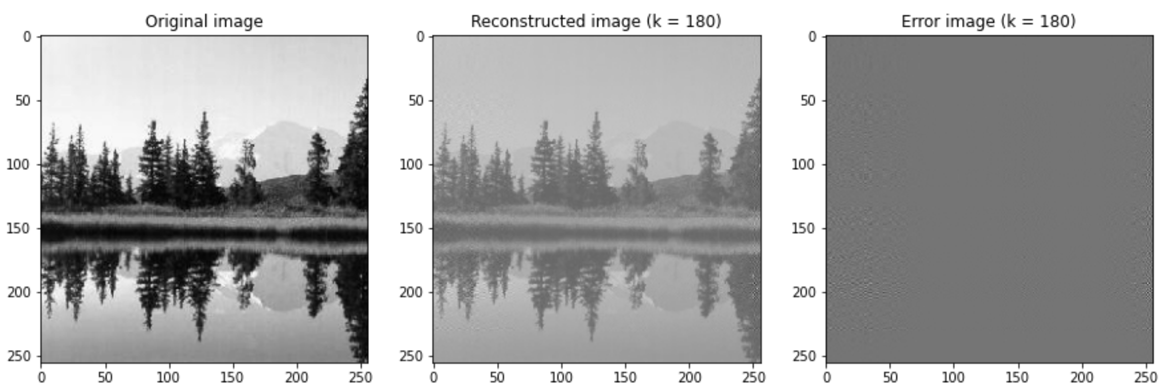

Figure 2: The figure shows the original, and reconstructed images along with the error matrix for k = 180

Results for SVD:

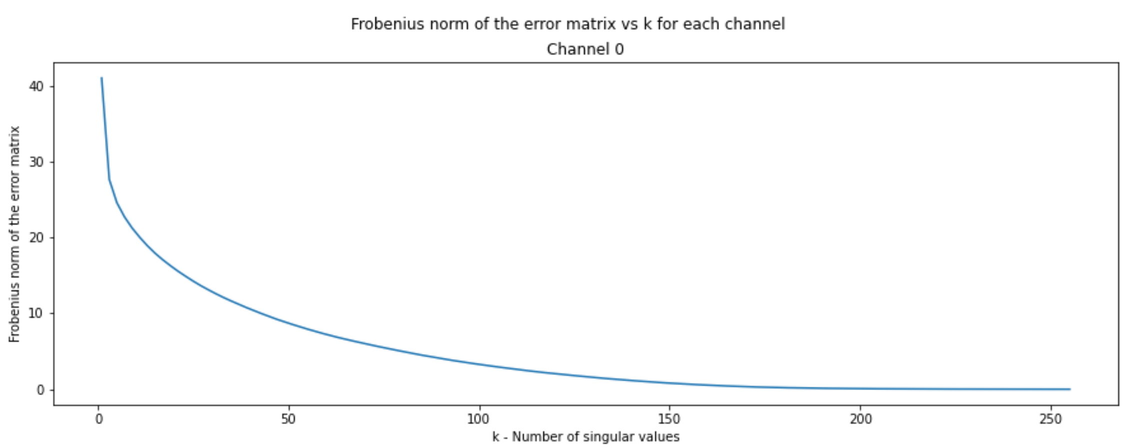

Figure 3: Plot of the Frobenius norm of the error matrix (image - reconstruction) against the order of approximation (k - the number of singular values retained)

The best trade-off between reconstruction quality (low error in reconstruction) and compression (lower order of approximation) is obtained when **k = 50**. The Frobenius norm of the reconstruction error matrix obtained using SVD is **8.706**.

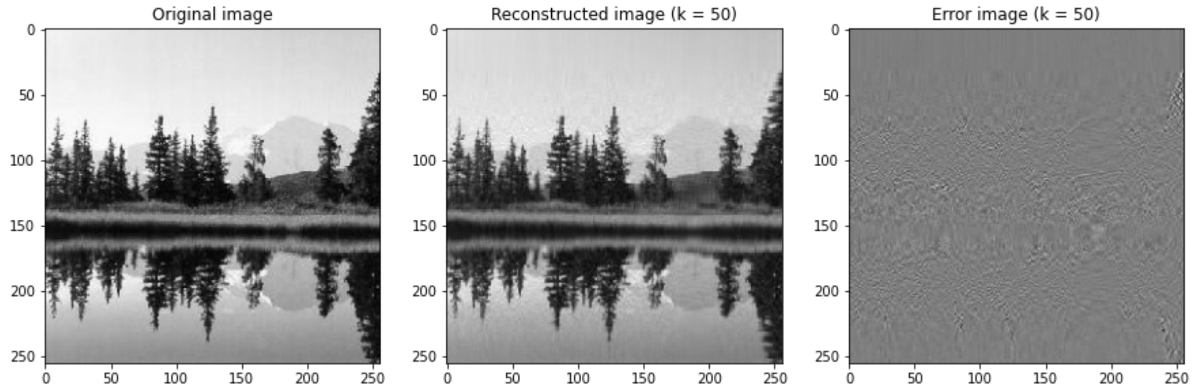

Figure 4: The figure shows the original, and reconstructed images along with the error matrix for k = 50

Inference:

From figures 1 and 3, we can see that the error drops quickly and monotonically when we use SVD for matrix reconstruction. This fact can also be inferred from figures 2 and 4. The number of singular/eigenvalues to be retained to obtain a visually perceivable reconstruction is greater for EVD-based image reconstruction. A qualitative explanation for the steady decrease in error could be the orthogonality of **U** and **V** matrices. While increasing the value of k during SVD-based reconstruction, we are including a new singular value to reconstruct the original matrix. This also means that we are including the left and right singular vectors corresponding to the singular value, which are orthogonal to the columns in **Uk** and **Vk**, respectively. In other words, orthogonality ensures the addition of "information". Hence, the error decreases steadily as we increase the order of approximation (evident from figure 3). Since orthogonality of the matrix **Q** is not ensured in EVD, increasing the order of approximation does not necessarily lead to a decrease in reconstruction error. This explains the fluctuations observed in figure 4. However, in the long run, the reconstruction error shows a decreasing trend when the order of approximation is increased to large values (~200). Also, we observe that a large amount of variance in the original image can be explained with the help of the first few singular values (sorted in descending order of their magnitude). Qualitatively, the magnitude of a singular value is directly proportional to the amount of variance it can explain in the original image. This explains the quick decrease in reconstruction error (evident from figure 3).

This folder contains two gifs that show the evolution of reconstructed images using EVD and SVD. This clearly shows the effectiveness of SVD over EVD in image reconstruction.