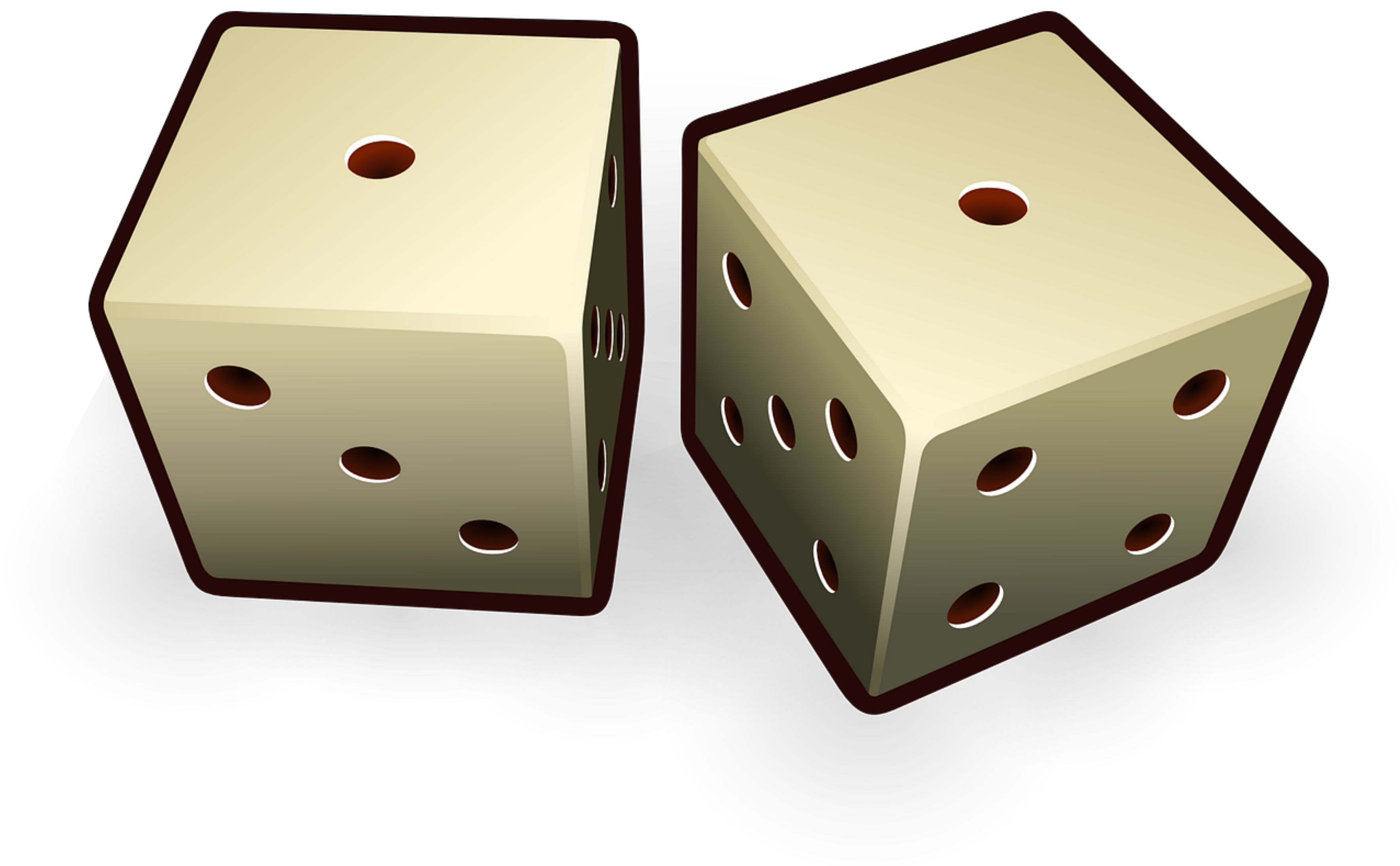


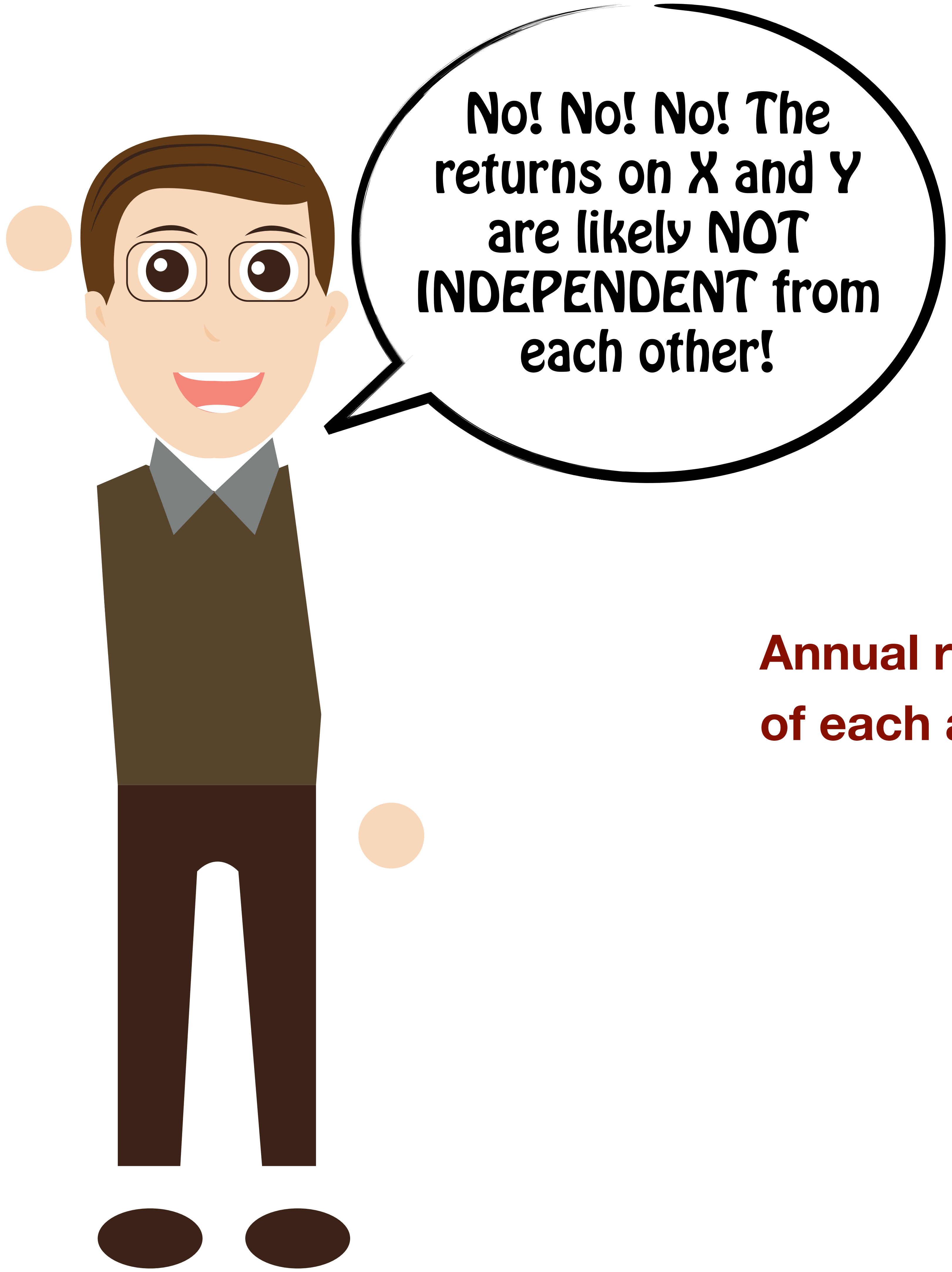
Probability Concepts

Portfolio Return and Variance, Covariance and Correlation

1. Portfolio Return & Variance
2. Covariance
3. Correlation

$$E(w_x X + w_y Y)$$

$$= w_x E(X) + w_y E(Y)$$



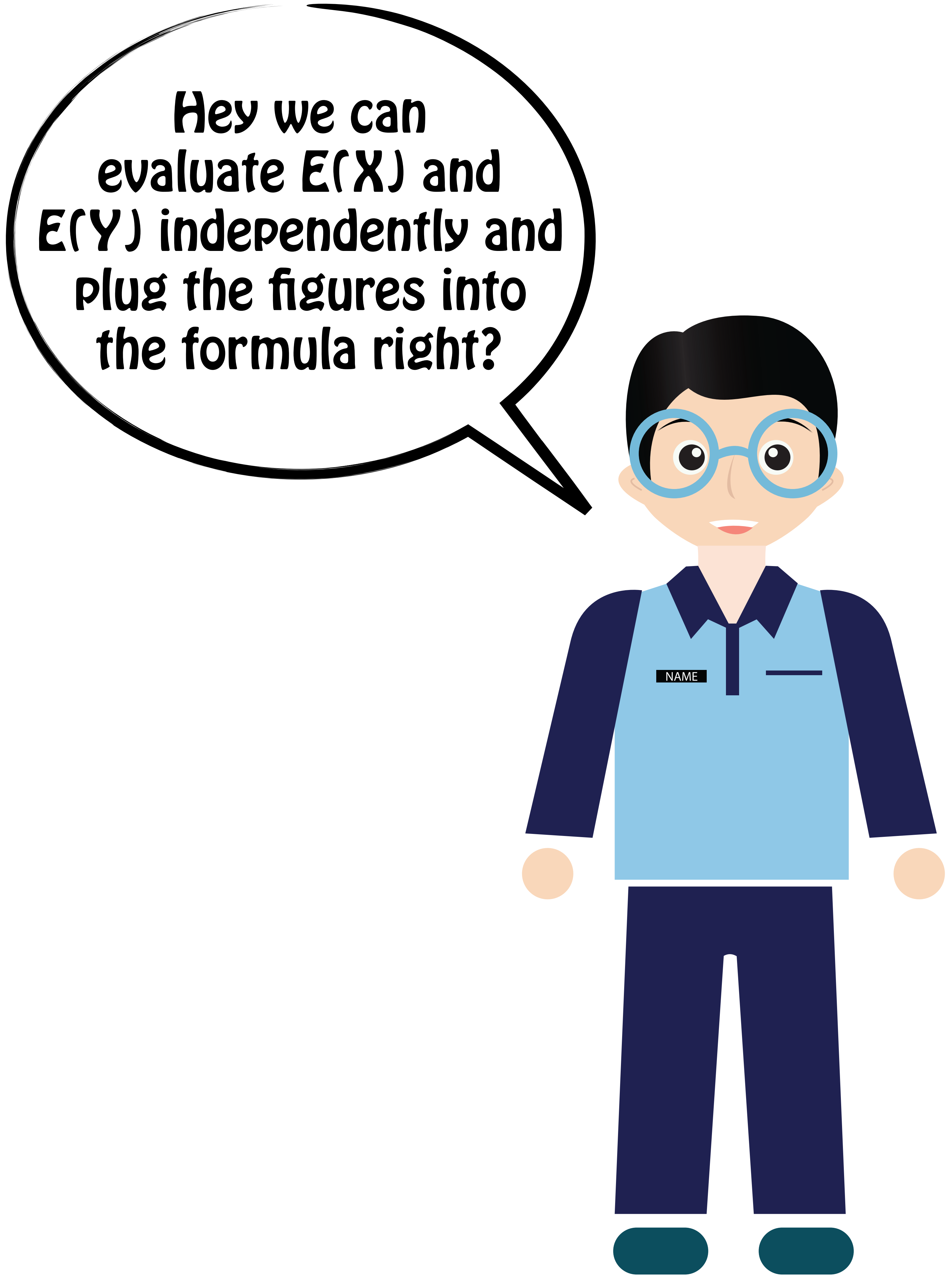
2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	w_X	E(X)
Y	w_Y	E(Y)

Annual return of each asset

$$E(\, w_X X + w_Y Y \,)$$

$$= \; w_X E(X) + w_Y E(Y)$$



Joint Probability Table

	Y_1: Y=0%	Y_2: Y=4%	Y_3: Y=10%
X_1: X=20%	0.4	0	0
X_2: X=0%	0	0.5	0
X_3: X= -15%	0	0	0.1

$\sum = 1$

$P(X_1Y_1) = 0.4$ $P(X_2Y_2) = 0.5$ $P(X_3Y_3) = 0.1$

Joint Probability Table

	Y_1: Y=0%	Y_2: Y=4%	Y_3: Y=10%
X_1: X=20%	0.4	0	0
X_2: X=0%	0	0.5	0
X_3: X= -15%	0	0	0.1

$$E(X) = 20x0.4 + 0x0.5 + (-15)x0.1 = 6.5\%$$

$$E(Y) = 0x0.4 + 4x0.5 + 10x0.1 = 3\%$$

2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	0.7	6.5%
Y	0.3	3%

$$E(w_x X + w_y Y)$$

$$= w_x E(X) + w_y E(Y)$$

$$= 0.7 \times 6.5 + 0.3 \times 3$$

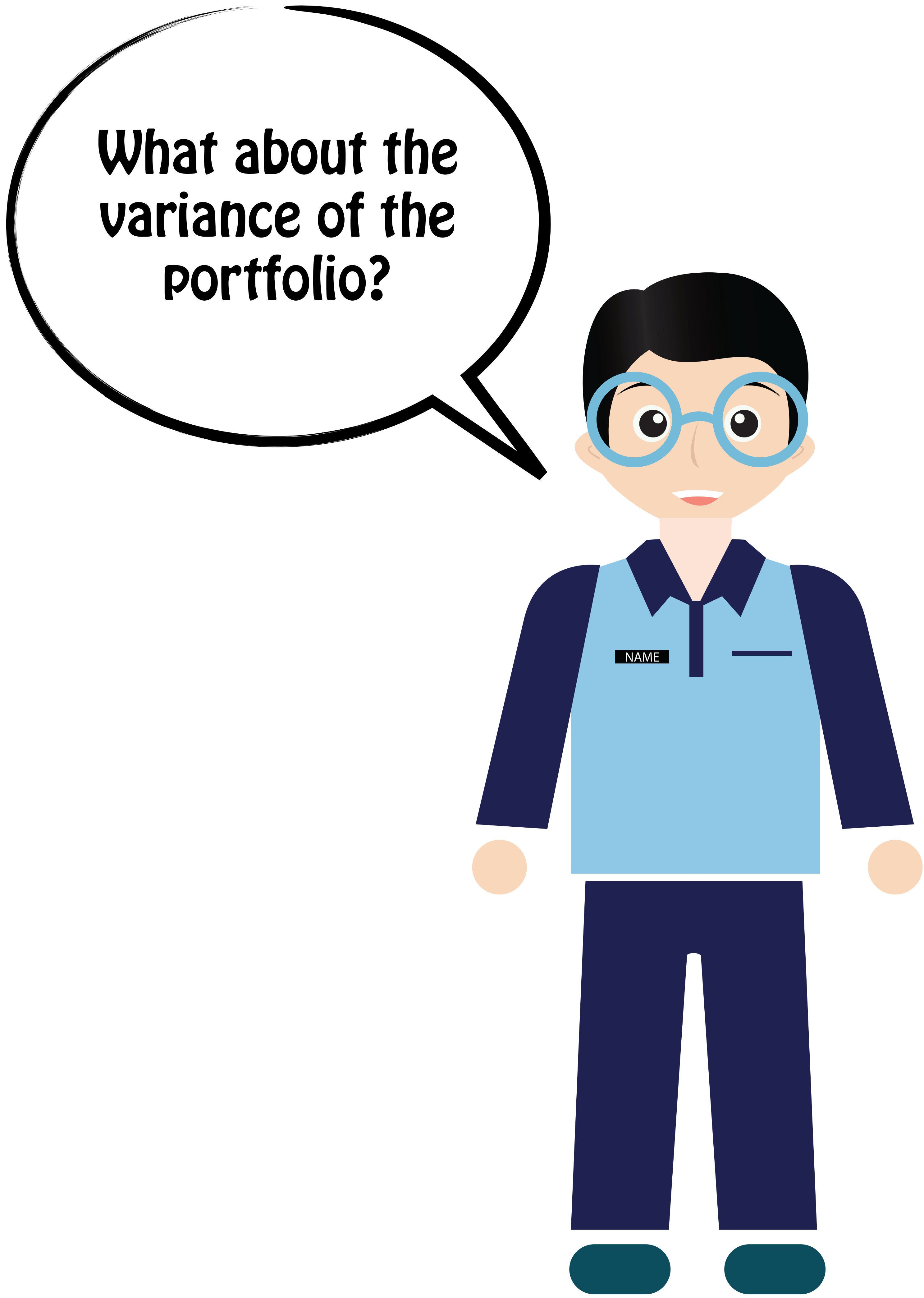
$$= 5.45\%$$

2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	0.7	6.5%
Y	0.3	3%

$$E(w_x X + w_y Y) = 5.45\%$$

$$\sigma^2 (w_x X + w_y Y) = ???$$



Joint Probability Table

	Y_1: Y=0%	Y_2: Y=4%	Y_3: Y=10%
X_1: X=20%	0.4	0	0
X_2: X=0%	0	0.5	0
X_3: X= -15%	0	0	0.1

How does X move with Y?

2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	0.7	6.5%
Y	0.3	3%

Joint Probability Table

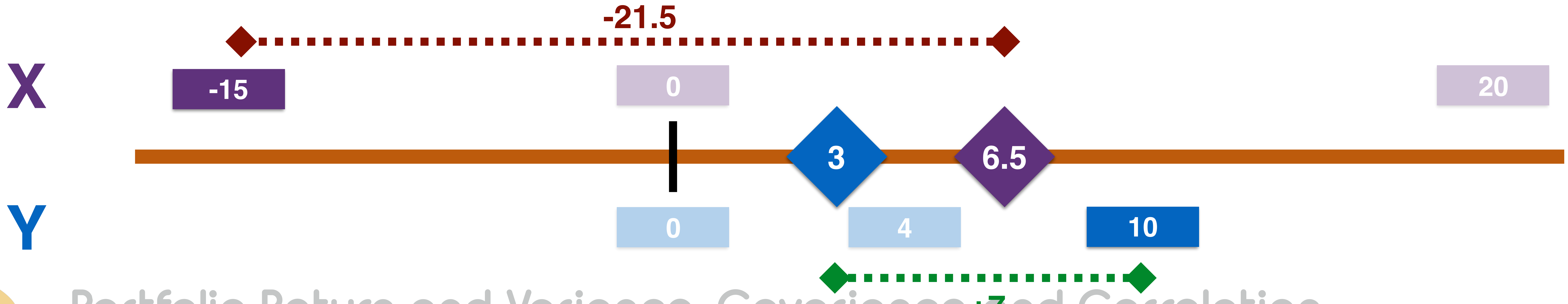
	Y_1: Y=0%	Y_2: Y=4%	Y_3: Y=10%
X_1: X=20%	0.4	0	0
X_2: X=0%	0	0.5	0
X_3: X= -15%	0	0	0.1

2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	0.7	6.5%
Y	0.3	3%

$$Cov(X,Y) = E\{\, [X-E(X)]\, [Y-E(Y)] \,\}$$

$$= 0.4 \times 13.5 \times (-3) \;+\; 0.5 \times (-6.5) \times 1 \;+\; 0.1 \times (-21.5) \times 7$$



Joint Probability Table

	Y_1: Y=0%	Y_2: Y=4%	Y_3: Y=10%
X_1: X=20%	0.4	0	0
X_2: X=0%	0	0.5	0
X_3: X= -15%	0	0	0.1

2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	0.7	6.5%
Y	0.3	3%

$$Cov(X,Y) = E\{ [X-E(X)]\ [Y-E(Y)] \}$$
$$= -34.5$$

$$Var(X) = Cov(X,X)$$
$$= 0.4 \times 13.5^2 + 0.5 \times (-6.5)^2 + 0.1 \times (-21.5)^2$$
$$= 140.25$$

Properties of Covariance

1. General representation of variance
2. Cov (X,X) = Var(X)

Joint Probability Table

	Y_1: Y=0%	Y_2: Y=4%	Y_3: Y=10%
X_1: X=20%	0.4	0	0
X_2: X=0%	0	0.5	0
X_3: X= -15%	0	0	0.1

2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	0.7	6.5%
Y	0.3	3%

$$Cov(X,Y) = E\{ [X-E(X)]\,[Y-E(Y)] \}$$
$$= -34.5$$

$$Var(X) = Cov(X,X) = 140.25$$

$$Var(Y) = Cov(Y,Y)$$
$$= 0.4x(-3)^2+0.5x1^2+0.1x7^2$$
$$= 9$$

Properties of Covariance

1. General representation of variance
2. Cov (X,X) = Var(X)

Joint Probability Table

	Y_1: Y=0%	Y_2: Y=4%	Y_3: Y=10%
X_1: X=20%	0.4	0	0
X_2: X=0%	0	0.5	0
X_3: X= -15%	0	0	0.1

2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	0.7	6.5%
Y	0.3	3%

$$\text{Cov}(X,Y) = E\{ [X-E(X)]\,[Y-E(Y)] \}$$
$$= -34.5$$

Returns move in opposite directions

$$\text{Var}(X) = \text{Cov}(X,X) = 140.25$$

$$\text{Var}(Y) = \text{Cov}(Y,Y) = 9$$

Properties of Covariance

1. General representation of variance
2. Cov (X,X) = Var(X)
3. Range from -∞ to +∞

Joint Probability Table

	Y_1: Y=0%	Y_2: Y=4%	Y_3: Y=10%
X_1: X=20%	0.4	0	0
X_2: X=0%	0	0.5	0
X_3: X= -15%	0	0	0.1

2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	0.7	6.5%
Y	0.3	3%

$$Cov(X,Y) = E\{ [X-E(X)]\ [Y-E(Y)] \}$$
$$= -34.5$$

$$Var(X) = Cov(X,X) = 140.25$$

$$Var(Y) = Cov(Y,Y) = 9$$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\ Var(Y)}}$$

Correlation Coefficient

$$= (-34.5) / (140.25x9)^{½}$$

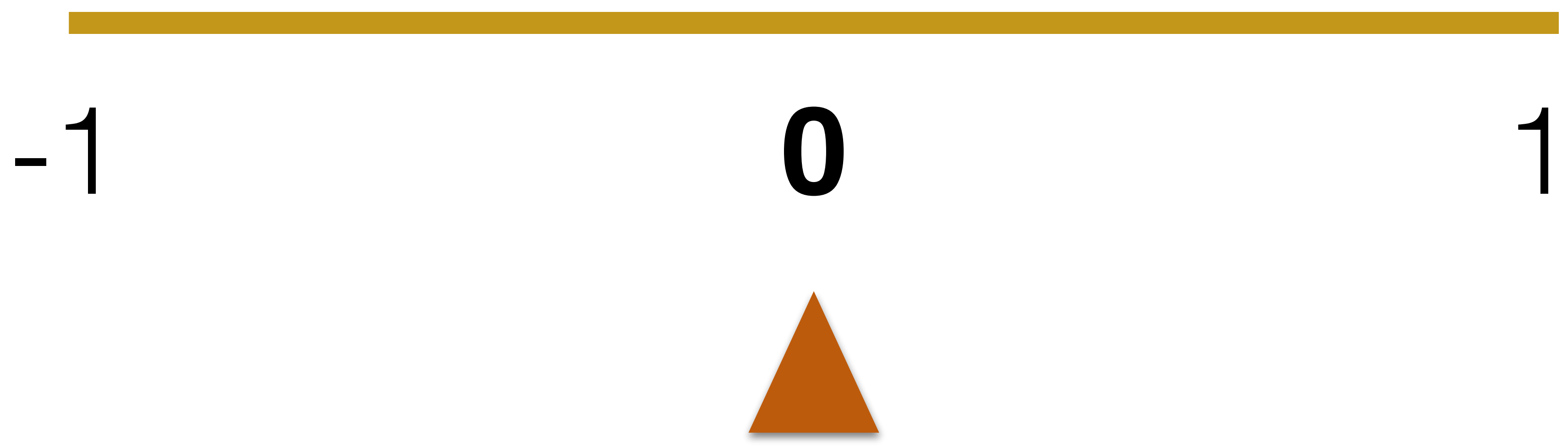
$$= -0.97$$

Joint Probability Table

	Y_1: Y=0%	Y_2: Y=4%	Y_3: Y=10%
X_1: X=20%	0.4	0	0
X_2: X=0%	0	0.5	0
X_3: X= -15%	0	0	0.1

2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	0.7	6.5%
Y	0.3	3%



No linear relationship

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\ \mathrm{Var}(Y)}}$$

$$= (-34.5) / (140.25 \mathrm{x} 9)^{½}$$

$$= -0.97$$

Joint Probability Table

	Y_1: Y=0%	Y_2: Y=4%	Y_3: Y=10%
X_1: X=20%	0.4	0	0
X_2: X=0%	0	0.5	0
X_3: X= -15%	0	0	0.1

2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	0.7	6.5%
Y	0.3	3%

-1 0 1

Perfect positive correlation

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\ \text{Var}(Y)}}$$

$$= (-34.5) / (140.25\text{x}9)^{½}$$

$$= -0.97$$

Joint Probability Table

	Y_1: Y=0%	Y_2: Y=4%	Y_3: Y=10%
X_1: X=20%	0.4	0	0
X_2: X=0%	0	0.5	0
X_3: X= -15%	0	0	0.1

2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	0.7	6.5%
Y	0.3	3%

-1 0 1

Perfect negative correlation

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\ \text{Var}(Y)}}$$

$$= (-34.5) / (140.25\text{x}9)^{1/2}$$

$$= -0.97$$

When one outperforms, the other will likely underperform

$$\mathrm{Cov}(X,Y) = -34.5$$

$$\mathrm{Var}(X) = 140.25$$

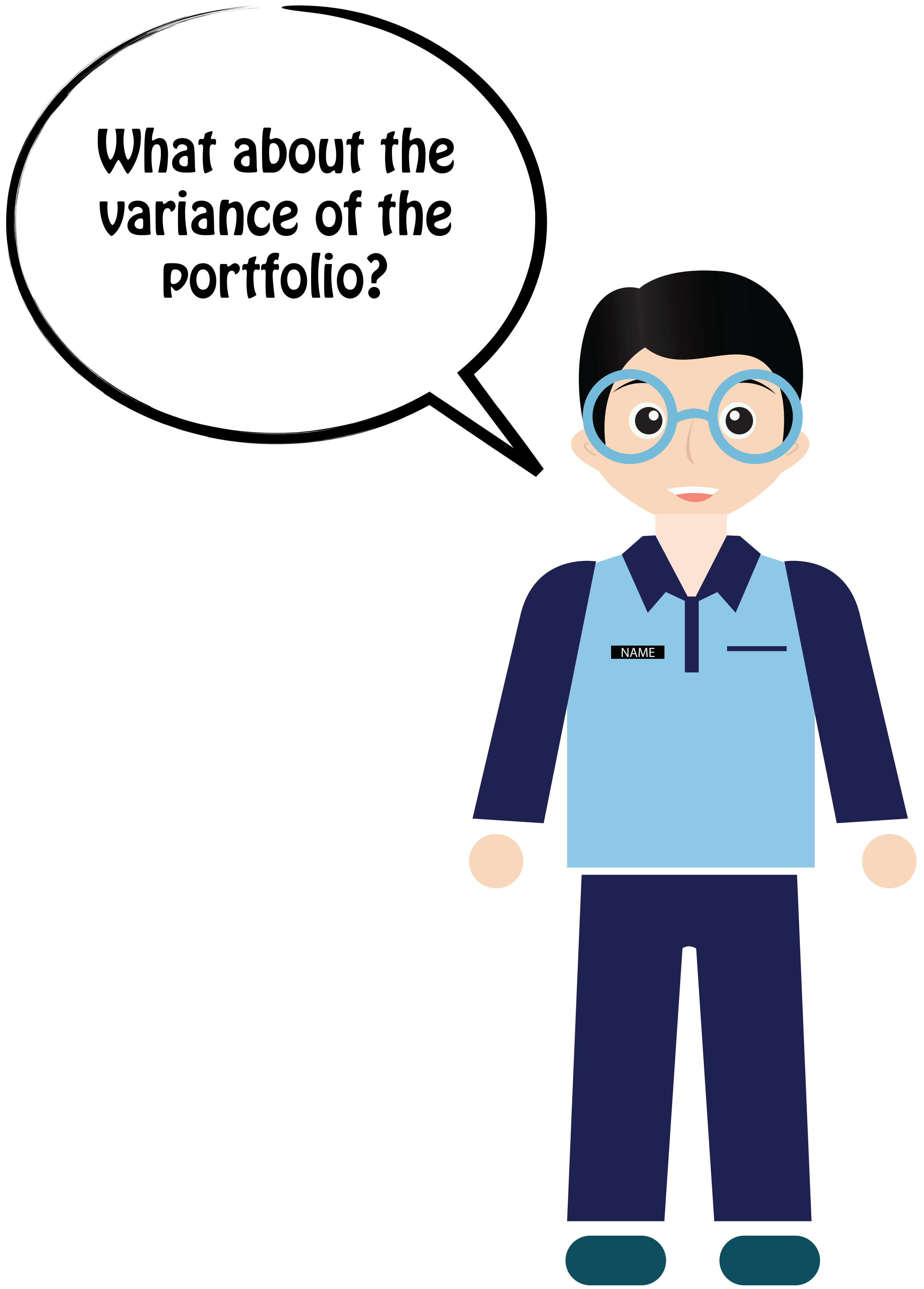
$$\mathrm{Var}(Y) = 9$$

2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	0.7	6.5%
Y	0.3	3%

$$E(\,w_xX + w_yY\,) = 5.45\%$$

$$\sigma^2\,(\,w_xX + w_yY\,) = ???$$



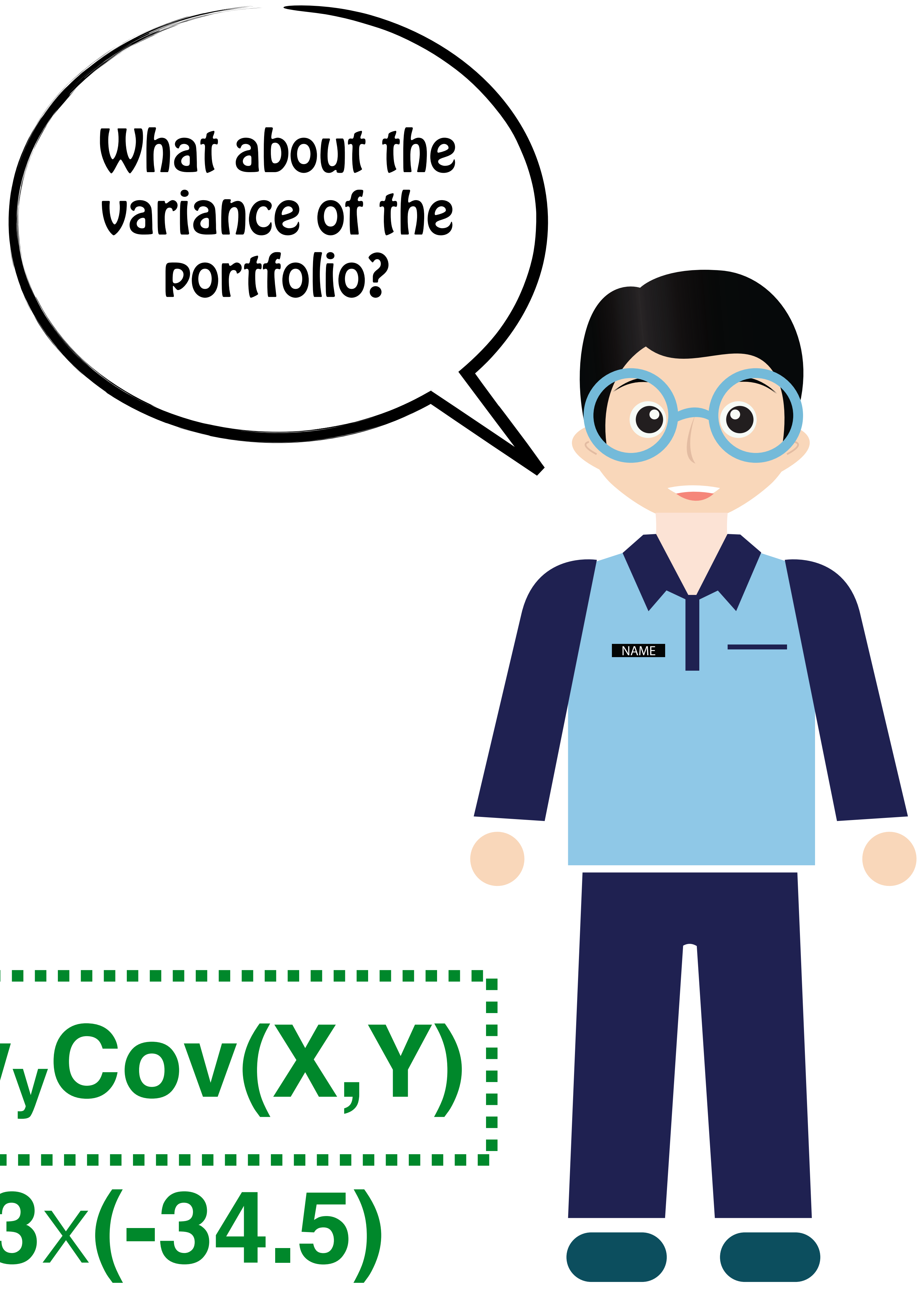
$Cov(X,Y) = -34.5$

$Var(X) = 140.25$

$Var(Y) = 9$

2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	0.7	6.5%
Y	0.3	3%



$$E(\, w_x X + w_y Y \,) = 5.45\%$$

$$\boxed{\sigma^2 (\, w_x X + w_y Y \,) = w_x{}^2\sigma^2(X) + w_y{}^2\sigma^2(Y) + 2w_x w_y Cov(X,Y)}$$

$$= 0.7^2 \times 140.25 + 0.3^2 \times 9 + 2 \times 0.7 \times 0.3 \times (-34.5)$$

$$= 55.04$$

Portfolio Return and Variance, Covariance and Correlation

1. Portfolio Return & Variance 2. Covariance 3. Correlation

Covariance Matrix

	X	Y
X	140.25	-34.5
Y	-34.5	9

Correlation Matrix

	X	Y
X	1	-0.97
Y	-0.97	1

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\ Var(Y)}} = \frac{-34.5}{\sqrt{140.25 \times 9}} = -0.97$$

Given the 3-asset portfolio as shown, calculate (a) the expected return of the portfolio and (b) the variance of the portfolio return.

Asset	Value ($)	Expected Return (%)
X	50,000	14
Y	80,000	4
Z	70,000	8

Covariance matrix

	X	Y	Z
X	82.1	18.8	0
Y	18.8	11.3	-16.2
Z	0	-16.2	34

(a) $w_x = 50/200 = 0.25$ $w_y = 80/200 = 0.4$ $w_z = 70/200 = 0.35$

$$\text{Expected Portfolio Return} = w_xE(X) + w_yE(Y) + w_zE(Z)$$
$$= 0.25 \times 14 + 0.4 \times 4 + 0.35 \times 8$$
$$= 7.9\%$$

Given the 3-asset portfolio as shown, calculate (a) the expected return of the portfolio and (b) the variance of the portfolio return.

Asset	Value ($)	Expected Return (%)
X	50,000	14
Y	80,000	4
Z	70,000	8

Covariance matrix

	X	Y	Z
X	82.1	18.8	0
Y	18.8	11.3	-16.2
Z	0	-16.2	34

(b) $w_x = 50/200 = 0.25$ $w_y = 80/200 = 0.4$ $w_z = 70/200 = 0.35$

Variance of 2-asset portfolio

$= w_x^2\sigma^2(X) + 2w_xw_y\mathrm{Cov}(X,Y)$

$+ w_y^2\sigma^2(Y)$

Given the 3-asset portfolio as shown, calculate (a) the expected return of the portfolio and (b) the variance of the portfolio return.

Asset	Value ($)	Expected Return (%)
X	50,000	14
Y	80,000	4
Z	70,000	8

Covariance matrix

	X	Y	Z
X	82.1	18.8	0
Y	18.8	11.3	-16.2
Z	0	-16.2	34

(b) $w_x = 50/200 = 0.25$ $w_y = 80/200 = 0.4$ $w_z = 70/200 = 0.35$

Variance of 3-asset portfolio

$$= w_x^2\sigma^2(X) + 2w_xw_yCov(X,Y) + w_y^2\sigma^2(Y) + 2w_yw_zCov(Y,Z) + w_z^2\sigma^2(Z) + 2w_zw_xCov(Z,X)$$

$$= 0.25^2 x 82.1 + 2x0.25x0.4 x 18.8 + 0.4^2 x 11.3 + 2x0.4x0.35 x(-16.2) + 0.35^2 x 34 + 2x0.35x0.25x0$$

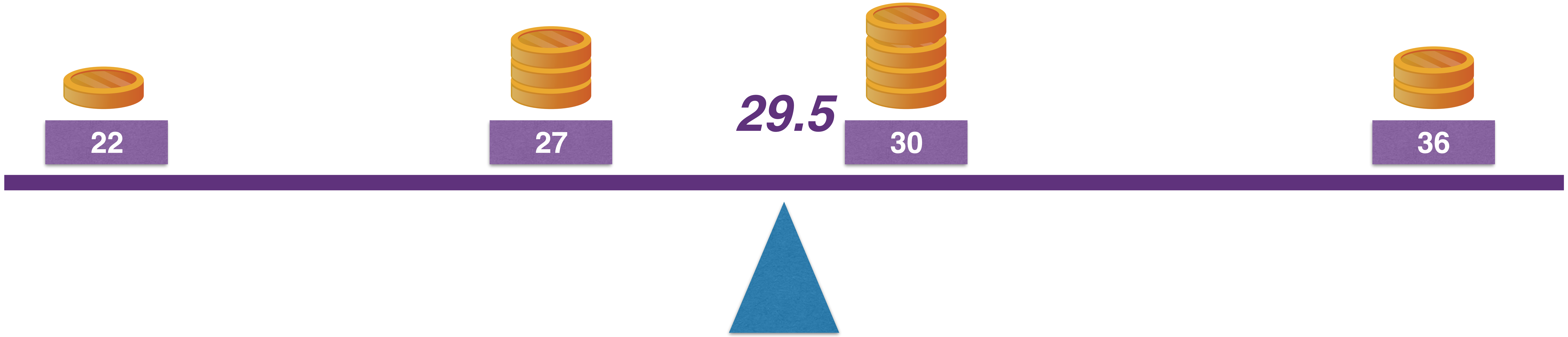
$$= 10.32$$

Expected Value

$$E(X) = \sum P(X_i)\, X_i$$

$P(X_i)$	X_i: EPS ($)	$P(X_i)\, X_i$
0.2	36	7.2
0.4	30	12
0.3	27	8.1
0.1	22	2.2

$\Sigma = 29.5$



PREP nuggets

Expected Value

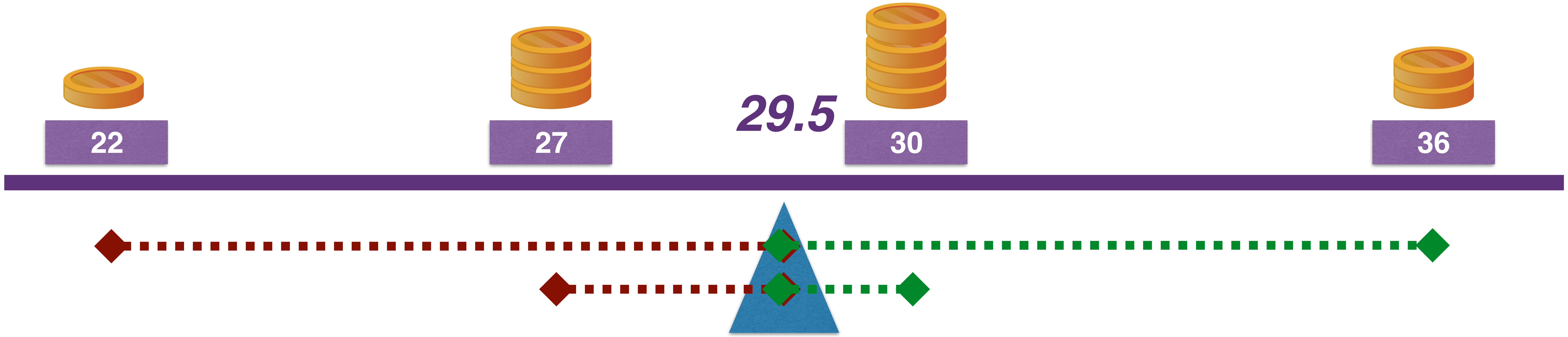
$$E(X) = \sum P(X_i)\, X_i$$

Variance

$$\sigma^2(X) = \sum P(X_i)\, [X_i - E(X)]^2$$

$P(X_i)$	X_i: EPS ($)	$P(X_i)\, X_i$	$P(X_i)\, [X_i-E(X)]^2$
0.2	36	7.2	8.45
0.4	30	12	0.1
0.3	27	8.1	1.875
0.1	22	2.2	5.625

$\sum = 16.05$



Expected Value

$$E(X) = \sum P(X_i)\, X_i$$
$$E(X \mid S) = \sum P(X_i \mid S)\, X_i$$

Variance

$$\sigma^2(X) = \sum P(X_i)\, [X_i - E(X)]^2$$

	$P(X_i)$	X_i: EPS ($)
No Recession (R^C)	0.2	36
	0.4	30
Recession (R)	0.3	27
	0.1	22

PREP nuggets

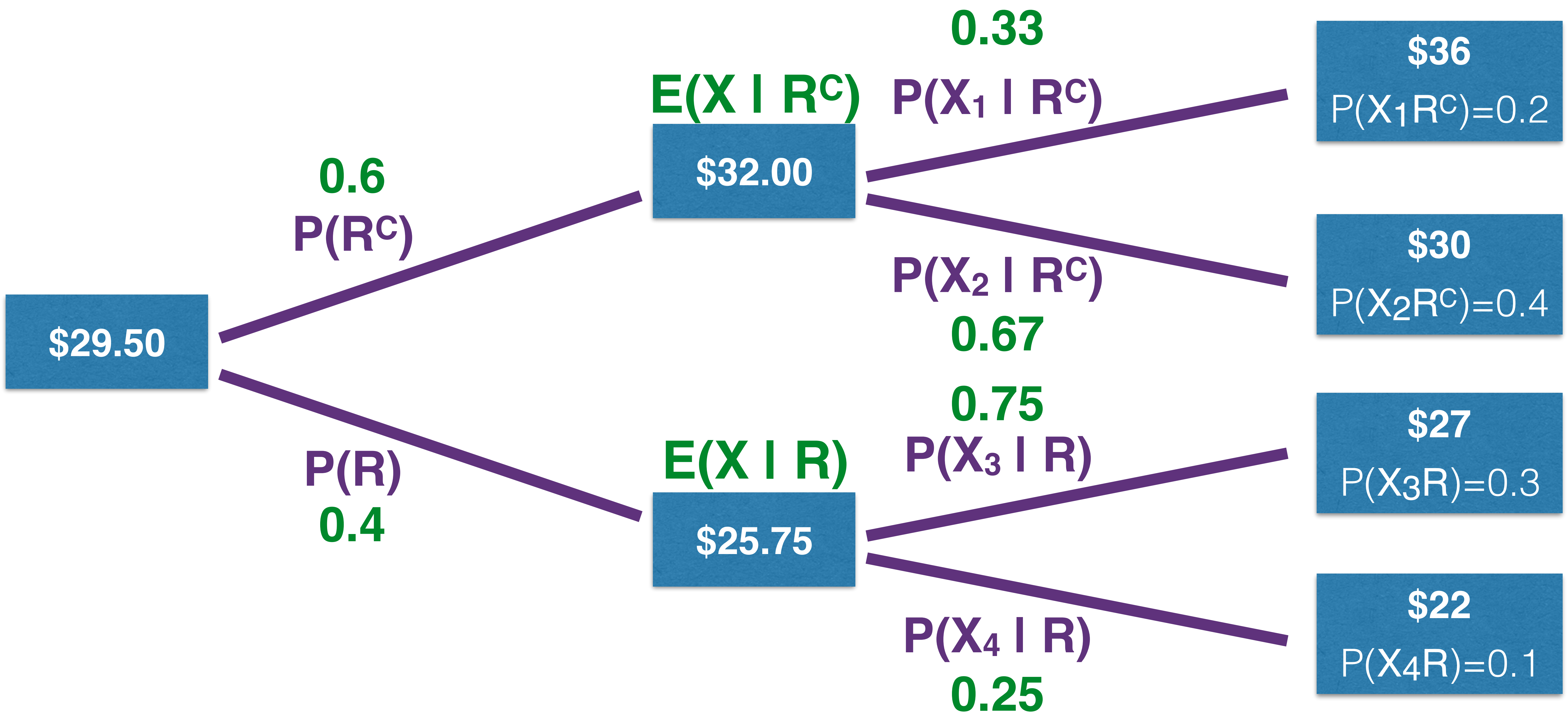
Expected Value

$$E(X) = \sum P(X_i)\, X_i$$

$$E(X \mid S) = \sum P(X_i \mid S)\, X_i$$

$$E(X) = E(X \mid R)\, P(R) + E(X \mid R^C)\, P(R^C)$$

	$P(X_i)$	X_i: EPS ($)
No Recession (R^C)	0.2	36
	0.4	30
Recession (R)	0.3	27
	0.1	22



2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	w_x	$E(X)$
Y	w_y	$E(Y)$

$$E(\,w_x X + w_y Y\,)$$

$$= w_x E(X) + w_y E(Y)$$

PREP nuggets

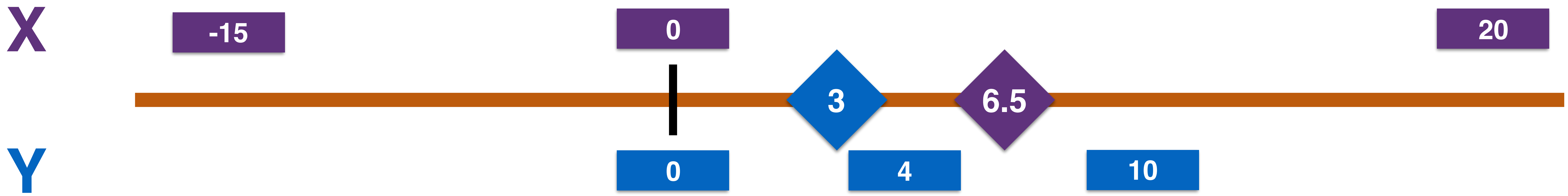
Joint Probability Table

	Y_1: Y=0%	Y_2: Y=4%	Y_3: Y=10%
X_1: X=20%	0.4	0	0
X_2: X=0%	0	0.5	0
X_3: X= -15%	0	0	0.1

2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	0.7	6.5%
Y	0.3	3%

$$Cov(X,Y) = E\{ [X-E(X)]\ [Y-E(Y)] \}$$



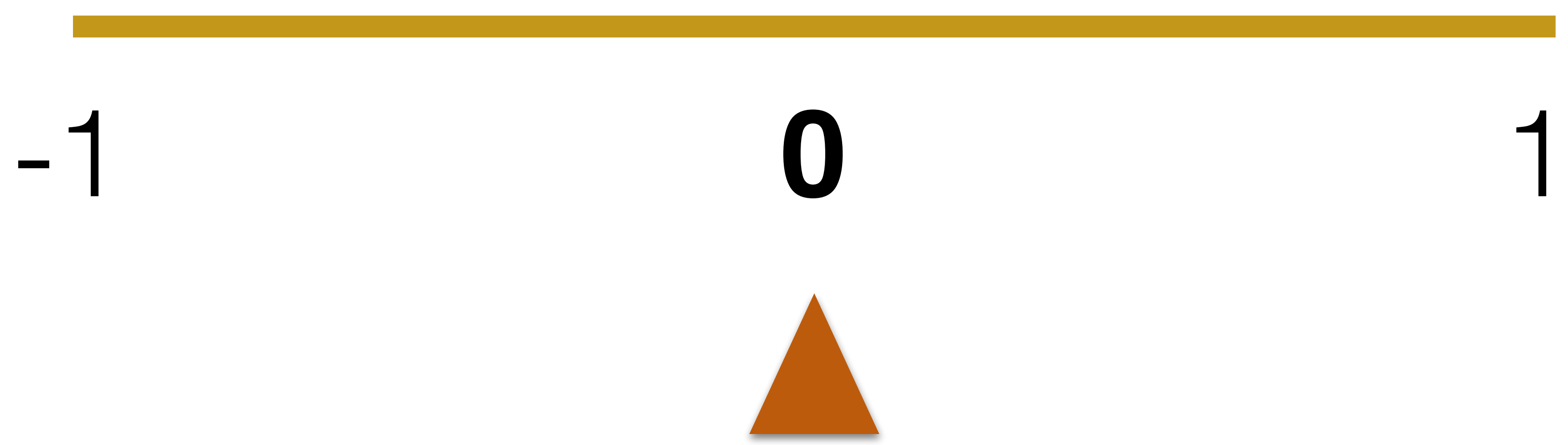
Joint Probability Table

	Y_1: Y=0%	Y_2: Y=4%	Y_3: Y=10%
X_1: X=20%	0.4	0	0
X_2: X=0%	0	0.5	0
X_3: X= -15%	0	0	0.1

2-Asset Investment Portfolio

Asset	Weight	Expected Return
X	0.7	6.5%
Y	0.3	3%

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\ \mathrm{Var}(Y)}}$$



Variance of 2-asset portfolio

$$= w_x^2\sigma^2(X) + 2w_xw_yCov(X,Y)$$

$$+ w_y^2\sigma^2(Y)$$

Variance of 3-asset portfolio

$$= w_x^2\sigma^2(X) + 2w_xw_y\mathrm{Cov}(X,Y)$$

$$+ w_y^2\sigma^2(Y) + 2w_yw_z\mathrm{Cov}(Y,Z)$$

$$+ w_z^2\sigma^2(Z) + 2w_zw_x\mathrm{Cov}(Z,X)$$

PREP nuggets

PREP
nuggets