Probability and Statistics with Reliability, Queuing and Computer Science Applications:

Second edition

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Publisher-John Wiley & Sons

Chapter 3: Continuous Random Variables

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Definitions

- Distribution function:

$$F_X(x) = P(X \leq x), \; -\infty < x < \infty$$

- If $F_X(x)$ is a continuous function of *x*, then *X* is a continuous random variable.
  - $F_X(x)$ *:* grows only by jumps → Discrete rv
  - $F_X(x)$ *:* both jumps and continuous growth → Mixed rv

. **(F1)** $0 \leq F_X(x) \leq 1, \; -\infty < x < \infty$

. **(F2)** $F_X(x)$: monotonically non-decreasing in x

. **(F3)** $x \xrightarrow{lim} -\infty \; F_X(x) = 0$ and $x \xrightarrow{lim} \infty \; F_X(x) = 1$

. **(F4')** $P(X = c) = P(c \leq X \leq c) = \int_c^c f_Y(y)dy = 0$

Note

- We will also allow defective distributions. Defective distributions, also known as improper distributions will be covered later and are very useful in computer science applications
- These distributions satisfy F1, F2 and a modified version of F3:

. **(F3')** $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) < 1$

- Unless otherwise specified, we will assume all distributions to be non-defective

Definitions (Contd.)

Equivalence:

- CDF (Cumulative Distribution Function)
- Probability Distribution Function (PDF)

  but avoid this name as it can be confused with

  pdf (prob. density function)
- Distribution function
- $F_X(x)$ or $F_X(t)$ or $F(t)$

probability density function (pdf)

- X: continuous rv, then, $f(x) = \frac{dF(x)}{dx}$ is the pdf of X.
- *CDF and pdf can be derived from each other*

$$P(X \leq x) \;=\; F(x) = \int_{-\infty}^{x} f(u)du, \;\; -\infty < x < \infty$$

$$P(X \in (a,b]) \;=\; P(a < X \leq b) = \int_{a}^{b} f_X(u)du.$$

- *pdf* properties:

  . **(f1)** $f(x) \geq 0$ for all x.

  . **(f2)** $\int_{-\infty}^{\infty} f(x)dx = 1.$

Definitions (Continued)

- Equivalence: pdf
  - probability density function
  - density function
  - density
  - $f(t) = \frac{dF}{dt}$

$$F(t) = \int_{-\infty}^{t} f(x)dx$$

$$= \int_{0}^{t} f(x)dx$$

, for a non-negative random variable

Example 3.1

- Random variable X : time (years) to complete a project

$$f_X(x) = \begin{cases} kx(1-x), & 0 \le x \le 1, \\ 0, & \text{otherwise} \end{cases}$$

- f_X clearly satisfies property **(f1)**.
- To be a *pdf*, it must also satisfy **(f2)**,

$$\int_0^1 kx(1-x)dx = 1 \overset{gives}{\longrightarrow} k\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = 1 \qquad \text{or, } k = 6$$

- Prob. of completing project in less than 4 months,

$$P(X < 4/12) = F_X(1/3) = \int_0^{1/3} f_X(x)dx = \frac{7}{27}, \text{ or } 26\%$$

Exponential Distribution

- Arises commonly in reliability & queuing theory.
- A non-negative continuous random variable.
- It exhibits memoryless property.
- Related to (discrete) Poisson distribution
- Often used to model
  - Interarrival times between two IP packets (or voice calls)
  - Service time distribution
  - Time to failure, time to repair etc.

Exponential Distribution

- The use of exponential distribution is an assumption that needs to be validated based on experimental data; if the data does not support the assumption, other distributions may be used
- For instance, Weibull distribution is often used to model time to failure; Markov modulated Poisson process is used to model arrival of IP packets

Exponential Distribution

$$F(x) = \begin{cases} 1 - \mathrm{e}^{-\lambda x}, & \text{if } 0 \le x < \infty \\ 0, & \text{otherwise} \end{cases}$$

where the base of natural logarithm, $e = 2.7182818284$

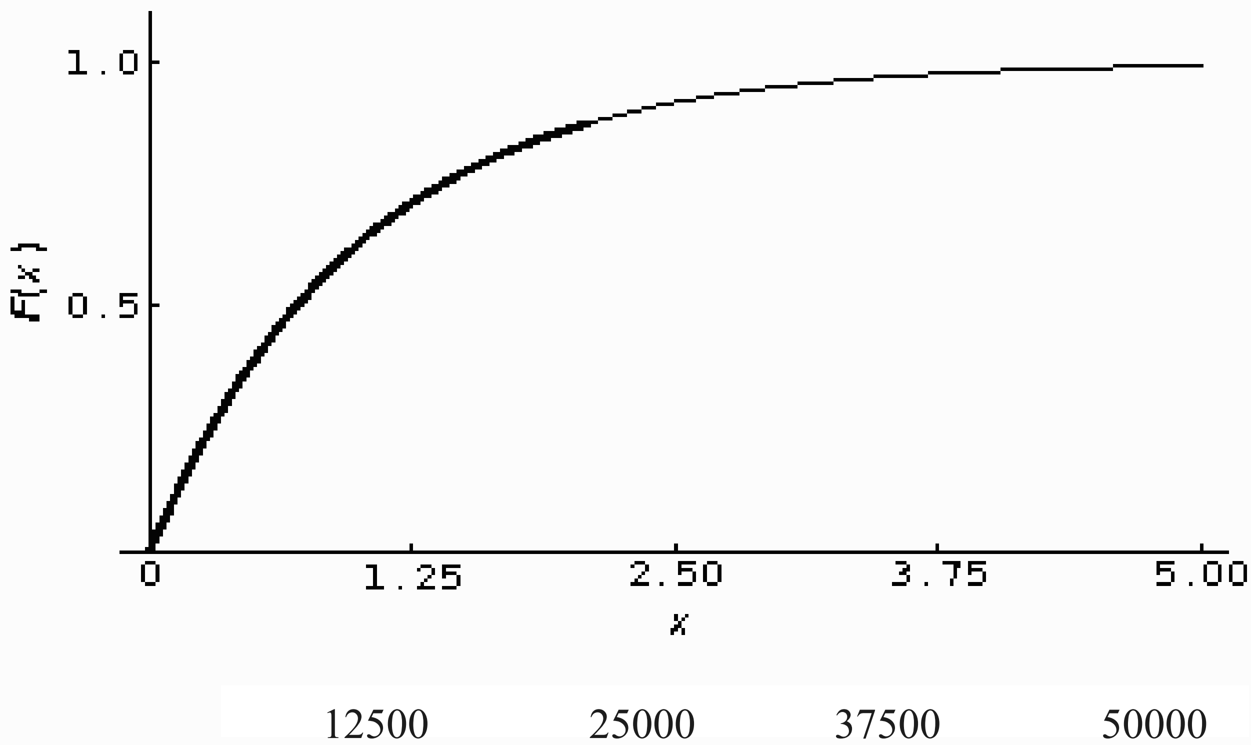
$$f(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

- Also $P(X > t) = \int_t^{\infty} f(x)dx = \mathrm{e}^{-\lambda t}$

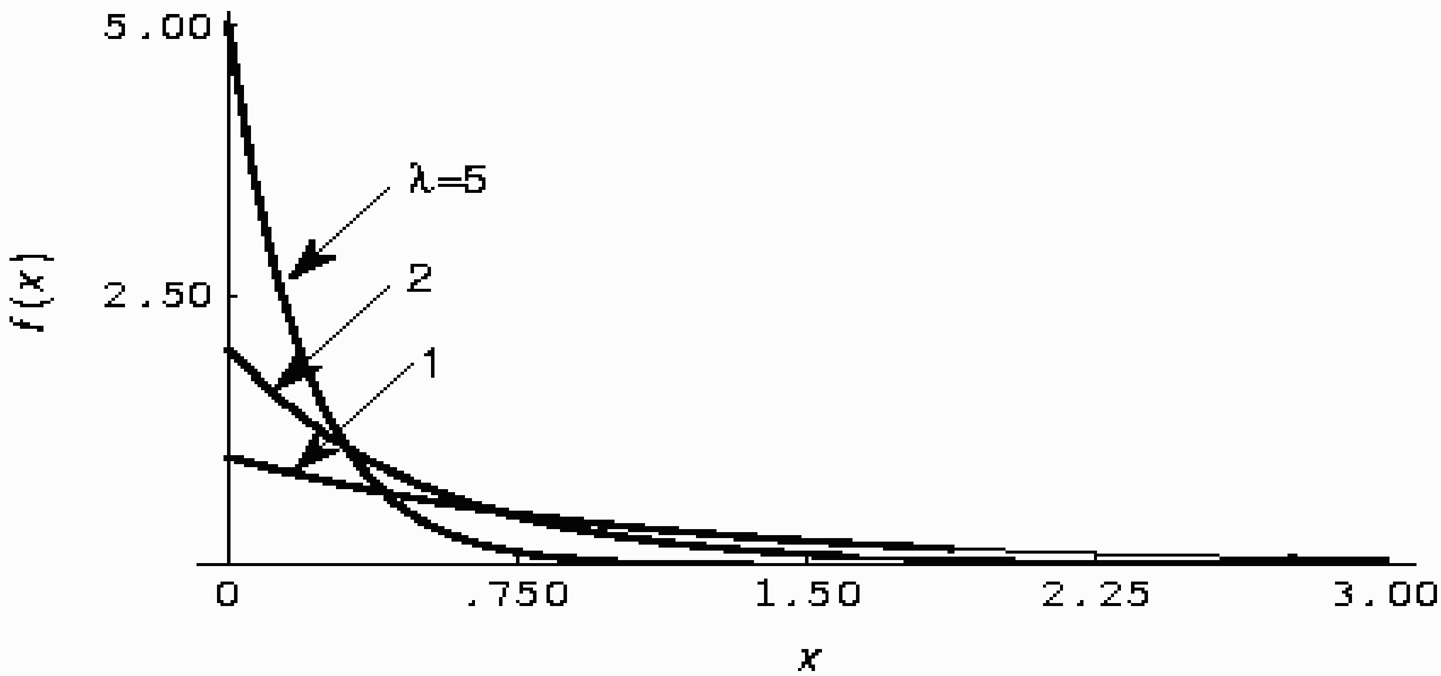
  and,

  $$P(a < X \le b) = \int_a^b f(x)dx = F(b) - F(a)$$
  $$= \mathrm{e}^{-\lambda a} - \mathrm{e}^{-\lambda b}$$

CDF of exponentially distributed random variable with $\lambda = 0.0001$



Exponential Density Function (pdf)



Memoryless property

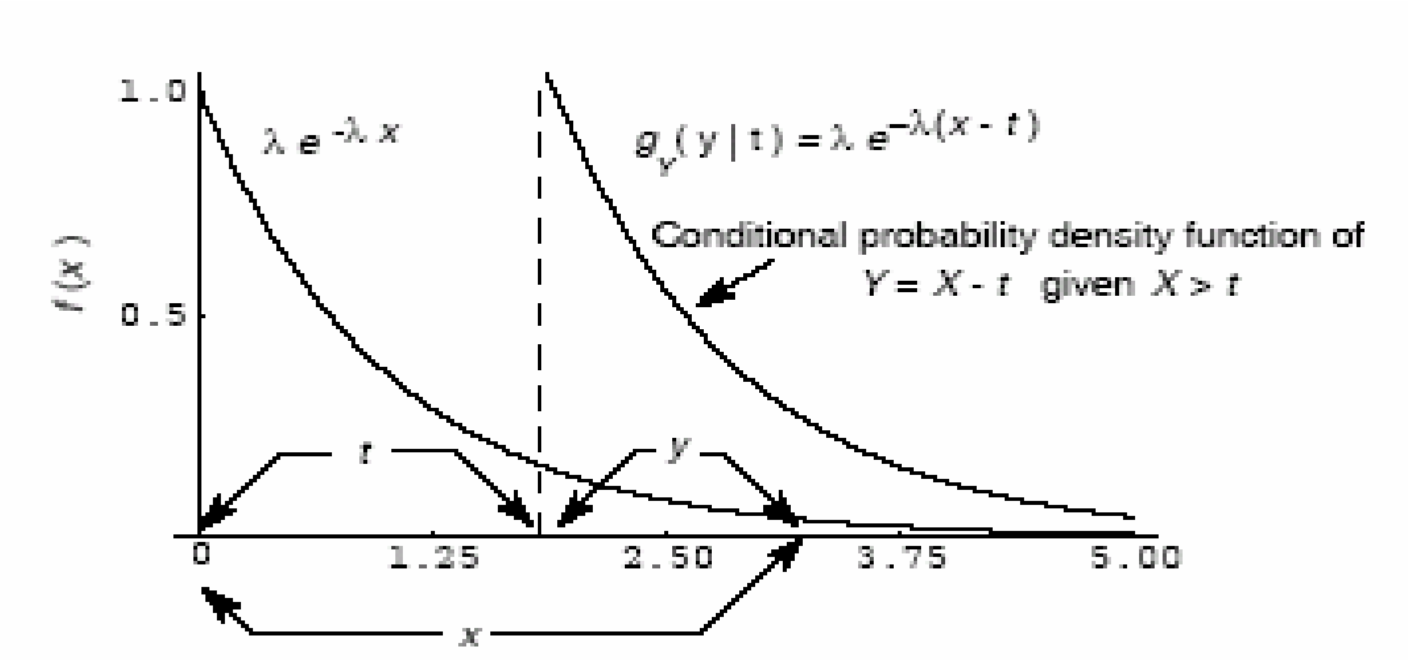
- Assume $X > t$, *i.e.*, We have observed that the component has not failed until time t.
- Let $Y = X - t$, the remaining (residual) lifetime

$$
\begin{aligned}
G_Y(y \mid t) &= P(Y \le y \mid X > t) \\
&= P(X \le y + t \mid X > t) \\
&= \frac{P(t < X \le y + t)}{P(X > t)} = 1 - e^{-\lambda y}
\end{aligned}
$$

Memoryless property

- Thus $G_Y(y/t)$ is independent of t and is identical to the original exponential distribution of X.
- The distribution of the remaining life does not depend on how long the component has been operating.
- Its eventual breakdown is the result of some suddenly appearing failure, not of gradual deterioration.

Memoryless property



Only Continuous Distribution with Memoryless property

X is a nonnegative R.V. with Memoryless property:

$$\frac{P(t < X \le y + t)}{P(X > t)} = P(X \le y) = P(0 < X \le y),$$

$$F_X(y+t) - F_X(t) = [1 - F_X(t)][F_X(y) - F_X(0)].$$

Since $F_X(0) = 0$,

$$\frac{F_X(y+t) - F_X(y)}{t} = \frac{F_X(t)[1 - F_X(y)]}{t}.$$

Taking the limit as t approaches zero,

$$F'_X(y) = F'_X(0)[1 - F_X(y)],$$

$$R'_X(y) = R'_X(0) R_X(y).$$

Only Continuous Distribution with Memoryless property

Solution to the differential equation is given by

$$R_X(y) = K e^{R'_X(0)y}$$

where K is the const. and $-R'_X(0) = F'_X(0) = f_X(0)$

since $R_X(0) = 1$ and denoting $f_X(0)$ by constant λ

$$R_X(y) = e^{-\lambda y}$$

$$F_X(y) = 1 - e^{-\lambda y}, \quad y > 0.$$

Therefore X must have the exponential distribution.

Example 3.2

- A discrete rv N_t : number of jobs arriving to a file server in the interval *(0, t]*
- N_t be *Poisson* distributed (parameter = λt)
- X : time to next arrival.

$$P(X > t) = P(N_t = 0)$$
$$= \frac{\mathrm{e}^{-\lambda t}(\lambda t)^0}{0!} = \mathrm{e}^{-\lambda t}$$

- Therefore,

$$F_X(t) = 1 - \mathrm{e}^{-\lambda t}$$

- X is exponentially distributed with parameter λ

Example 3.3

- Web server: time to next request is random
- Average rate of requests, $\lambda = 0.1$ reqs/sec.
- Number of request arrivals per sec is *Poisson distributed*
- Or inter-arrival times are *EXP(λ)*. Therefore,

$$P(X \geq 10) = \int_{10}^{\infty} 0.1\mathrm{e}^{-0.1t}dt = \lim_{t\to\infty}[\mathrm{e}^{-0.1t}] - (-\mathrm{e}^{-1})$$

$$= \mathrm{e}^{-1} = 0.368$$

Reliability as a Function of Time

- Reliability *R(t):* prob. that no failure occurs during the interval *(0,t)*. Let *X* be the lifetime of a component subject to failures.

$$R(t) = P(X > t) = 1 - F(t)$$

- *Let* N_0*=* total no. of components (fixed); $N_s(t)$*=* surviving ones; $N_f(t)$*= no.* failed by time *t.*

$$R(t) \approx \frac{N_s(t)}{N_0} = \frac{N_0 - N_f(t)}{N_0} = 1 - \frac{N_f(t)}{N_0}$$

$$R'(t) \approx -\frac{1}{N_0} N'_f(t) = -f_X(t)$$

Definitions (Contd.)

Equivalence:

- Reliability
- Complementary distribution function
- Survivor function
- $R(t) = 1 - F(t)$

Failure Rate or Hazard Rate

- Instantaneous failure rate: *h(t)*

  (#failures/time unit)

$$h(t) = \lim_{x \to 0} \frac{F(t+x) - F(t)}{xR(t)} = \lim_{x \to 0} \frac{R(t) - R(t+x)}{xR(t)} = \frac{f(t)}{R(t)}$$

- As a special case let the *rv X* be *EXP(λ)*. Then the failure rate is time or age independent:

$$h(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda \quad (\rightarrow \text{ CFR})$$

- This is the only continuous distribution with a constant failure rate (CFR)

Hazard Rate and the pdf

$$h(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1 - F(t)}$$

- $h(t)\,\Delta t$ = conditional prob. of system failing in $(t, t + \Delta t]$ given that it has survived until time t.
- $f(t)\,\Delta t$ = unconditional prob. of system failing in $(t, t + \Delta t]$.
- Analogous to difference between:
  - probability that someone will die between 90 and 91, given that he lives to 90
  - probability that someone will die between 90 and 91

Reliability from Failure Rate

- In the general case, reliability R(t) can be related to the hazard rate in the following way
- Using simple calculus the following applies to any rv,

$$\int_0^t h(x)dx = \int_0^t \frac{f(x)}{R(x)}dx = \int_0^t \frac{-R'(x)}{R(x)}dx = -\int_{R(0)}^{R(t)} \frac{dR}{R} = -\ln R(t)$$

$$\text{or, } R(t) = e^{-\int_0^t h(x)dx}$$

Failure-Time Distributions

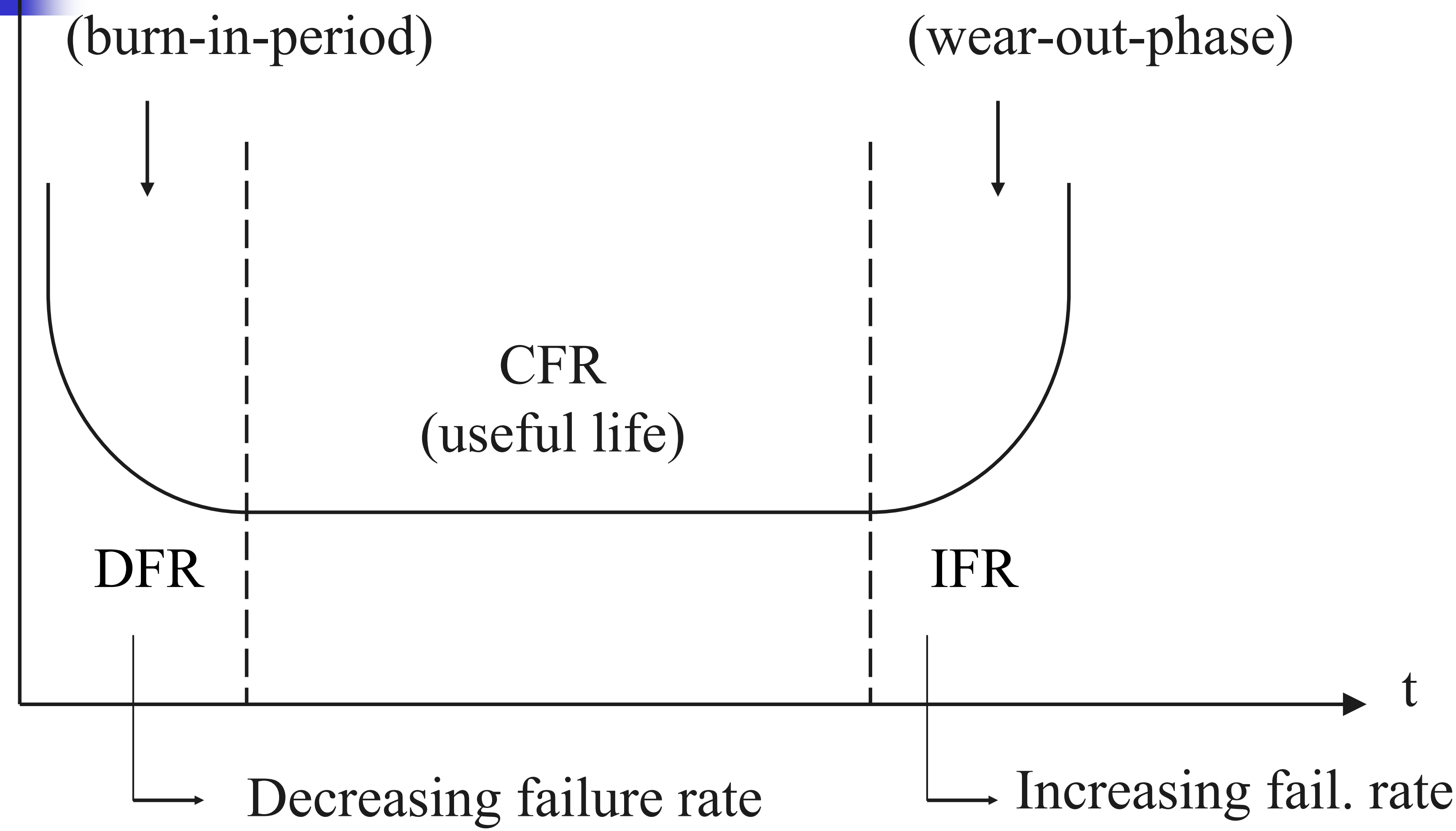
- Relationships

	$f(t)$	$F(t)$	$R(t)$	$h(t)$
$f(t)$	1	$F'(t)$	$-R'(t)$	$h(t)e^{-\int_0^t h(u)du}$
$F(t)$	$\int_0^t f(u)du$	1	$1-R(t)$	$1-e^{-\int_0^t h(u)du}$
$R(t)$	$\int_t^\infty f(u)du$	$1-F(t)$	1	$e^{-\int_0^t h(u)du}$
$h(t)$	$\frac{f(t)}{\int_t^\infty f(u)du}$	$\frac{F'(t)}{(1-F(t))}$	$-\frac{d}{dt}\log_e R(t)$	1

Bathtub curve

h(t)

DFR phase: Initial design, constant bug fixes
CFR phase: Normal operational phase
IFR phase: Aging behavior



Weibull Distribution

- Frequently used to model fatigue failure, ball bearing failure etc. (very long tails)

$$F(t) = 1 - e^{-\lambda t^{\alpha}}$$

$$h(t) = \lambda \alpha t^{\alpha - 1}$$

- Reliability: $R(t) = e^{-\lambda t^{\alpha}}$ $\qquad t \geq 0$
- Weibull distribution is capable of modeling DFR ($\alpha < 1$), CFR ($\alpha = 1$) and IFR ($\alpha > 1$) behavior.
- α is called the shape parameter and λ is the scale parameter.

Weibull Distribution (alternate form)

- Some texts use a slightly different form for Weibull:

$$F(t) = 1 - e^{-(\lambda t)^{\alpha}}$$

$$h(t) = \lambda^{\alpha} \alpha t^{\alpha - 1}$$

- Reliability: $R(t) = e^{-(\lambda t)^{\alpha}}$ $\qquad t \geq 0$
- In this text we will use the definition on the previous slide

Example 3.4

- Life time X: Weibull distributed with $\alpha = 2$
- Observation: 15% components last 90 hrs, but fail before 100 hrs., *i.e.,*

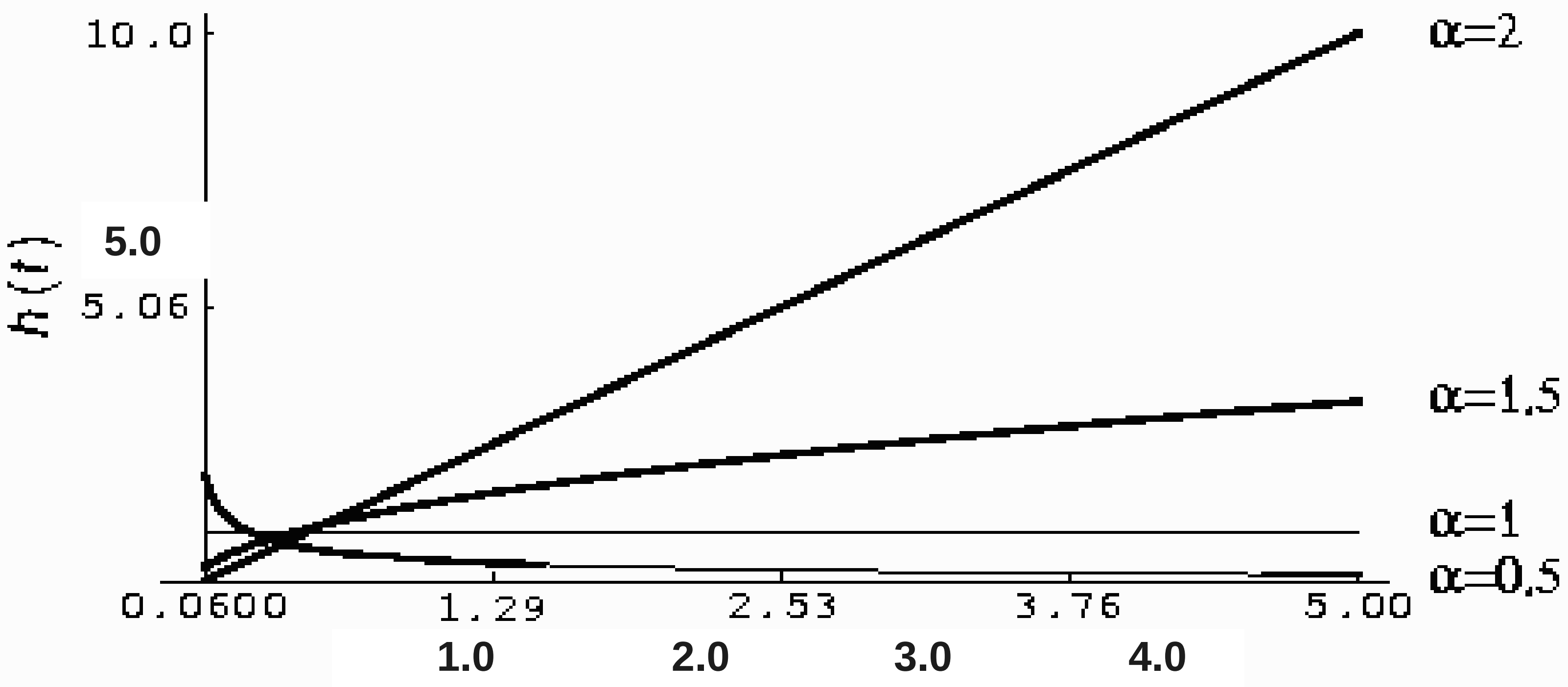
$$P(X < 100|X > 90) = 0.15$$

- Find scale parameter λ for this Weibull distribution:

$$\begin{aligned} P(X < 100|X > 90) &= \frac{P(90 < X < 100)}{P(X > 90)} \\ &= \frac{F_X(100) - F_X(90)}{1 - F_X(90)} \\ &= \frac{e^{-\lambda(90)^2} - e^{-\lambda(100)^2}}{e^{-\lambda(90)^2}} = 0.15 \end{aligned}$$

$$\text{solving above eq., } \lambda = -\frac{\ln(0.85)}{1900} = \frac{0.1625}{1900} = 0.00008554.$$

Failure rate of the Weibull distribution with various values of α and $\lambda = 1$



Three parameter Weibull Distribution

- Sometimes a more complex version of Weibull is used so that the image of the random variable is in the interval (θ,∞):

$$F(t) = 1 - \mathrm{e}^{-\lambda(t-\theta)^{\alpha}},\ t \geq \theta\ (\theta\text{: location parameter})$$

Infant Mortality Effects in System Modeling

- Bathtub curves
  - Early-life period
  - Steady-state period
  - Wear out period
- Failure rate models

Early-life Period

- Also called infant mortality phase or reliability growth phase or decreasing failure rate (DFR phase).
- Caused by undetected hardware/software defects that are being fixed resulting in reliability growth.
- Can cause significant prediction errors if steady-state failure rates are used.
- Availability models can be constructed and solved to include this effect.
- DFR Weibull Model can be used.

Steady-state Period

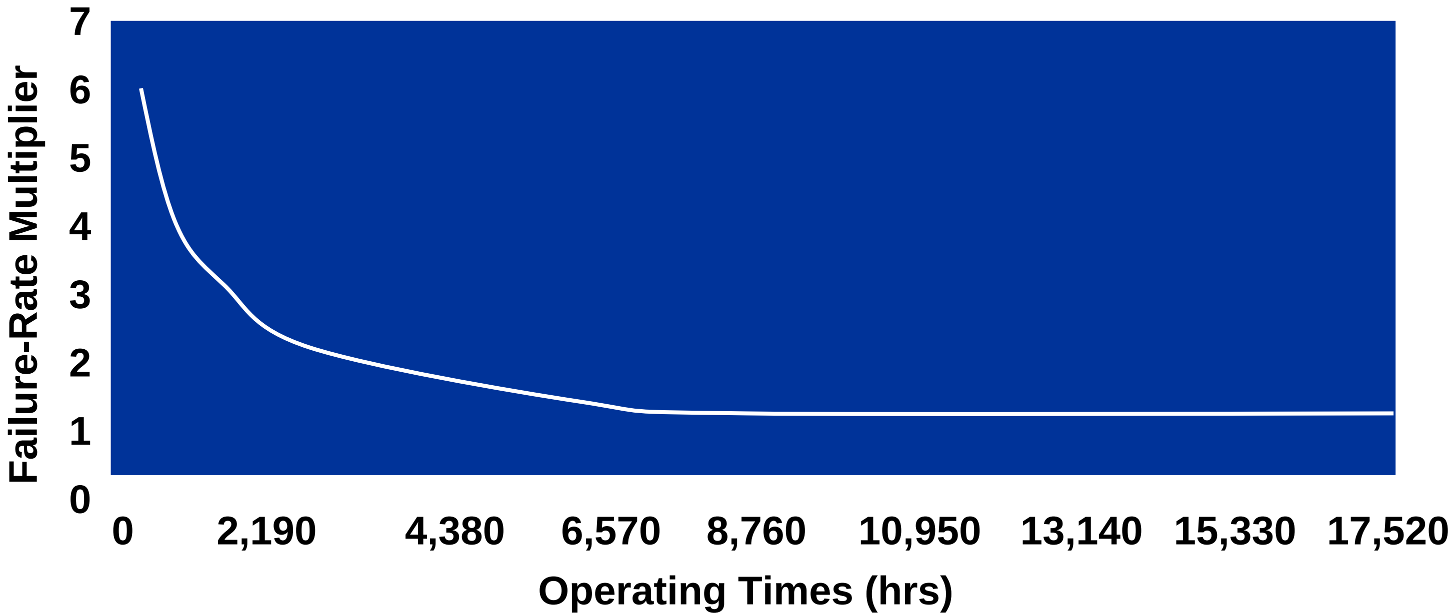
- Failure rate much lower than in early-life period.
- Either constant (CFR) (age independent) or slowly varying failure rate.
- Failures caused by environmental shocks.
- Arrival process of environmental shocks can be assumed to be a Poisson process.
- Hence time between two shocks has exponential distribution.

Wear out Period

- Failure rate increases rapidly with age (IFR phase).
- Properly qualified electronic hardware do not exhibit wear out failure during its intended service life (as per Motorola).
- Applicable for mechanical and other systems.
- Again (IFR) Weibull Failure Model can be used for capturing such behavior.

Failure Rate Models

- We use a truncated Weibull Model



- Infant mortality phase modeled by DFR Weibull and the steady-state phase by the exponential.

Failure Rate Models (cont.)

- This model has the form:

$$h_W(t) = C_1 t^{-\alpha} \qquad 1 \le t \le 8{,}760$$
$$= h_{SS} \qquad t > 8{,}760$$

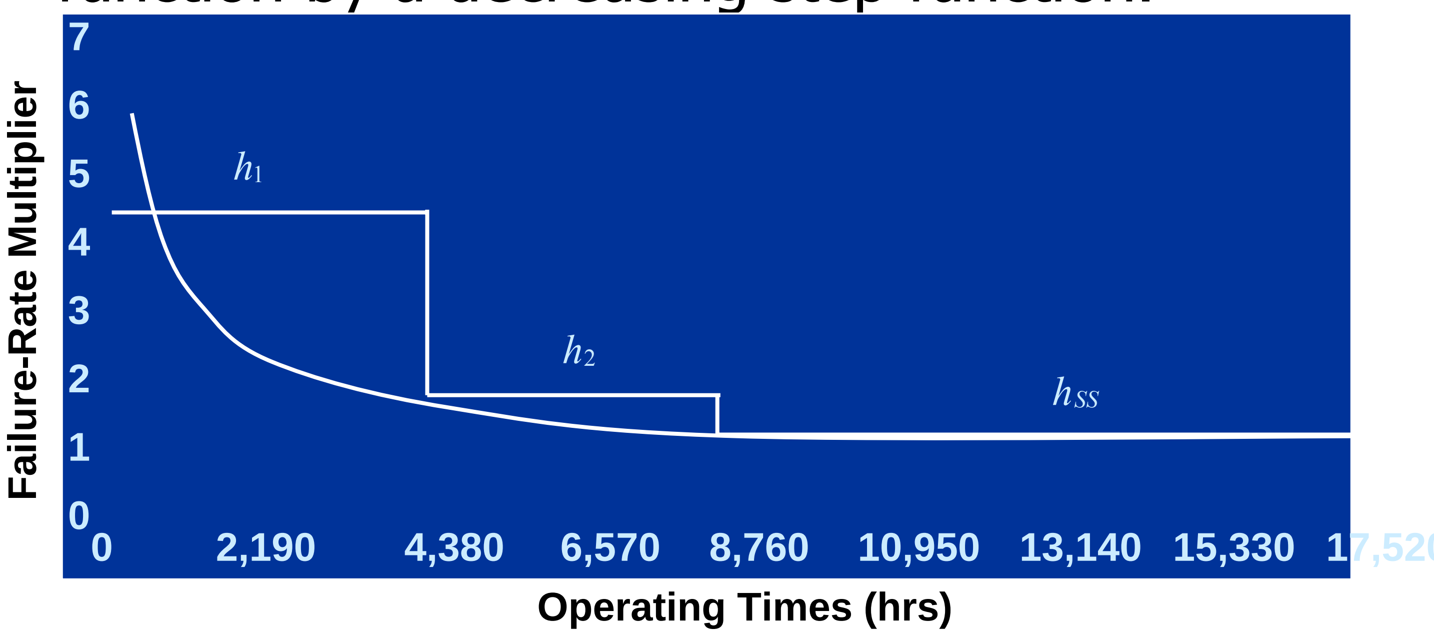
- where:

$C_1 = h_W(1), h_{SS} =$ steady-state failure rate

- α is the Weibull shape parameter
- Failure rate multiplier = $h_W(t)/h_{SS}$

Failure Rate Models (cont.)

- There are several ways to incorporate time dependent failure rates in availability models.
- The easiest way is to approximate a continuous function by a decreasing step function.



Failure Rate Models (contd.)

- Here the discrete failure-rate model is defined by:

$$h_W(t) = h_1 \qquad 0 \le t < 4{,}380$$
$$\quad\;\; = h_2 \qquad 4{,}380 \le t < 8{,}760$$
$$\quad\;\; = h_{ss} \qquad t \ge 8{,}760$$

- The approximation can be improved by taking smaller time steps.

HypoExponential (HYPO)

- HypoExp: multiple Exp stages in series.
- 2-stage HypoExp denoted as *HYPO*(λ_1, λ_2). The density, distribution and hazard rate function are:

$$f(t) = \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}),\ t > 0$$

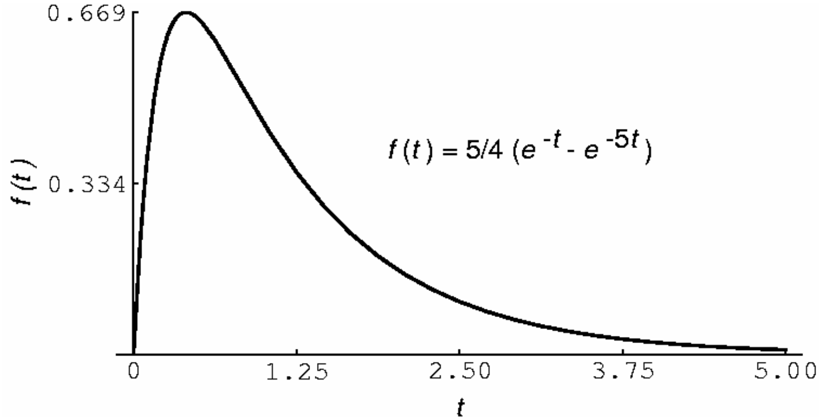
$$F(t) = 1 - \frac{\lambda_2}{\lambda_2-\lambda_1}e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2-\lambda_1}e^{-\lambda_2 t},\ t \geq 0$$

$$h(t) = \frac{\lambda_1\lambda_2(e^{-\lambda_1 t}-e^{-\lambda_2 t})}{\lambda_2 e^{-\lambda_1 t}-\lambda_1 e^{-\lambda_2 t}}.$$

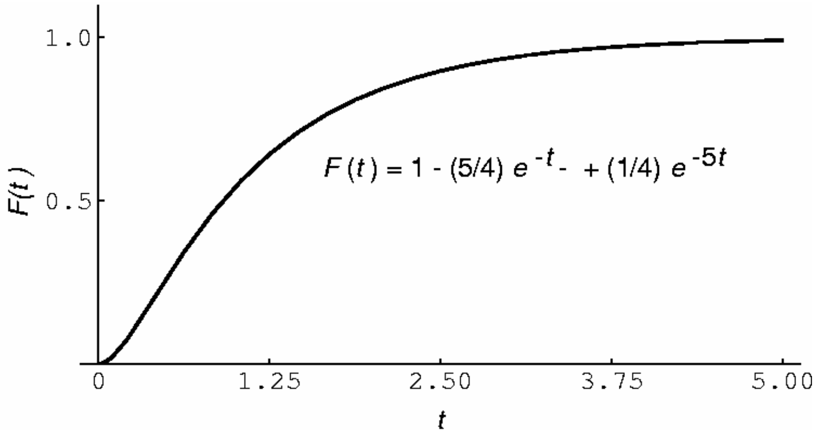
- HypoExp is an IFR as its *h(t): 0 → min*{λ_1, λ_2}
- Disk service time may be modeled as a 3-stage Hypoexponential as the overall time is the sum of the seek, the latency and the transfer time.

HypoExponential pdf and CDF

- Hypo(1,5)



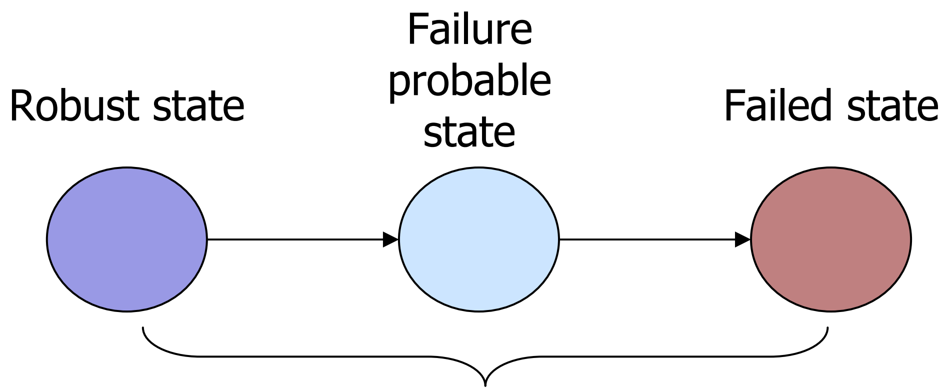
Hypo(1,5) pdf



Hypo(1,5) CDF

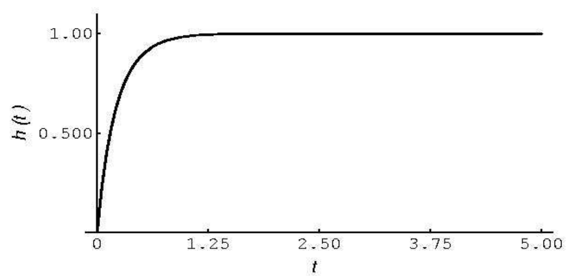
HypoExponential used in software rejuvenation models

- **Preventive maintenance is useful only if failure rate is increasing**
- **A simple and useful model of increasing failure rate:**



Time to failure: Hypo-exponential distribution

Increasing failure rate → aging



Erlang Distribution

- Special case of HYPO: All stages have same rate.

$$f(t) = \frac{\lambda^r t^{r-1} e^{-\lambda t}}{(r-1)!}, \; t > 0, \; \lambda > 0, \; r = 1, 2, \ldots$$

$$F(t) = 1 - \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \; t \geq 0, \; \lambda > 0, \; r = 1, 2, \ldots$$

$$h(t) \;=\; \frac{\lambda^r t^{r-1}}{(r-1)! \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!}}, \; t > 0, \; \lambda > 0, \; r = 1, 2, \ldots$$

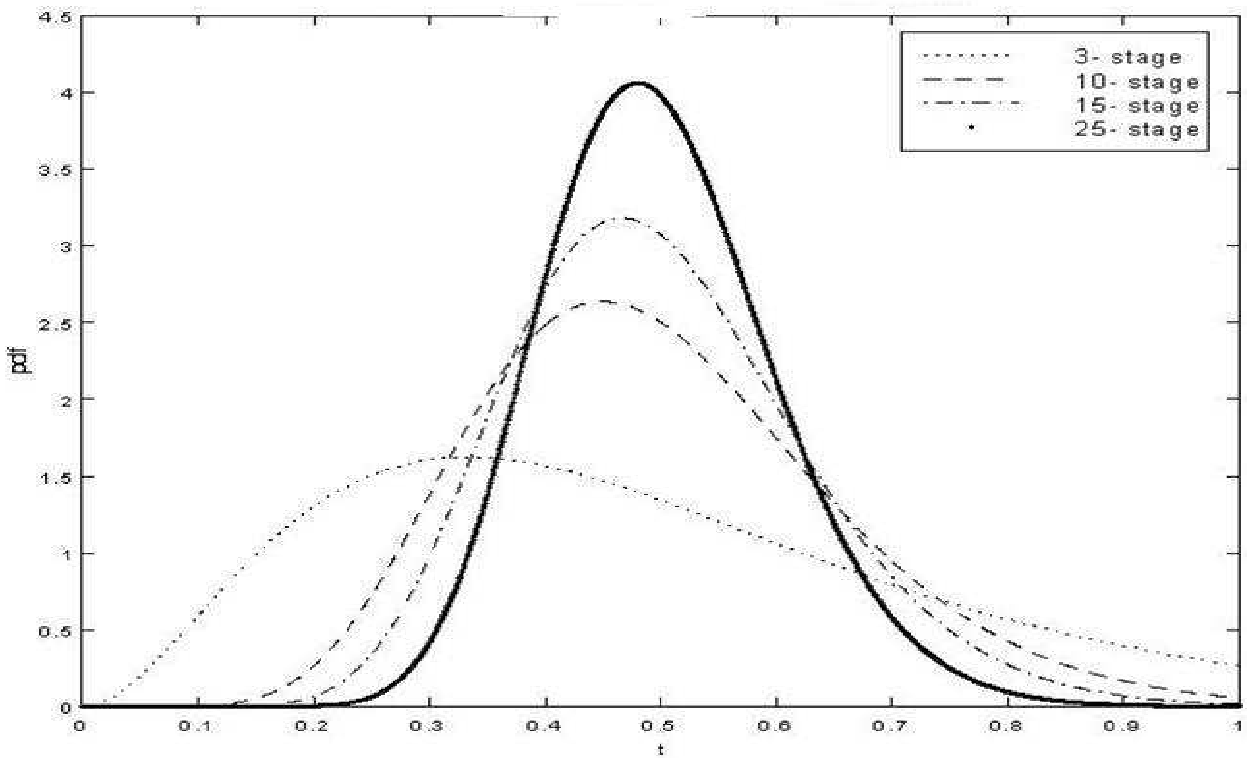
- $[X > t] = [N_t < r]$ (N_t : no. of stresses applied in $(0,t]$ and N_t is Poisson (parameter: λt). This interpretation gives,

$$R(t) = e^{-\lambda t} \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!}$$

Erlang Distribution

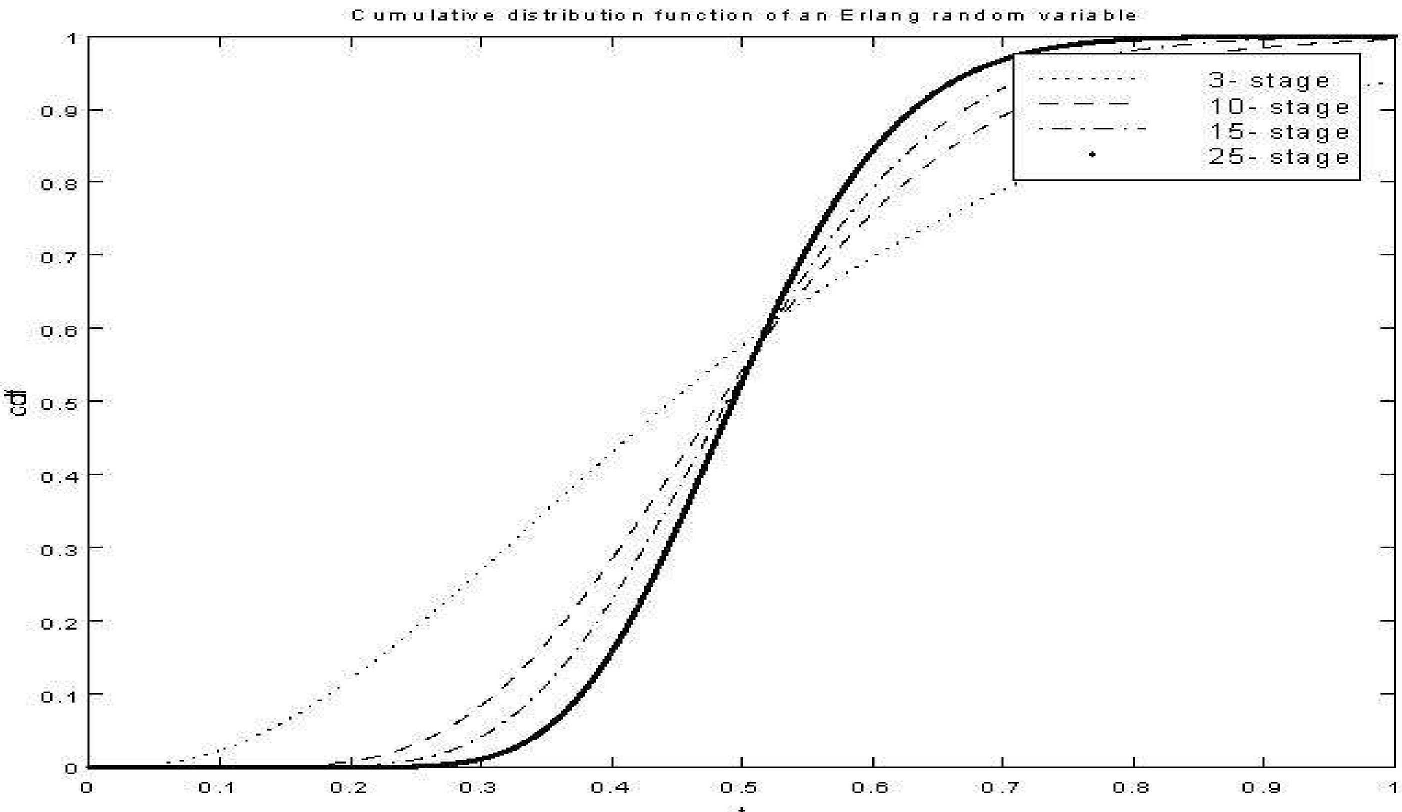
- If we set the parameter $r=1$, we get the exponential distribution
- Erlang distribution can be used to approximate the deterministic variable, since if the mean is kept same but number of stages are increased, the pdf approaches the delta (impulse) function in the limit.

Erlang density function



If we vary r keeping r/λ constant, pdf of r-stage Erlang approaches an impulse function at r/λ.

Erlang Cumulative Distribution Function



And the cdf approaches a step function at r/λ. In other words r-stage Erlang can approximate a deterministic variable.

Gamma Random Variable

- A basic distribution of statistics for non-negative variables (see Section 3.9 and Chapter 10)
- Gives distribution of time required for exactly r independent events to occur, assuming events take place at a constant rate (p. 131 of text). Used frequently in queuing theory, reliability theory
- Example: Distribution of time between re-calibrations of instrument that needs re-calibration after r uses; time between inventory restocking, time to failure for a system with cold standby redundancy (Ex. 3.25)
- Erlang, exponential, and chi- square distributions are special cases.

Gamma Random Variable

- Gamma density function is,

$$f(t) = \frac{\lambda^{\alpha} t^{\alpha-1} \mathrm{e}^{-\lambda t}}{\Gamma(\alpha)}, \left(\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} \mathrm{e}^{-x} dx\right), \;\; \alpha > 0, \;\; t > 0$$

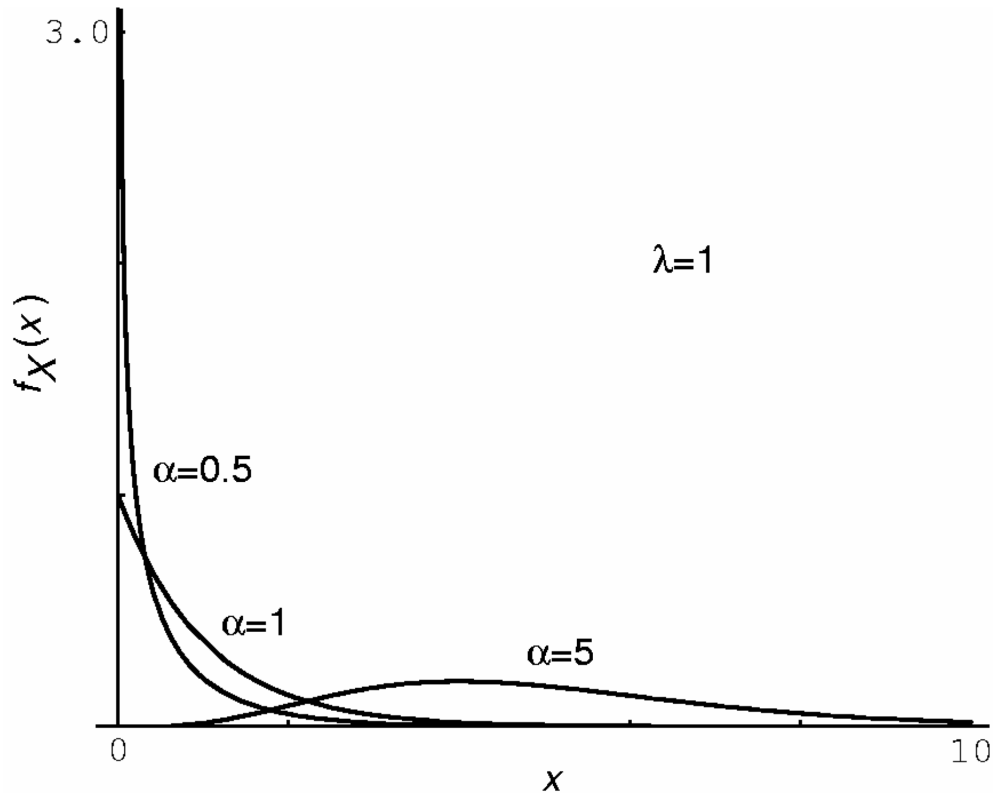
- $\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$; $\Gamma(1/2)=\sqrt{\pi}$
- Because $\Gamma(1)=1$, it follows that $\Gamma(r)=(r-1)\,\Gamma(r-1)=\ldots=(r-1)!$ So gamma with an integer valued shape parameter is the Erlang distribution
- Gamma with shape parameter $\alpha = ½$ and scale parameter $\lambda = n/2$ is known as the **chi-square** random variable with n degrees of freedom.

Gamma distribution: failure rate

- Gamma distribution can capture all three types failure rate behavior, viz. DFR, CFR or IFR depending on the value of the shape paramter α

$\alpha = 1$: CFR
$\alpha < 1$: DFR
$\alpha > 1$: IFR

Gamma density function



HyperExponential Distribution (HyperExp)

- Hypo or Erlang have sequential Exp( ) stages.
- When there are alternate Exp( ) stages it becomes Hyperexponential.

$$f(t) = \sum_{i=1}^{k} \alpha_i \lambda_i \mathrm{e}^{-\lambda_i t}, \ t > 0, \ \lambda_i > 0, \ \alpha_i > 0, \ \sum_{i=1}^{k} \alpha_i = 1$$

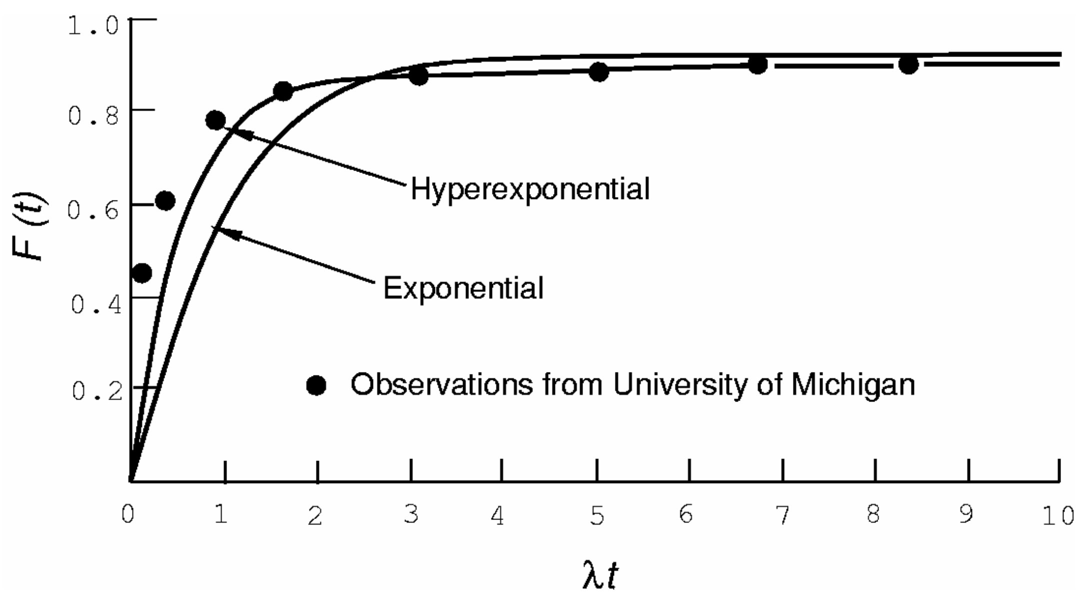
$$F(t) = \sum_{i} \alpha_i (1 - \mathrm{e}^{-\lambda_i t}), \ t \geq 0$$

$$h(t) = \frac{\sum_i \alpha_i \lambda_i \mathrm{e}^{-\lambda_i t}}{\sum_i \alpha_i \mathrm{e}^{-\lambda_i t}}, \ t \geq 0$$

- CPU service time may be modeled by HyperExp.
- In workload based software rejuvenation model we found the sojourn times in many workload states have this kind of distribution.

Hyper Exponential Vs. Exponential CDF

- Distribution of measured CPU service time may be better described by the HyperExp( ) as compared to the EXP( ) distribution.



Log-logistic Distribution

- Log-logistic can model more complex failure rate behavior than simple CFR, IFR, DFR cases.

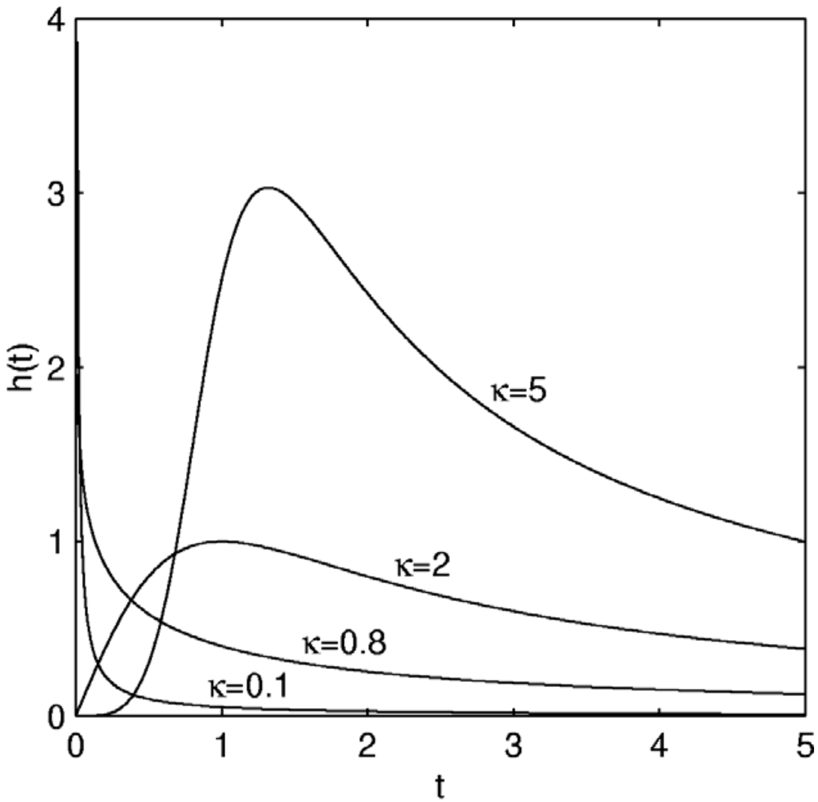
  $f(t) = \frac{\lambda\kappa(\lambda t)^{\kappa-1}}{[1+(\lambda t)^{\kappa}]^2},\ t \geq 0$ (λ: scale, κ:shape parameter)

  $F(t) = 1 - \frac{1}{(\lambda t)^{\kappa}}$

  $h(t) = \frac{\lambda\kappa(\lambda t)^{\kappa-1}}{1+(\lambda t)^{\kappa}}$

- For, $\kappa > 1$, the failure rate first increases with t; after momentarily leveling off, it decreases with time. This is known as the **inverse bath tub shape curve.**
- Useful in modeling software reliability growth .

Log-logistic failure rate



Gaussian (Normal) Random Variable

- A basic distribution of statistics. Many applications arise from **central limit theorem** (average of values of n observations approaches normal distribution, irrespective of form of original distribution under quite general conditions).
- Consequently, appropriate model for many, but not all, physical phenomena.
- Example: Distribution of physical measurements on living organisms, intelligence test scores, product dimensions, average temperatures, and so on.
- Many methods of statistical analysis presume normal distribution.
- In a normal distribution, about 68% of the values are within one standard deviation of the mean and about 95% of the values are within two standard deviations of the mean.

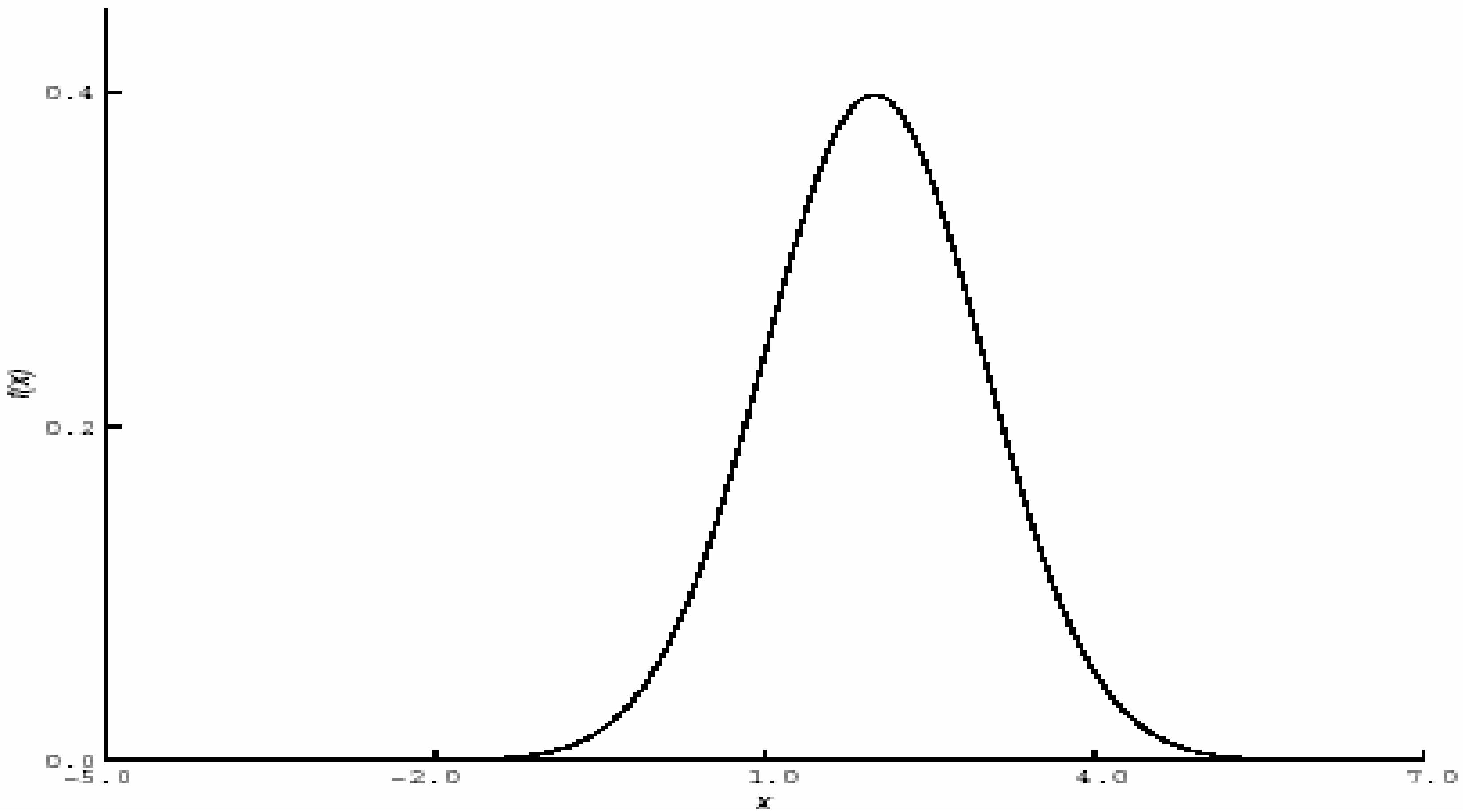
Gaussian (Normal) Random Variable

- Bell shaped and symmetrical pdf – intuitively pleasing!
- Central Limit Theorem: sum *of a large number of mutually independent rv's (having arbitrary distributions) starts following Normal distribution as n $\rightarrow$ ∞*

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \; -\infty < x < \infty$$

- μ: mean, σ: std. deviation, σ^2: variance *($N(\mu, \sigma^2)$)*
- μ and σ completely describe the rv. This is significant in statistical estimation/signal processing/communication theory etc.
- Mean, median and mode are all equal; infinite range

Normal Density with parameter μ=2 and σ=1

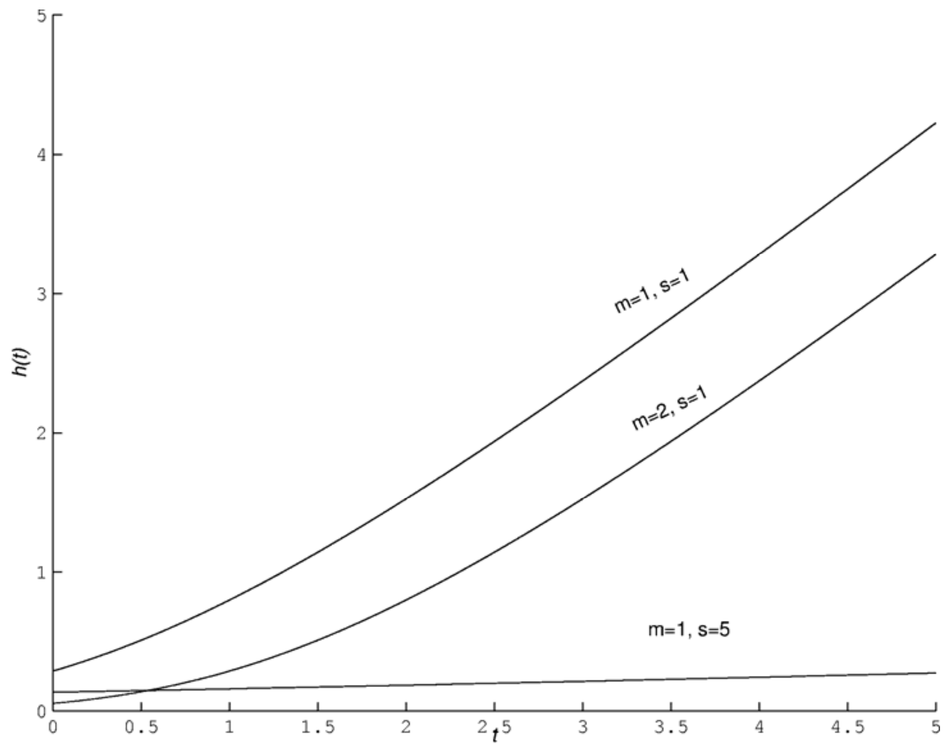


Normal Distribution (contd.)

- Failure rate $h(t)$ follows IFR behavior.
  - Hence, normal distribution is suitable for modeling long-term wear or aging related failure phenomena.
- See page 138-140 for Examples.

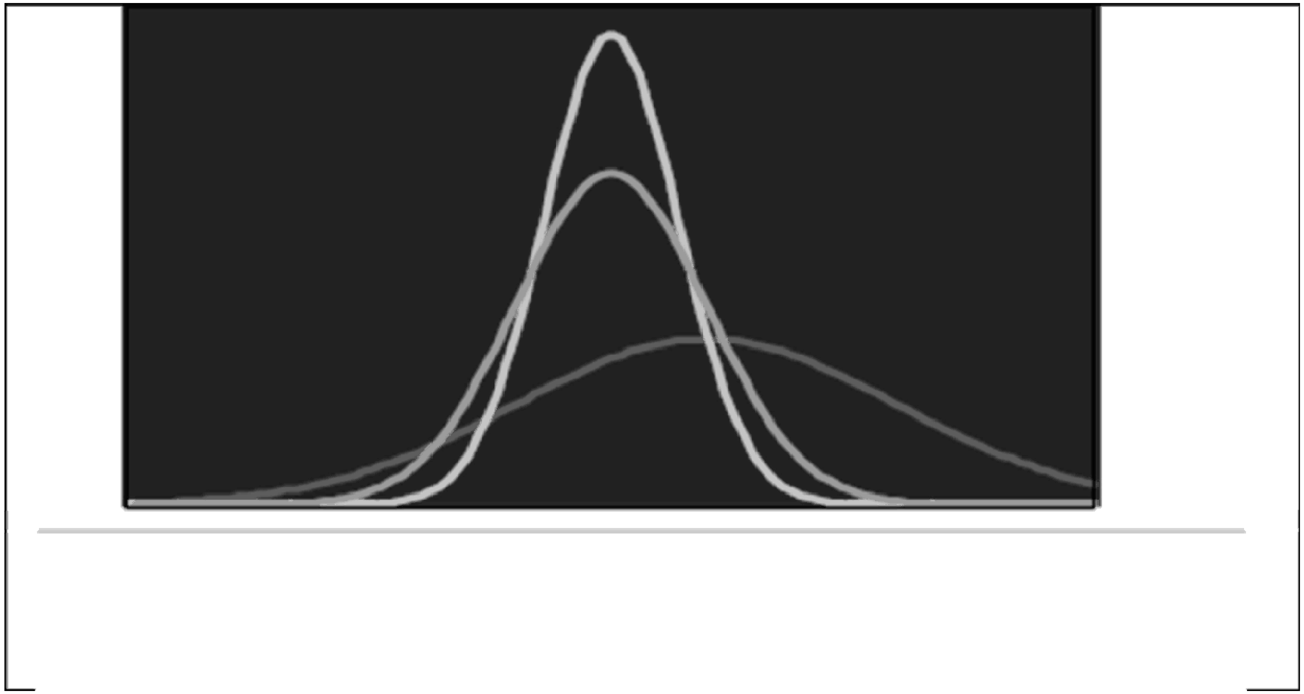
Failure rate for Normal distribution

- h(t) for normal distribution is IFR



Infinitely many normal pdfs

- By changing the two parameters, we can get infinitely many normal densities



Normal Distribution (contd.)

- No closed form for the CDF; how do we determine $P(a < X < b)$?
- Answer*:* use tables after a transformation to standard normal
- $N(0,1)$ is called *standard normal distribution.*
- $X \sim N(\mu, \sigma^2))$ then $Z=(X-\mu)/\sigma$ is $N(0,1)$
- $N(0,1)$ is symmetric i.e.
  - $f(x)=f(-x)$
  - $F(-z) = 1-F(z)$.

Example 3.5

- X: amplitude of an analog signal at a detector
- X has a normal distribution $N(200,256)$
- Find $P(X > 240)$

$$\begin{aligned}
P(X > 240) &= 1 - P(X \leq 240) \\
&= 1 - F_Z\left(\frac{240 - 200}{16}\right), \qquad \text{using equation (3.40)} \\
&= 1 - F_Z(2.5) \\
&\simeq 0.00621.
\end{aligned}$$

Example 3.5 (contd.)

- Find *P(X>240 | X>210)*

$$
\begin{aligned}
P(X > 240 | X > 210) &= \frac{P(X > 240)}{P(X > 210)} \\
&= \frac{1 - F_Z\left(\frac{240-200}{16}\right)}{1 - F_Z\left(\frac{210-200}{16}\right)} \\
&= \frac{0.00621}{0.26599} \\
&\simeq 0.02335.
\end{aligned}
$$

Example 3.6

- *X:* Wearout phase lifetime of a subsystem is normal $N(10^5, 10^6)$ (in hour units)
- Find $R_{9,000}(500)$ and $R_{11,000}(500)$

$$R_{9000}(500) = \frac{R(9500)}{R(9000)} = \frac{\int_{9500}^{\infty} f(t)dt}{\int_{9000}^{\infty} f(t)dt}.$$

since, $\mu - 0.5\sigma = 9500$ and $\mu - \sigma = 9000$,

$$\begin{aligned} R_{9000}(500) &= \frac{\int_{\mu-0.5\sigma}^{\infty} f(t)dt}{\int_{\mu-\sigma}^{\infty} f(t)dt} = \frac{1 - F_X(\mu - 0.5\sigma)}{1 - F_X(\mu - \sigma)} \\ &= \frac{1 - F_Z(-0.5)}{1 - F_Z(-1)} = \frac{F_Z(0.5)}{F_Z(1)} = \frac{0.6915}{0.8413} = 0.8219. \end{aligned}$$

Example 3.6 (contd.)

- Similarly,

since, $\mu + 1.5\sigma = 11,500$ and $\mu + \sigma = 11,000$,

$$
\begin{aligned}
R_{11,000}(500) &= \frac{1 - F_X(\mu + 1.5\sigma)}{1 - F_X(\mu + \sigma)} \\
&= \frac{0.0668}{0.1587} \\
&= 0.4209.
\end{aligned}
$$

Uniform Random Variable

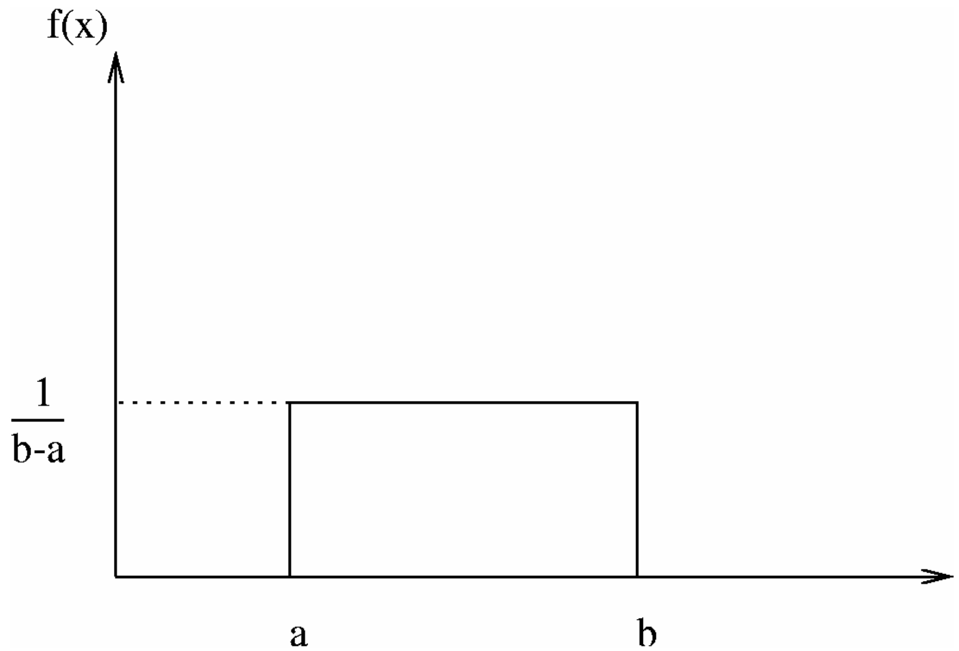
- *Unif(a,b)* → pdf constant over the interval (a,b) and CDF is the ramp function

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

- All (pseudo) random number generators generate random deviates of *Unif(0,1)* distribution; that is, if a large number of random variables are generated and their empirical distribution function is plotted, it will approach this distribution in the limit.

Uniform density function

- Uniform *pdf* – Unif(a, b)

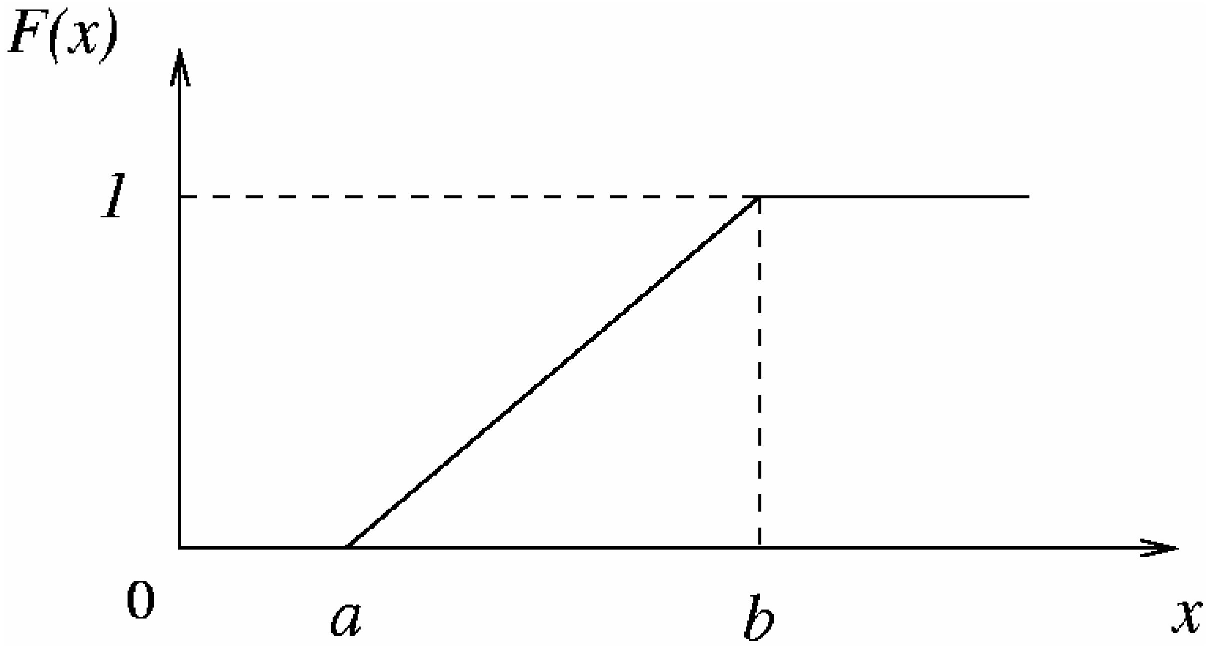


Uniform distribution

- The distribution function is given by:

$$F(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x < b \\ 1, & b \le x \end{cases}$$

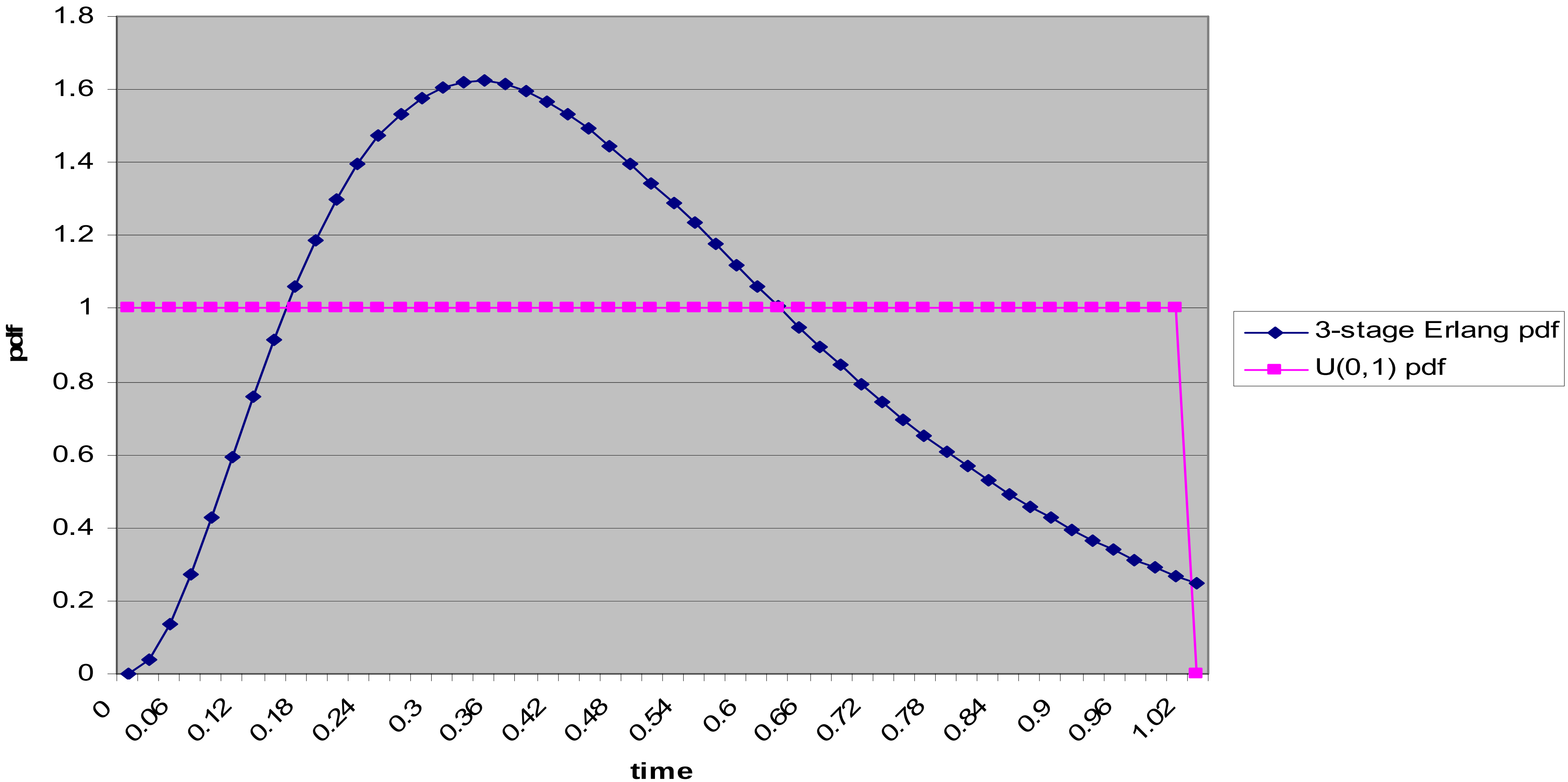
Uniform CDF



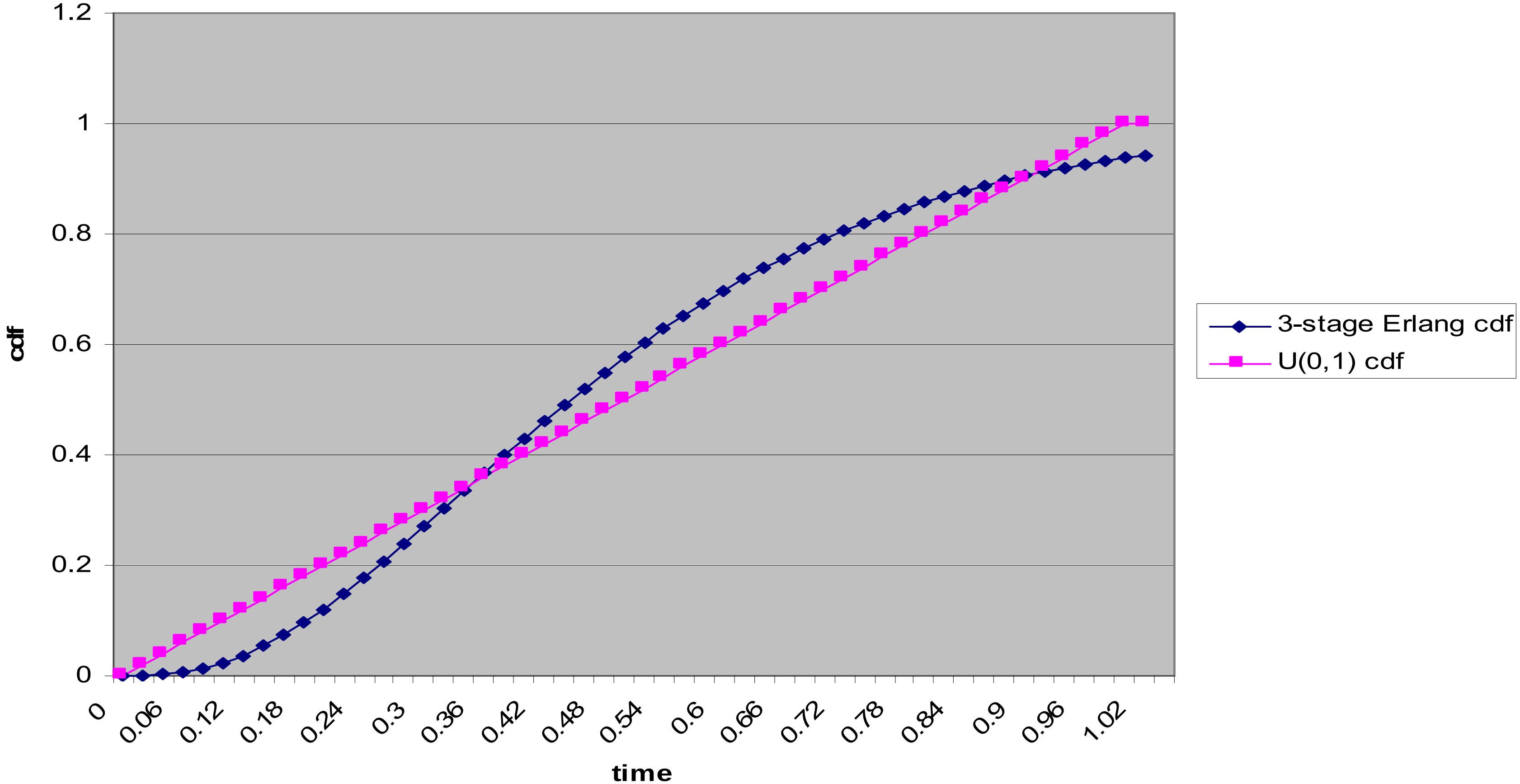
Erlang to approximate Uniform

- Uniform random variable is sometimes approximated by a Erlang random variable
- We will see an example of this in Chapter 8
- In the next two slides, pdfs and CDFs of Unif(0,1) and 3-stage Erlang with parameter $\lambda=6$ are compared

Comparison of probability density functions (pdf)



Comparison of cumulative distribution functions (cdf)



Pareto Distribution

- Also known as the power law or long-tailed distribution also, heavy-tailed distribution.
- Found to be useful in modeling of
  - CPU time consumed by a request.
  - Web file size on internet servers.
  - Number of data bytes in FTP bursts.
  - Thinking time of a Web browser.

Pareto Distribution (Contd.)

- The density is given by

$$f(x) = \alpha k^{\alpha} x^{-\alpha-1} \quad , x \geq k, \alpha, k > 0$$

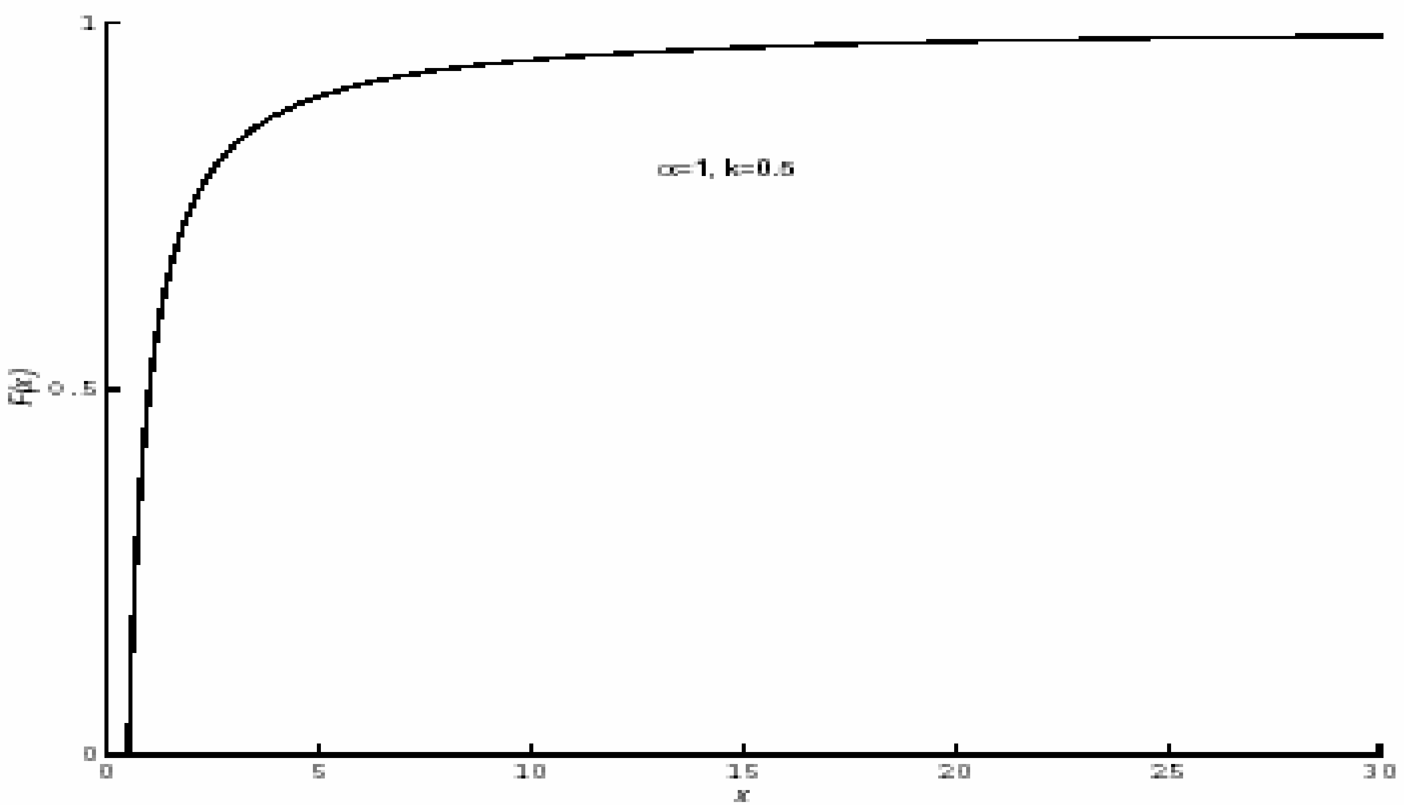
- The Distribution is given by

$$F(x) = \begin{cases} 1 - (\frac{k}{x})^{\alpha}, & x \geq k \\ 0, & x < k \end{cases}$$

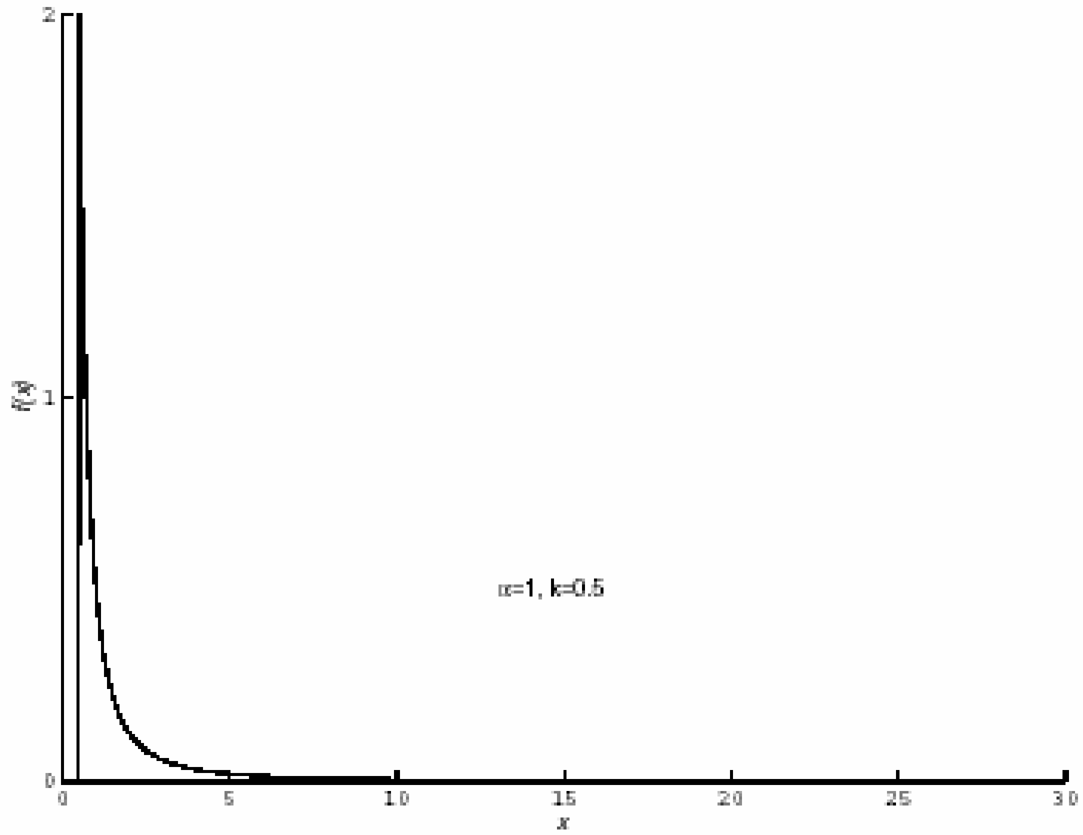
- And the failure rate is given by

$$h(x) = \begin{cases} \frac{\alpha}{x}, & x \geq k \ , \\ 0, & x < k \ . \end{cases}$$

The CDF of Pareto Distribution



The pdf of Pareto Distribution



Log-normal

- Permits representation of random variable whose logarithm follows normal distribution. Model for a process arising from many small multiplicative errors. Appropriate when the value of an observed variable is a random proportion of the previously observed value.
- In the case where the data are log-normally distributed, the geometric mean acts as a better data descriptor than the mean.The more closely the data follow a lognormal distribution, the closer the geometric mean is to the median
- Example: Repair time distribution; life distribution of some transistor types.
- pdf is given by:

$$f_Y(y) = \frac{1}{\sigma y \sqrt{2\pi}} \exp\left[-\frac{(\ln y - \mu)^2}{2\sigma^2}\right], \qquad y > 0.$$

Defective Distribution

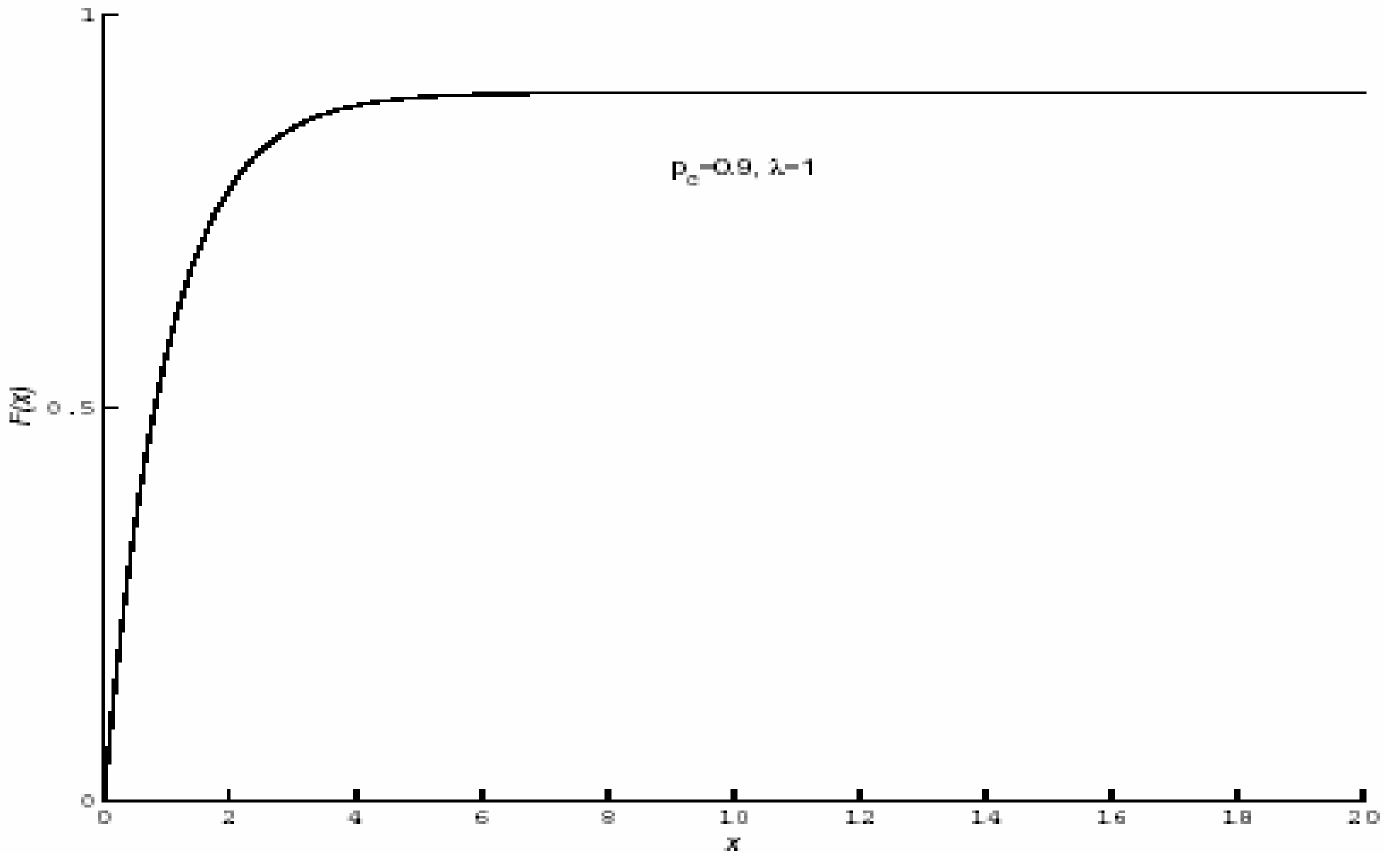
- Recall that $\lim_{x \to \infty} F_X(x) < 1 \iff$ Defective distribution
- Example:

$$If \;\; p_c < 1, then, F_X(x) = p_c(1 - e^{-\lambda x}) \; is \; a$$

$$\text{defective exponential distribution.}$$

- This defect (also known as the mass at infinity) could represent the probability that the program will not terminate (*1-p_c*). Continuous part can model completion time of program; we will see many examples in later chapters.
- There can also be a mass at origin.

The CDF of a Defective random variable



Functions of Random Variables

- Often, rv's need to be transformed/operated upon.
  - $Y = \Phi(X)$: so, what is the distribution or the density of Y?
  - Example: $Y = X^2$
  - Example: $Y = -\lambda^{-1} \ln(1-X)$

Example 3.8

- Distribution for Y = Φ(X) = X²

$$
\begin{aligned}
F_Y(y) &= 0, \text{ for } y \le 0 \\
\text{For } y > 0, & \\
F_Y(y) &= P(Y \le y) \\
&= P(X^2 \le y) \\
&= P(-\sqrt{y} \le X \le \sqrt{y}) \\
&= F_X(\sqrt{y}) - F_X(-\sqrt{y}),
\end{aligned}
$$

The *pdf* can be obtained by differentiation,

$$
f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right], & y > 0, \\ 0, & \text{otherwise.} \end{cases}
$$

Example 3.9

- In Example 3.8, assume *X* to be N(0,1):

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad -\infty < x < \infty.$$

- Using result from Example 3.8:

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} \left(\frac{1}{\sqrt{2\pi}} e^{-y/2} + \frac{1}{\sqrt{2\pi}} e^{-y/2} \right), & y > 0, \\ 0, & y \le 0, \end{cases}$$

or,

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-y/2}, & y > 0, \\ 0, & y \le 0. \end{cases}$$

$i.e.,$

Y has a $gamma$ distribution with $\alpha = 1/2$ and $\lambda = 1/2$

- Which is also known as chi-square distribution with 1 degree of freedom

Example 3.10

- Let *X* be uniformly distributed, *Unif(0,1)*
- Then, if $Y = -\lambda^{-1} \ln(1-X)$ is $EXP(\lambda)$.

for $y \le 0$, $F_Y(y) = 0$

for $y > 0$,

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P[-\lambda^{-1}\ln(1-X) \le y] \\ &= P[\ln(1-X) \ge -\lambda y] \\ &= P[(1-X) \ge e^{-\lambda y}] \quad \text{(since } e^x \text{ is an increasing function of } x,) \\ &= P(X \le 1 - e^{-\lambda y}) \\ &= F_X(1 - e^{-\lambda y}). \end{aligned}$$

Since X is $U(0,1)$, $F_X(x) = x, 0 \le x \le 1$. Therefore,

$$F_Y(y) = 1 - e^{-\lambda y} \Rightarrow Y \text{ is } EXP(\lambda)$$

- This transformation is used to generate a random variate (or deviate) of the *Exp(λ)* distribution

Theorem: pdf for a transformed RV

X: a continuous random variable with density f_X that is nonzero on a subset I of real numbers [i.e., $f_X(x) > 0$, $x \in I$ and $f_X(x) = 0$, $x \notin I$].

Φ: a differentiable monotone function whose domain is I and whose range is the set of reals.

Then $Y = \Phi(X)$ is a continuous random variable with the density, f_Y, given by

$$f_Y(y) = \begin{cases} f_X[\Phi^{-1}(y)][|(\Phi^{-1})'(y)|], & y \in \Phi(I), \\ 0, & \text{otherwise}, \end{cases}$$

Proof:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P[\Phi(X) \leq y] \\ &= P[X \leq \Phi^{-1}(y)], \quad \text{(since } \Phi \text{ is monotone increasing)} \\ &= F_X[\Phi^{-1}(y)]. \end{aligned}$$

Taking derivatives & using chain rule gives the result

Random Variate Generation

- Random variate is defined as a typical value sampled from a given distribution. If we take a large number (ideally infinite) of them and plot a histogram, it will approach the original pmf or pdf.
- Goal: Study methods of generating random deviates of a given distribution, assuming a routine to generate uniformly distributed random numbers is available.
- Note that distribution of interest can be discrete or continuous.

Some generation Methods

- Some popular methods of generating random variate are:
  - Inverse Transform Method
  - Convolution Method
  - Direct Transformation of Normal Distribution.
  - Acceptance-Rejection Method

Inverse Transform

- Based on the following idea:
  - If $F(x)$ is strictly monotonic distribution function and U is uniformly distributed over the interval (0, 1).

  →Then the new random variable $X=F^{-1}(U)$ *has the* the CDF F.
- Method:
  - A random number u from a uniform distribution over (0, 1) is generated and then the F is inverted to generate random deviate x of X.
  - $F^{-1}(u)$ gives the required value of x.

Example 3.11

- Generate random variate x with distribution $F=F_X$
- Let, $Y = F(X)$
- $F_Y(u) = F_Y(F^{-1}(u)) = u$, *or*, $Y=F(X)$ has *pdf*,

$$f_Y(y) = \begin{cases} 1, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

- Hence, to generate random variate (deviate) with distribution F,
  1. Generate random number u
  2. Find $x = F^{-1}(u)$ and x will be a random deviate with distribution F
  3. If $x = \lambda^{-1} \ln(1-u)$, then x will be a random deviate of $EXP(\lambda)$ distribution.
- Inversion can be done in closed form, graphically or using a table

Inverse Transform

- Variates of Exponential, Uniform, Weibull, Pareto, Rayleigh, Triangular, Log-logistic and many others can be generated by this method.
- Variates of empirical and discrete distributions like Bernoulli and Geometric can also be generated using this idea.
- It is most useful when the inverse of the CDF, *F(.)* can be easily computed in closed form although a numerical or tabular method can also be used.

Some Examples

Exponential Distribution

- CDF

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0$$

where λ is its failure rate ($1/\lambda$ is the mean).

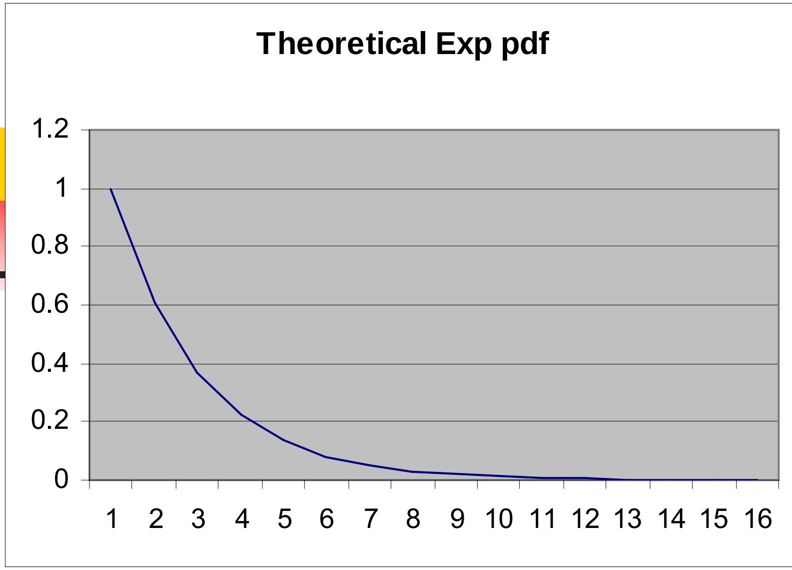
- Random Variate *x*

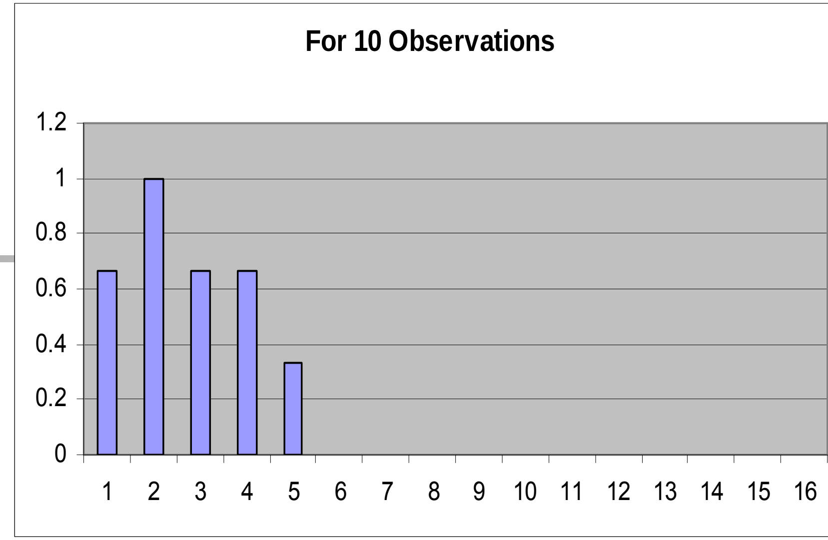
$$x = -\frac{\ln(1-u)}{\lambda}$$

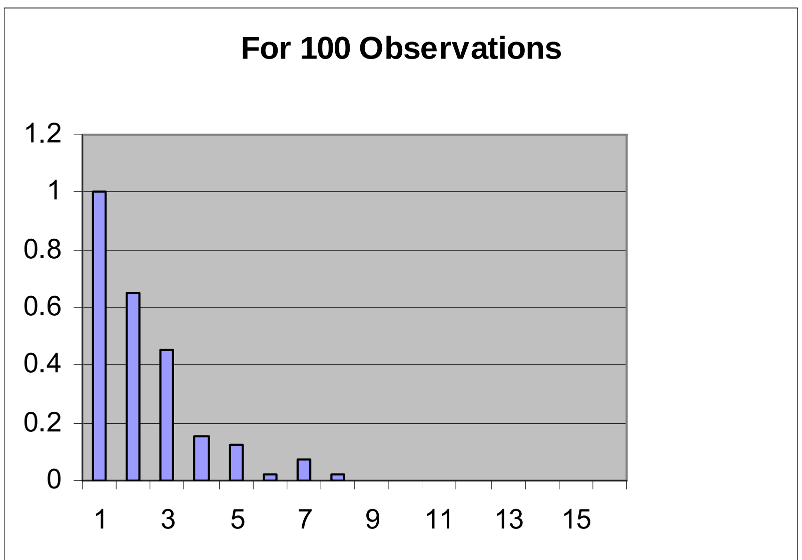
- *where u* is drawn from uniform distribution *Unif(0,1)*.
- Since *(1-u)* is also a random number , use the simpler formula $x = -\frac{\ln(u)}{\lambda}$

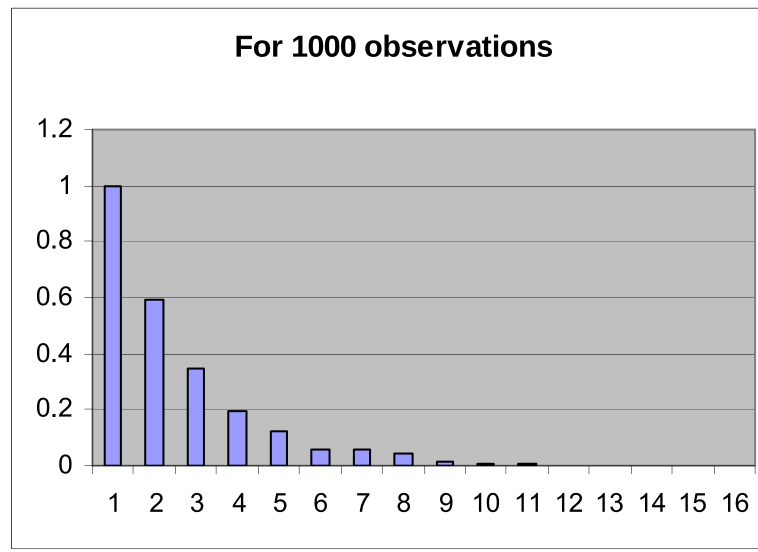
Exponential variate

- Next we will show that if enough variates are sampled then the sample of generated numbers is sufficient to describe the pdf of the distribution.
- We see that as we increase number of observations the plot becomes closer and closer to the theoretical pdf $f(t) = e^{-t} \ \ for\ t \geq 0$
- pdf of exponential distribution with mean 1 is plotted
- Three cases are taken with 10, 100 and 1000 observations.


Theoretical Exp pdf
1.2
1
0.8
0.6
0.4
0.2
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16


For 10 Observations
1.2
1
0.8
0.6
0.4
0.2
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16


For 100 Observations
1.2
1
0.8
0.6
0.4
0.2
0
1 3 5 7 9 11 13 15


For 1000 observations
1.2
1
0.8
0.6
0.4
0.2
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

Some Examples: Example 3.12

- Weibull Distribution
  - CDF $F_X(x) = 1 - e^{-\lambda x^{\alpha}}$

  where λ is the scale parameter and α is the shape parameter.
  - Random Variate *x*

$$x = \left(\frac{-\ln(1-u)}{\lambda} \right)^{\frac{1}{\alpha}}$$

  - *where u* is drawn from uniform distribution *Unif(0,1)*.
  - Simplified formula

$$x = \left(-\frac{\ln(u)}{\lambda} \right)^{\frac{1}{\alpha}}$$

Extensions to Example 3.12

- Write down the random deviate formual for the alternate for of the Weibull Distribution
  - CDF

$$F_X(x) = 1 - e^{-(\lambda x)^{\alpha}}$$

- And for the three parameter Weibull distribution:
  - CDF

$$F(t) = 1 - \mathrm{e}^{-\lambda(t-\theta)^{\alpha}},\ t \geq \theta\ (\theta\text{: location parameter})$$

Some Examples: Pareto

- Pareto Distribution
  - CDF $$F_X(x) = 1 - \left(\frac{k}{x}\right)^{\alpha} \qquad x \ge k$$
  $$= 0 \qquad x \le k$$

  where $k>0$ is location parameter and α is shape parameter.
  - Random Variate *x* $$x = \frac{k}{(1-u)^{\frac{1}{\alpha}}}$$
  - *where u* is drawn from uniform distribution *Unif(0,1)*.
  - Simplified Formula $$x = \frac{k}{u^{\frac{1}{\alpha}}}$$

Some Examples: Rayleigh

- Rayleigh Distribution
  - CDF $F_X(x) = 1 - e^{-x^2/2\sigma^2} \quad x \geq 0$

$$= 0 \qquad otherwise$$

  where σ^2 is the variance.
  - Random Variate *x*

$$x = \sqrt{-2\sigma^2 \ln(1-u)}$$

  - *where u* is drawn from uniform distribution *Unif(0,1)*.
  - Simplified Formula $x = \sqrt{-2\sigma^2 \ln(u)}$

Some Examples: Log-logistic

- Log-Logistic Distribution
  - CDF $$F_X(x) = 1 - \frac{1}{1+(\lambda x)^\kappa}$$

  where $\lambda>0$ is the scale parameter and $\kappa>0$ is shape parameter.
  - Random Variate *x* $$x = \frac{1}{\lambda}\left(\frac{u}{1-u}\right)^{\frac{1}{\kappa}}$$

  where u is drawn from uniform distribution *Unif(0,1)*.

Random variate Table

Name	Density $f(x)$	$F(x)$	$X=F^{-1}(u)$	Simpl. form
Expo	$\lambda e^{-\lambda x} \quad x>0$	$1-e^{-\lambda x} \quad x>0$	$x=-\frac{\ln(1-u)}{\lambda}$	$x=-\frac{\ln(u)}{\lambda_1}$
Weibull	$\lambda e^{-\lambda x^{\alpha}} \quad x>0$	$1-e^{-\lambda x^{\alpha}}$	$x=\left(\frac{-\ln(1-u)}{\lambda}\right)^{\frac{1}{\alpha}}$	$x=\left(-\frac{\ln(u)}{\lambda_1}\right)^{\frac{1}{\alpha}}$
Pareto	$\alpha k^{\alpha} x^{-\alpha-1} \quad x>k, \quad \alpha, k>0$	$1-\left(\frac{k}{x}\right)^{\alpha}$	$x=\frac{k}{(1-u)^{\frac{1}{\alpha}}}$	$x=\frac{k}{u^{\frac{1}{\alpha}}}$
Rayleigh	$\frac{x}{\sigma^2}\exp\left[-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right]$	$1-e^{-x^2/2\sigma^2} \quad x\geq 0$	$x=\sqrt{-2\sigma^2\ln(1-u)}$	$x=\sqrt{-2\sigma^2\ln(u)}$

Random variate Table

Name	Density $f(x)$	$F(x)$	$X=F^{-1}(u)$	Simplified form
Log-Logistic	$\dfrac{\lambda\kappa(\lambda t)^{\kappa-1}}{[1+(\lambda t)^{\kappa}]^2} \quad t \geq 0$	$1-\dfrac{1}{1+(\lambda t)^{\kappa}}$	$x=\dfrac{1}{\lambda}\left(\dfrac{u}{1-u}\right)^{\frac{1}{\kappa}}$	$x=\dfrac{1}{\lambda}\left(\dfrac{u}{1-u}\right)^{\frac{1}{\kappa}}$
Cauchy	$\dfrac{\sigma}{\pi(x^2+\sigma^2)}$	$\dfrac{1}{2}+\dfrac{1}{\pi}\arctan\left(\dfrac{x}{\sigma}\right)$	$\sigma\tan\left(\pi(u-\frac{1}{2})\right)$	$\sigma\tan(\pi u)$
Triangular	$\dfrac{2}{a}\left(1-\dfrac{x}{a}\right)$	$\dfrac{2}{a}\left(x-\dfrac{x^2}{2a}\right)$	$a(1-\sqrt{1-u})$	$a(1-\sqrt{u})$

For Discrete distributions

- We want to generate *X* having pmf

$$p_j = P(X = x_j), \;\; j = 0,1,\ldots$$

- and distribution $F(x_j) = \sum_{i=0}^{j} p_i$

- If *u* is the uniform random number then

$$x = x_j, \; if \;\; F(x_{j-1}) < u \le F(x_j)$$

- This is as

$$P(X = x_j) = P(\sum_{i=1}^{j-1} p_i < U \le \sum_{i=1}^{j} p_i) = p_j$$

For discrete distributions

- Bernoulli distribution
- CDF with parameter *(1-q)*

$$
\begin{aligned}
F_X(x) &= 0 \quad x < 0 \\
&= q \quad 0 \le x < 1 \\
&= 1 \quad x \ge 1
\end{aligned}
$$

- Inverse function is given by

$$
\begin{aligned}
F^{-1}(u) &= 0 \quad 0 < u \le q \\
&= 1 \quad q < u \le 1
\end{aligned}
$$

- By generating *u* from *Unif(0,1)* function, we can get Bernoulli random variate from above.

For Geometric Distributions

- Geometric pmf is given by $p(1-p)^{j-1} \quad j \geq 1$
- Distribution is given by $F(j) = 1-(1-p)^j$
- So geometric random variate *j* satisfies

$$1-(1-p)^{j-1} < u \leq 1-(1-p)^j$$

$$(1-p)^{j-1} < 1-u \leq (1-p)^j$$

• Hence Geometric random variate *x*

$$x = \min\{ i : (1-p)^i < 1-u \}$$

$$= \min\{ i : i > \frac{\log(1-u)}{\log(1-p)} \}$$

$$= \left\lceil \frac{\log(1-u)}{\log(1-p)} \right\rceil$$

- Since (*1-u*) is also uniform random number it becomes $\left\lceil \frac{\log(u)}{\log(1-p)} \right\rceil$

Example 3.13

- Scaling a random variable X
- Let $f_X(x) = \lambda e^{-\lambda x}$ and $Y = rX$

$$f_Y(y) = \frac{1}{r}\lambda e^{-\lambda y/r}, \qquad y > 0.$$

- Hence, Y is also EXP( ) with parameter λ/r
- Thus the exp distribution is closed under a multiplication by a scalar

Example 3.14

- Distribution for $Y = \Phi(X) = e^X$, given X is $N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$\Phi^{-1}(y) = \ln(y) \;\Rightarrow\; [\Phi^{-1}]'(y) = \frac{1}{y}$$

Therefore,

$$\begin{aligned} f_Y(y) &= \frac{f(\ln y)}{y} \\ &= \frac{1}{\sigma y\sqrt{2\pi}} \exp\left[-\frac{(\ln y-\mu)^2}{2\sigma^2}\right], \qquad y > 0. \end{aligned}$$

- Random variable Y is said to have a *log-normal* distribution.
- Repair times are often found to follow this distribution

Jointly Distributed RVs

- Two cont. rv's X and Y on the same (S, F, P).
  Then, event, $[X \leq x, Y \leq y] = [X \leq x] \cap [Y \leq y]$

  made of sample points $\{s \in S\colon\ X(s) \leq x \text{ and } Y(s) \leq y\}$

- Joint Distribution Function:
  $$F_{X,Y}(x, y) = P(X \leq x, Y \leq y),\ -\infty < x, y < \infty$$
- Independent rv's: *iff* the following holds:
  $$F_{X,Y}(x, y) = F_X(x) F_Y(y),\ -\infty < x, y < \infty$$
- Independent rv's: *iff* the following holds:
  $$f_{X,Y}(x, y) = f_X(x) f_Y(y),\ -\infty < x, y < \infty$$

Joint Distribution Properties

1) $0 \leq F(x, y) \leq 1, \quad -\infty < x, y < \infty$

2) $F(x, y)$: monotonic increasing in x and y

3) If either x or $y \to -\infty$, then, $F(x, y) \to 0$

4) If both x and $y \to \infty$, then, $F(x, y) \to 1$

5) $F(x, y)$ is right-continuous. If X and Y are both continuous $\to$ $F(x, y)$ is continuous

6) $P(a < X \leq b, c < Y \leq d) =$

$$F(b, d) - F(a, d) - F(b, c) + F(a, c)$$

7) $F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(u, v) du dv$

Joint Distribution Properties (contd.)

8) $f(x,y) = \dfrac{\partial^2 F(x,y)}{\partial x \partial y}$

9) $F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(u,y)\,dy\,du$

10) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

11) $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

Order statistics, '*k of n*', TMR

Order Statistics: '*k of n*'

- $X_1, X_2, \ldots, X_n$ iid (independent and identically distributed) random variables with a common distribution function *F and common density f.*
- Let $Y_1, Y_2, \ldots, Y_n$ be random variables obtained by permuting the set $X_1, X_2, \ldots, X_n$ so as to be in increasing order.
- To be specific:

$$Y_1 = \min\{X_1, X_2, \ldots, X_n\} \quad \text{and}$$

$$Y_n = \max\{X_1, X_2, \ldots, X_n\}$$

Order Statistics: *k of n* (Continued)

- The random variable Y_k is called the k^{th} ORDER STATISTIC.
- If X_i is the lifetime of the i^{th} component in a system of n components. Then:
  - Y_1 will be the overall series system lifetime.
  - Y_n will denote the lifetime of a parallel system.
  - Y_{n-k+1} will be the lifetime of an *k-of-n* system.

Order Statistics: *k of n* (Continued)

- To derive the distribution function of Y_k, we note that the probability that exactly j of the X_i's lie in $(-\infty, y]$ and $(n-j)$ lie in (y, ∞) is (n Bernoulli trials; $p=F(y)$):

$$\binom{n}{j} F^j(y)[1-F(y)]^{n-j} \qquad -\infty < y < \infty$$

hence

$$F_{Y_k}(y) = \sum_{j=k}^{n} \binom{n}{j} F^j(y)[1-F(y)]^{n-j}, -\infty < y < \infty$$

Overview: General iid Random Variables

Let $Y_1, \ldots, Y_n$ denote the order statistics of the random variables $X_1, \ldots, X_n$, which are iid with common distribution function F.

Y_1		
Y_{n-k+1}	$F_{Y_{n-k+1}}(y) = \sum_{j=n-k+1}^{n} \binom{n}{j} F^j(y)[1-F(y)]^{n-j}$	$R_{k\|n}(t) = 1 - F_{Y_{n-k+1}}(t) = \sum_{j=k}^{n} \binom{n}{j} R^j(t)[1-R(t)]^{n-j}$
Y_n		

Overview: General iid Random Variables

Let Y_1 ,,…, Y_n denote the order statistics of the random variables X_1 ,…, $X_{n,}$, which are iid with common distribution function F.

Y_1		
Y_{n-k+1}	$F_{Y_{n-k+1}}(y) = \sum_{j=n-k+1}^{n} \binom{n}{j} F^j(y)[1-F(y)]^{n-j}$	$R_{k\|n}(t) = 1 - F_{Y_{n-k+1}}(t) = \sum_{j=k}^{n} \binom{n}{j} R^j(t)[1-R(t)]^{n-j}$
Y_n	$F_{Y_n}(y) = [F(y)]^n$	$R_{parallel}(t) = 1 - F_{Y_n}(t) = 1-[F(t)]^n = 1-[1-R(t)]^n$

Overview: General iid Random Variables

Let $Y_1, \ldots, Y_n$ denote the order statistics of the random variables $X_1, \ldots, X_n$, which are iid with common distribution function F.

Y_1	$F_{Y_1}(y) = 1 - [1 - F(y)]^n$	$R_{series}(t) = 1 - F_{Y_1}(t)$ $= [1 - F(t)]^n = [R(t)]^n$
Y_{n-k+1}	$F_{Y_{n-k+1}}(y)$ $= \sum_{j=n-k+1}^{n} \binom{n}{j} F^j(y)[1 - F(y)]^{n-j}$	$R_{k\|n}(t) = 1 - F_{Y_{n-k+1}}(t)$ $= \sum_{j=k}^{n} \binom{n}{j} R^j(t)[1 - R(t)]^{n-j}$
Y_n	$F_{Y_n}(y) = [F(y)]^n$	$R_{parallel}(t) = 1 - F_{Y_n}(t)$ $= 1 - [F(t)]^n = 1 - [1 - R(t)]^n$

Applications of order statistics

- Reliability of a *k* out of *n* system

$$R_{kofn}(t) = \sum_{j=k}^{n} \binom{n}{j} [R(t)]^j [1 - R(t)]^{n-j}$$

- Series system:

$$R_{series}(t) = [R(t)]^n \quad or \quad \prod_{i=1}^{n} R_i(t)$$

- Parallel system:

$$R_{parallel}(t) = 1 - [1 - R(t)]^n \, or \, 1 - \prod_{i=1}^{n} (1 - R_i(t))$$

- Minimum of *n* EXP random variables is special case of $Y_1 = \min\{X_1, \ldots, X_n\}$ where $X_i \sim EXP(\lambda_i)$ so $Y_1 \sim EXP(\Sigma\, \lambda_i)$

- Thus the exponential is closed under series composition but not under the parallel composition .

Example 3.16

- Series system lifetime distribution
- i^{th} component's life time distribution ~ *EXP*(λ_i)

$$R_{\text{series}}(t) = \exp\left[-(\sum_{i=1}^{n} \lambda_i)t\right]$$

- Lifetime distribution of series system of components with each component having *EXP*( ) distribution is also *EXP*(λ_s) with

$$\lambda_s = \sum_{i=1}^{n} \lambda_i$$

Example 3.17 (Parts count method)

- Assuming that times to failure of all chip types are exponentially distributed with the following failure rates:

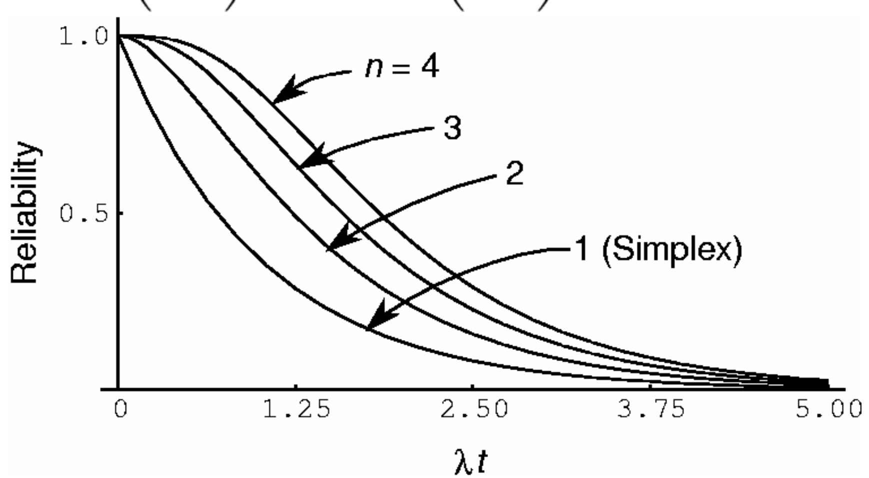
Chip type	Number of chips, n_i	Failure rate per chip (number of failures/10^6 h) λ_i
SSI	1,202	0.1218
MSI	668	0.242
ROM	58	0.156
RAM	414	0.691
MOS	256	1.0602
BIP	2,086	0.1588

$$\begin{aligned}
\lambda &= \sum_{\text{all chip types}} n_i \lambda_i \\
&= 146.40 + 161.66 + 9.05 + 286.07 + 261.41 + 331.27 \\
&= 1205.85 \text{ failures per } 10^6 \text{ hours.}
\end{aligned}$$

Example 3.18

- Hence series system of Example 3.17 has a constant failure rate.
- What about a parallel system of *n* such components:

$$
\begin{aligned}
R_p(t) &= 1-(1-e^{-\lambda t})^n \\
&= \binom{n}{1} e^{-\lambda t} - \binom{n}{2} e^{-2\lambda t} + \cdots + (-1)^{n-1} e^{-n\lambda t}.
\end{aligned}
$$



Example 3.20

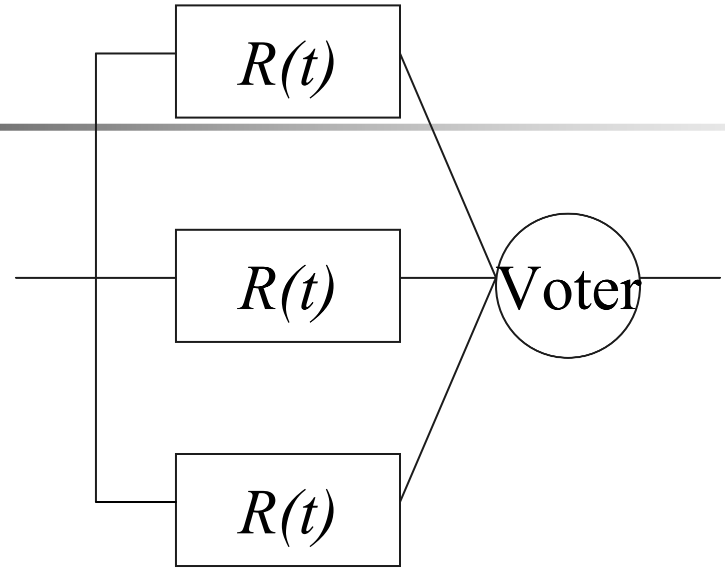
- Arrivals from n sources : s_i generates $N_i(t)$ tasks in time t . $N_i(t)$ Poisson with parameter $\lambda_i t$ $(1 \le i \le n)$
- X_i : time between two successive arrivals from s_i has $EXP(\lambda_i)$ distribution.
- Total no. of jobs $N(t) = \sum_{i=1}^{n} N_i(t)$
- is also Poisson with rate parameter $\lambda = \sum_{i=1}^{n} \lambda_i$
- The jobs arrive in the pooled stream with interarrival time,

$$Y_1 = \min\{X_1, X_2, \ldots, X_n\} \quad \text{and } F_{X_i}(t) = 1 - e^{-\lambda_i t}$$

$$F_{Y_1}(t) \;=\; 1 - \prod_{i=1}^{n} [1 - F_{X_i}(t)] = 1 - \prod_{i=1}^{n} e^{-\lambda_i t} = 1 - \exp[-\sum_{i=1}^{n} \lambda_i t].$$

Thus Y_1 has exponential distribution $EXP(\sum_{i=1}^{n} \lambda_i)$

Triple Modular Redundancy (TMR)



- An interesting case of order statistics occurs when we consider the Triple Modular Redundant (TMR) system ($n = 3$ and $k = 2$). Y_2 then denotes the time until the second component fails. We get:

$$R_{TMR}(t) = 3R^2(t) - 2R^3(t)$$

TMR (Continued)

- Assuming that the reliability of a single component is given by,

  $$R(t) = e^{-\lambda t}$$

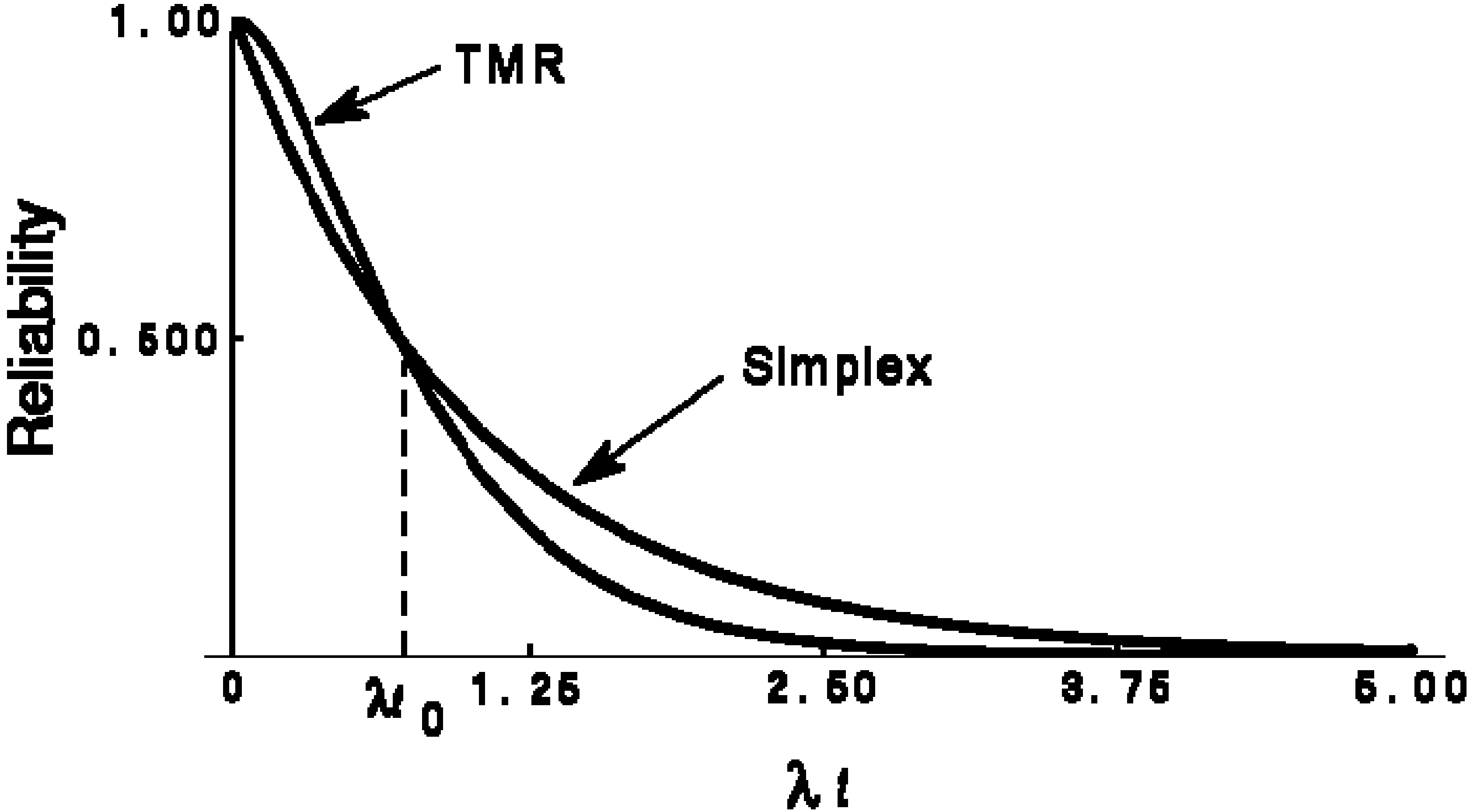
  we get:

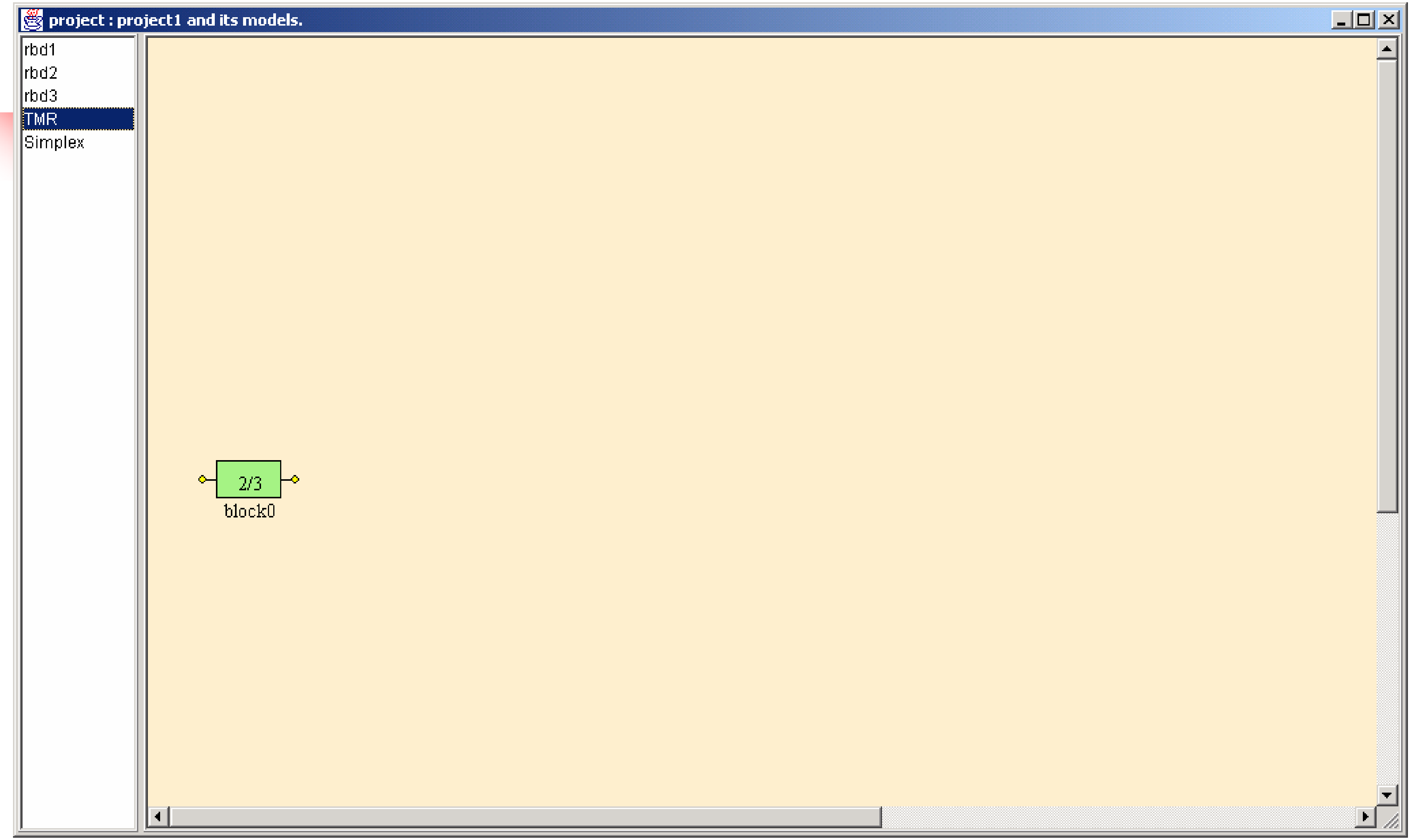
  $$R_{TMR}(t) = 3e^{-2\lambda t} - 2e^{-3\lambda t}$$

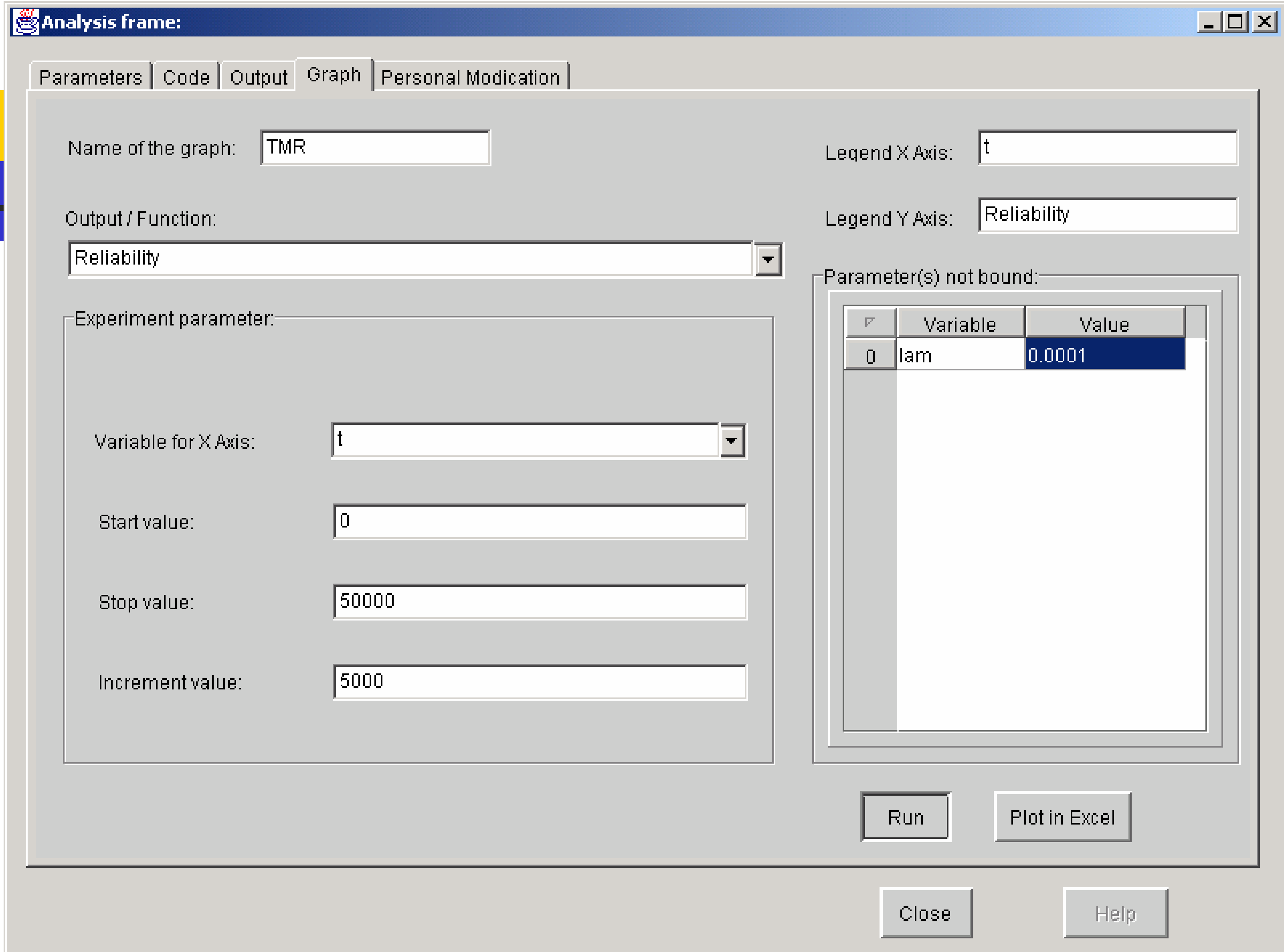
TMR (Continued)

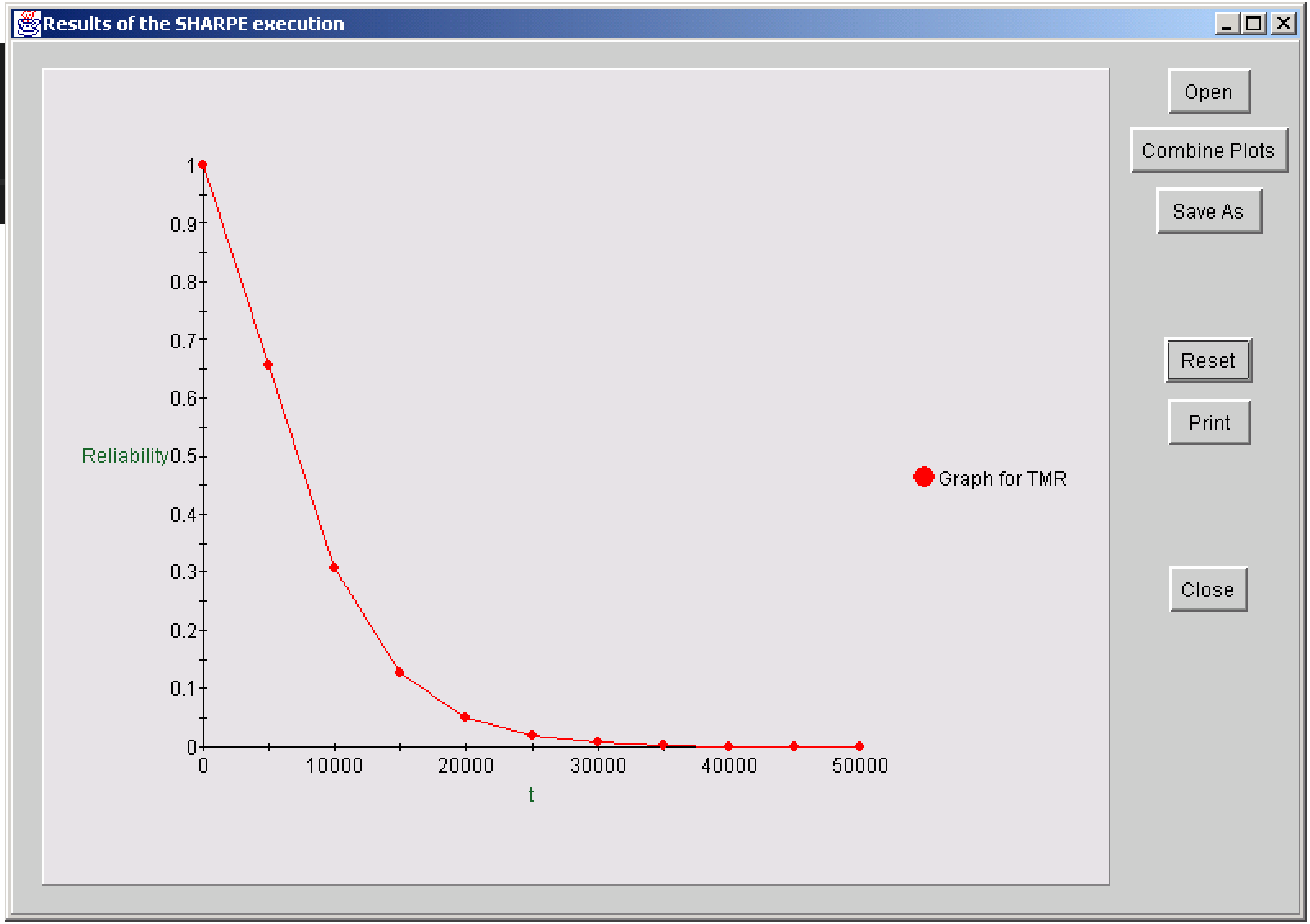
- In the following figure, we have plotted $R_{TMR}(t)$ vs. t as well as $R(t)$ vs. t.
- Also graphs have been plotted for comparison between TMR and TMR/Simplex using SHARPE GUI. A step by step procedure has been shown.
- We see that TMR improves reliability over the simplex for short mission times (defined by $t < \ln 2/\lambda$); for longer mission times, TMR has *lower reliability* than simplex

TMR (Continued)




project : project1 and its models.
rbd1
rbd2
rbd3
TMR
Simplex
2/3
block0


Analysis frame:
Parameters
Code
Output
Graph
Personal Modication
Name of the graph:
TMR
Legend X Axis:
t
Legend Y Axis:
Reliability
Output / Function:
Reliability
Parameter(s) not bound:
Variable
Value
0
lam
0.0001
Experiment parameter:
Variable for X Axis:
t
Start value:
0
Stop value:
50000
Increment value:
5000
Run
Plot in Excel
Close
Help



Results of the SHARPE execution
Open
Combine Plots
Save As
Reset
Print
Close
Reliability
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0.9
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0.7
0.6
0.5
0.4
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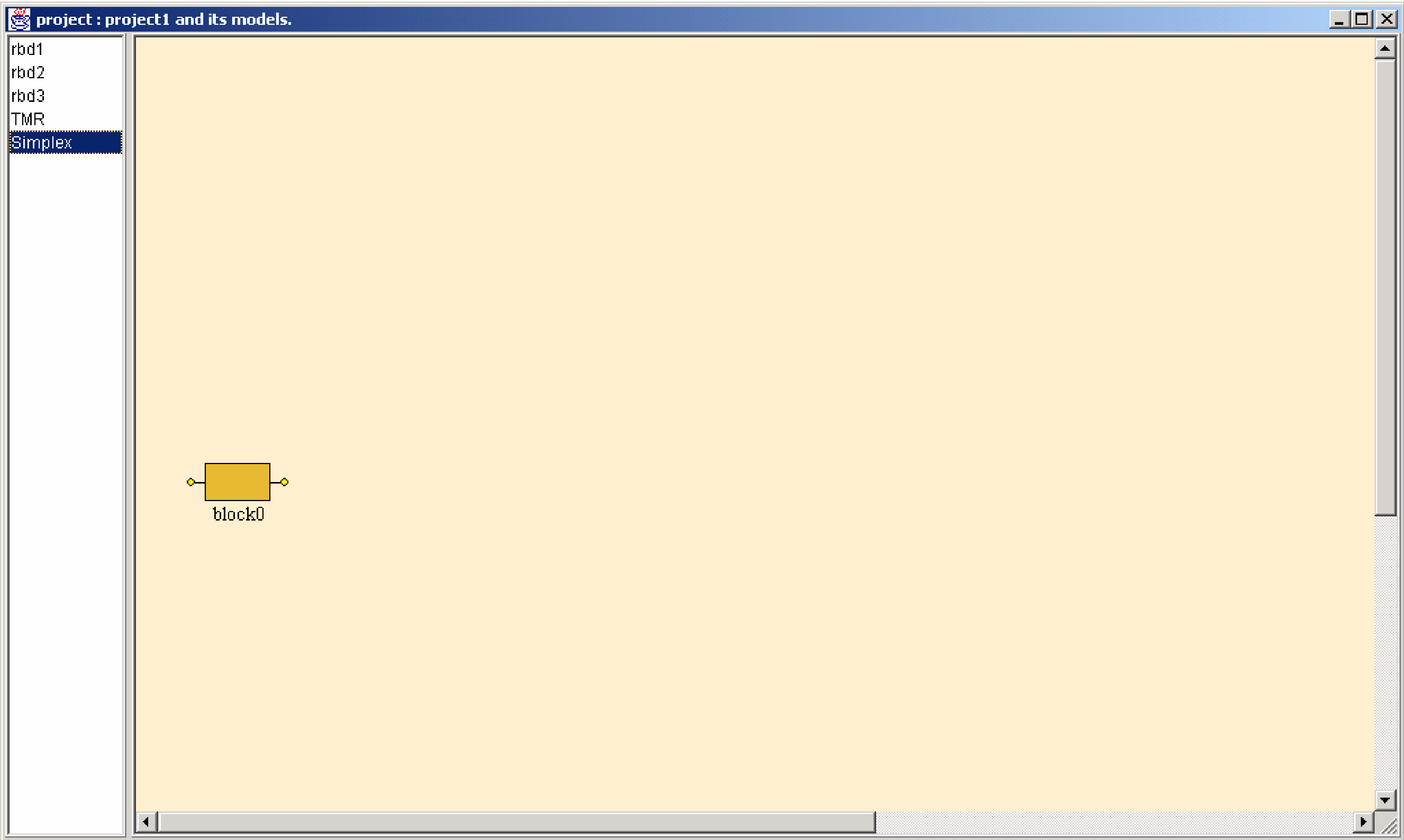
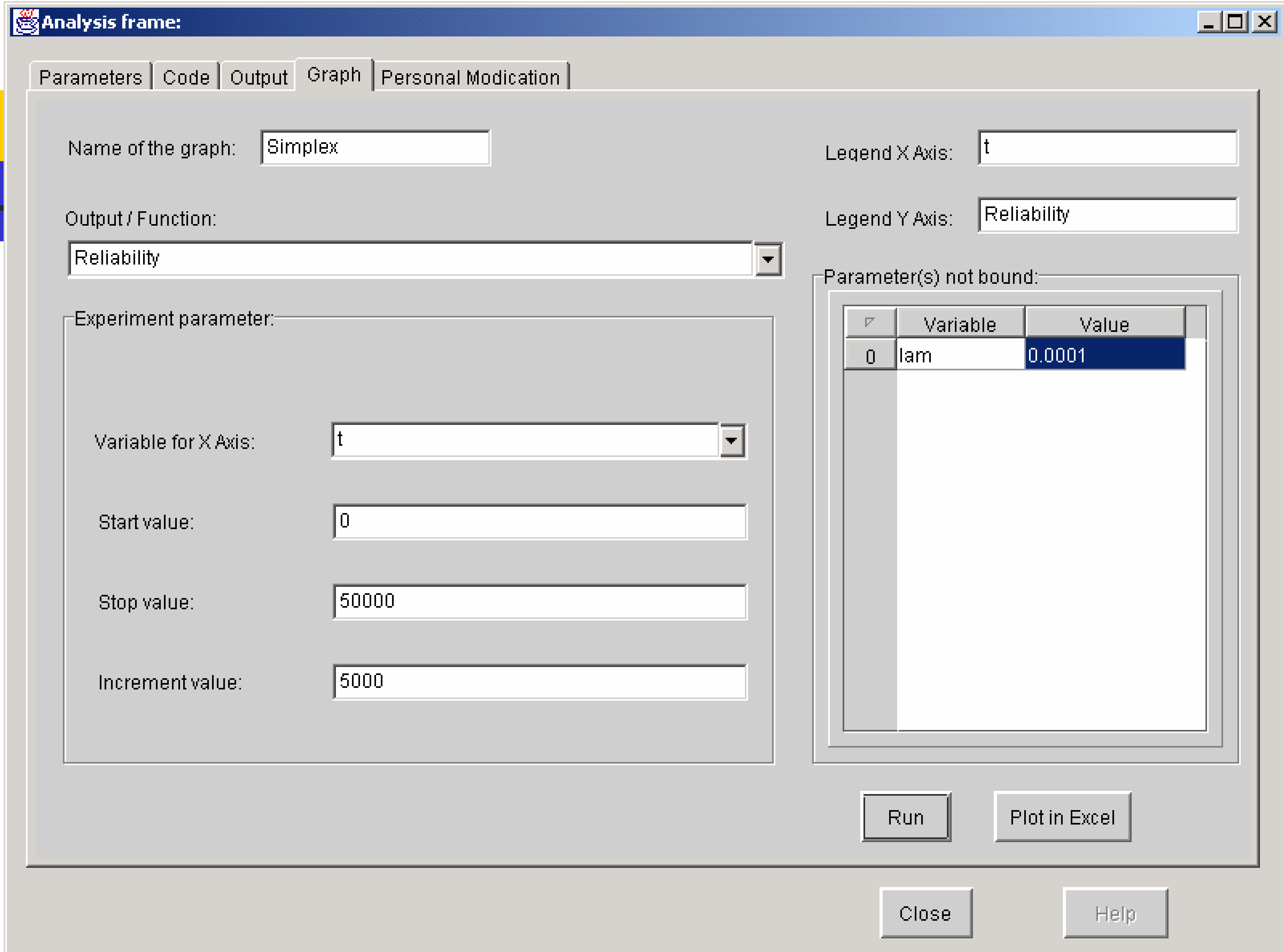
20000
30000
40000
50000
t
Graph for TMR


project : project1 and its models.
rbd1
rbd2
rbd3
TMR
Simplex
block0


Analysis frame:
Parameters
Code
Output
Graph
Personal Modication
Name of the graph:
Simplex
Legend X Axis:
t
Legend Y Axis:
Reliability
Output / Function:
Reliability
Parameter(s) not bound:
Variable
Value
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lam
0.0001
Experiment parameter:
Variable for X Axis:
t
Start value:
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Stop value:
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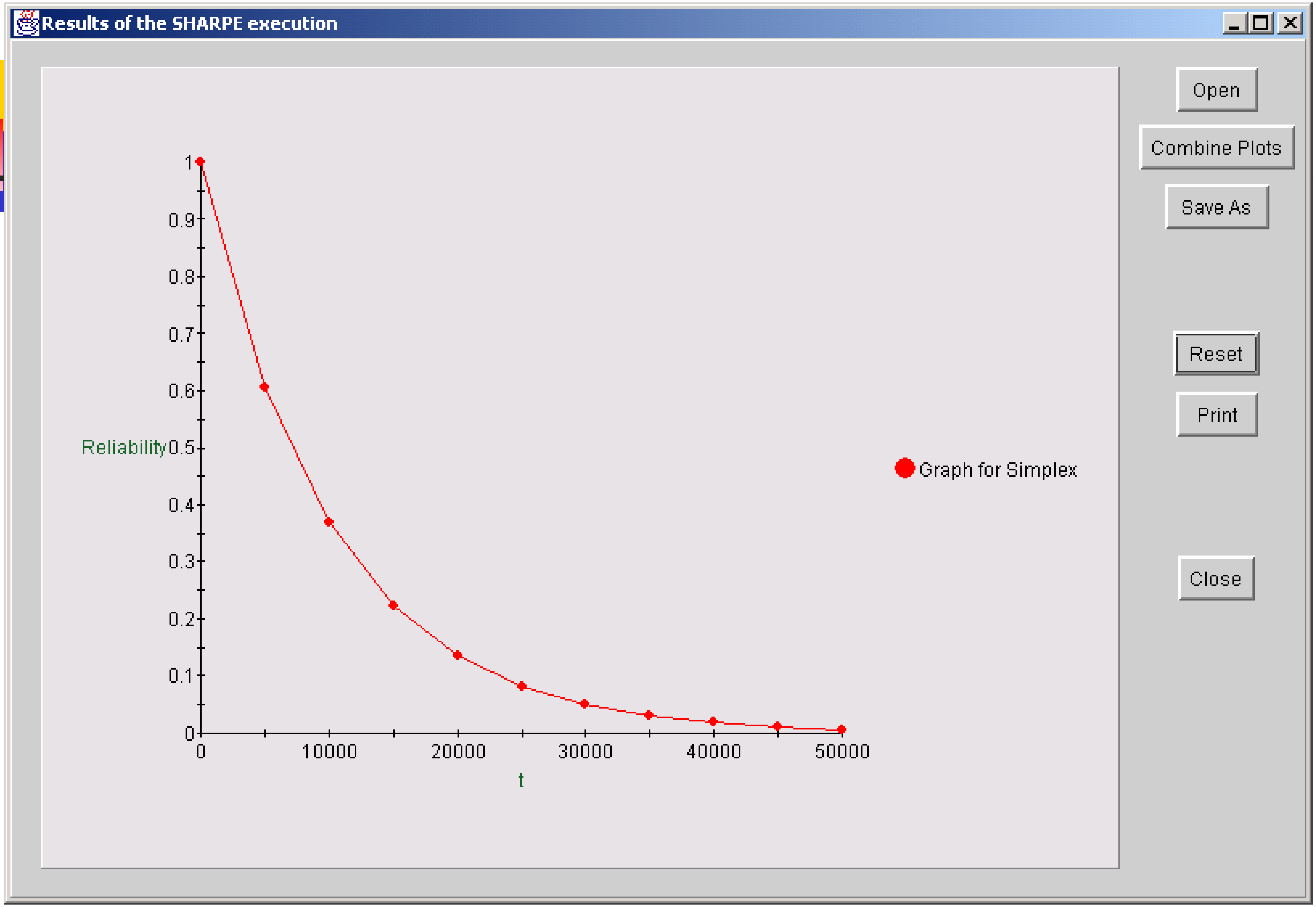
Increment value:
5000
Run
Plot in Excel
Close
Help


Results of the SHARPE execution
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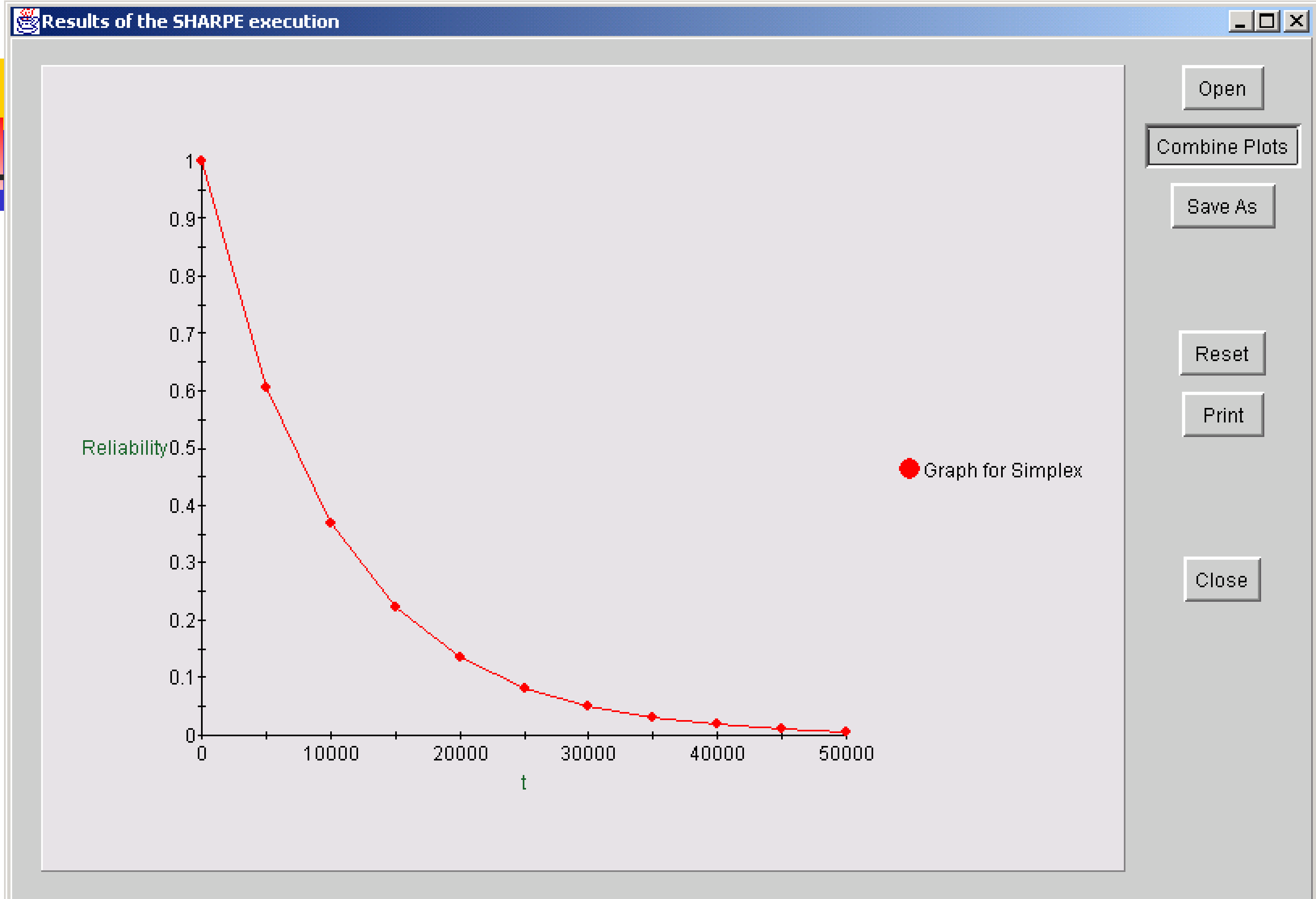
Graph for Simplex


Results of the SHARPE execution
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Combine Plots
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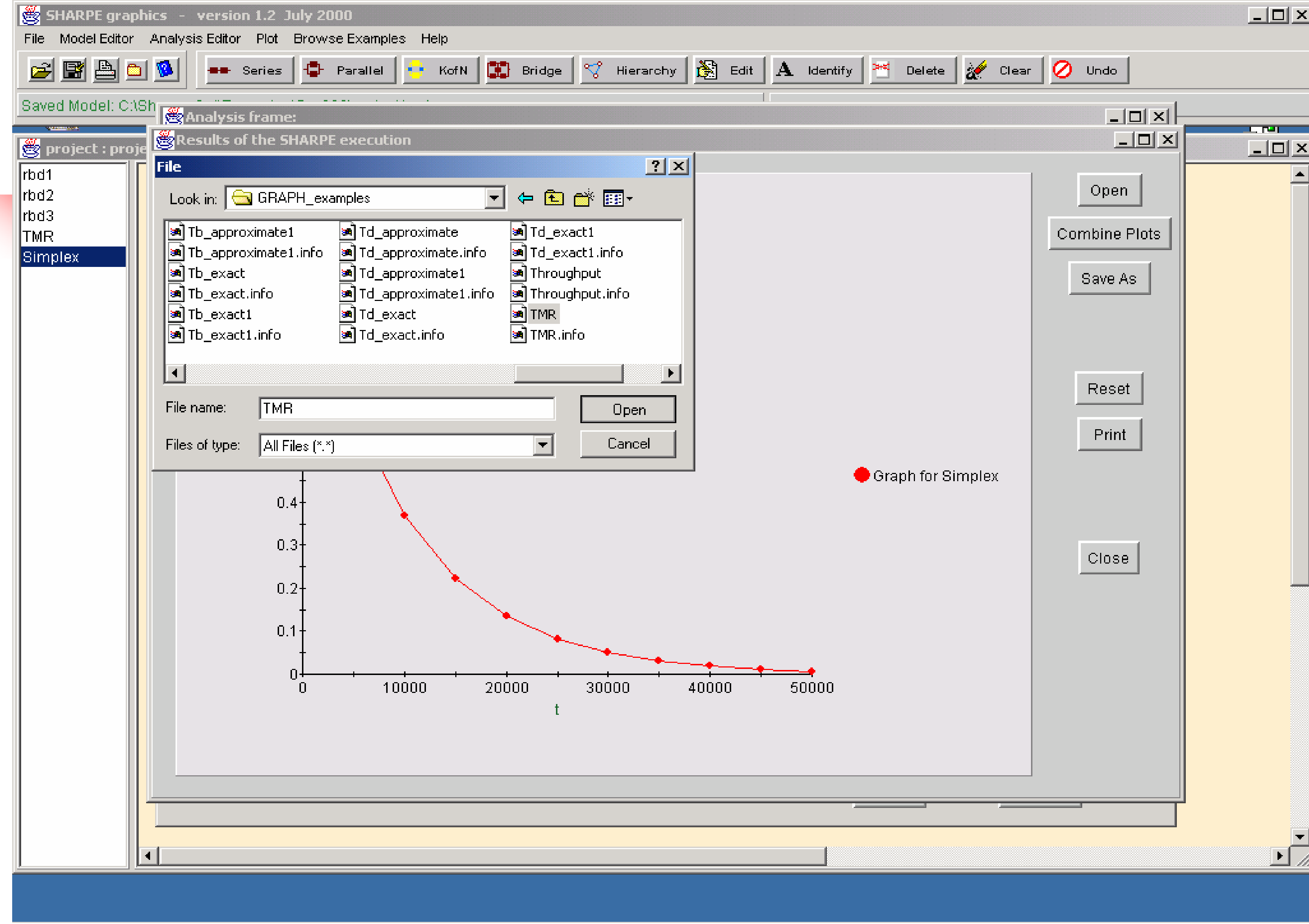
Graph for Simplex


SHARPE graphics - version 1.2 July 2000
File
Model Editor
Analysis Editor
Plot
Browse Examples
Help
Series
Parallel
KofN
Bridge
Hierarchy
Edit
Identify
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Undo
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Analysis frame:
project : proje
rbd1
rbd2
rbd3
TMR
Simplex
Results of the SHARPE execution
File
Look in: GRAPH_examples
Tb_approximate1
Tb_approximate1.info
Tb_exact
Tb_exact.info
Tb_exact1
Tb_exact1.info
Td_approximate
Td_approximate.info
Td_approximate1
Td_approximate1.info
Td_exact
Td_exact.info
Td_exact1
Td_exact1.info
Throughput
Throughput.info
TMR
TMR.info
File name: TMR
Files of type: All Files (*.*)
Open
Cancel
Open
Combine Plots
Save As
Reset
Print
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Graph for Simplex
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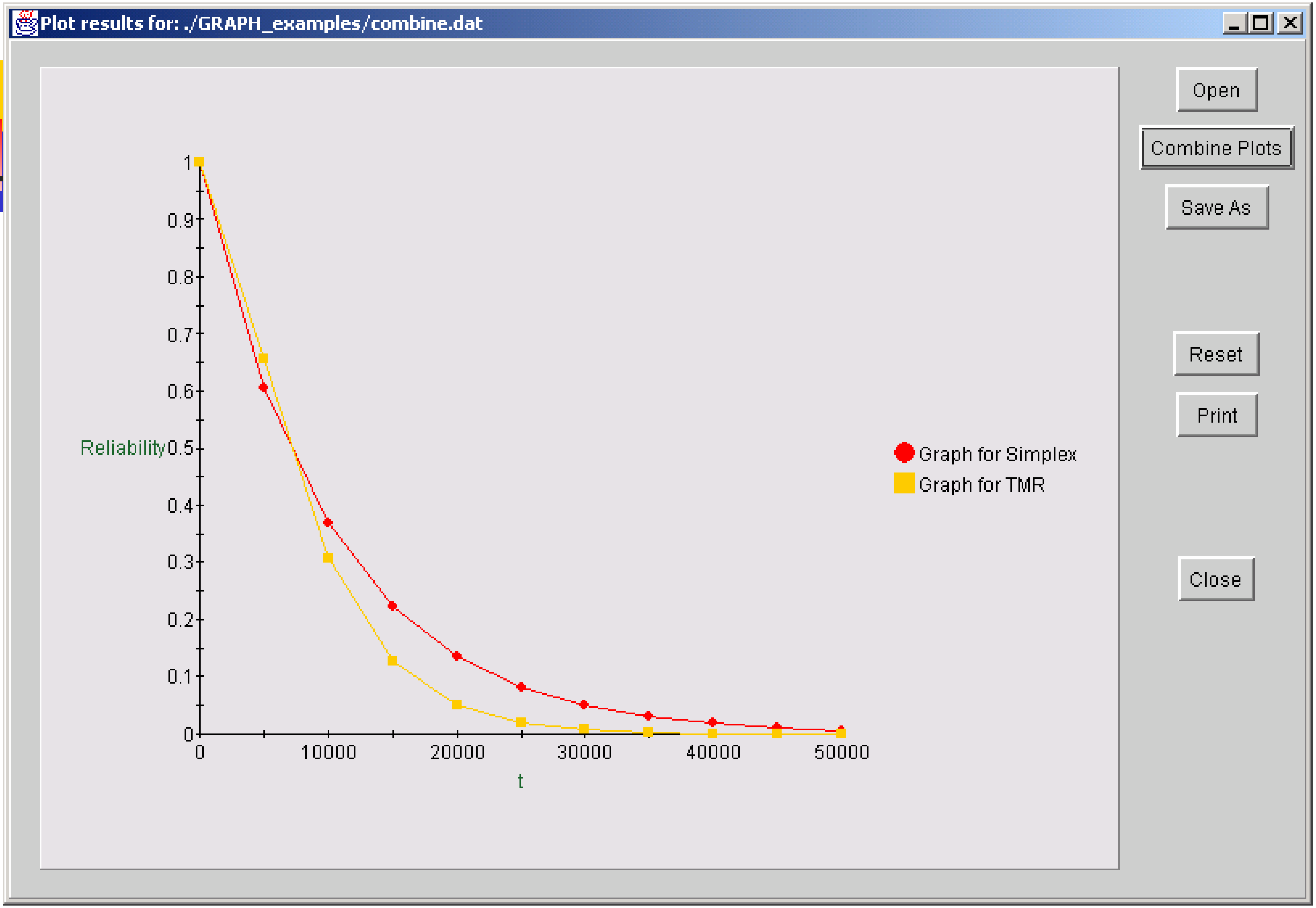
Plot results for: ./GRAPH_examples/combine.dat
Open
Combine Plots
Save As
Reset
Print
Close
Reliability
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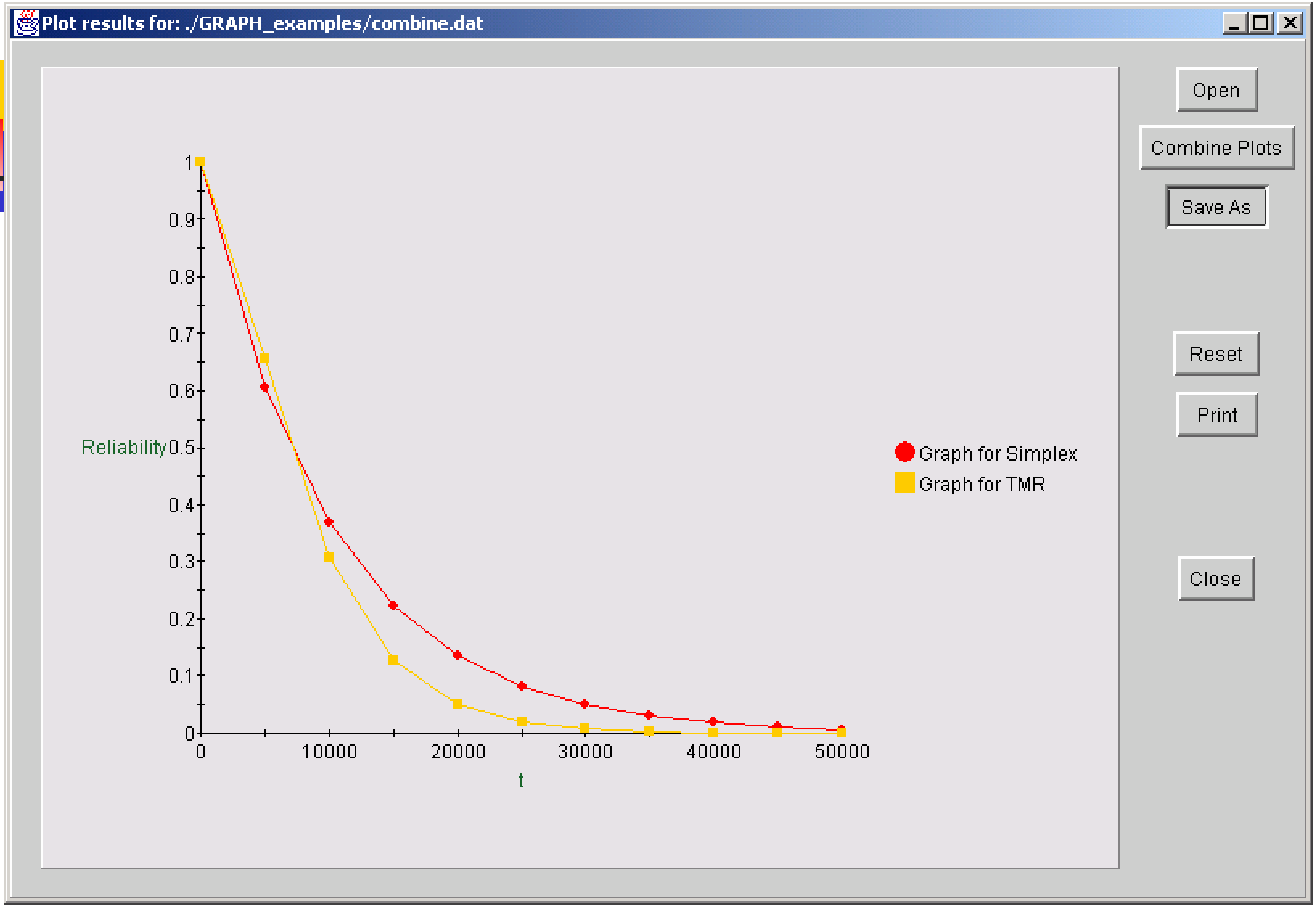
Graph for Simplex
Graph for TMR



Plot results for: ./GRAPH_examples/combine.dat
Open
Combine Plots
Save As
Reset
Print
Close
Reliability
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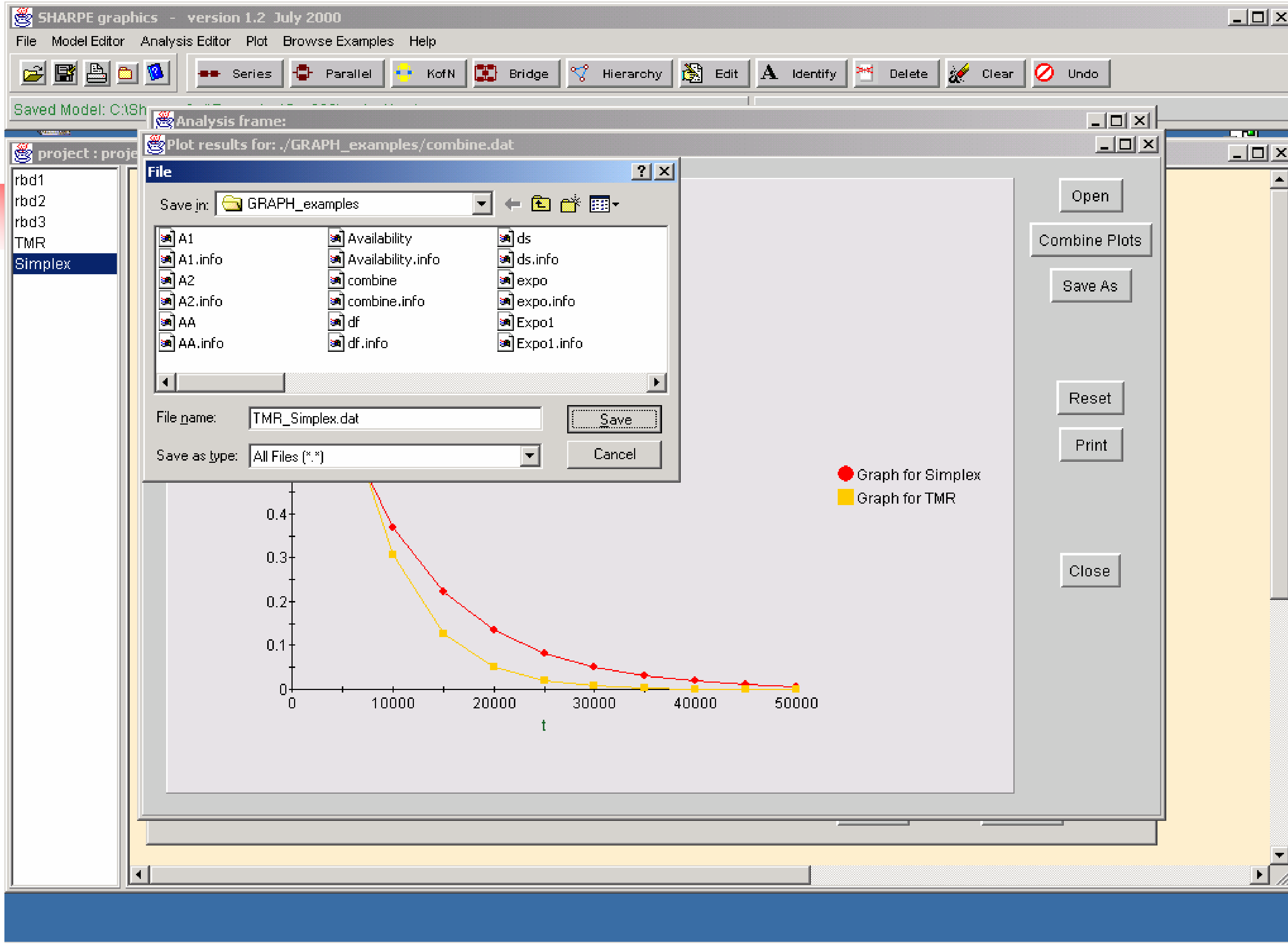
Graph for Simplex
Graph for TMR


SHARPE graphics - version 1.2 July 2000
File
Model Editor
Analysis Editor
Plot
Browse Examples
Help
Series
Parallel
KofN
Bridge
Hierarchy
Edit
Identify
Delete
Clear
Undo
Analysis frame:
Plot results for: ./GRAPH_examples/combine.dat
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rbd1
rbd2
rbd3
TMR
Simplex
File
Save in: GRAPH_examples
A1
A1.info
A2
A2.info
AA
AA.info
Availability
Availability.info
combine
combine.info
df
df.info
ds
ds.info
expo
expo.info
Expo1
Expo1.info
File name: TMR_Simplex.dat
Save as type: All Files (*.*)
Save
Cancel
Open
Combine Plots
Save As
Reset
Print
Close
Graph for Simplex
Graph for TMR
0.4
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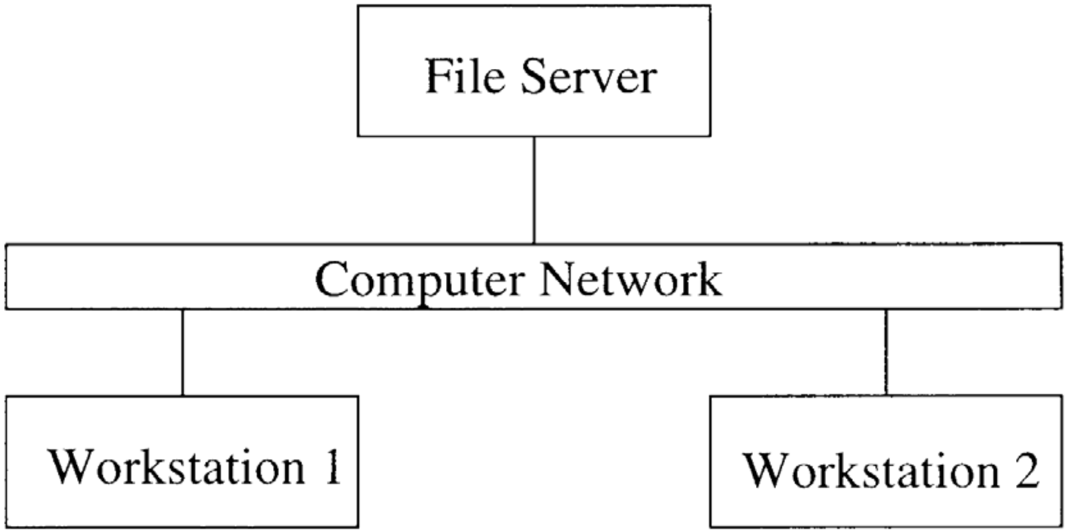
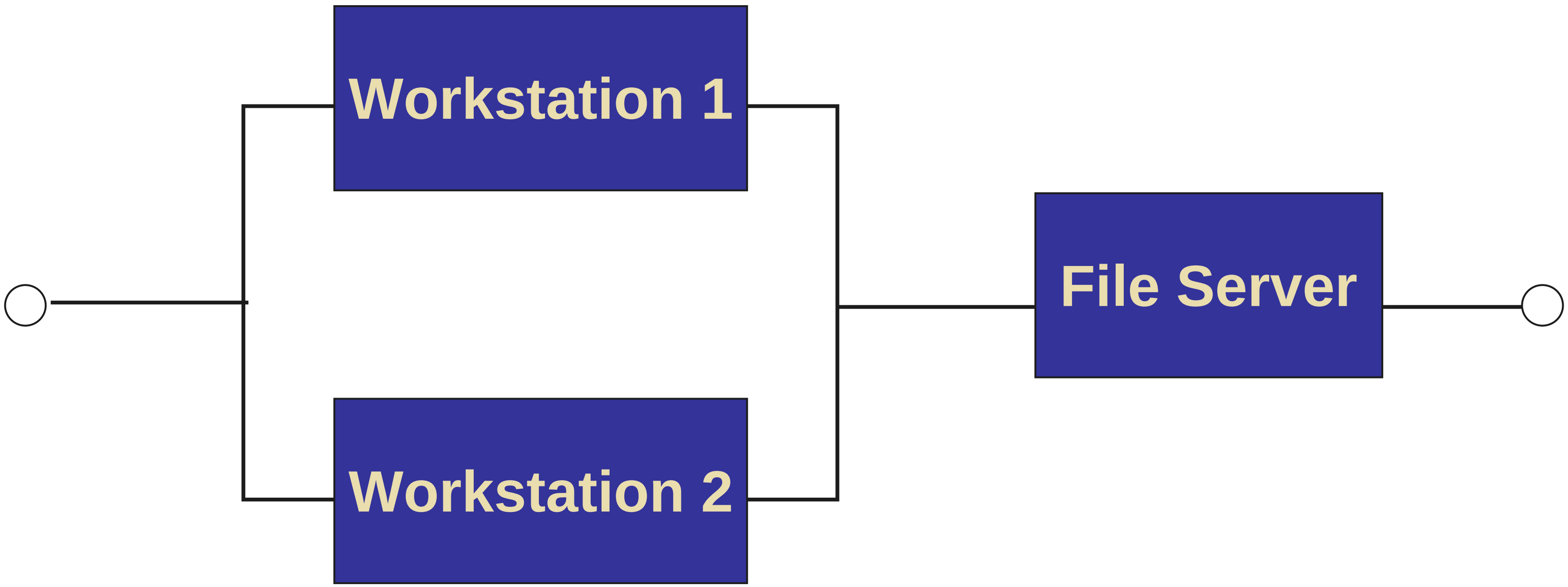
Workstations & File server (WFS) Example – RBD Approach

The WFS Example



- Computing system consisting of:
  - A file-server
  - Two workstations
  - Computing network connecting them

RBD for the WFS Example



RBD for the WFS Example

$R_w(t)$: workstation reliability

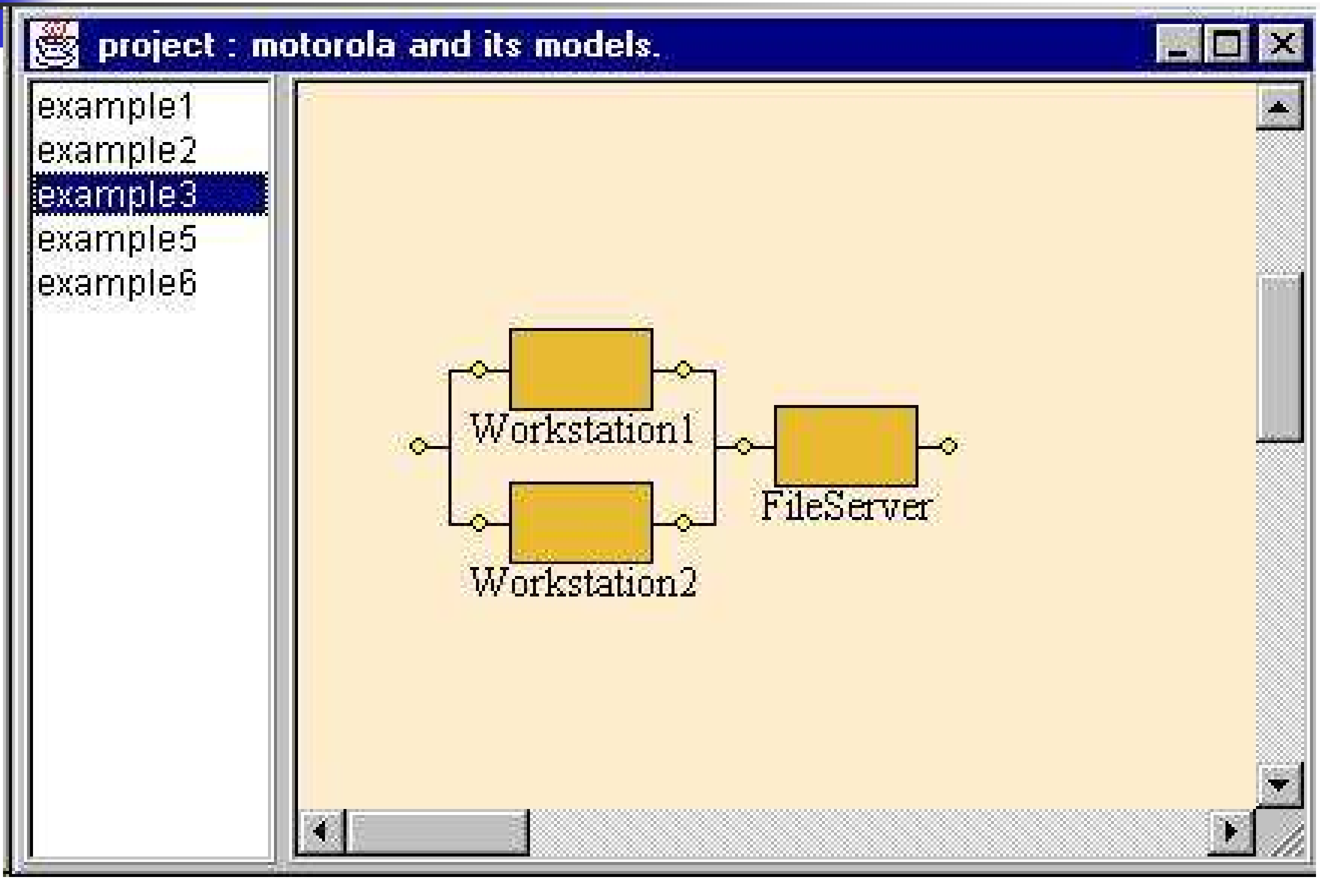
$R_f(t)$: file-server reliability

System reliability $R(t)$ is given by:

$$R(t) = [1 - (1 - R_w(t))^2]\, R_f(t)$$

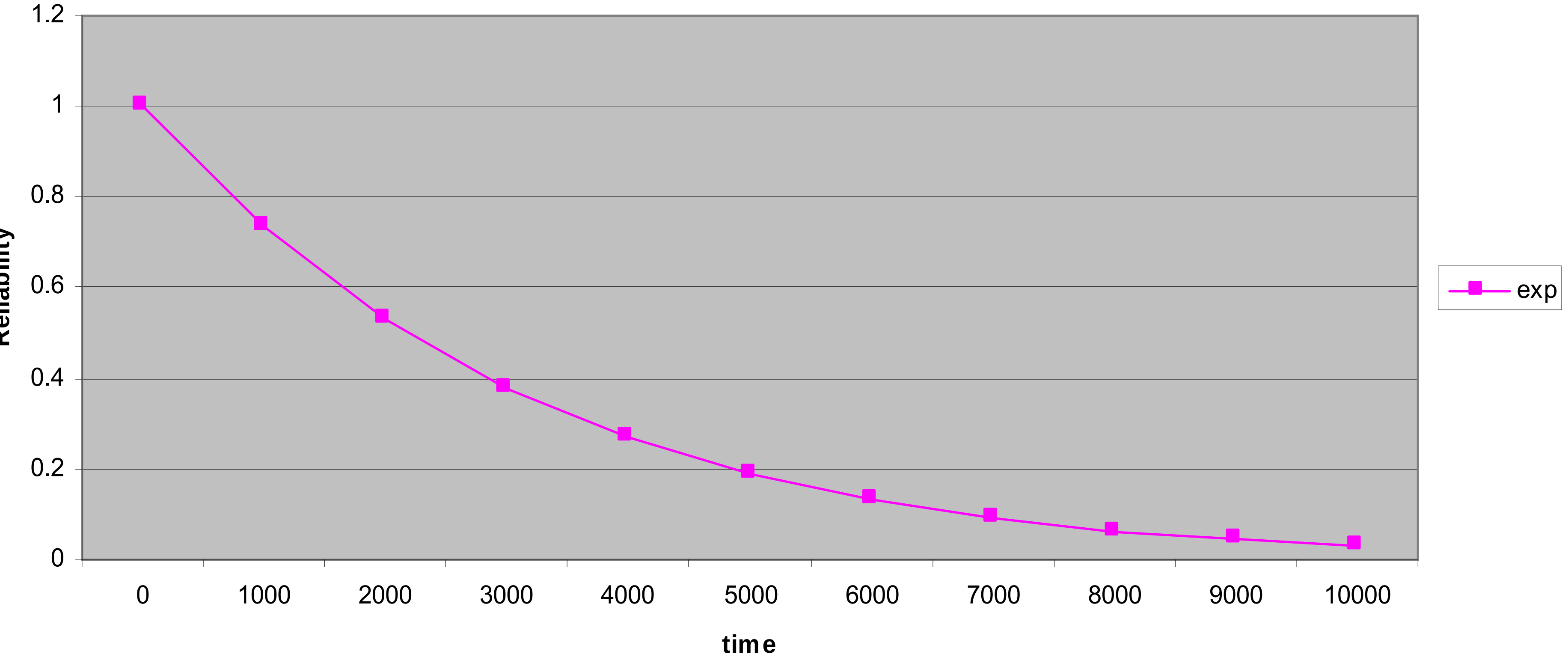
Note: applies to any time-to-failure distributions

Snapshot of the GUI



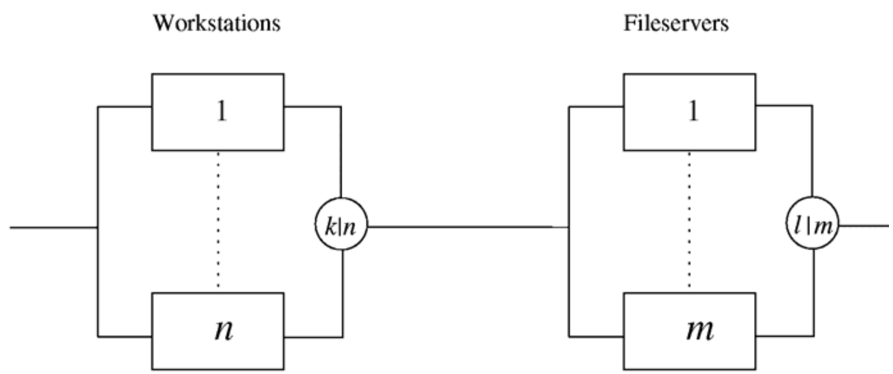
```
bind
lambdaW 0.0001
lambdaF 0.0003
end
block wfs1
* each component is non-restorable and has exp time to fail dist
comp Workstation exp(lambdaW)
comp FileServer exp(lambdaF)
parallel work Workstation Workstation
series sys work FileServer
end
* define function R(t) for reliability at time t
func R(t)  1-tvalue(t;wfs1)
* vary the time t from t=0 to 10000 in steps of 1000 hours and print R(t)
loop t,0,10000,1000
  expr R(t)
end
end
```

R(t) vs. time


1.2
1
0.8
0.6
0.4
0.2
0
Reliability
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000
time
exp

Example 3.21

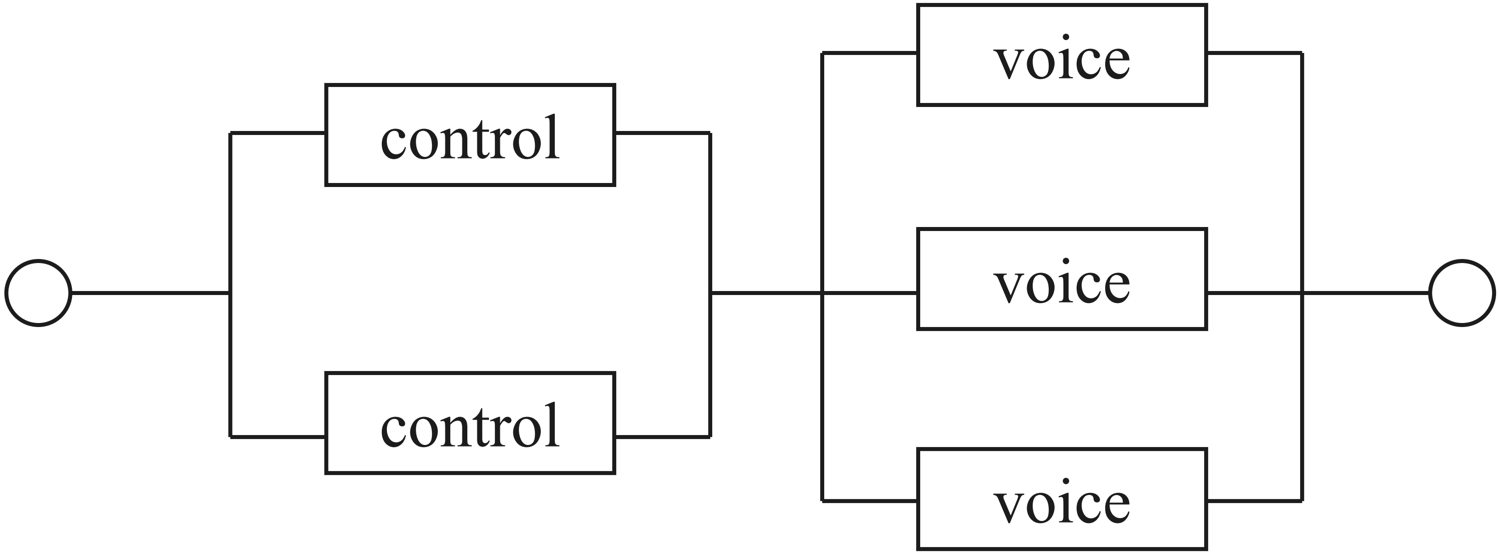
- A system with *n* workstations and *m* file servers



- System Operational: *k* workstation & *j* file servers
- WS reliability is $R_w(t)$ and FS reliability is $R_f(t)$
- Assuming all devices fail independently,

$$R(t) = \sum_{j=k}^{n} \binom{n}{j} [R_w(t)]^j [1 - R_w(t)]^{n-j} \sum_{j=l}^{m} \binom{m}{j} [R_f(t)]^j [1 - R_f(t)]^{m-j}.$$

2 Control and 3 Voice Channels Example

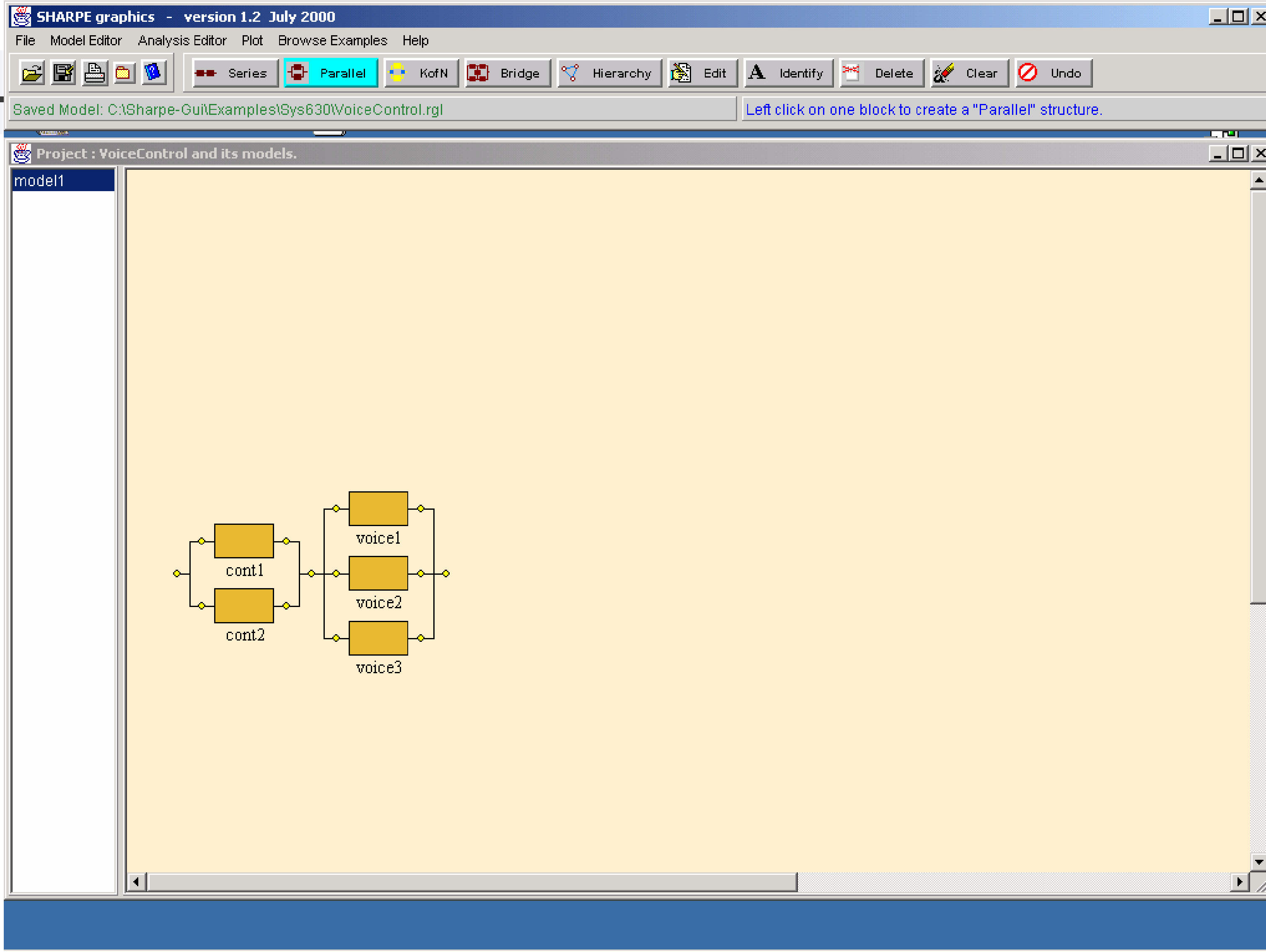


Description

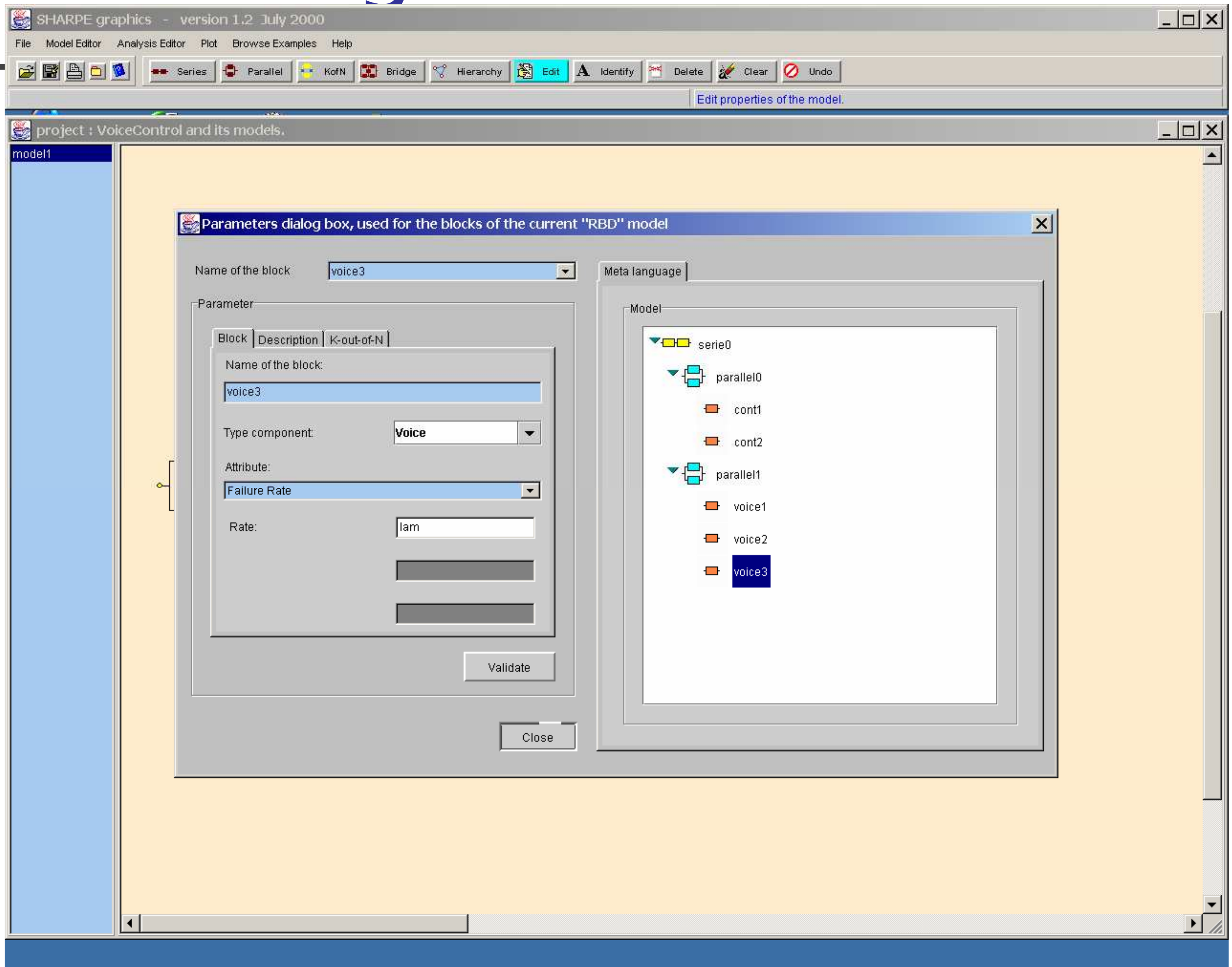
- Each control channel has a reliability $R_c(t)$
- Each voice channel has a reliability $R_v(t)$
- System is up if at least one control channel and at least 1 voice channel are up.
- Reliability:

$$R(t) = [1-(1-R_c(t))^2][1-(1-R_v(t))^3]$$

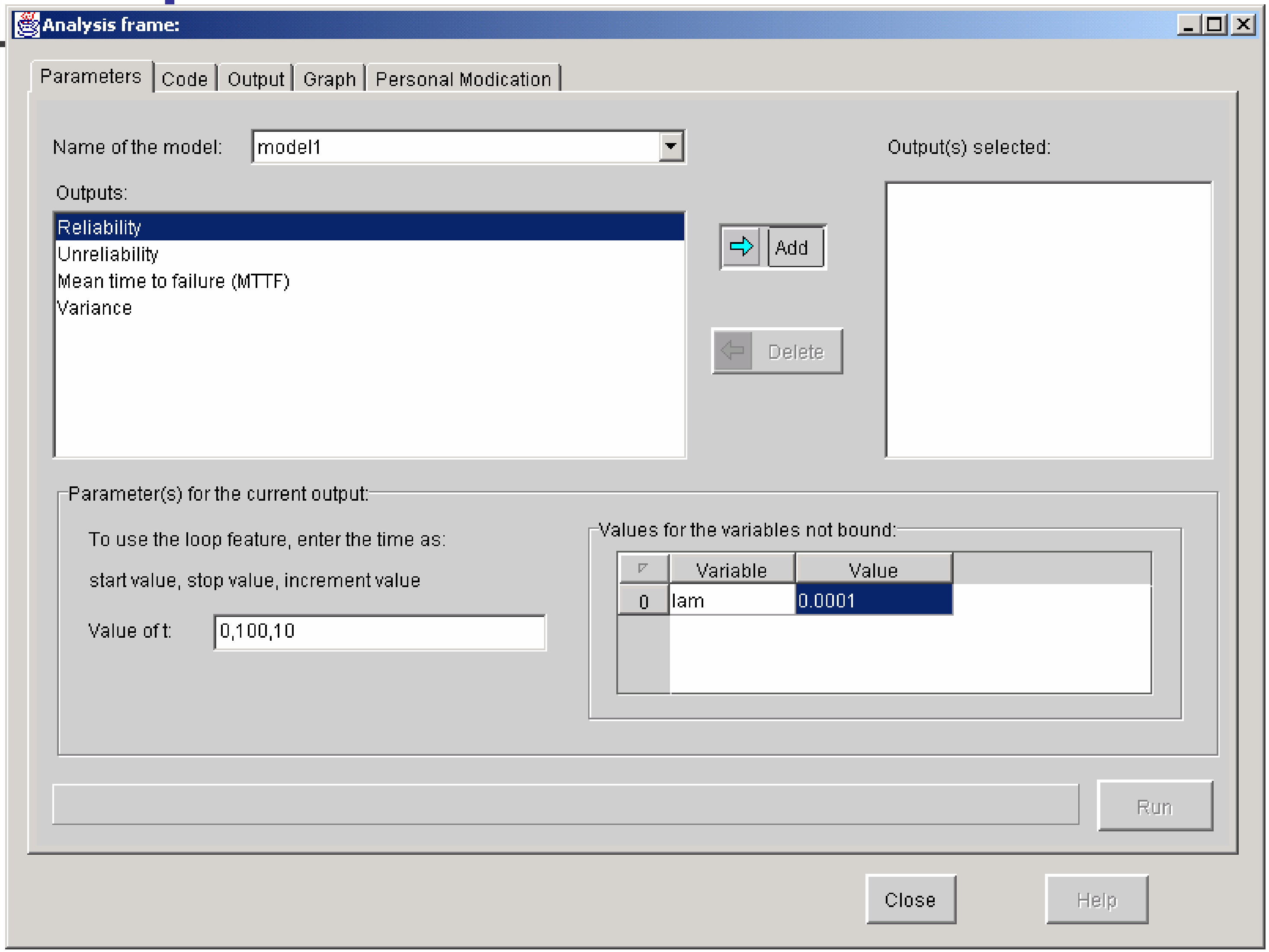
Reliability block diagram model in SHARPE GUI



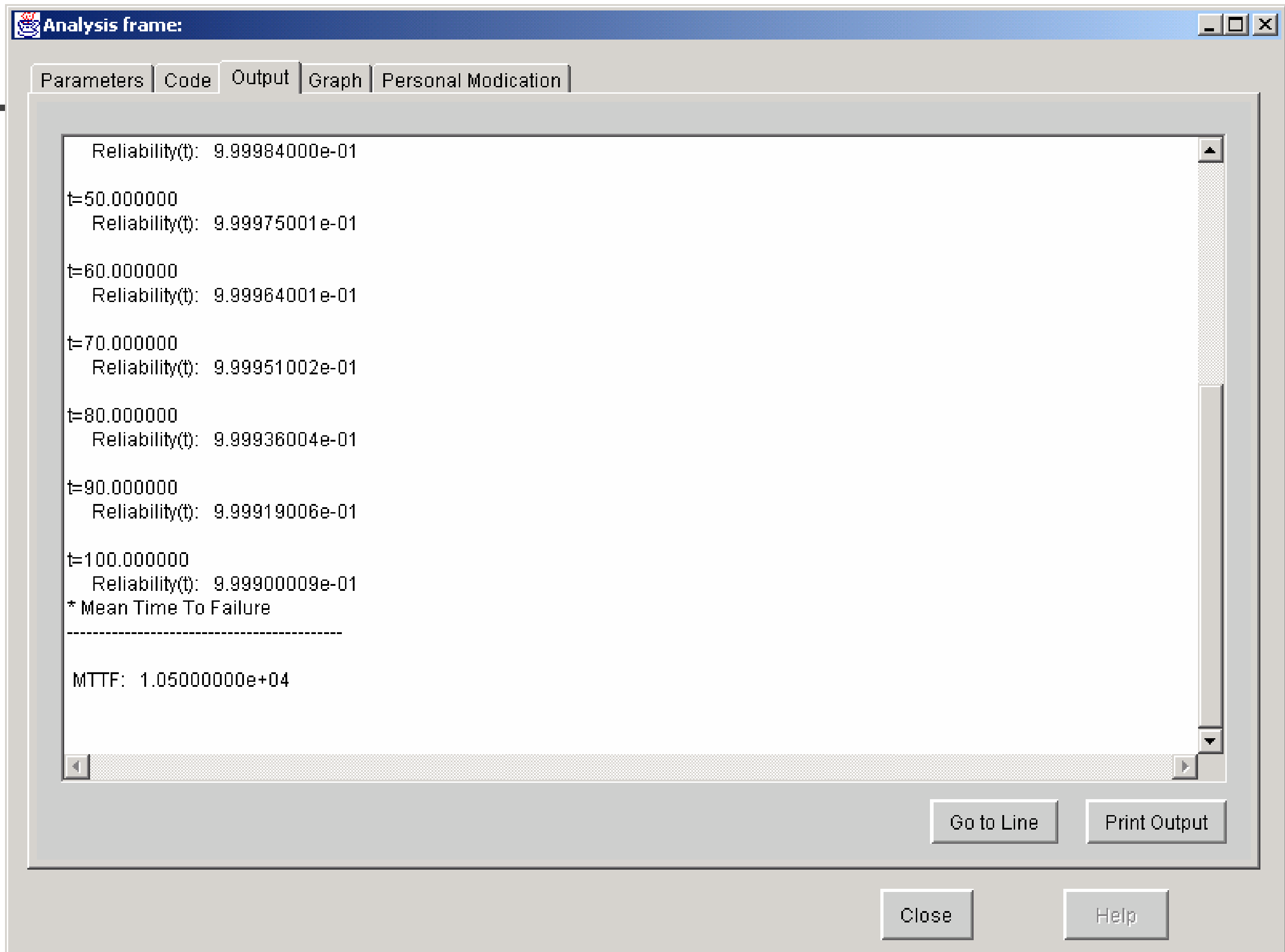
Define the components in the block diagram model



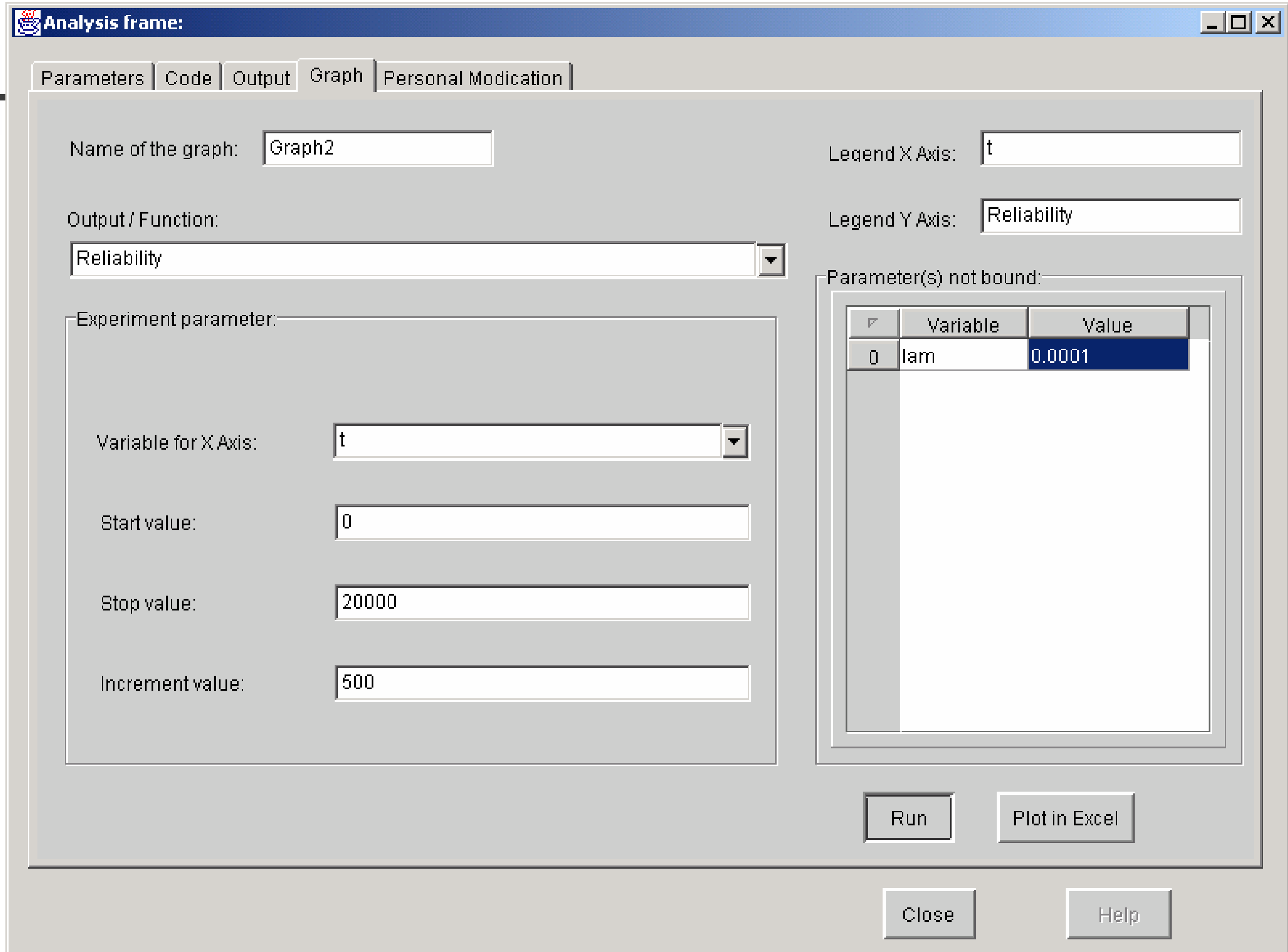
Output selection



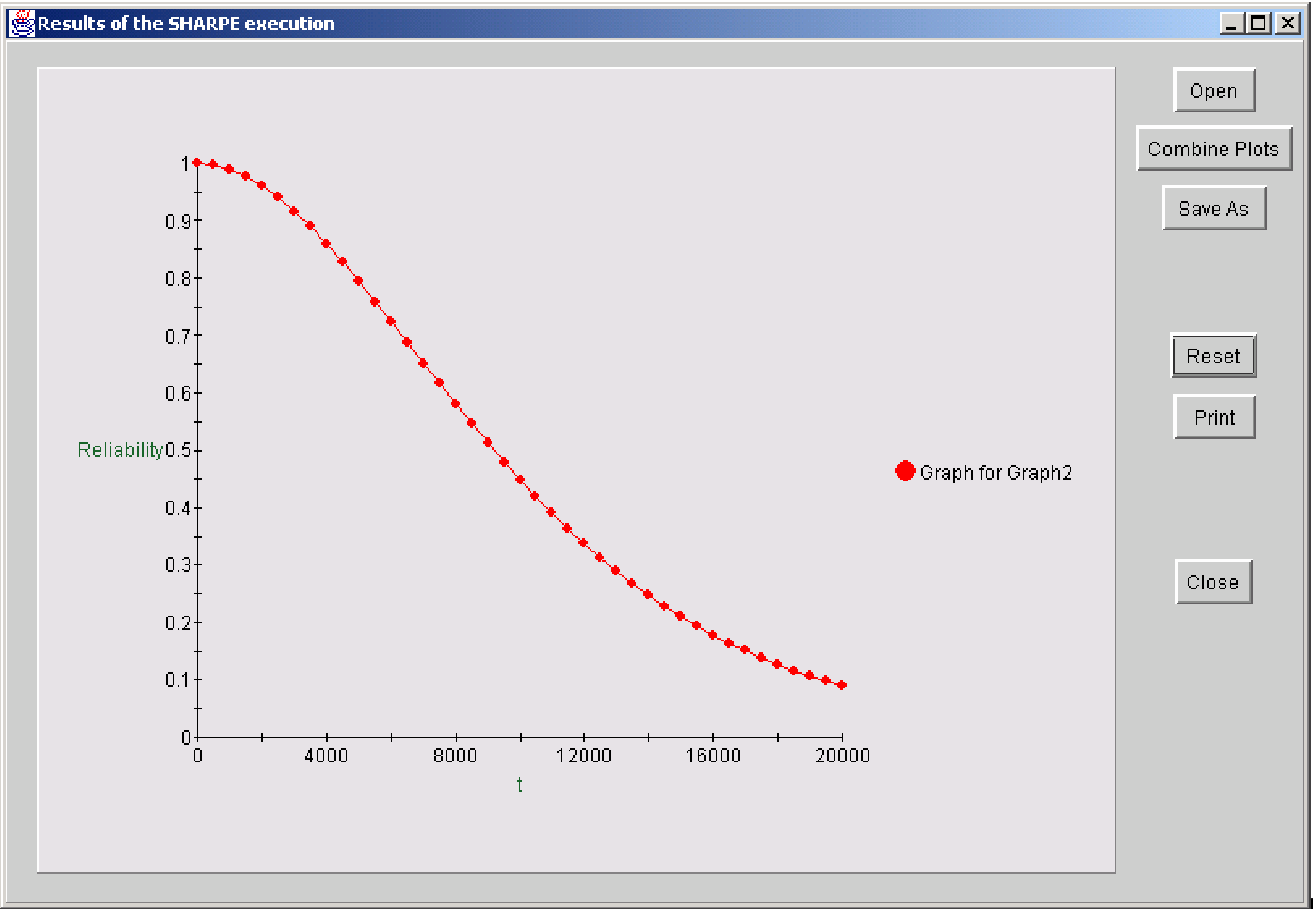
Results from SHARPE



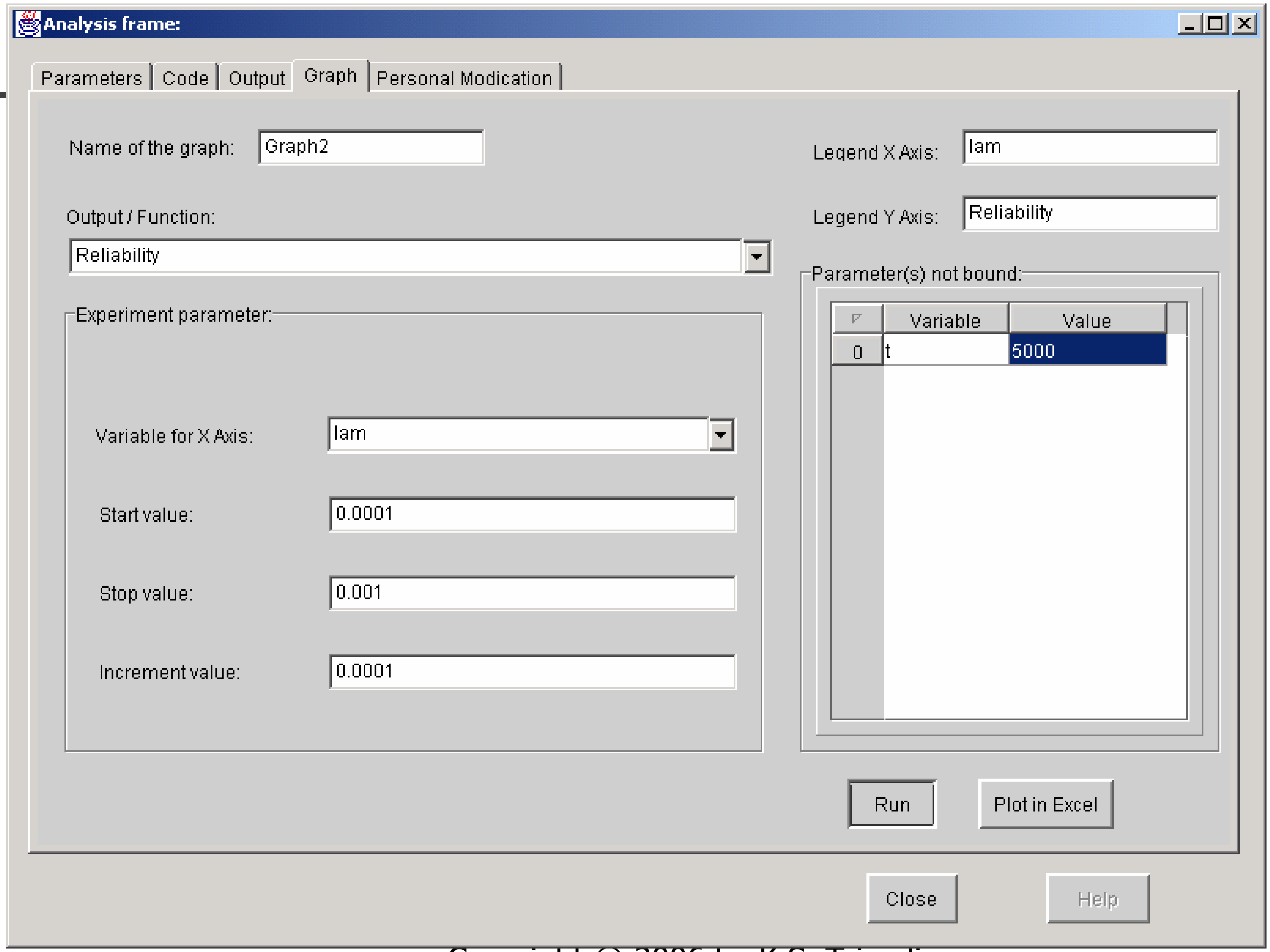
Plot definition



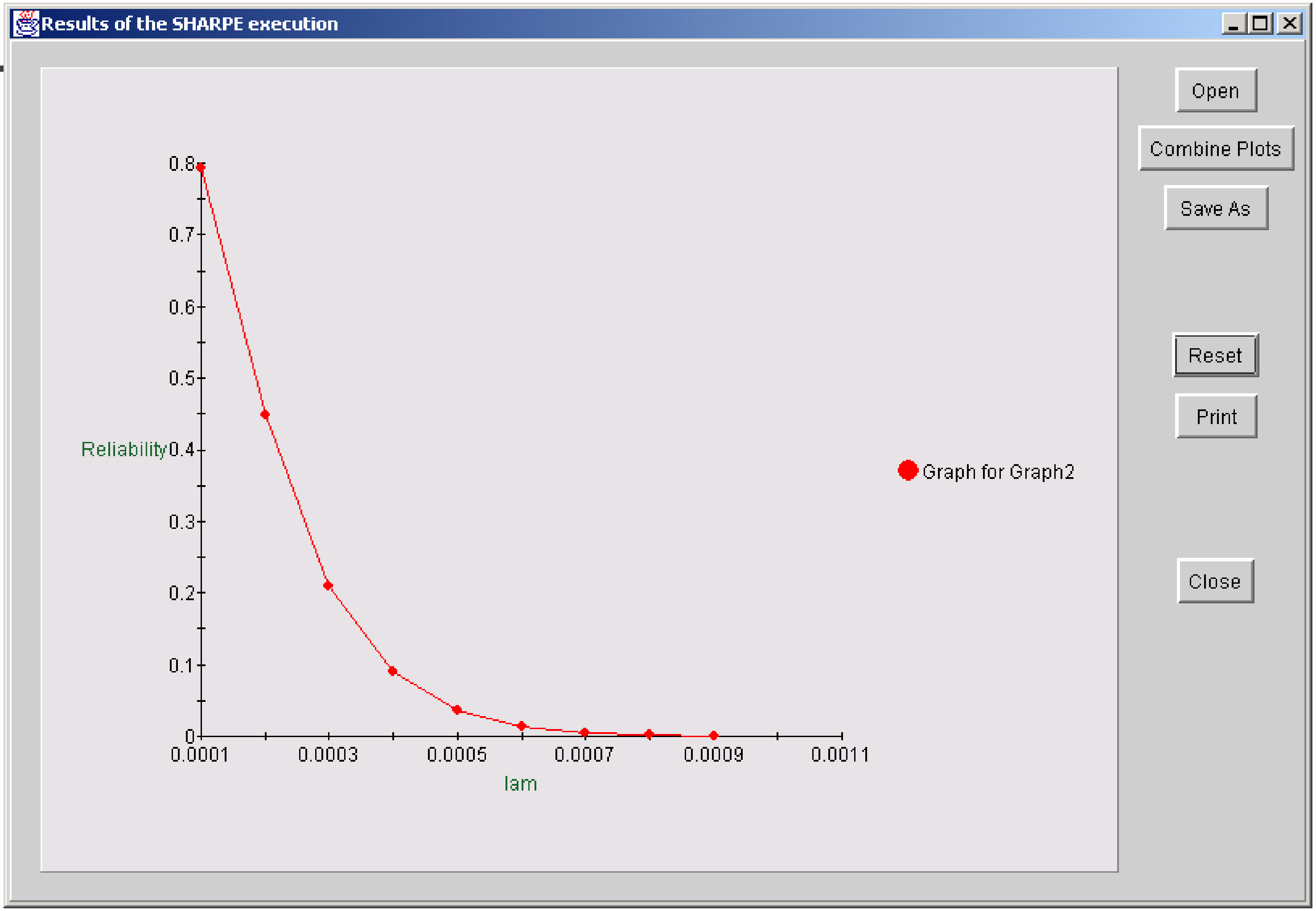
Reliability vs. time



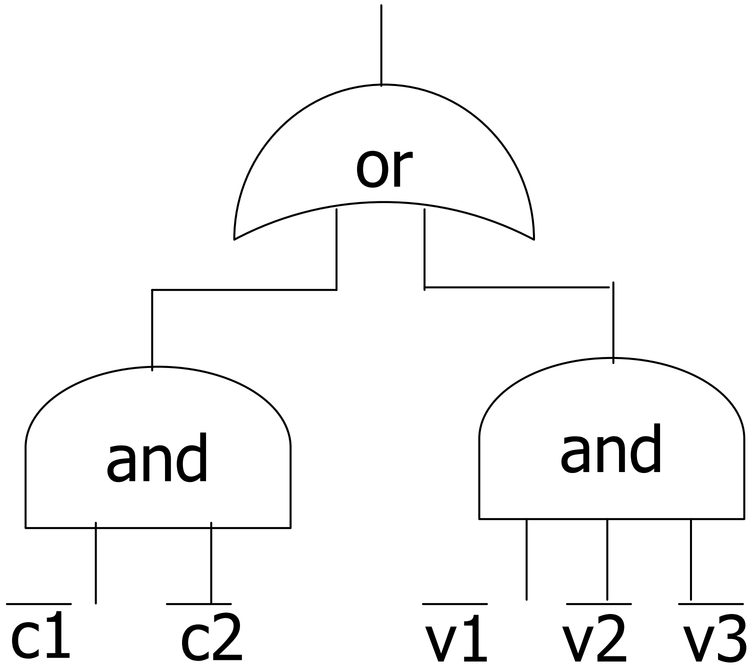
Definition of another plot



Reliability vs. lambda



2c3v as a Fault Tree



- Structure Function:

$$\overline{\phi} = \overline{c_1} \cdot \overline{c_2} + \overline{v_1} \cdot \overline{v_2} \cdot \overline{v_3}$$

2 Control and 3 Voice Channels Example

Fault Tree Example (contd.)

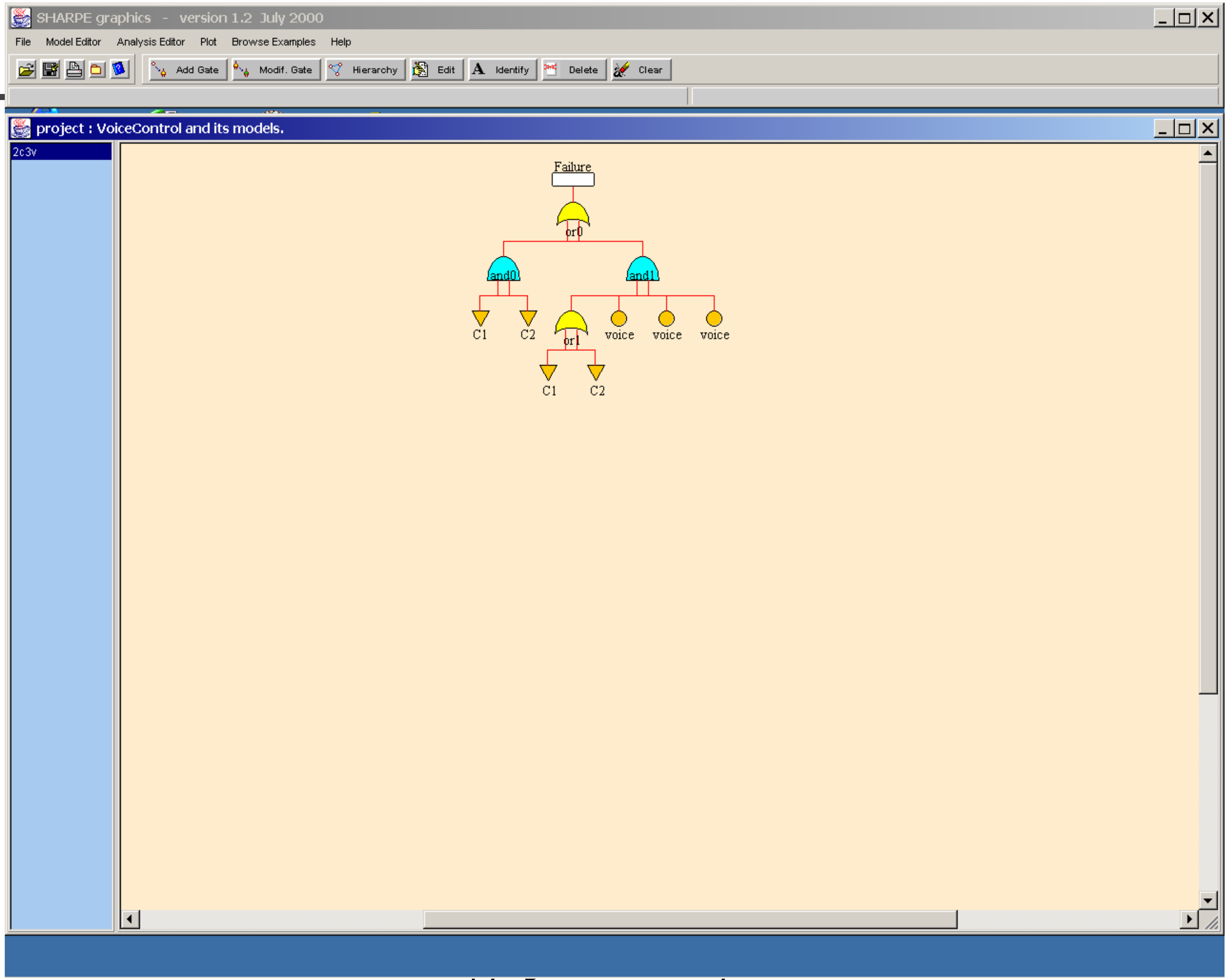
- Reliability of the system:

Assume $R_c(t) = e^{-\lambda_c t}$ *and* $R_v(t) = e^{-\lambda_v t}$,

$$R(t) = [1-(1-R_c(t))^2][1-(1-R_v(t))^3]$$

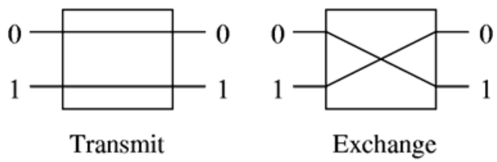
$$= (2e^{-\lambda_c t} - e^{-2\lambda_c t})(3e^{-\lambda_v t} - 3e^{-2\lambda_v t} + e^{-3\lambda_v t})$$

Fault tree input in SHARPE GUI

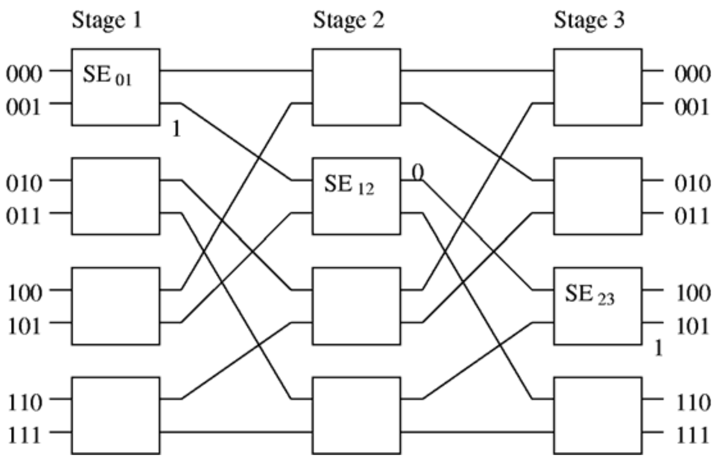


Example 3.22

- Shuffle exchange network (SEN) with $N = 2^n$ inputs
- $(N/2)$ switching elements/stage; $\log_2 N$ such stages



Single switch element



8 x 8 SEN

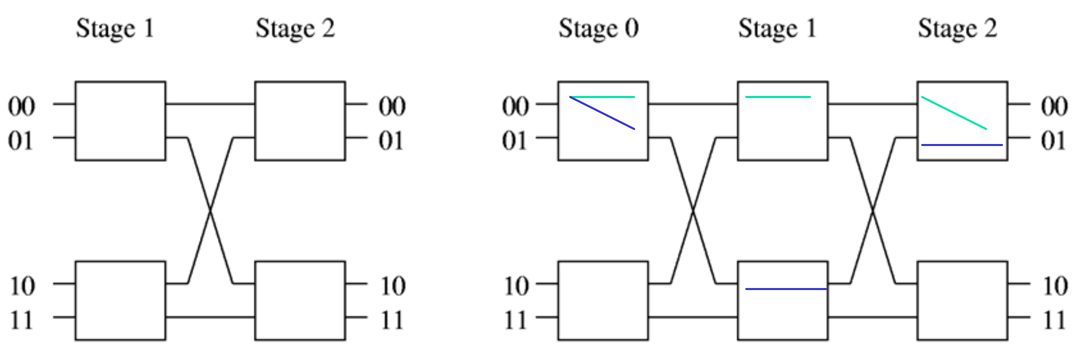
Example 3.22 (contd.)

- We are interested in finding the reliability $R_{SEN}(t)$ of this SEN given individual switch reliability $r_{SE}(t)$

$$\begin{aligned} R_{\text{SEN}}(t) &= \text{Reliability of a series system of} \\ &\quad (N/2)\log_2 N \text{ elements} \\ &= [r_{\text{SE}}(t)]^{(N/2)\log_2 N} \end{aligned}$$

Example 3.23

- SEN+ has an extra stage of N/2 switching elements to increase reliability; 00 → 01 has two paths.



(a) (b)

- $r_{SE}(t)$: time-dependent reliability of an SE
  - SEN: (N/2) $\log_2 N$ = 4 elements

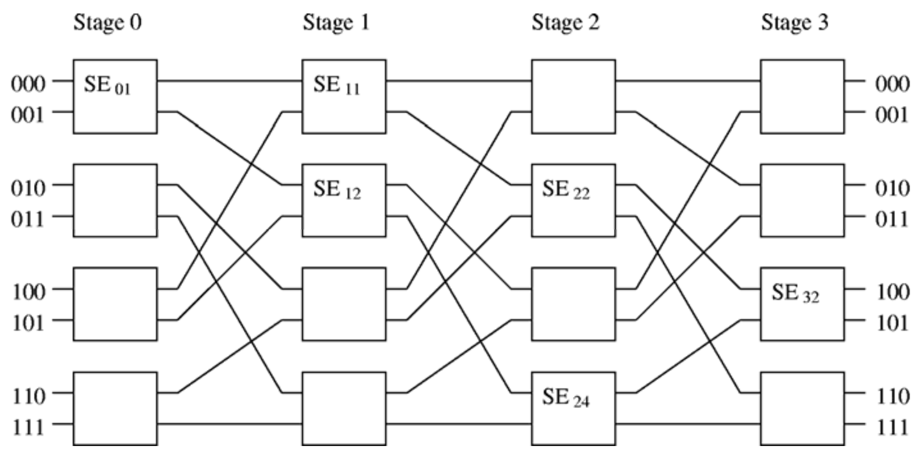
$$R_{\mathsf{SEN}}(t) = [r_{\mathsf{SE}}(t)]^4$$

Example 3.23 (contd)

- SEN+ reliability in this case is worse that SEN

$$R_{\text{SEN+}}(t) = [r_{\text{SE}}(t)]^4[1-(1-[r_{\text{SE}}(t)]^2].$$

- For 8X8 case, SEN+ reliability works out to be better than SEN
- 8x8 SEN+



$$R_{\text{SEN+}}(t) = 2[r_{\text{SE}}(t)]^{12}+4[r_{\text{SE}}(t)]^{14}-8[r_{\text{SE}}(t)]^{15}+3[r_{\text{SE}}(t)]^{16} \geq [r_{\text{SE}}(t)]^{12} = R_{\text{SEN}}(t).$$

Overview: Exponential iid Random Variables

Let $Y_1, \ldots, Y_n$ denote the order statistics of the random variables $X_1, \ldots, X_n$, which are iid with common distribution function $F(x) = 1 - \exp(-\lambda x)$.

Y_1	$F_{Y_1}(y) = 1 - [\exp(-\lambda y)]^n$ $= 1 - \exp(-n\lambda y)$	$R_{series}(t) = 1 - F_{Y_1}(t)$ $= \exp(-n\lambda t)$	$Y_1 \sim \text{EXP}(n\lambda)$
Y_{n-k+1}	$F_{Y_{n-k+1}}(y) = \sum_{j=n-k+1}^{n} \binom{n}{j} [1 - \exp(-\lambda y)]^j \times \exp(-(n-j)\lambda y)$	$R_{k\|n}(t) = 1 - F_{Y_{n-k+1}}(t)$	
Y_n	$F_{Y_n}(y) = [1 - \exp(-\lambda y)]^n$	$R_{parallel}(t) = 1 - F_{Y_n}(t)$ $= 1 - [1 - \exp(-\lambda t)]^n$	

Overview: Exponential iid Random Variables

Let $Y_1, \ldots, Y_n$ denote the order statistics of the random variables $X_1, \ldots, X_n$, which are iid with common distribution function $F(x) = 1 - \exp(-\lambda x)$.

Y_1	$F_{Y_1}(y) = 1 - [\exp(-\lambda y)]^n$ $= 1 - \exp(-n\lambda y)$	$R_{series}(t) = 1 - F_{Y_1}(t)$ $= \exp(-n\lambda t)$	$Y_1 \sim \mathrm{EXP}(n\lambda)$
Y_{n-k+1}	$F_{Y_{n-k+1}}(y)$ $= \sum_{j=n-k+1}^{n} \binom{n}{j} [1 - \exp(-\lambda y)]^j$ $\times \exp(-(n-j)\lambda y)$	$R_{k\|n}(t) = 1 - F_{Y_{n-k+1}}(t)$	Y_{n-k+1} $\sim \mathrm{HYPO}(n\lambda,$ $(n-1)\lambda, \ldots, k\lambda)$
Y_n	$F_{Y_n}(y) = [1 - \exp(-\lambda y)]^n$	$R_{parallel}(t) = 1 - F_{Y_n}(t)$ $= 1 - [1 - \exp(-\lambda t)]^n$	$Y_n \sim \mathrm{HYPO}(n\lambda,$ $(n-1)\lambda, \ldots, \lambda)$

Overview: Exponential Independent Random Variables

Let Y_1 ,,..., Y_n denote the order statistics of the independent random variables X_1 ,..., X_n. The distribution function of X_i is $F(x_i)=1-\exp(\lambda_i x)$.

Y_1	$F_{Y_1}(y)=1-\prod_{i=1}^{n}\exp(-\lambda_i y)$ $=1-\exp(-y\sum_{i=1}^{n}\lambda_i)$	$R_{series}(t)=1-F_{Y_1}(t)$ $=\exp(-t\sum_{i=1}^{n}\lambda_i)$	$Y_1 \sim \mathrm{EXP}(\sum_{i=1}^{n}\lambda_i)$
Y_{n-k+1}	*Complicated…*	*Complicated…*	*Complicated…*
Y_n	$F_{Y_n}(y)=\prod_{i=1}^{n}F_i(y)$ $=\prod_{i=1}^{n}[1-\exp(-\lambda_i y)]$	$R_{parallel}(t)=1-F_{Y_n}(t)$ $=1-\prod_{i=1}^{n}[1-\exp(-\lambda_i t)]$	*Complicated…*

Sum of Random Variables

- $Z = \Phi(X, Y)$

$$F_Z(z) = P(Z \le z) = \int\int_{A_z} f(x,y)dxdy$$

$$A_z \subset \Re^2 = \{(x,y)|\Phi(x,y) \le z\} = \Phi^{-1}((-\infty, z])$$

$$A_z = \{(x,y)|x + y \le z\}$$

- For the special case, $Z = X + Y$
- The resulting *pdf is (assuming independence),*

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx, \ -\infty < z < \infty$$

- Convolution integral (modify for the non-negative case)

Convolution (non-negative case)

$Z = X + Y$, X & Y are independent random variables (in this case, non-negative)

$$f_Z(t) = \int_0^t f_X(x)\, f_Y(t-x)\, dx$$

- The above integral is often called the convolution of f_X and f_Y. Thus the density of the sum of two non-negative independent, continuous random variables is the convolution of the individual densities.

Example 3.24: Multithreaded program performance

- Three independent computer tasks τ_1, τ_2 and τ_3
- Precedence relationship: τ_3 has to wait for both τ_1 and τ_2 to complete
- T_1, T_2 and T_3: respective random execution times
- Total execution time = $\max\{T_1, T_2\} + T_3 = M + T_3$
- T_1 and $T_2 \sim$Unif$(t_1 - t_0, t_1 + t_0)$; $T_3 \sim$Unif$(t_3 - t_0, t_3 + t_0)$
- Find Probability that $T > t_1 + t_3$

Example 3.24 (contd)

- The *pdf*s are:

$$
\begin{aligned}
f_{T_1}(t) &= f_{T_2}(t) \\
&= \begin{cases} \dfrac{1}{2t_0}, & t_1 - t_0 < t < t_1 + t_0, \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}
$$

$$
f_{T_3}(t) = \begin{cases} \dfrac{1}{2t_0}, & t_3 - t_0 < t < t_3 + t_0, \\ 0, & \text{otherwise.} \end{cases}
$$

CDF for M is

$$
\begin{aligned}
F_M(m) &= P(M \le m) = P(\max\{T_1, T_2\} \le m) \\
&= P(T_1 \le m \text{ and } T_2 \le m) \\
&= P(T_1 \le m)P(T_2 \le m) \quad \text{by independence} \\
&= F_{T_1}(m)F_{T_2}(m).
\end{aligned}
$$

Example 3.24 (contd)

CDF for T_1 (and T_2) is,

$$F_{T_1}(t) = \begin{cases} 0, & t < t_1 - t_0, \\ \dfrac{t - t_1 + t_0}{2t_0}, & t_1 - t_0 \le t < t_1 + t_0 \\ 1, & \text{otherwise.} \end{cases}$$

$F_M(m)$ can now be written as,

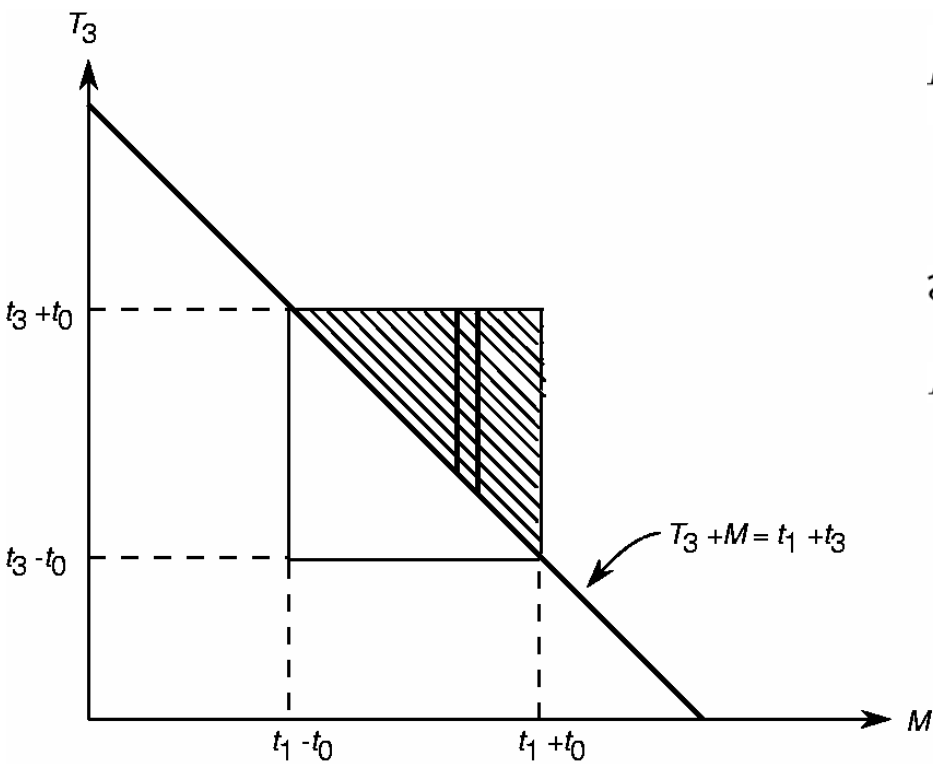
$$F_M(m) = \begin{cases} 0, & m < t_1 - t_0, \\ \dfrac{(m - t_1 + t_0)^2}{4t_0^2}, & t_1 - t_0 \le m < t_1 + t_0, \\ 1, & \text{otherwise.} \end{cases}$$

or,

$$f_M(m) = \begin{cases} \dfrac{m - t_1 + t_0}{2t_0^2}, & t_1 - t_0 < m < t_1 + t_0, \\ 0, & \text{otherwise.} \end{cases}$$

Example 3.24 (contd)

- We need to find $P(T > t_1 + t_3)$; 'A' : shaded area



$$P(T > t_1 + t_3) = \iint_A f_{M,T_3}(m,t)\ dm\ dt$$
$$= \iint_A f_M(m) f_{T_3}(t)\ dm\ dt$$

Since M and T_3 are independent,

$$P(T > t_1 + t_3) =$$
$$\int_{t_1 - t_0}^{t_1 + t_0} \left(\int_{t_1 + t_3 - m}^{t_3 + t_0} \frac{m - t_1 + t_0}{4t_0^3}\ dt \right)\ dm = \frac{2}{3}$$

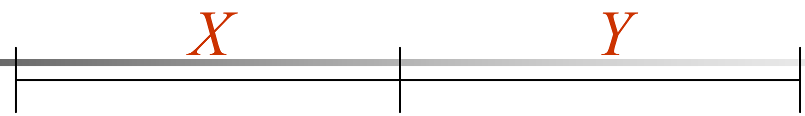
Reliability Modeling Examples

- Sums of exponential random variables appear naturally in reliability modeling
  - Cold-standby redundancy
  - Warm-standby redundancy
  - Hot-standby redundancy
  - Triple Modular Redundancy (TMR)
  - TMR/Simplex
  - *k-out-of-n* Redundancy

Two component system

- With respective lifetime random variable, X and Y, assumed independent
  - Series system ($Z=min\{X,Y\}$)
  - Parallel System ($Z=max\{X,Y\}$)
  - Cold standby: the lifetime $Z=X+Y$

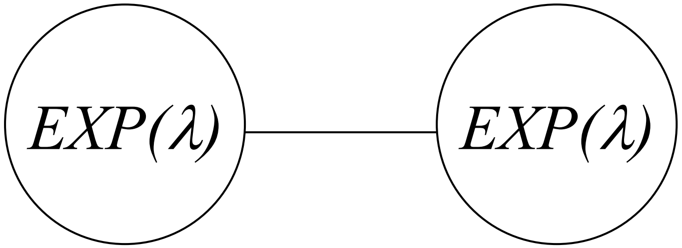
Cold standby (standby redundancy)



Lifetime of Active $EXP(\lambda)$

Lifetime of Spare $EXP(\lambda)$

Total lifetime 2-Stage Erlang



$$R(t) = (1 + \lambda t)e^{-\lambda t}$$

Assumptions (to be relaxed later):

- Detection & Switching perfect;
- Spare does not fail.

Cold standby derivation

- *X* and *Y* are both *EXP(λ)* and independent.
- Then $$f_Z(t) = \int_0^t \lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)} dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^t dx$$

$$= \lambda^2 t e^{-\lambda t}, \quad t > 0$$

Cold standby derivation (Continued)

- Z is two-stage Erlang Distributed

$$F_Z(t) = \int_0^t f_Z(z)dz = 1 - (1 + \lambda t)e^{-\lambda t}$$

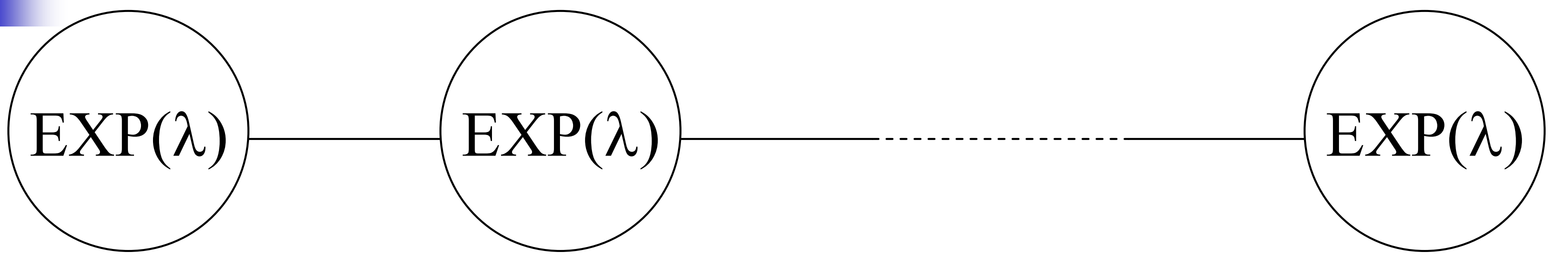
$$R(t) = 1 - F(t)$$

$$= (1 + \lambda t)e^{-\lambda t}, \quad t \geq 0$$

Convolution: r-stage Erlang

- The general case of *r*-stage Erlang Distribution
  - When *r* sequential phases have independent identical exponential distributions, then the resulting random variable is known as *r*-stage (or *r*-phase) Erlang and is given by:

Convolution: Erlang (Continued)



$$f(t) = \frac{\lambda^r t^{r-1} e^{-\lambda t}}{(r-1)!}$$

$$F(t) = 1 - \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Standby Sparing Example 3.26

- System with *n* processors whose lifetimes are iid, following an exponential distribution with failure rate λ.
- Two modes
  - Only *1* of *n* is active, others are cold standby
  - All *n* are active, working in parallel
- We can see that: $R_{standby}(t) \geq R_{parallel}(t)$

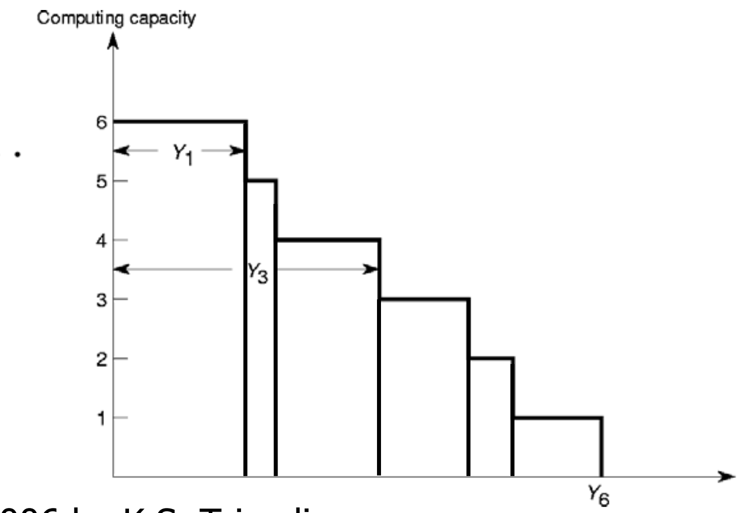
since
$$\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \geq 1-(1-e^{-\lambda t})^n$$

- However, parallel arrangement delivers more capacity.

Derivation of the result

- Let $X_1, X_2, \ldots, X_n$ be the times to failure random variables of the n processors
- At time $Y_1 = \min\{X_1, X_2, \ldots, X_n\}$, one processor has failed and remaining $(n-1)$ are working
- Computing capacity will also drop to $(n-1)$,

$$C_n = nY_1 + (n-1)(Y_2 - Y_1) + \ldots$$
$$+ (n-j)(Y_{j+1} - Y_j) + \ldots$$
$$+ (Y_n - Y_{n-1})$$



Derivation of the result (contd)

- From the diagram, C_n is the area under the curve and we wish to find distribution for C_n.
- First find distribution for $Y_{j+1}-Y_j$
- Assume that all processor lifetimes are $EXP(\lambda)$, then we assert, $(Y_{j+1}-Y_j) \sim EXP[(n-j)\,\lambda]$.
- Assume $Y_0 = 0$,

  $$(Y_1-Y_0)=Y_1=\min\{X_1, X_2, \ldots, X_n\} \sim EXP(n\lambda)$$

  Hence, assertion is true for $j=0$.
- After j procs have failed, the residual lifetimes are W_1, $W_2, \ldots, W_{n-j}$, *each of which is* $EXP(\lambda)$ *due to the memoryless property of the exponential distribution.*

Derivation of the result (contd2)

- $(Y_{j+1}-Y_j)$ is then given by,

  $$(Y_{j+1}-Y_j) = \min\{W_1, W_2, \ldots, W_{n-j}\}$$

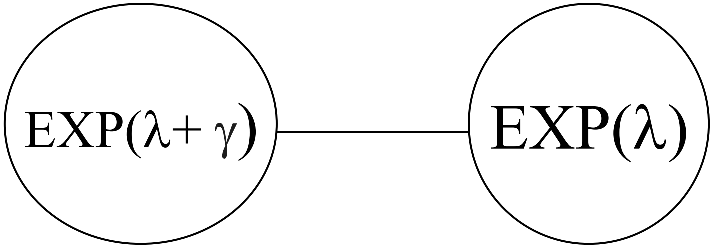
- $(Y_{j+1}-Y_j) \sim EXP[(n-j)\,\lambda]$ using result of Example 3.16
- Using the result of Example 3.13,

  $$(n-j)\,(Y_{j+1}-Y_j) \sim EXP(\lambda).$$

- Therefore, C_n is the sum of n independent identically distributed exponential rv's or C_n is n-stage Erlang.
- Thus the *total* computing capacity delivered before failure has the same distribution in both the modes of operation.

Warm standby

- With Warm spare, we have:
  - Active unit time-to-failure: EXP(λ)
  - Spare unit time-to-failure: EXP(γ)



2-stage hypoexponential distribution

Warm standby derivation

- First event to occur is that either the active or the spare will fail. Time to this event is $\min\{EXP(\lambda), EXP(\gamma)\}$ which is $EXP(\lambda + \gamma)$.
- Then due to the memoryless property of the exponential, remaining lifetime is still $EXP(\lambda)$.
- Hence system lifetime has a two-stage hypoexponential distribution with parameters

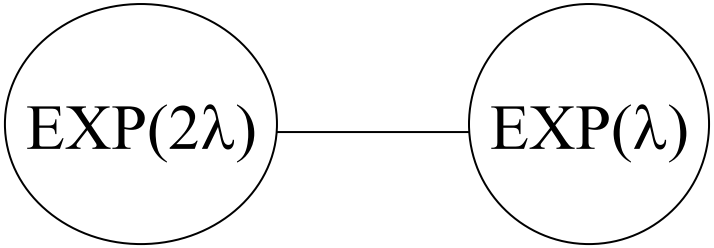
  $\lambda_1 = \lambda + \gamma$ and $\lambda_2 = \lambda$.

Warm standby derivation (Continued)

- X is EXP(λ_1) and Y is EXP(λ_2) with $\lambda_1 \neq \lambda_2$; X and Y are independent.
- Then $$f_Z(t) = \int_0^t \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 (t-x)} dx$$ $$= \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_2 t} + \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t}.$$
- This is the density of the 2-stage hypo-exponential distribution with parameters λ_1 and λ_2.

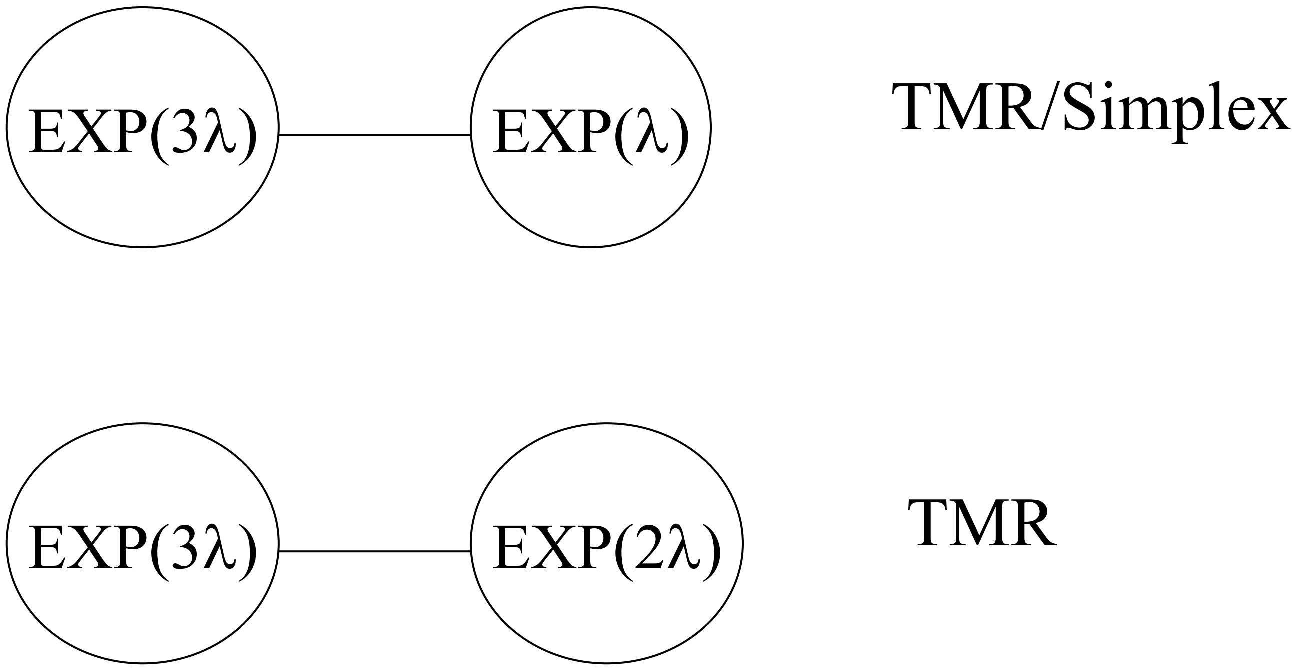
Hot standby (Active/Active)

- With hot spare, we have:
  - Active unit time-to-failure: EXP(λ)
  - Spare unit time-to-failure: EXP(λ)



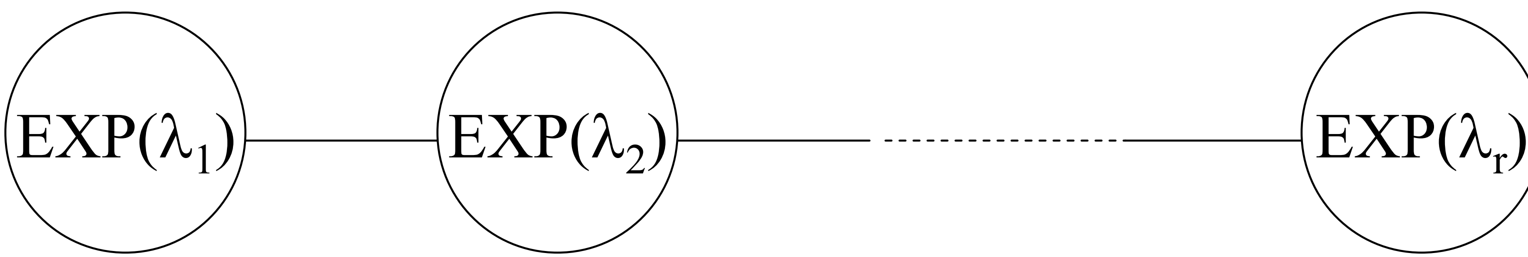
2-stage hypoexponential

TMR and TMR/simplex as hypoexponentials



Hypoexponential: general case

- $Z = \sum_{i=1}^{r} X_i$, where $X_1, X_2, \ldots, X_r$ are mutually independent and X_i is exponentially distributed with parameter λ_i where $\lambda_i \neq \lambda_j \; for \; i \neq j$
- Then Z is a r-stage hypoexponentially distributed random variable.



Hypoexponential: general case

- Density function:

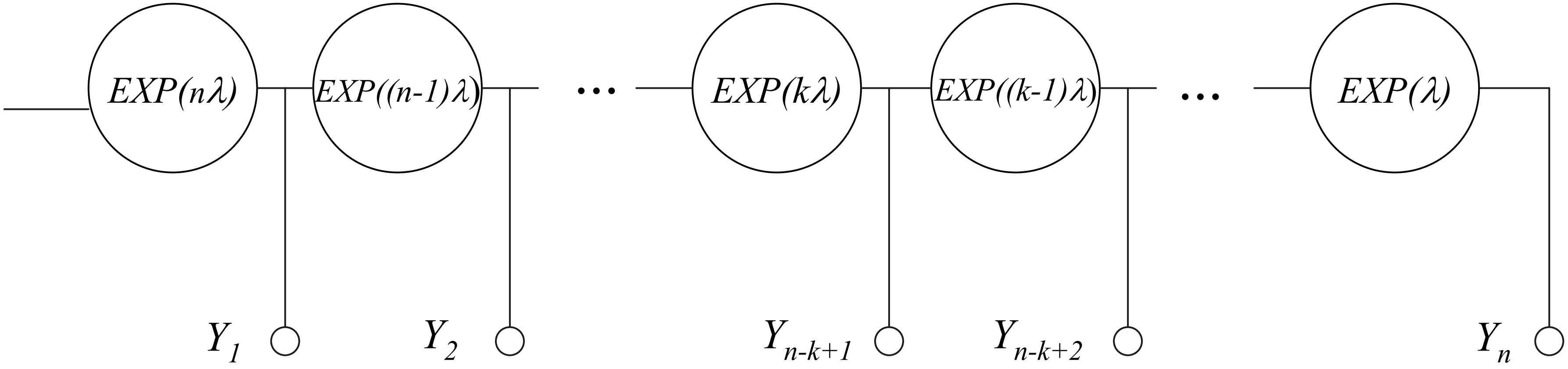
$$f_Z(z) = \sum_{i=1}^{k} a_i \lambda_i e^{-\lambda_i z}, \quad z > 0,$$

where $a_i = \prod_{\substack{i=1 \\ j \neq i}}^{k} \frac{\lambda_j}{\lambda_j - \lambda_i}, \quad 1 \leq i \leq k,$

See Page 174, Theorem 3.4 of the Text.

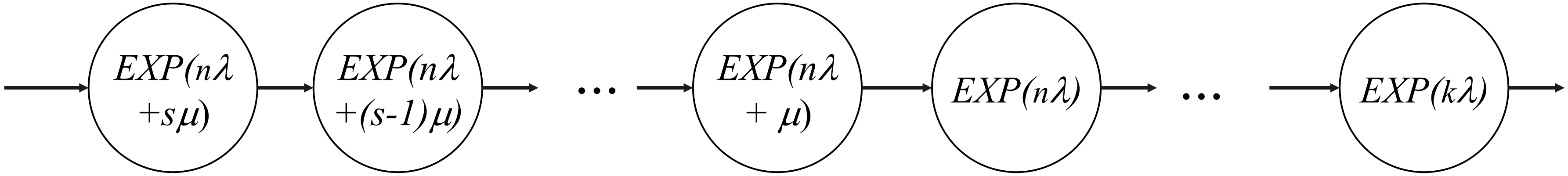
'*k of n*' system lifetime, as a hypoexponential

At least, k out of n units should be operational for the system to be up. Here failure rate of each unit is λ.



‘*k of n*’ with warm spares

- At least, k out of $n + s$ units should be operational for the system to be up. Initially n units are active and s units are warm spares. The failure rate of a unit when active is λ and the failure rate of a unit when spare is μ.



Sums of Normal Random Variables

- $X_1, X_2, .., X_n$ are mutually independent normal rv's, then, the rv Z = $(X_1 + X_2 + .. + X_n)$ is also normal with $$\mu_Z = \sum_{i=1}^{n} \mu_i \text{ and } \sigma_Z^2 = \sum_{i=1}^{n} \sigma_i^2$$

  $\Rightarrow$ The sum of mutually independent normal random variables is also normal.

- $X_1, \ldots, X_n$ are independent standard normal. Then $Y = \sum_{i=1}^{n} X_i^2$ follows the gamma distribution $\Gamma(\frac{1}{2}, \frac{n}{2})$ or the χ^2 distribution with n degrees of freedom.

Example 3.34

- A sequence of independent, identically distributed random variables, $X_1, X_2, \ldots, X_n$, is known as a *random sample* of size *n*.
- In many problems of statistical sampling theory, it is reasonable to assume that the underlying distribution is the normal distribution.
- Thus let $X_i \sim N(\mu, \sigma^2),\ i = 1, 2, \ldots, n.$
- Then from last slide, we obtain

$$S_n = \sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2)$$

Example 3.34 (contd)

One important function known as the sample mean is quite useful in problems of statistical inference. *Sample mean* $\overline{X}$ is given by

$$\overline{X} = \frac{S_n}{n} = \sum_{i=1}^{n} \frac{X_i}{n}.$$

To obtain the pdf of the sample mean $\overline{X}$, we use equation (3.54) to obtain

$$f_{\overline{X}} = n f_{S_n}(nx).$$

But since $S_n \sim N(n\mu, n\sigma^2)$, we have

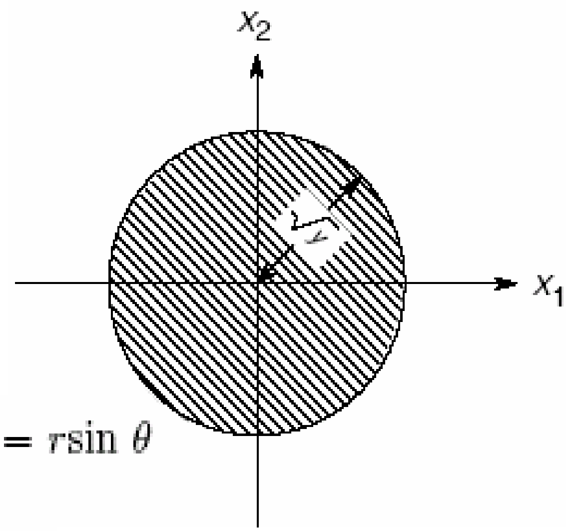
$$\begin{aligned} f_{\overline{X}}(x) &= n \frac{1}{\sqrt{2\pi}(\sqrt{n}\sigma)} e^{-\frac{(nx-n\mu)^2}{2n\sigma^2}} \\ &= \frac{1}{\sqrt{2\pi}[\sigma(n)^{-1/2}]} e^{-\frac{(x-\mu)^2}{2(\sigma^2/n)}}, \qquad -\infty < x < \infty. \end{aligned}$$

It follows that $\overline{X} \sim N(\mu, \sigma^2/n)$. Similarly, it can be shown that the random variable $(\overline{X} - \mu)\sqrt{n}/\sigma$ has the standard normal distribution, $N(0,1)$.

Example 3.35

X_1 and X_2 are independent, and $X_1 \sim N(0,1)$, $X_2 \sim N(0,1)$

$$F_Y(y) = P(X_1^2 + X_2^2 \le y)$$
$$= \int\int_{x_1^2+x_2^2 \le y} f_{X_1X_2}(x_1,x_2)dx_1dx_2.$$



$$F_Y(y) = \int\int_{x_1^2+x_2^2 \le y} \frac{1}{2\pi}e^{-\frac{x_1^2+x_2^2}{2}}dx_1dx_2.$$

Variables to polar co-ordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$

$$F_Y(y) = \int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{y}} \frac{r}{2\pi}e^{-r^2/2}dr\ d\theta$$
$$= \begin{cases} 1 - e^{-y/2}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$Y = X_1^2 + X_2^2$ is exp distributed with parameter 0.5

Example 3.36

- Assume that $X_1, X_2, \ldots, X_n$ are mutually independent identically distributed normal random variables such that

$$X_i \sim N(\mu, \sigma^2)$$

- Then $Z_i = \dfrac{X_i - \mu}{\sigma}$ has the standard normal distribution.
- Therefore, $Y = \sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \left(\dfrac{X_i - \mu}{\sigma}\right)^2$

  has the the χ^2 distribution with n degrees of freedom.
- The random variable

$$\sum_{i=1}^{n} \frac{(X_i - \mu)^2}{n}$$

  may be used as an estimator of the variance σ^2.

Example 3.37

- However, the mean of the population μ is often unknown.
- σ^2 can then be estimated from the sample variance

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n-1} = \frac{\sigma^2}{n-1} \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma} \right)^2$$

- It can be shown that the random variable

$$W = \frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma} \right)^2$$

has the χ^2 distribution with n–1 degrees of freedom.

Example 3.39

- Assume that $X_1, X_2, \ldots, X_n$ are mutually independent identically distributed normal random variables such that

$$X_i \sim N(\mu, \sigma^2)$$

- Then $V = \frac{(\overline{X} - \mu)\sqrt{n}}{\sigma}$ has the standard normal distribution.
- Also $\frac{(n-1)S^2}{\sigma^2} = W = \sum_{i=1}^{n} \left[\frac{X_i - \overline{X}}{\sigma}\right]^2$ has the χ^2_{n-1} distribution.
- Therefore,

$$\begin{aligned} T &= \frac{V}{\sqrt{\frac{W}{(n-1)}}} = \frac{(\overline{X} - \mu)\sqrt{n}/\sigma}{\left[S\frac{\sqrt{n-1}}{\sigma}\right]} \cdot \sqrt{n-1} \\ &= \frac{\overline{X} - \mu}{S/\sqrt{n}} \end{aligned}$$

has the t distribution with $(n-1)$ degrees of freedom.