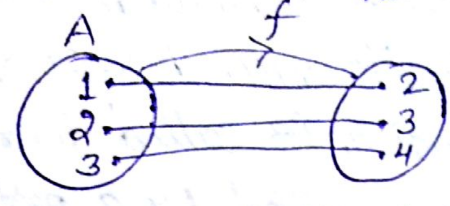
Random variables

Real valued function:- If the domain and co-domain of a function are subset of R (set of all real numbers). It is called a real valued function or real function.

Ex



Random variable:→ A random variable is a real valued-function which maps the sample space on to the real line.

OR

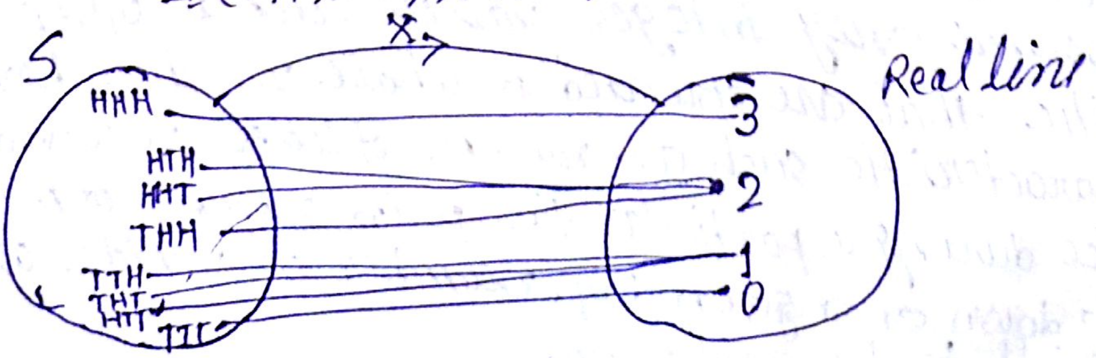
The random variable is a real valued function defined on a sample space whose range is a non-empty set of real number.

Random variable is also called chance variable, stochastic variable or variate.

Ex- Let X be a random variable which is the number of heads obtained in three independent tosses of a fair coin.

here $S = \{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\}$,

Then $X(HHH) = 3$, $X(HHT) = 2$; $X = (HTH) = 2$, $X(THH) = 2$,
$X(TTT) = 0$, $X(TTH) = 1$, $X(THT) = 1$, $X(HTT) = 1$



Types of Random variable:-

(i) Discrete Random variables (Σ)

(ii) Continuous Random variables. ($\int$)

* Discrete Random variable :- A random variable X, which can take only a finite number of values in an interval of the domain is called discrete random variable.

below another definition

Ex- number of letters received by a post office during a particular time period, the number of mistake in a page, Number of defective items in a lot etc.

Continuous Random varible :→ A random variable X which can take every value in the domain or when its range R is an interval then X is continuous random variable.

OR

A random variable is said to be continuous random variable, where the range space is a continuum of numbers, such as an interval or a union of intervals.

Ex- Age, Height, weight, Temperature etc.

Probability Distribution functions :-

Probability distribution functions (Pdf) can be classified into two categories:

(i) Discrete probability distribution

(ii) Continuous probability distribution

* Discrete random variable :→ A discrete random variable can assume only integer values such as 0, 1, 2, ... Quantities that are counted in whole numbers have such characteristic such as: number of letters received by a post office during a particular time period, number of machine breaking down on a given day, number of vehicles arriving at a toll bridge and so on.

Types of Random variables:→

A random variable may be either discrete or Continuous.

Discrete Random variable :→ A variable that is allowed to take on only integer values. ex- number of letters received by a post office during a particular time period, number of items sold.

Continuous Random variable:→ A variable that is allowed to take on any value with in a given range.

or

A continuous random variable can take both integer and non-integer value over a range of value.

ex- time, weight, distance, height of individuals and soon

Probability distribution function :→ ~~A Probability~~ ~~or~~ Probability distribution function (pdf) can be classified into two categories:

(i) Discrete probability distributions

(ii) continuous probability distributions

Discete probability Distribution:→ A probability distribution in which the random variable is permitted to take only integer values.

continuous Probability distribution :→ Probability distribution in which the random variable is permitted to take any value within a given range.

Discrete probability Distribution :→

If a random variable X can assume a discrete set of values say $x_1, x_2, \cdots, x_n$ with respect to probabilities $p_1, p_2, \cdots, p_n$ such that $p_1+p_2+\cdots+p_n=1$ i.e $\sum_{i=1}^{n} p_i = 1$ then occurrences of values x_i with respective probabilities p_i is called the discrete probability distribution of X.

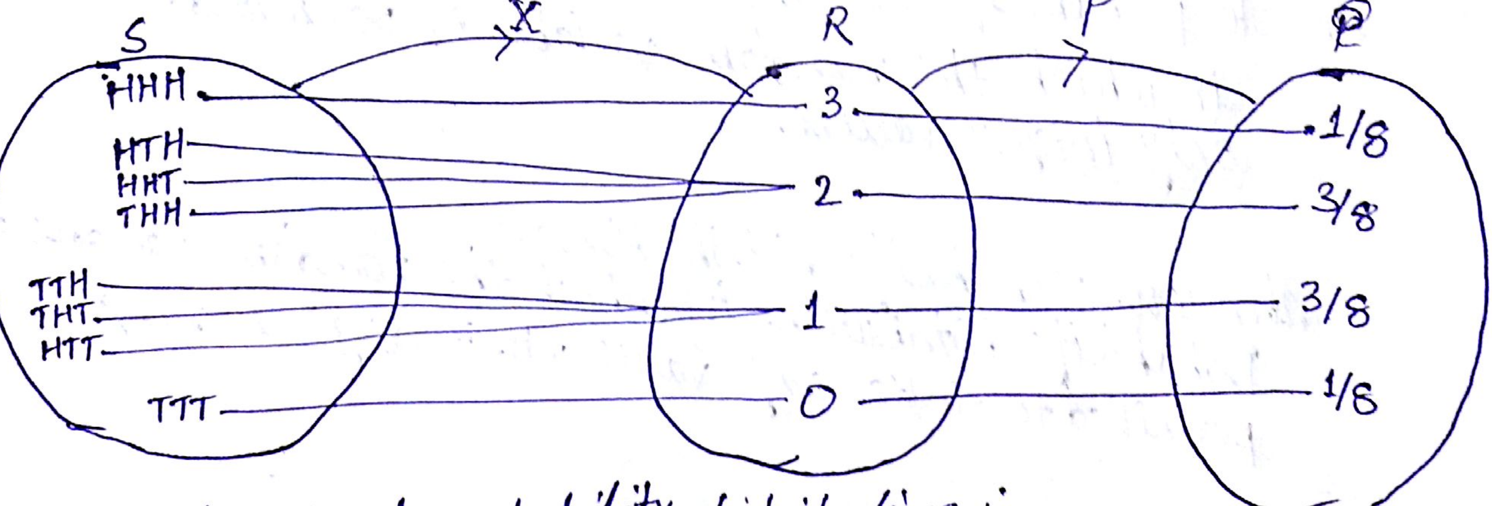
Ex we consider a three coin tossing experiment whose sample space is $S=\{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$ the random variable X, which counts the number of heads that turn up, has the range $\{0,1,2,3\}$ and the four possible events are:

(i) The event which no head turn up $= E(0)=\{TTT\}$

(ii) " " " one " " " $=E(1)=\{HTT, THT, TTH\}$

(iii) " " " Two " " " $=E(2)=\{HHT, HTH, THH\}$

(iv) " " " Three " " " $= E(3) = \{HHH\}$



So, the discrete probability distribution is

x	0	1	2	3
$p(x)$	1/8	3/8	3/8	1/8

Probability mass function or Probability function or Point Probability function :→ If X is the discrete random variable which can take the values $x_1, x_2, x_3, \ldots$ such that $P(X=x_i)=p_i$ then p_i is called Probability function or probability mass function or point probability function provided $p_i\,(i=1,2,\ldots)$ satisfying the following condition

(i) $p_i \geq 0 \quad \forall i$

(ii) $\sum p_i = 1$

Distribution function / cumulative distribution function :→
The function $F(x)$ which gives the probabilities that the discrete random variable (variate) X takes the value less than equal to x, is called cumulative distribution function or distribution function.

$$F(x) = P(X \leq x) = \sum_{i=1}^{x} P(x_i) \;, \quad (X \leq x_i)$$

Ex The probability density function (Probability distribution) of a variate X is

X_i:	0	1	2	3	4	5	6
$P(X)$:	K	3K	5K	7K	9K	11K	13K

(i) Find K (ii) Evaluate $P(X<4)$, $P(X \geq 5)$ and $P(3 < x \leq 6)$

Solⁿ :→ (i) since X is a random variable, then

$$\sum_{i=0}^{6} P(x_i) = 1$$

$\Rightarrow K+3K+5K+7K+9K+11K+13K = 1$

$\Rightarrow 49K = 1$

$\Rightarrow K = \frac{1}{49}$

(ii) Now $P(X<4) = K+3K+5K+7K = 16K = \frac{16}{49}$

$P(X \geq 5) = 11K + 13K = 24K = \frac{24}{49}$

$P(3 < X \leq 6) = 9K + 11K + 13K = 33K = \frac{33}{49}$

(19) Ex A random variable X has the following probability fun[ction]

x	0	1	2	3	4	5	6	7
P_i	0	c	$2c$	$3c$	$2c$	c^2	$2c^2$	$7c^2+c$

(i) Find the value c

(ii) Evaluate $P(X \geq 5)$ and $P(X < 3)$

Solⁿ:→ we know that the sum of $P(x)$ for all possible value of X is equal to unity.

$\Sigma P_i = 1$

$\Rightarrow 0+c+2c+3c+2c+c^2+2c^2+7c^2+c = 1$

$\Rightarrow 10c^2+9c-1=0$

$\Rightarrow 10c^2+10c-c-1=0$

$\Rightarrow 10c(c+1)-1(c+1)=0$

$\Rightarrow (c+1)(10c-1)=0$

$c+1=0 \;\Rightarrow c=-1$ | $10c-1=0$, $10c=1$, $\Rightarrow c=\frac{1}{10}$

since probability is always +ve

so $c=\frac{1}{10}$

(ii) $P(X\geq 5) = c^2+2c^2+7c^2+c$

$= 10c^2+c = 10\times\frac{1}{100}+\frac{1}{10} = \frac{2}{10}$

$P(X<3) = 0+c+2c = 3c = \frac{3}{10}$ Ans

Probability density function (P.d.f): →

For a continuous random variable X the function f(x) satisfying the following, is known as the probability density function (P.d.f) or simply density function.

(i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

(iii) $P(a<x<b) = \int_a^b f(x)dx$ = area under f(x) between ordinates $x=a$ & $x=b$.

Cumulative distribution function

The Distribution function or cumulative distribution function F(x) usually be written as $F(x) = P\{X(e) \leq x\} = \int_{-\infty}^{x} f(x)dx$

Where $P\{X(e) \leq x\}$ is the probability of an event whose, each out comes e is uniquely associated with all real numbers ... not more than x by continuous random variable X.

Ex - A continuous random variable X has the probability desity function given by $f(x) = kx^2, 0 \leq x \leq 1$. Find the value of K. With this value of k, find $P(X<\frac{1}{2})$ and $P(X \geq \frac{3}{4})$

Soln → since f(x) is the p.d.f.

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_0^1 f(x)dx = 1$$

$$\Rightarrow \int_0^1 Kx^2 dx = 1$$

$$\Rightarrow K\left[\frac{x^3}{3}\right]_0^1 = 1$$

$$\Rightarrow \frac{K}{3} = 1$$

$$\Rightarrow K = 3$$

$$\therefore f(x) = 3x^2, 0 \leq x \leq 1.$$

Now $P(X<\frac{1}{2}) = P(0<X<\frac{1}{2})$

$$= \int_0^{1/2} f(x)dx = \int_0^{1/2} 3x^2 dx$$

$$= 3 \times \left[\frac{x^3}{3}\right]_0^{1/2} = \frac{1}{8}$$

$P(X \geq \frac{3}{4}) = P(\frac{3}{4} \leq X \leq 1)$

$$= \int_{3/4}^{1} f(x)dx = \int_{3/4}^{1} 3x^2 dx$$

$$= \left[3\frac{x^3}{3}\right]_{3/4}^{1} = 1 - \frac{27}{64} = \frac{37}{64}.$$

(2) Ex Is the function $f(x)$ defined as follows, a density function.

$$f(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{18}(3+2x), & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$$

Sol $\rightarrow$ $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx + \int_{4}^{\infty} f(x)\,dx$$

$$= 0 + \int_{2}^{4} \frac{1}{18}(3+2x)\,dx + 0 = \int_{2}^{4} \frac{1}{18}[3+2x]\,dx$$

$$= \frac{1}{18}\left[3x+x^2\right]_2^4 = 1.$$

Mathematical Expectation :- If the discrete random variable X takes n mutually exclusive values $x_1, x_2, \dots, x_n$, and no others, with respective probabilities $p_1, p_2, \dots, p_n$ the expected value of X is given by

$$E(X) = p_1x_1 + p_2x_2 + \dots + p_nx_n$$
$$= \sum_{i=1}^{n} p_i x_i$$

Similarly for continuous random variables, the expected value of X is given by

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

where $f(x)$ is the probability density function.

In other words;
The mean or expectation of a random variable X is the sum of the products of all possible values of X by their respective probabilities.

Note :→ Expectation is also the Mean values of the random variable X probabilities distribution of a random variable X

$$E(X) = \mu = \frac{\sum_{i=1}^{n} p_i x_i}{\sum p_i}$$

$\boxed{\text{Mean} = \frac{\sum f_i x_i}{\sum f_i}}$

Variance :→ Variance is the expected value of $(X-\mu)^2$, where μ is the mean.

$$\text{Variance } (\sigma^2) = E\{(X-\mu)^2\}$$
$$= E\{X^2 + \mu^2 - 2X\mu\}$$
$$= E(X^2) + \mu^2 - 2\mu E(X)$$
$$= E(X^2) + \mu^2 - 2\mu.\mu$$
$$= E(X^2) - \mu^2$$

Case-I For discrete distribution

$$\sigma^2 = \sum_i (x_i - \mu)^2 p_i \quad \text{(definition)}$$
$$= \sum_i x_i^2 p_i - \mu^2 \quad \text{(in computation)}$$

Case II :- For continuous distribution

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \quad \text{[definition]}$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2 \quad \text{[in computation]}$$

Note :- The +ve square root of variance gives the Standard deviation (σ)

$$S.D. = +\sqrt{V(X)} = +\sqrt{\sigma^2}$$

Another Definition of variance :-

Let X be a random variable whose possible values $x_1, x_2, \ldots, x_n$ occur with probabilities $p(x_1), p(x_2), \ldots, p(x_n)$ respectively.

Let $\mu = E(x)$ be the mean of X. The variance of X, denoted by $Var(X)$ or σ^2 is defined as

$$\sigma^2 = Var(X) = \sum_{i=1}^{n} (x_i - \mu)^2 \cdot p(x_i)$$

or equivalently $\sigma^2 = E(X-\mu)^2$

The non-negative number

$$\sigma = \sqrt{Var(X)} = \sqrt{\sum_{i=1}^{n} (x_i - \mu)^2 P_i}$$

is called the standard deviation of the random ~~ex~~ variable X.

Another formula to find the variance of a random variab

$$Var(X) = \sum_{i=1}^{n} (x_i - \mu)^2 p(x_i)$$

$$= \sum_{i=1}^{n} (x_i^2 + \mu^2 - 2\mu x)\, p(x_i)$$

$$= \sum_{i=1}^{n} x_i^2 p(x_i) + \sum_{i=1}^{n} \mu^2 p(x_i) - \sum_{i=1}^{n} 2\mu x_i\, p(x_i)$$

$$= \sum_{i=1}^{n} x_i^2 P(x_i) + \mu^2 \sum_{i=1}^{n} P(x_i) - 2\mu \sum_{i=1}^{n} x_i P(x_i)$$

$$= \sum_{i=1}^{n} x_i^2 P(x_i) + \mu^2 - 2\mu^2 \quad \left[\text{Since } \sum_{i=1}^{n} P(x_i) = 1 \ \& \ \mu = \sum_{i=1}^{n} x_i P(x_i)\right]$$

$$= \sum_{i=1}^{n} x_i^2 P(x_i) - \mu^2$$

$$\Rightarrow \text{Var}(X) = \sum_{i=1}^{n} x_i^2 P(x_i) - \left(\sum_{i=1}^{n} x_i P(x_i)\right)^2$$

or $\text{Var}(X) = E(X^2) - [E(X)]^2$, where $E(X^2) = \sum_{i=1}^{n} x_i^2 P(x_i)$.

Similarly for continuous distribution

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \quad [\text{definition}]$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2 \quad [\text{in computation}]$$

Ex:- Find the mean & variance of the number obtained on a throw of an unbiased die.

Sol^n:→ The sample space of the experiment is $S = \{1, 2, 3, 4, 5, 6\}$
let X denote the number obtained on the throw. Then X is a random variable which can take values 1, 2, 3, 4, 5, or 6.

Also $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$

Therefore, the probability distribution of X is

X	1	2	3	4	5	6
P(X)	1/6	1/6	1/6	1/6	1/6	1/6

$$\therefore \mu = E(X) = \sum_{i=1}^{n} x_i p_i = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$$

$$= \frac{21}{6}$$

Also, $E(x^2) = \sum x_i^2 \cdot p(x_i)$

$$= 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6} = \frac{92}{6}$$

Thus, $$\mathrm{var}(X) = E(x^2) - [E(x)]^2$$

$$= \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{91}{6} - \frac{441}{36} = \frac{35}{12}$$

Ex- During the course of a day, a machine turns out either 0, 1, or 2 defective pens with probability 1/6, 2/3 and 1/6 respectively. Calculate the mean value and the variance of the ~~variance of the~~ defective pens produced by the machine in a day.

Soln:-> The probability distribution is given by

$P(0) = \frac{1}{6}$, $P(1) = \frac{2}{3}$, $P(2) = \frac{1}{6}$

x_i	0	1	2
p_i	1/6	2/3	1/6

$\therefore$ Mean value $= \mu = \sum_{i=0}^{2} p_i x_i = \frac{1}{6} \times 0 + \frac{2}{3} \cdot 1 + \frac{1}{6} \times 2 = 1$

$$\mathrm{var}(X) = \sum_{i=0}^{2} x_i^2 p_i - \mu^2 = \frac{1}{6} \cdot 0^2 + \frac{2}{3} \cdot 1^2 + \frac{1}{6} \times 2^2 - 1^2$$

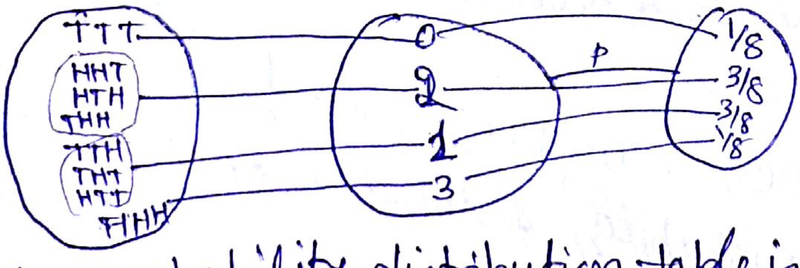
$$= \frac{2}{3} + \frac{2}{3} - 1 = \frac{4-3}{3} = \frac{1}{3}$$

Ex- Find the mean number of heads in three tosses of a coin.

Soln:-> The sample space $S = \{HHH, HHT, HTH, THH, THH, HTH, HHT, TTT\}$

Let X denotes a random variable which is the no. of heads obtained in three tosses of a coin.

Clearly X takes the values 0, 1, 2, 3.



$\therefore$ $P(X=0) = \frac{1}{8}$, $P(X=1) = \frac{3}{8}$

$P(X=2) = \frac{3}{8}$, $P(X=3) = \frac{1}{8}$

The probability distribution table is

X_i	0	1	2	3
P_i	1/8	3/8	3/8	1/8

$$\text{Mean} = \sum P_i X_i = 0 + \frac{3}{8} \cdot 1 + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$$

$$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}$$

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(26) (5) (27A)

Example: (1) A random variable x has the following probability funct

Values of x :	-2	-1	0	1	2	3
$P(x)$:	0.1	K	0.2	$2K$	0.3	K

Find the value of K and Calculate mean & Variance.

Sol^n → Since we know that the probability distribution will be valid when $\sum_{i=1}^{n} p_i = 1$

$$0.1 + K + 0.2 + 2K + 0.3 + K = 1$$

$$\Rightarrow 0.6 + 4K = 1$$

$$\Rightarrow 4K = 0.4$$

$$\Rightarrow K = 0.1 \quad \underline{\text{Ans}}$$

Since mean $= \mu =$ expected value of X

$$= E(X) = \sum px = (0.1)\times(-2) + K(-1) + (0.2)\times 0 + (0.3)\times 2 + (1\times 2K) + K3$$

$$= -.2 + (0.1)(-1) + 0 + 0.6 + (0.1)3 + 2(0.1)$$

$$= -0.2 - 0.1 + 0.6 + 0.3 + 0.2 = 0.8$$

Now variance $(\sigma^2) = \sum_{i=1}^{n} p_i (x_i - \mu)^2$

$$= 0.1(-2-0.8)^2 + (0.1)(-1-0.8)^2 + (0.2)(0-0.8)^2 + (0.2)(1-0.8)^2 + (0.3)(2-0.8)^2 + 0.1(3-0.8)^2$$

$$= (0.1)\cdot(-2.8)^2 + (0.1)(-1.8)^2 + (0.2)(-0.8)^2 + (0.2)(0.2)^2 + (0.3)(1.2)^2 + (0.1)(2.2)^2$$

$$= 0.784 + 0.324 + 0.128 + 0.008 + 0.432 + 0.484 = 2.160 \text{ Ans}$$

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Ex (7) The diameter of an electric cable is assumed to be a Continuous variate with P.d.f. $f(x) = 6x(1-x)$, $0 \le x \le 1$. verify that the above is a. P.d.f. A
Also find the mean and Variance.

(22) Sol^n → Verification of $f(x) = 6x(1-x);\ 0 \le x \le 1$ to be P.d.f :→

since $\int_0^1 f(x)\,dx = 6\int_0^1 x(1-x)\,dx = 6\left\{\int_0^1 x\,dx - \int_0^1 x^2\,dx\right\}$

$$= 6\left\{\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right\} = 6\left\{\frac{1}{2} - \frac{1}{3}\right\} = \frac{6(3-2)}{6} = 1.$$

$\Rightarrow f(x) = 6x(1-x);\ 0 \le x \le 1$ is a probability density function.

Since we know that for a continuous random distribution mean i.e. expected value of variate x is given by $\int_{-\infty}^{\infty} x f(x)\,dx$

Here

$$\text{Mean} = \int_0^1 x \cdot 6x(1-x)\,dx$$

$$= 6\left[\int_0^1 x^2\,dx - \int_0^1 x^3\,dx\right] = 6\left[\left\{\frac{x^3}{3}\right\}_0^1 - \left\{\frac{x^4}{4}\right\}_0^1\right]$$

$$= 6\left[\frac{1}{3} - \frac{1}{4}\right] = 6\left[\frac{4-3}{12}\right] = \frac{1}{2}$$

Also for continuous distribution the variance is given by

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

here,

$$\sigma^2 = \int_0^1 \left(x - \frac{1}{2}\right)^2 6x(1-x)\,dx = \int_0^1 \left(x^2 - x + \frac{1}{4}\right)(x - x^2)\,dx$$

$$= 6\int_0^1 \left(-x^4 + 2x^3 - \frac{5}{4}x^2 + \frac{x}{4}\right)dx$$

$$= -6\left[\frac{x^5}{5}\right]_0^1 + 12\left[\frac{x^4}{4}\right]_0^1 - \frac{15}{2}\left[\frac{x^3}{3}\right]_0^1 + \frac{3}{2}\left[\frac{x^2}{2}\right]_0^1$$

$$= -6 \cdot \frac{1}{5} + \cancel{12}^{\,3} \cdot \frac{1}{\cancel{4}} - \frac{\cancel{15}^{\,5}}{2} \cdot \frac{1}{\cancel{3}} + \frac{3}{2} \cdot \frac{1}{2}$$

$$= -\frac{6}{5} + 3 - \frac{5}{2} + \frac{3}{4} = \frac{-24 + 60 - 50 + 15}{20} = \frac{1}{20} \text{ Ans}$$

Ex A random variable X has the following prob. function

x:	0	1	2	3	4	5	6	7
$p(x)$:	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

(i) Find the value of K.

(ii) Evaluate $P(X \ge 6)$.

Exp:→ Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean, variance and standard deviation of the number of kings.

Sol:→ Let X denote the number of kings in a draw of two cards. X is a random variable which can assume the values 0, 1 or 2.

Now, $P(X=0) = P(\text{no king}) = \dfrac{{}^{48}C_2}{{}^{52}C_2} = \dfrac{\dfrac{\lfloor 48}{\lfloor 2 \lfloor 46}}{\dfrac{\lfloor 52}{\lfloor 2 \lfloor 52-2}} = \dfrac{48\times47}{52\times51} = \dfrac{188}{221}$

$P(X=1) = P(\text{one king \& one non-king})$

$= \dfrac{{}^{4}C_1 \cdot {}^{48}C_1}{{}^{52}C_2} = \dfrac{4\times48\times2}{52\times51} = \dfrac{32}{221}$

and $P(X=2) = P(\text{two kings}) = \dfrac{{}^{4}C_2}{{}^{52}C_2} = \dfrac{4\times3}{52\times51} = \dfrac{1}{221}$

Thus, the probability distribution of X is

X	0	1	2
$P(X)$	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

Mean of $X = E(X) = \sum_{i=1}^{n} x_i p(x_i)$

$= 0\times\dfrac{188}{221} + 1\times\dfrac{32}{221} + 2\times\dfrac{1}{221} = \dfrac{34}{221}$

Also $E(X^2) = \sum_{i=1}^{n} x_i^2 p(x_i) = 0^2\times\dfrac{188}{221} + 1^2\times\dfrac{32}{221} + 2^2\times\dfrac{1}{221} = \dfrac{36}{221}$

$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2$

$= \dfrac{36}{221} - \left(\dfrac{34}{221}\right)^2 = \dfrac{6800}{(221)^2}$

Therefor $S.D\,(\sigma) = \sqrt{\text{Var}(X)} = \dfrac{\sqrt{6800}}{221} = 0.37.$