A1 (a) Step 0: $K = 2$ unknown parameters

Step 1:

For $i=1$, $\hat{\alpha}_1 = \frac{1}{n}\sum_{j=1}^{n}(X_j)^1 = \overline{X}$ (sample mean)

For $i=2$, $\hat{\alpha}_2 = \frac{1}{n}\sum_{j=1}^{n}(X_j)^2$

Step 2:

For $i=1$, $\alpha_1 = xy$ (theoretical 1st moment)

For $i=2$,

We know that:

$$Var(X) = E[X^2] - (E[X])^2$$

From the question:

$$xy^2 = E[X^2] - (xy)^2$$

Here $E[X^2]$ is the theoretical 2nd moment which comes out to be:-

$$E[X^2] = xy^2 + x^2y^2 = xy^2(1+x)$$

Step 3:

Equating the estimated and theoretical moments to obtain:-

$$xy = \overline{X}$$

$$\hat{x}_{MME} = \frac{\overline{X}}{\hat{y}} = \frac{\sum_{j=1}^{n} X_j}{n\hat{y}} \quad (1)$$

$$\frac{1}{n}\sum_{j=1}^{n}(X_j)^2 = \hat{x}\hat{y}^2(1+\hat{x})$$

$\hat{x} = \frac{\overline{X}}{\hat{y}}$ from (1), we have:

$$\frac{1}{n}\sum_{j=1}^{n}(X_j)^2 = \frac{\overline{X}}{\hat{y}}\cdot\hat{y}^2\left(1+\frac{\overline{X}}{\hat{y}}\right) \Rightarrow \frac{1}{n}\left(\sum_{j=1}^{n}X_j^2\right) = \frac{\overline{X}\hat{y}^2}{\hat{y}}\left(1+\frac{\overline{X}}{\hat{y}}\right)$$

$$\Rightarrow \frac{1}{n}\sum_{j=1}^{n}X_j^2 = \overline{X}\hat{y} + (\overline{X})^2$$

$$\therefore \bar{X}\hat{y} = \frac{1}{n}\sum_{j=1}^{n} X_j^2 - (\bar{X})^2$$

$$\hat{y}_{MME} = \frac{\frac{1}{n}\sum_{j=1}^{n} X_j^2 - (\bar{X})^2}{\bar{X}}$$

From ①:

$$\hat{x}_{MME} = \frac{\bar{X}}{\hat{y}} = \frac{\bar{X}^2}{\frac{1}{n}\sum_{j=1}^{n} X_j^2 - (\bar{X})^2}$$

Answer:

$$\hat{x}_{MME} = \frac{\bar{X}^2}{\frac{1}{n}\sum_{j=1}^{n} X_j^2 - (\bar{X})^2}, \quad \hat{y}_{MME} = \frac{\frac{1}{n}\sum_{j=1}^{n} X_j^2 - (\bar{X})^2}{\bar{X}}$$

(b) Step 0: $k = 2$ unknown parameters

Step 1:

For $k=1$, $\hat{\alpha}_1 = \frac{1}{n}\sum_{j=1}^{n} X_j = \bar{X}$ — ①

For $k=2$, $\hat{\alpha}_2 = \frac{1}{n}\sum_{j=1}^{n} (X_j)^2$ — ②

Step 2

For $i = 1$, $\alpha_1 = E[X]$

$$f_X(x) = \begin{cases} 0 & x \le a \\ 1/b-a & a < x < b \\ 0 & x \ge b \end{cases}$$

$$E[X] = \int_a^b x f(x)\,dx = \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2} \quad \text{— ③}$$

For $i=2$,

$$\alpha_2 = E[X^2] = \int_a^b x^2 f(x)\,dx = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{b^3-a^3}{3(b-a)}$$

$$\alpha_2 = \frac{(b-a)(b^2+a^2+ab)}{3(b-a)}$$

$$\alpha_2 = \frac{a^2+b^2+ab}{3} \quad \text{(4)}$$

Step 3:

Equating (1) and (3):

$$\frac{a+b}{2} = \bar{X} \quad \text{(5)}$$

Equating (2) and (4):

$$\frac{1}{n}\sum_{j=1}^{n}(x_j)^2 = \frac{a^2+b^2+ab}{3} \quad \text{(6)}$$

From (5):

$$a+b = 2\bar{X} \Rightarrow \hat{a}_{MME} = 2\bar{X} - \hat{b} \quad \text{(7)}$$

Substituting in (6), we get:

$$\frac{(2\bar{X}-b)^2 + b^2 + (2\bar{X}-b)b}{3} = \frac{1}{n}\sum_{j=1}^{n}(x_j)^2$$

$$4\bar{X}^2 + b^2 - 4b\bar{X} + \cancel{b^2} - \cancel{b^2} + 2b\bar{X} = \frac{3}{n}\sum_{j=1}^{n}(x_j)^2$$

$$b^2 - 2b\bar{X} + 4\bar{X}^2 - \frac{3}{n}\underbrace{\sum_{j=1}^{n}(x_j)^2}_{X_{(2)}} = 0$$

The above is a quadratic equation in b. Solving, we get:

$$\hat{b}_{MME} = \frac{-(-2\bar{X}) \pm \sqrt{(-2\bar{X})^2 - 4.1(4\bar{X}^2 - 3X_{(2)})}}{2}$$

(~~Assuming both a > 0 and b > 0~~)

$$\hat{b}_{MME} = \bar{X} \pm \frac{1}{2}\sqrt{4\bar{X}^2 - 4.4.\bar{X}^2 + 4.3\,X_{(2)}}$$

$$\hat{b}_{MME} = \bar{X} \pm \frac{1 \times 2}{2}\sqrt{\bar{X}^2 - 4\bar{X}^2 + 3X_{(2)}}$$

$$\hat{b}_{MME} = \bar{X} \pm \sqrt{3(X_{(2)} - \bar{X}^2)}$$

$$\hat{b}_{MME} = \bar{X} \pm \sqrt{3\bar{S}^2}$$

$$\boxed{\hat{b}_{MME} = \bar{X} \pm \sqrt{3}\,\bar{S}}$$

Substituting in 7

$$\hat{a}_{MME} = 2\bar{X} - \hat{b}_{MME} = 2\bar{X} - (\bar{X} \pm \sqrt{3}\,\bar{S})$$

$$\boxed{\hat{a}_{MME} = \bar{X} \mp \sqrt{3}\,\bar{S}}$$

For any uniform distribution

$b > a$; $f(x) = \begin{cases} 0 & x \le a \\ 1/b-a & a < x < b \\ 0 & x \geq b \end{cases}$

Answers: ∴ Final solution will only be:-

$$\boxed{\hat{a}_{MME} = \bar{X} - \sqrt{3}\,\bar{S}, \quad \hat{b}_{MME} = \bar{X} + \sqrt{3}\,\bar{S}}$$

Ans 2 $X \sim$ Exponential $(1/\lambda)$

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{\lambda} e^{\frac{-x}{\lambda}} & x \geq 0 \end{cases}$$

$$L(\lambda) = \prod_{i=1}^{n} f_X(X_i)$$

$$= \prod_{i=1}^{n} \frac{1}{\lambda} e^{\frac{-X_i}{\lambda}} = \frac{1}{\lambda^n} \cdot \text{ex} \frac{1}{e^{\sum_{i=1}^{n} \frac{X_i}{\lambda}}}$$

$$l(\lambda) = \ln(L(\lambda)) = \ln\left(\frac{1}{\lambda^n}\right) + \ln\left(e^{-\sum_{i=1}^{n} \frac{X_i}{\lambda}}\right)$$

$$l(\lambda) = -n \ln \lambda - \frac{1}{\lambda} \sum_{i=1}^{n} X_i$$

$$\frac{d(l(\lambda))}{d\lambda} = 0$$

$$\frac{-n}{\lambda} + \sum_{i=1}^{n} X_i \times \frac{1}{\lambda^2} = 0$$

$$\frac{\sum_{i=1}^{n} X_i}{\lambda^{\cancel{2}}} = \frac{n}{\cancel{\lambda}}$$

$$\hat{\lambda}_{MLE} = \frac{\sum_{i=1}^{n} X_i}{n} = \bar{X}$$

$$\hat{\lambda}_{MLE} = \bar{X}$$

$$E[\hat{\lambda}_{MLE}] = E\left[\frac{\sum_{i=1}^{n} X_i}{n}\right] = \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right]$$

$$\stackrel{LOE}{=} \frac{1}{n} \sum_{i=1}^{n} E[X_i] \stackrel{iid}{=} \frac{1}{n} \cdot n \cdot E[X_1] = E[X].$$

$$E[X] = \int_0^\infty x f(x)\,dx = \int_0^\infty \frac{x \cdot e^{-x/\lambda}}{\lambda}\,dx$$

$$= \int_0^\infty \underset{f}{x} \cdot \underbrace{\frac{e^{-X/\lambda}}{\lambda}\,dx}_{dg} \qquad dg = e^{-x/\lambda}/\lambda \qquad g = -e^{-x/\lambda}$$

$f = x,\ g = -e^{-x/\lambda}$

Applying integration by parts:

$$\int f \cdot dg = fg - \int g\,df$$

$$\Rightarrow \int_0^\infty x \cdot d\left(-e^{-x/\lambda}\right) = x\left(-e^{-x/\lambda}\right)\Big|_0^\infty + \int_0^\infty e^{-x/\lambda}\,dx$$

$$= xe^{-x/\lambda}\Big|_\infty^0 + \frac{e^{-x/\lambda}}{-1/\lambda}\Big|_0^\infty$$

$$= xe^{-x/\lambda}\Big|_\infty^0 + \lambda e^{-x/\lambda}\Big|_\infty^0$$

$$= 0 - xe^{-x/\lambda}\Big|_{x\to\infty} + \lambda\left[e^0 - \cancel{\frac{1}{e^{x/\lambda}}}^{\,0}\,_{x\to\infty}\right]$$

$$= \underbrace{-\frac{x}{e^{x/\lambda}}}_{\downarrow} + \lambda$$

$$= \frac{1}{\frac{1}{\lambda}e^{x/\lambda}}\Big|_{x\to\infty} + \lambda = 0 + \lambda = \lambda$$

$$E[\hat{\lambda}_{MLE}] = \lambda$$

$$\text{bias}(\hat{\lambda}) = E[\hat{\lambda}_{MLE}] - \lambda = \lambda - \lambda = 0$$

Computing $se(\hat{\lambda}_{MLE}) = \sqrt{Var(\hat{\lambda}_{MLE})}$

$$\sqrt{Var\left(\frac{\sum_{i=1}^{n} X_i}{n}\right)} = \sqrt{\frac{1}{n^2} Var(\sum X_i)}$$

By linearity of variance:

$$= \sqrt{\frac{1}{n^2} \cdot \sum Var(X_i)} \stackrel{iid}{=} \sqrt{\frac{1}{n^2} \times n \times Var(X_1)}$$

$$= \sqrt{\frac{Var(X)}{n}} \quad \text{.. (3)}$$

$$Var(X) = E[X^2] - (E[X])^2 \quad \text{.. (2)}$$

Now $E[X^2] = \int_0^\infty \underbrace{x^2}_{f} \cdot \underbrace{\frac{e^{-x/\lambda}}{\lambda} dx}_{dg}$

$\hookrightarrow g = -e^{-x/\lambda}$

Applying integration by parts:

$$\int_0^\infty x^2 \cdot \frac{e^{-x/\lambda}}{\lambda} dx = x^2(-e^{-x/\lambda})\Big|_0^\infty + \int e^{-x/\lambda} \cdot 2x \, dx.$$

$$= x^2 \cdot e^{-x/\lambda}\Big|_\infty^0 + 2\int_0^\infty x \cdot e^{-x/\lambda} dx.$$

$\hookrightarrow$ (1)

In the previous integral, evaluation for $E[X]$, we obtained

$$\frac{1}{\lambda}\int_0^\infty x \cdot e^{-x/\lambda} dx = \lambda$$

$$\therefore \int_0^\infty x e^{-x/\lambda} dx = \lambda^2$$

$\therefore$ From (1):

$$E[X^2] = \frac{x^2}{e^{x/\lambda}}\Big|_\infty^0 + 2\lambda^2 = 0 - \left[\frac{2x}{\frac{1}{\lambda} \cdot e^{x/\lambda}}\right]_{x \to \infty} + 2\lambda^2$$

$$E[X^2] = 0 - \left.\frac{2}{\left(\frac{1}{\lambda^2}\right) e^{x/\lambda}}\right|_{x \to \infty} + 2\lambda^2$$

$$= 0 - 0 + 2\lambda^2$$

$$E[X^2] = 2\lambda^2$$

$$Var(X) = 2\lambda^2 - (\lambda)^2 = \lambda^2 \qquad [\text{from } ②]$$

$$\therefore se(\hat{\lambda}_{MLE}) = \sqrt{\frac{Var(X)}{n}} = \sqrt{\frac{\lambda^2}{n}} = \frac{\lambda}{\sqrt{n}}$$

As $n \to \infty$,

$$se(\hat{\lambda}_{MLE}) = 0$$

$\therefore$ Both the bias($\hat{\lambda}_{MLE}$) and se($\hat{\lambda}_{MLE}$) tends to 0 as $n \to \infty$

$\therefore$ $\hat{\lambda}_{MLE}$ is a consistent estimator of λ

$$\hat{\lambda}_{MLE} \xrightarrow{n \to \infty} \lambda$$

MLE($\hat{\lambda}$) will converge to the unknown parameter λ as $n \to \infty$

3. a) pmf for Poisson distribution is given by:

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$L(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-\lambda n} \cdot \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

$$l(\lambda) = \ln(L(\lambda)) = -n\lambda + \sum_{i=1}^{n} x_i \ln\lambda - \ln\left(\prod_{i=1}^{n} x_i!\right)$$

$$\frac{d(l(\lambda))}{d\lambda} = 0$$

$$l(\lambda) = -n\lambda + \ln\lambda \sum_{i=1}^{n} x_i - \sum_{i=1} \ln(x_i!)$$

$$\frac{d(l(\lambda))}{d\lambda} = -n + \frac{\sum_{i=1} x_i}{\lambda} - 0 = 0$$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{\lambda} = n \Rightarrow \lambda = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\boxed{\hat{\lambda}_{MLE} = \frac{\sum_{i=1} x_i}{n} = \overline{X}}$$

Ans 3. (b) $f(x) = \frac{1}{2} e^{-(|x-\theta|)}$

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{2} e^{-|x-\theta|} = \frac{1}{2^n} \prod_{i=1}^{n} e^{-(|x_i-\theta|)} = \frac{1}{2^n} e^{-\sum_{i=1}^{n}(|x_i-\theta|)}$$

log likelihood:

$$l(\theta) = \log\left(1/2^n\right) - \sum_{i=1}^{n}(|x_i-\theta|) \ln e$$

Differentiating w.r.t. θ:

$$l'(\theta) = 0 - \frac{d\sum(x_i-\theta) I(x_i>\theta)}{d\theta} - \frac{d\sum(x_i-\theta) I(x_i<\theta)}{d\theta}$$

(where I is an indicator fn)

$$l'(\theta) = \sum_{i=1}^{n} I(x_i>\theta) - \sum_{i=1}^{n} I(x_i<\theta)$$

Now, for $l'(\theta)$ to be 0:

$$\sum_{i=1}^{n} I(x_i>\theta) = \sum_{i=1}^{n} I(x_i<\theta)$$

$\Rightarrow$ the no. of samples (in X) that are greater than θ should be equal the no. of samples less than θ.

$\Rightarrow$ Hence, $\hat{\theta}_{MLE}$ should be the sample median

[no. of samples < sample median is equal to no. of samples > sample median]

$\therefore$ $\boxed{\hat{\theta}_{MLE} = \text{SAMPLE MEDIAN}}$

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3 (c) $X_1, X_2 \ldots X_n \sim \text{Normal}(\theta, 1)$

Let $\delta = E[I_{X_1>0}]$

From lecture; $\hat{\theta}_{MLE} = \dfrac{\sum X_i}{n} = \bar{X}$ (for a normal distribution)

$$\delta = E[I_{X_1>0}] = P(X_1>0).1 + 0 = P(X_1>0)$$

$$= 1 - P(X_1 \leq 0)$$

$$= 1 - F_{X_1}(0)$$

$$= 1 - Pr\left(Z \leq \frac{0-\mu}{\sigma}\right)$$ (Transformation property $Z \sim Nor(0,1)$)

$$= 1 - Pr\left(Z \leq \frac{0-\theta}{1}\right)$$

$$= 1 - \Phi(-\theta)$$

$$= 1 - (1 - \Phi(\theta))$$

$$= \Phi(\theta)$$

Now, By equivariance property [If $\hat{\theta}_{MLE}$ is MLE of θ, then for any function $\phi(\theta)$, the MLE of $\phi(\theta)$ is $\phi(\hat{\theta})$]

$\therefore$ MLE of $\delta = \phi(\theta)$ is $\phi(\bar{X}) = \phi\left(\dfrac{\sum_{i=1}^{n} X_i}{n}\right)$

Ay. (a) $X = \begin{cases} 2 & \text{with prob } \theta \\ 3 & \text{otherwise} \end{cases}$ ie with prob $1-\theta$

Step 0: $k = 1$ unknown parameter

Step 1: For $i = 1$, $\hat{\alpha}_1 = \frac{1}{n}\sum_{j=1}^{n} X_j = \bar{X}$

Step 2: For $i = 1$,

$$E[X(\theta)] = 2.\theta + 3(1-\theta)$$
$$= 2\theta + 3 - 3\theta$$
$$= 3 - \theta$$

Step 3: $\bar{X} = 3 - \hat{\theta}$ $\Rightarrow$ $\boxed{\hat{\theta}_{MME} = 3 - \bar{X}}$ ..(1)

$$\hat{\theta}_{MME} = 3 - \bar{X} = 3 - \frac{2+3+2}{3} = 3 - \frac{7}{3} = \frac{2}{3}$$

$\therefore$ $\boxed{\hat{\theta}_{MME} = \frac{2}{3}}$ Ans

(b) $se(\hat{\theta}_{MME}) = \sqrt{Var(\hat{\theta}_{MME})} = \sqrt{Var(3-\bar{X})}$ [from (1)]

We know: $Var(c - X) = Var(X)$ $\left[\begin{array}{l} Var(c + (-X)) \overset{LoV}{=} Var(c) + Var(-X) \\ \text{Now, } Var(-X) = Var((-1)X) = (-1)^2 . Var(X) \\ \qquad = Var(X) \\ \therefore Var(c - X) = Var(c) + Var(X) = Var(X) \\ \qquad \text{as } Var(c) = 0 \end{array} \right]$

$$se(\hat{\theta}_{MME}) = \sqrt{Var(\bar{X})} = \sqrt{Var\left(\frac{\Sigma X_j}{n}\right)} = \sqrt{\frac{1}{n^2} Var(\Sigma X_j)}$$

By linearity of variance,

$$se(\hat{\theta}_{MME}) = \sqrt{\frac{1}{n^2}\Sigma Var(X_j)} \overset{i.i.d}{=} \sqrt{\frac{1}{n^2} . n . Var(X_1)}$$

$$se(\hat{\theta}_{MME}) = \sqrt{\frac{1}{n} . Var(X_1)} = \sqrt{\frac{Var(X)}{n}} \quad .(2)$$

Now,

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = \sum_{\omega} \omega^2 p(\omega) = 4(\theta) + 9(1-\theta) = 4\theta + 9 - 9\theta$$

$$E[X^2] = 9 - 5\theta$$

$$(E[X])^2 = (3-\theta)^2 = 9 + \theta^2 - 6\theta$$

$$Var(X) = 9 - 5\theta - 9 - \theta^2 + 6\theta = \theta - \theta^2 = \theta(1-\theta)$$

$$Var(X) = \theta(1-\theta) \longrightarrow (3)$$

Now, from (2) and

$$se(\hat{\theta}_{MME}) = \sqrt{\frac{\theta(1-\theta)}{n}}$$

As se cannot be computed directly as θ is true value.
∴ computing the estimate the se as:-

Now, $$\boxed{\hat{se}(\hat{\theta}) = \sqrt{\frac{\hat{\theta}_{MME}(1-\hat{\theta}_{MME})}{n}}}$$

Substituting values.

$$\hat{se}(\hat{\theta}) = \sqrt{\frac{\frac{2}{3} \times \left(1 - \frac{2}{3}\right)}{3}} = \sqrt{\frac{2}{3} \times \frac{1}{3} \times \frac{1}{3}}$$

$$\boxed{\hat{se}(\hat{\theta}) = \frac{1}{3}\sqrt{\frac{2}{3}}}$$ Ans

(c)

$$X = \begin{cases} 2 & w.p\ \theta \\ 3 & w.p\ 1-\theta \end{cases}$$

writing the pmf in closed form we will get:

$$\boxed{P_X(x) = (1-\theta)^{x-2}\ \theta^{3-x}}$$

Now, Let $X_1, X_2, \ldots X_n$ are i.i.d data samples [Here $n=3$]

$$L(\theta) = \prod_{i=1}^{n} (1-\theta)^{X_i - 2}\ \theta^{3-X_i}$$

$$L(\theta) = \theta^{3n - \Sigma X_i} (1-\theta)^{\Sigma X_i - 2n}$$

$$l(\theta) = \ln(L(\theta)) = \ln(\theta^{3n-\Sigma X_i}) + \ln((1-\theta)^{\Sigma X_i - 2n})$$

$$= (3n - \Sigma X_i)\ln\theta + (\Sigma X_i - 2n)\ln(1-\theta)$$

$$\frac{d(l(\theta))}{d\theta} = 0$$

$$\frac{3n - \Sigma X_i}{\theta} - \frac{\Sigma X_i - 2n}{1-\theta} = 0$$

$$\frac{3n - \Sigma X_i}{\theta} = \frac{\Sigma X_i - 2n}{1-\theta}$$

$$(1-\theta)(3n - \Sigma X_i) = \theta(\Sigma X_i - 2n)$$

$$3n - \Sigma X_i - 3n\theta + \cancel{\theta \Sigma X_i} = \cancel{\theta \Sigma X_i} - 2n\theta$$

$$3n - \Sigma X_i = n\theta$$

$$\hat{\theta}_{MLE} = \frac{3n - \Sigma X_i}{n} = 3 - \frac{\Sigma X_i}{n}$$

$$\hat{\theta}_{MLE} = 3 - \frac{2+3+2}{3} = 3 - \frac{7}{3} = \frac{2}{3}$$

$$\therefore \boxed{\hat{\theta}_{MLE} = \frac{2}{3}}$$ Ans

5. (a) $X = Exp(\lambda)$

$D = \{x_1, x_2, \ldots, x_n\} \overset{iid}{\sim} X$

First moment →

Plugin estimator $= \frac{1}{n}\sum_{i=1}^{n} x_i$

Theoretical moment $= E[X]$ → [For Exponential $= \frac{1}{\lambda}$]

$= \frac{1}{\lambda_{MME}}$

$$\frac{1}{\lambda_{MME}} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\boxed{\hat{\lambda}_{MME} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}}$$ → sample mean

(b) $X = Exp(\lambda)$

$D = \{x_1, x_2, \ldots, x_n\} \overset{iid}{\sim} X$

$$L(\lambda) = \prod_{i=1}^{n} f_X(x_i)$$

$$= \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

$$L(\lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$l(\lambda) = \ln(L(\lambda)) = \ln(\lambda^n e^{-\lambda\sum_{i=1}^{n} x_i})$$

$$= \ln(\lambda^n) + \ln(e^{-\lambda\sum_{i=1}^{n} x_i})$$

$$l(\lambda) = n\ln(\lambda) - \lambda\sum_{i=1}^{n} x_i$$

$$\frac{d\, l(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$

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$$\Rightarrow \frac{n}{\lambda} = \sum_{i=1}^{n} x_i$$

$$\Rightarrow \boxed{\hat{\lambda}_{MLE} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{X}}}$$

sample mean

(c) MME →

$\hat{\mu}_{MME} = 15.568$
$\hat{\sigma}^2_{MME} = 7.586$
} Acceleration dataset (normal)

$\hat{a}_{MME} = 69.614$
$\hat{b}_{MME} = 82.406$
} Model dataset (uniform)

$\hat{\lambda}_{MME} = 0.043$ } MPG dataset (exponential)

(d) MLE →

$\hat{\mu}_{MLE} = 15.568$
$\hat{\sigma}^2_{MLE} = 7.568$
} Acceleration dataset (normal)

$\hat{a}_{MLE} = 70.000$
$\hat{b}_{MLE} = 82.000$
} Model dataset (uniform)

$\hat{\lambda}_{MLE} = 0.043$ } MPG dataset (exponential)

(e) MLE is better than MME as it provides more optimal solution. From our answer in (c) and (d) we can see that in case of uniform distribution the MLE ~~is~~ for a and b is minimum and maximum value which is more appropriate description of the sample data. The MME of a and b for uniform distribution has value 66 and 85 which is lesser than and ~~most~~ more than the data in our sample. So, MLE is more accurate.

6.a) Given: Null hypothesis: $H_0: \theta = \theta_0$

true value $= \theta_*$

Type II error : False Negative.

$\therefore$ Probability (Type II Error) = P (False Negative) = P(accept H_0 | H_0 false)

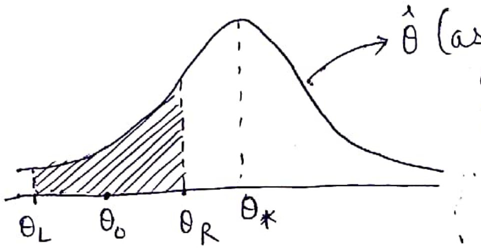
In Wald's test, we accept H_0 if $\left|\frac{\hat{\theta}-\theta_0}{\hat{se}(\hat{\theta})}\right| \leq z_{\alpha/2}$

i.e. $-z_{\alpha/2} \leq \frac{\hat{\theta}-\theta_0}{\hat{se}(\hat{\theta})} \leq z_{\alpha/2}$

$\theta_0 - z_{\alpha/2}\cdot\hat{se}(\hat{\theta}) \leq \hat{\theta} \leq \theta_0 + z_{\alpha/2}\cdot\hat{se}$ } for these values of $\hat{\theta}$ we accept H_0

Let us assume, $\theta_0 < \theta_*$,

We need to calculate the $Pr(\hat{\theta} \in [\underbrace{\theta_0 - z_{\alpha/2}\cdot\hat{se}(\hat{\theta})}_{\theta_L}, \underbrace{\theta_0 + z_{\alpha/2}\cdot\hat{se}(\hat{\theta})}_{\theta_R}])$



$\hat{\theta}$ (as $\hat{\theta}$ is Asymptotically Normal) assumption of Wald's test.

$\hat{\theta} \approx Nor(\theta_*, \hat{se}^2)$

$\therefore$ We need to calculate the shaded region above.

$\theta_L = \theta_0 - z_{\alpha/2}\cdot\hat{se}$, $\theta_R = \theta_0 + z_{\alpha/2}\cdot\hat{se}$

$\therefore Pr(\theta_L \leq \hat{\theta} \leq \theta_R) = Pr(\hat{\theta} < \theta_R) - Pr(\hat{\theta} < \theta_L)$ [Ignoring = as it is continuous distribution]

$= Pr\left(\frac{\hat{\theta}-\theta_*}{\hat{se}} < \frac{\theta_R-\theta_*}{se}\right) - Pr\left(\frac{\hat{\theta}-\theta_*}{\hat{se}} < \frac{\theta_L-\theta_*}{\hat{se}}\right)$ [Converting to std. normal form]

$= \Phi\left(\frac{\theta_R-\theta_*}{\hat{se}}\right) - \Phi\left(\frac{\theta_L-\theta_*}{\hat{se}}\right)$

$= \Phi\left(\frac{\theta_0-\theta_*}{\hat{se}} + z_{\alpha/2}\right) - \Phi\left(\frac{\theta_0-\theta_*}{\hat{se}} - z_{\alpha/2}\right)$

Hence Proved.

6.b) Test results : $|W| = 11.009$

$Z_{\alpha/2} = 1.96$

$\therefore |W| > Z_{\alpha/2}$

$\therefore H_0$ is rejected, μ_X & μ_Y are not same

Since for Wald's test for 2 population, we assume that both estimators i.e. μ_X & μ_Y are Asymptotically Normal, therefore, Wald's 2 population test is applicable because both X & Y are independent Normal distributions.

$H_0 : \mu_x = \mu_y$ $\qquad$ $H_1 : \mu_x \neq \mu_y$

6 (c) Value of $T = \frac{\bar{D}}{S_D / \sqrt{n}}$, $D = X_i - Y_i$

$\boxed{\bar{D} = -1.084, S_D = 3.11}$

$= 11.023$ > 1.962

So, the null hypothesis is rejected

The test is applicable here because both X and Y distributions are drawn from normal distributions which is the only assumption of T-test.
Since X and Y are normally distributed, D will also be normally distributed by transformation property.

7. a) 1. $H_0 : \mu = 1.5$ vs $H_1 : \mu \neq 1.5$

since data samples are normally distributed and also n is small, we have :-

2. t-statistic $T = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}}$ where $\bar{X}$ is sample mean and s is the sample std. deviation

$$\bar{X} = \frac{\Sigma X_i}{n} = \frac{1.87 + 1.29 + 2.01 + 0.93 + 1.02 + 2.78 + 2.33 + 1.65 + 0.50 + 0.99}{10}$$

$$\bar{X} = 1.537$$

$$s = \sqrt{\frac{1}{n} \Sigma (X_i - \bar{X})^2}$$

$$= \sqrt{\frac{1}{10}\left[(1.87 - 1.537)^2 + (1.29 - 1.537)^2 + (2.01 - 1.537)^2 + (0.93 - 1.537)^2 + (1.02 - 1.537)^2 + (2.78 - 1.537)^2 + (2.33 - 1.537)^2 + (1.65 - 1.537)^2 + (0.5 - 1.537)^2 + (0.99 - 1.537)^2\right]}$$

$$s = \sqrt{\frac{4.59261}{10}} = \sqrt{0.459261} = 0.677688$$

t-statistic $T = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}} = \dfrac{1.537 - 1.5}{\sqrt{\dfrac{4.59261}{10}} / \sqrt{10}} = \dfrac{0.037}{\sqrt{0.459261} / \sqrt{10}}$

$$T = \frac{0.037}{0.2143} = \cancel{0.017265}\ 0.17265$$

for $\alpha = 0.05$, critical value $C = 2.228$

3. $0.17265 < t_{n-1, \alpha/2} = 2.228$

Fail to reject $H_0 : \mu = 1.5$.

7. (b) $X_1, X_2, \ldots\ldots X_{100} \sim \text{Bernoulli}(p)$

$\rightarrow$ $H_0: p = 0.5$ vs. $H_1: p \neq 0.5$

By MLE, $\hat{P}_{MLE} = \bar{X} = \frac{46}{100} = 0.46$ [$\hat{p}_{MLE}$ is Asymptotic normal and $\therefore$ we can apply the Walds test]

$\rightarrow$ Statistic $W = \dfrac{\hat{\theta} - \theta_0}{\hat{se}(\hat{\theta})} = \dfrac{\hat{P}_{MLE} - 0.5}{\hat{se}(\hat{p}_{MLE})}$ [If $|W| > z_{\alpha/2}$ → reject H_0 else fail to reject H_0]

$$se(\hat{\theta}) = \sqrt{Var(\hat{\theta})} = \sqrt{Var(\bar{X})} = \sqrt{Var\left(\frac{\Sigma X_i}{n}\right)}$$

By Linearity of variance

$$se(\hat{\theta}) = \sqrt{\frac{1}{n^2} \Sigma Var(X_i)} \overset{iid}{=} \sqrt{\frac{1}{n^{\cancel{2}}} \times \cancel{n} \times \Sigma Var(X_1)} = \sqrt{\frac{Var(X)}{n}}$$

$$se(\hat{\theta}) = \sqrt{\frac{p(1-p)}{n}}$$

$$\hat{se}(\hat{\theta}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.46 \times 0.54}{100}} = 0.0498397$$

$$W = \frac{0.46 - 0.5}{0.0498397} = -0.80257$$

$|W| = 0.80257 < z_{\alpha/2} = 1.96$ [for $\alpha = 0.05$, $z_{\alpha/2} = 1.96$]

$\Rightarrow$ Fail to rejelt $H_0: p = 0.5$

Now,
Consider $H_0: p = 0.7$ ~~and~~ vs $H_1: p \neq 0.7$

$$W = \frac{\hat{\theta} - \theta_0}{\hat{se}(\hat{\theta})} = \frac{0.46 - 0.7}{0.0498397}$$ [from the values obtained ~~above~~ above and $\theta_0 = 0.7$]

$W = -4.8154$ $\because |W| = 4.8154 > 1.96$ ($z_{\alpha/2}$) $\Rightarrow$ Reject $H_0: p = 0.7$