CSE 544, Spring 2020: Probability and Statistics for Data Science

Assignment 6: Bayesian Inference and Regression Due: 04/29, 2:30pm, via google forms

(5 questions, 60 points total)

I/We understand and agree to the following:

(a) Academic dishonesty will result in an 'F' grade and referral to the Academic Judiciary.

(b) Late submission, beyond the 'due' date/time, will result in a score of 0 on this assignment.

(write down the name of all collaborating students on the line below)

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1. Posterior for Normal

(Total 10 points)

Let $X_1, X_2, \ldots, X_n$ be distributed as Normal(θ, σ²), where σ is assumed to be known. You are also given that the prior for θ is Normal(a, b²).

(a) Show that the posterior of θ is Normal(x, y²), such that: (6 points)

$x = \frac{b^2\bar{X}+se^2 a}{b^2+se^2}$ and $y^2 = \frac{b^2 se^2}{b^2+se^2}$; where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $se^2 = \sigma^2/n$.

(Hint: less messier if you ignore the constants, but please justify why you can ignore them)

(b) Compute the (1-α) posterior interval for θ. (4 points)

(a) $X = \{X_1, X_2 \ldots X_n\}$

$\sigma^2 \to$ known, $\theta \to$ unknown (mean)

$f(\theta|x) = f(\theta|\{x_1, x_2 \ldots x_n\})$

$\overset{\text{Bayes}}{=} \frac{f(\{x_1, x_2 \ldots x_n\}|\theta) \cdot f(\theta)}{f(\{x_1 \ldots x_n\})}$

$\Rightarrow f(\theta|x) \propto f(\{x_1 \ldots x_n\}|\theta) \cdot f(\theta)$

posterior $\propto$ likelihood $\times$ prior

$\hookrightarrow f(\theta) \sim \text{Nor}(a, b^2)$

$\Rightarrow f(\theta) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\theta-a}{b}\right)^2}$

$f(\theta) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\theta-a}{b}\right)^2}$

Likelihood

$f(x|\theta)$ ~~$\propto$ ...~~ $= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\theta}{\sigma}\right)^2}$

$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^{n}\frac{1}{2}\left(\frac{X_i-\theta}{\sigma}\right)^2}$

$\Rightarrow f(\theta|x) \propto \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^{n}\frac{1}{2}\left(\frac{X_i-\theta}{\sigma}\right)^2} \cdot \frac{1}{\sqrt{2\pi}b} e^{-\frac{1}{2}\left(\frac{\theta-a}{b}\right)^2}$

Removing all the constants as we are considering proportionality → so only terms involving θ are relevant.

$$\Rightarrow f(\theta|X) \propto \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \left(\frac{1}{b\sqrt{2\pi}}\right) e^{\left\{-\frac{1}{2}\left(\frac{\theta-a}{b}\right)^2 - \sum_{i=1}^{n}\frac{1}{2}\left(\frac{x_i-\theta}{\sigma}\right)^2\right\}}$$

$$f(\theta|x) \propto e^{-\frac{1}{2}\underbrace{\left[\left(\frac{\theta-a}{b}\right)^2 + \sum_{i=1}^{n}\left(\frac{x_i-\theta}{\sigma}\right)^2\right]}_{t}}$$

Let $f(\theta|x) \propto e^{-\frac{1}{2}\cdot t}$

$$t = \left(\frac{\theta-a}{b}\right)^2 + \sum_{i=1}^{n}\left(\frac{X_i-\theta}{\sigma}\right)^2$$

$$t = \frac{\theta^2 + \overset{\text{const.}}{a^2} - 2a\theta}{b^2} + \frac{\overset{\text{Const}}{\sum X_i^2} - 2\sum_{i=1}^{n} X_i\theta + \sum_{i=1}^{n}\theta^2}{\sigma^2}$$

Removing all constant terms (that do not involve θ).

$$t \propto \frac{\sigma^2(\theta^2 - 2a\theta) + b^2\left(\cancel{\sum X_i^2} - 2\overset{n\bar{X}}{\sum_{i=1}^{n} X_i}\theta + \theta^2(n)\right)}{\sigma^2 b^2}$$

Replacing $\sum_{i=1}^{n} X_i$ by $n\bar{X}$:

$$t \propto \frac{\sigma^2\theta^2 - 2a\theta\sigma^2 + nb^2\theta^2 - 2n\bar{X}\theta b^2}{\sigma^2 b^2}$$

$$t \propto \frac{\theta^2(\sigma^2 + nb^2) - 2(\sigma^2 a + n\bar{X}b^2)\theta}{\sigma^2 b^2}$$

Dividing the numerator & denominator by $(\sigma^2 + nb^2)$:

$$t \propto \frac{\theta^2 - 2\theta\left(\frac{\sigma^2 a + nb^2\bar{X}}{\sigma^2 + nb^2}\right)}{\frac{\sigma^2 b^2}{\sigma^2 + nb^2}}$$

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Now, in order to complete this as a square term we can add & subtract a constant term to t as follows:

$$t \propto \frac{\theta^2 - 2\theta\left(\frac{\sigma^2 a + nb^2\bar{X}}{\sigma^2+nb^2}\right) + \left(\frac{\sigma^2 a + nb^2\bar{X}}{\sigma^2+nb^2}\right)^2 - \underbrace{\left(\frac{\sigma^2 a + nb^2\bar{X}}{\sigma^2+nb^2}\right)^2}_{\text{Const independent of } \theta}}{\frac{\sigma^2 b^2}{\sigma^2+nb^2}}$$

$$\Rightarrow t \propto \left(\frac{\left[\theta - \left(\frac{\sigma^2 a + nb^2\bar{X}}{\sigma^2+nb^2}\right)\right]^2}{\frac{\sigma^2 b^2}{\sigma^2+nb^2}}\right)$$

$$\Rightarrow f(\theta|X) \propto e^{-\frac{1}{2}\left[\frac{\left(\theta - \frac{\sigma^2 a + nb^2\bar{X}}{\sigma^2+nb^2}\right)^2}{\frac{\sigma^2 b^2}{\sigma^2+nb^2}}\right]}$$

The constants can be adjusted with the proportionality sign to show that $f(\theta|X)$ follows a Normal distribution : Nor (x, y^2) with

the mean, $x = \frac{\sigma^2 a + nb^2\bar{X}}{\sigma^2+nb^2}$

and std deviation, $y^2 = \frac{\sigma^2 b^2}{\sigma^2+nb^2}$

Let $se = \sigma^2/n$

$$X = \frac{\sigma^2 a + nb^2 \bar{x}}{\sigma^2 + nb^2}$$

$$= \frac{\frac{\sigma^2}{n} a + b^2 \bar{x}}{\frac{\sigma^2}{n} + b^2}$$

$$\boxed{X = \frac{se^2 a + b^2 \bar{x}}{se^2 + b^2}}$$

$$y^2 = \frac{\sigma^2 b^2}{\sigma^2 + nb^2}$$

$$= \frac{\frac{\sigma^2}{n} \cdot b^2}{\frac{\sigma^2}{n} + b^2}$$

$$\boxed{y^2 = \frac{se^2 b^2}{se^2 + b^2}}$$

$\therefore$ Posterior for Normal $= f(\theta|x)$

$= \text{Normal}(x, y^2)$.

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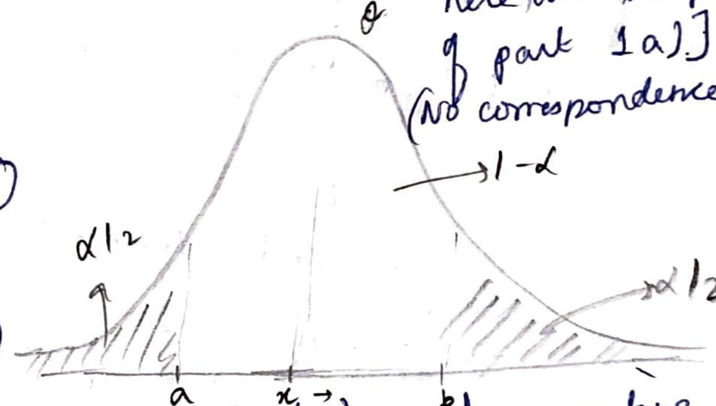
b) Compute the $(1-\alpha)$ posterior interval for θ

Let $D = \{X_1, X_2, \ldots X_n\}$

We want an ~~estimate~~ interval $[a, b]$ such that : [Note: a and b defined here are independent of part 1a)] (No correspondence)

$Pr(\theta \in [a,b]) \geq 1-\alpha$

$Pr(\theta < a | D) = \frac{\alpha}{2}$ ... (1)



$Pr(\theta > b | D) = \frac{\alpha}{2}$ ... (2)

Utilizing the result obtained in part 1a) and converting it to standard normal by subtracting the mean x and dividing by the standard deviation, y we get :-

$$Pr(\theta < a | D) = Pr\left(\frac{\theta - x}{y} < \frac{a-x}{y} \middle| D\right) = \frac{\alpha}{2}$$

$$\Rightarrow Pr\left(Z < \frac{a-x}{y}\right) = \frac{\alpha}{2} \quad \ldots (3)$$

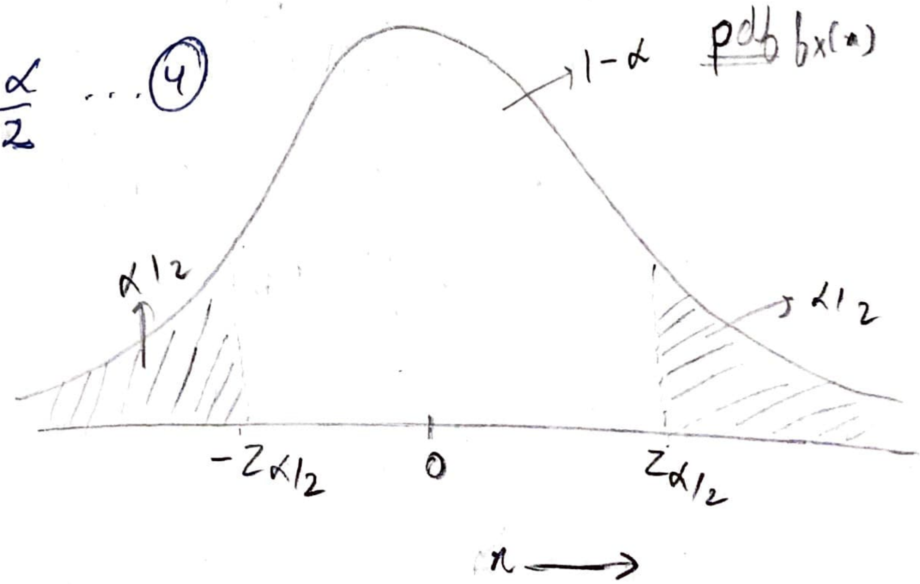
Z here refers to a std. normal distribution

We know that:

$\Rightarrow Pr(Z < -Z_{\alpha/2}) = \frac{\alpha}{2}$ ... (4)

From (3) and (4), we get :-

i.e $\frac{a-x}{y} = -Z_{\alpha/2}$



$\therefore \boxed{a = x - y Z_{\alpha/2}}$

$Pr(\theta > b | D) = \alpha/2$

Converting to std normal by subtracting the mean x and dividing by std. deviation, y we get:-

$$Pr\left(\frac{\theta - x}{y} > \frac{b - x}{y} \middle| D \right) = \frac{\alpha}{2}$$

$$Pr\left(Z > \frac{b-x}{y} \right) = \alpha/2 \quad \ldots (5)$$

We know that:

$$Pr(Z \leq Z_{\alpha/2}) = 1 - \frac{\alpha}{2}$$

$$\frac{\alpha}{2} = 1 - Pr(Z \leq Z_{\alpha/2})$$

$$\frac{\alpha}{2} = Pr(Z > Z_{\alpha/2}) \quad \ldots (6)$$

from (5) and (6), we conclude.

$$\frac{b-x}{y} = Z_{\alpha/2}$$

$$\boxed{b = x + y Z_{\alpha/2}}$$

$(1-\alpha)$ Posterior interval for θ = $[a, b] = [x - yZ_{\alpha/2}, x + yZ_{\alpha/2}]$

Substituting values for x and y as obtained in part 1a), we get.

$(1-\alpha)$ Posterior interval for θ is:

$$\left[\frac{b^2 \bar{X} + se^2 a}{b^2 + se^2} - Z_{\alpha/2}\left(\frac{b.se}{\sqrt{b^2 + se^2}} \right), \frac{b^2 \bar{X} + se^2 a}{b^2 + se^2} + Z_{\alpha/2}\left(\frac{b.se}{\sqrt{b^2 + se^2}} \right) \right]$$

↳ Answer

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A2.

a) for $\sigma = 3$,

In Bayesian Inference, we are first assuming prior as Nor (0, 1) → given. We then see new data and update our estimates at each step.

So, in step 1,

we have prior = Nor (0, 1)

We see the first row & calculate posterior

Posterior (calculated) = Nor (4.59, 0.083)

Now, in step 2,

we take our prior as ~~step~~ posterior observed from step 1.

∴ Prior = Nor (4.59, 0.083)

Posterior (calculated from seeing row 2 data)
= Nor (4.813, 0.043)

Similarly, the steps are below:-

Step 3 Prior = Nor (4.813, 0.043)

Posterior = Nor (4.921, 0.029) - calculated

Step 4:- Prior :- Nor (4.921, 0.029)

Posterior ~~f4~~ :- Nor (4.972, 0.022)

Step 5:- Prior :- Nor (4.972, 0.022)

Posterior :- Nor (4.9839, 0.017)

Tabulating the results as below:-

Step	Mean	Variance
1	4.59	0.082
2	4.8135	0.043
3	4.9212	0.029
4	4.972	0.022
5	4.9839	0.0176

Looking at the graphs attached, we observe that as we see more data, the distributions shift closer to the true ~~mean~~ ~~& var.~~ distribution faster as our confidence increases with each step. The fluctuations at each step are large.

b) For $\sigma = 100$,
Again we start with Prior Nor(0, 1) and at each step, ~~esti~~ calculate the new posterior values seeing data row by row.

Step 1:-
Prior :- Nor (0, 1) → given
Posterior :- Nor (0.058, 0.990) → calculated

Step 2:-
Prior = Posterior of Step 1 = Nor (0.058, 0.99)
Posterior :- Nor (0.095, 0.9803)

Step 3:-
Prior = Nor(0.095, 0.9803)
Posterior = Nor(0.138, 0.97)

Step 4:-

Prior = Nor (0.138, 0.97)

Posterior = Nor (0.171, 0.961)

Step 5:-

Prior :- Nor (0.171, 0.961)

Posterior :- Nor (0.2189, 0.9523)

Tabulating the results of posterior after each step:-

Step	Mean	Variance
1	0.0587	0.99
2	0.0950	0.98
3	0.1382	0.971
4	0.1712	0.9615
5	0.2189	0.952

Looking at the graphs, we observe that the distributions at each step are almost the same (even the mean & variance values above do not change much). Hence the change from prior at step 1:- Nor (0,1) is very less at each step and finally posterior after step 5 is Nor (0.218, 0.95). Thus, it will take more time-steps to converge.

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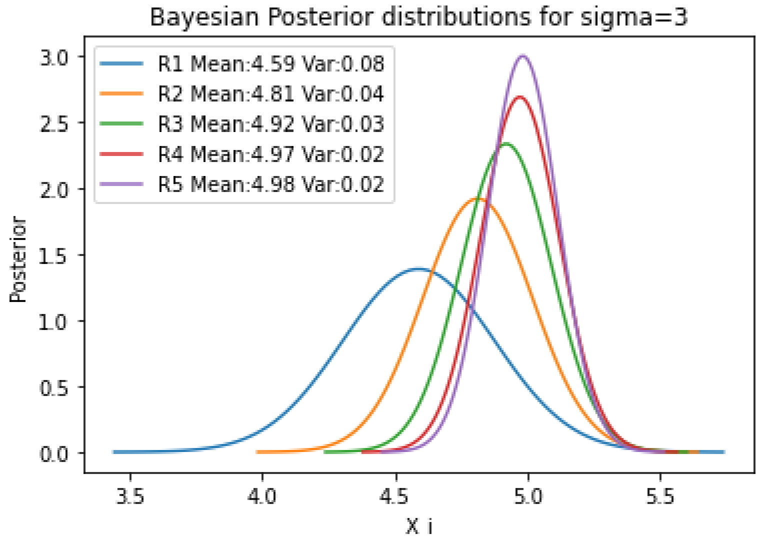
A2.c After each step in Bayesian Inference, we keep refining our estimate as we see new data. From part (a), we observe that for small value of $\sigma = 3$, our estimates quickly converge to true distribution (large changes in mean & variance after each step) & our confidence increases (height of peaks of distributions increases). Whereas for part (b), for a high value of $\sigma = 100$, we observe low fluctuations in mean & variances & hence it will take more steps to converge (our confidence in estimates increased marginally).

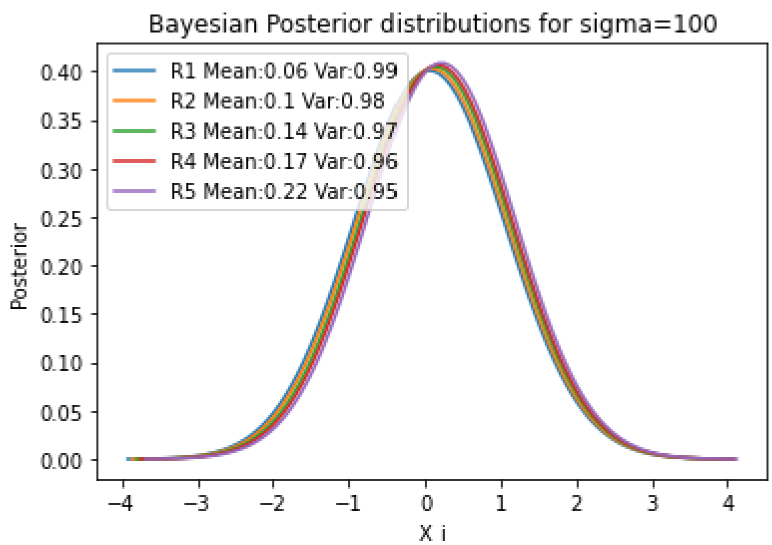
```
1 plotBayesianDistributions(sigma3, 3)
```

	Mean	Variance
0	4.590762	0.082569
1	4.813524	0.043062
2	4.921257	0.029126
3	4.972837	0.022005
4	4.983966	0.017682




```
1 plotBayesianDistributions(sigma100, 100)
```

	Mean	Variance
0	0.058716	0.990099
1	0.095009	0.980392
2	0.138226	0.970874
3	0.171219	0.961538
4	0.218918	0.952381



3. Regression Analysis **(Total 7 points)**

Assume Simple Linear Regression on n sample points $(Y_1, X_1), (Y_2, X_2), \ldots, (Y_n, X_n)$; that is, $Y = \beta_0 + \beta_1 X + \varepsilon_i$, where $E[\varepsilon_i] = 0$.

(a) Derive the estimates of β when minimizing the sum of squared errors and show that:

$\widehat{\beta_1} = \frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sum_{i=1}^{n}(X_i-\bar{X})^2}$ and $\widehat{\beta_0} = \bar{Y} - \widehat{\beta_1}\bar{X}$, where $\bar{X} = (\sum_{i=1}^{n} X_i)/n$ and $\bar{Y} = (\sum_{i=1}^{n} Y_i)/n$. (4 points)

(b) Show that the above estimators, given X_i's, are unbiased (Hint: Treat X's as constants) (3 points)

(a)

$$Y = \beta_0 + \beta_1 X + \varepsilon_i$$

$$y_i \mid x_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

calculating the expectation:

$$E[y_i \mid x_i] = E[\beta_0 + \beta_1 x_i + \varepsilon_i]$$

(assume $E[\varepsilon_i] = 0$)

$$E[y_i \mid x_i] = \beta_0 + \beta_1 x_i$$

(β_0 & $\beta_1 x_i$ are constants)

To estimate β_0 & β_1:

$$\hat{y}_i = E[y_i \mid x_i] = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

$$\hat{\varepsilon}_i = y_i - \hat{y}_i \quad \text{(the residual)}$$

$$\Rightarrow \hat{\varepsilon}_i = y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i) \quad \text{—— ①}$$

using the OLS (Ordinary Least Squares method):

$$S = \sum_{i=1}^{n} (\hat{\varepsilon}_i)^2$$

goal: find $\hat{\beta}_0$ and $\hat{\beta}_1$ such that S is minimized.

$$S = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} \left(y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)\right)^2$$

(from ①)

Now, taking the partial derivative of S w.r.t. $\hat{\beta_0}$

$$S = \sum_{i=1}^{n} \left(y_i - (\hat{\beta_0} + \hat{\beta_1} x_i)\right)^2$$

$$\frac{\partial S}{\partial \hat{\beta_0}} = \sum_{i=1}^{n} 2\left(y_i - (\hat{\beta_0} + \hat{\beta_1} x_i)\right)(-1) = 0.$$

(Equating to zero to get the minima)

Since x_i & y_i are constants:

$$\Rightarrow \sum_{i=1}^{n} \left(y_i - (\hat{\beta_0} + \hat{\beta_1} x_i)\right) = 0$$

$$\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \hat{\beta_0} - \sum_{i=1}^{n} \hat{\beta_1} x_i = 0$$

$$\sum_{i=1}^{n} y_i = n\hat{\beta_0} + \sum_{i=1}^{n} \hat{\beta_1} x_i$$

Dividing by n on both sides:

$$\Rightarrow \sum_{i=1}^{n} \frac{y_i}{n} = \hat{\beta_0} + \hat{\beta_1} \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\bar{y} = \hat{\beta_0} + \hat{\beta_1} \bar{x} \quad \text{--- (1)}$$

$$\Rightarrow \boxed{\hat{\beta_0} = \bar{y} - \hat{\beta_1} \bar{x}} \quad \text{--- (2)}$$

Now, taking the partial derivative of S wrt $\hat{\beta_1}$:

$$\frac{\partial S}{\partial \hat{\beta_1}} = \sum_{i=1}^{n} 2\left(y_i - (\hat{\beta_0} + \hat{\beta_1} x_i)\right)(-x_i) = 0$$

(equating to 0 to get the minima)

$$\Rightarrow \sum_{i=1}^{n} y_i x_i - \hat{\beta_0} \sum_{i=1}^{n} x_i - \hat{\beta_1} \sum_{i=1}^{n} x_i^2 = 0$$

substituting $\hat{\beta_0} = \bar{y} - \hat{\beta_1} \bar{x}$ from (2):

$$\sum_{i=1}^{n} y_i x_i - (\bar{y} - \hat{\beta_1} \bar{x}) \sum_{i=1}^{n} x_i - \hat{\beta_1} \sum_{i=1}^{n} x_i^2 = 0.$$

$$\Rightarrow \sum_{i=1}^{n} y_i x_i - \bar{y}\sum_{i=1}^{n} x_i + \hat{\beta}_1 \bar{x} \sum_{i=1}^{n} x_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 = 0.$$

$$\sum_{i=1}^{n} x_i y_i - \bar{y}\sum_{i=1}^{n} x_i - \bar{x}\sum_{i=1}^{n} y_i + \bar{x}\sum_{i=1}^{n} y_i = \hat{\beta}_1 \left[\sum_{i=1}^{n} x_i^2 - \bar{x}\sum_{i=1}^{n} x_i - \bar{x}\sum_{i=1}^{n} x_i + \bar{x}\sum_{i=1}^{n} x_i \right]$$

Adding and subtracting $\bar{x}\sum_{i=1}^{n} y_i$ on LHS and $\hat{\beta}_1 \bar{x} \sum_{i=1}^{n} x_i$ on RHS :

$$\sum_{i=1}^{n} x_i y_i - \bar{x}\sum_{i=1}^{n} y_i - \bar{y}\sum_{i=1}^{n} x_i + \bar{x}\sum_{i=1}^{n} y_i = \hat{\beta}_1 \left[\sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + \bar{x}\sum_{i=1}^{n} x_i \right]$$

On writing $\sum_{i=1}^{n} y_i$ as $n\bar{y}$ and $\sum_{i=1}^{n} x_i$ as $n\bar{x}$, we get :

$$\sum_{i=1}^{n} x_i y_i - \bar{x}\sum_{i=1}^{n} y_i - \bar{y}\sum_{i=1}^{n} x_i + \underbrace{n\bar{x}\bar{y}}_{\sum_{i=1}^{n} \bar{x}\bar{y}} = \hat{\beta}_1 \left[\sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + \underbrace{n\bar{x}\bar{x}}_{\sum_{i=1}^{n} \bar{x}^2} \right]$$

$$\Rightarrow \sum_{i=1}^{n} \left[x_i y_i - \bar{x} y_i - \bar{y} x_i + \bar{x}\bar{y} \right] = \hat{\beta}_1 \left[\sum_{i=1}^{n} \left(x_i^2 - 2\bar{x} x_i + \bar{x}^2 \right) \right]$$

$$\Rightarrow \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) = \hat{\beta}_1 \left(\sum_{i=1}^{n} (x_i - \bar{x})^2 \right)$$

Therefore,

$$\Rightarrow \boxed{\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sum_{i=1}^{n}(X_i-\bar{X})^2}} = \frac{S_{xy}}{S_{xx}}$$

$$\text{and} \quad \boxed{\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}} \quad \text{—} \ (3)$$

(b) <u>Expectation of $\hat{\beta}_0$ and $\hat{\beta}_1$</u>

$\boxed{\hat{\beta}_0}$ $\quad E[\hat{\beta}_0] = E[\bar{Y} - \hat{\beta}_1 \bar{X}]$

Using : $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$

$$E[\hat{\beta}_0] = E[\bar{Y} - \hat{\beta}_1 \bar{X}] = E[\beta_0 + \beta_1 \bar{X} + \varepsilon_i - \hat{\beta}_1 \bar{X}]$$

using the following assumption : $E[\varepsilon_i] = 0$.

$$\boxed{E[\hat{\beta}_0] = \beta_0 + \beta_1 \bar{X} - \bar{X} E[\hat{\beta}_1]} \quad \text{—} \ (4)$$

$\boxed{\hat{\beta}_1}$ $\quad E[\hat{\beta}_1] = E\left[\frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sum_{i=1}^{n}(X_i-\bar{X})^2}\right] = E\left[\frac{S_{xy}}{S_{xx}}\right]$

Using :

$Y_i = \beta_0 + \beta_1 X_i$ and $\bar{Y} = \hat{\beta}_0 - \hat{\beta}_1 \bar{X}$

from (3)

We know that:

Let $$\frac{\sum_{i=1}^{n}(X_i - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})}{S_{xx}} = 0$$

Let $$\frac{\sum_{i=1}^{n}(X_i - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2} = t$$, then the expectation:

$$E[\hat{\beta}_1] = E\left[\sum t_i (y_i - \bar{y})\right]$$

$t_i \to$ constant for given X_i's

$$= \sum_{i=1}^{n} t_i E[Y_i] - \cancel{\sum t_i E[\bar{Y}]}$$

$\bar{X} \to$ constant for given X_i's

$$= \sum_{i=1}^{n} t_i E[\beta_0 + \beta_1 X_i]$$

$$\boxed{E[\hat{\beta}_1] = \beta_1} \quad \text{— (5)}$$

$$E[\hat{\beta}_0] = \beta_0 + \beta_1 \bar{X} - \bar{X} E[\hat{\beta}_1] \quad \text{from (4)}.$$

substituting the value of $E[\hat{\beta}_1]$ from (5):

$$E[\hat{\beta}_0] = \beta_0 + \beta_1 \bar{X} - \beta_1 \bar{X}$$

$$\boxed{E[\hat{\beta}_0] = \beta_0}$$

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Bias of $\hat{\beta_0}$:

$$Bias(\hat{\beta_0}) = E[\hat{\beta_0}] - \beta_0$$

$$\boxed{Bias(\hat{\beta_0}) = 0}$$

Bias for $\hat{\beta_1}$:

$$Bias(\hat{\beta_1}) = E[\hat{\beta_1}] - \beta_1$$

$$\boxed{Bias(\hat{\beta_1}) = 0}$$

4a.

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Sum of Square Errors in case: 1 is: 28176831.90116
Sum of Square Errors in case: 2 is: 10656145.20472
Sum of Square Errors in case: 3 is: 43197449.81127
Sum of Square Errors in case: 4 is: 10948958.73407
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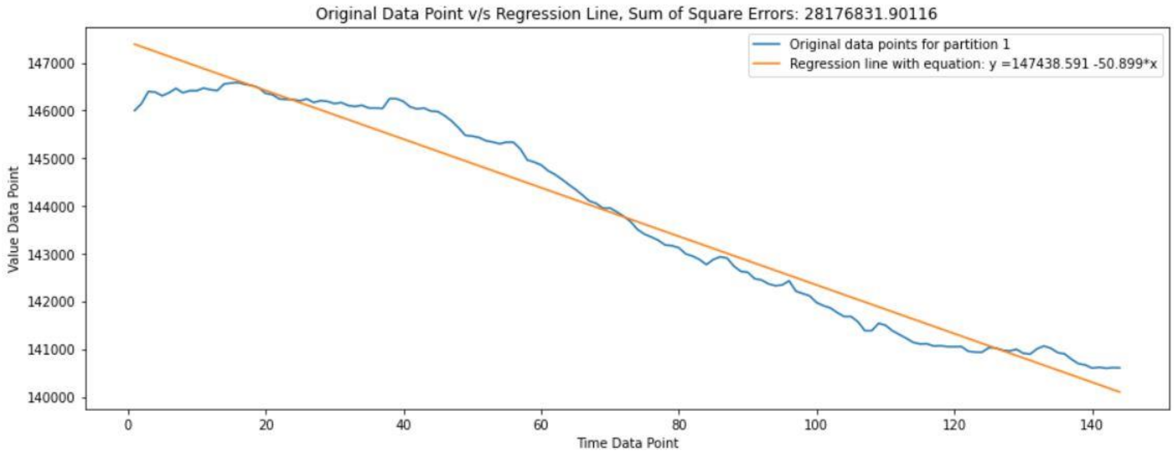
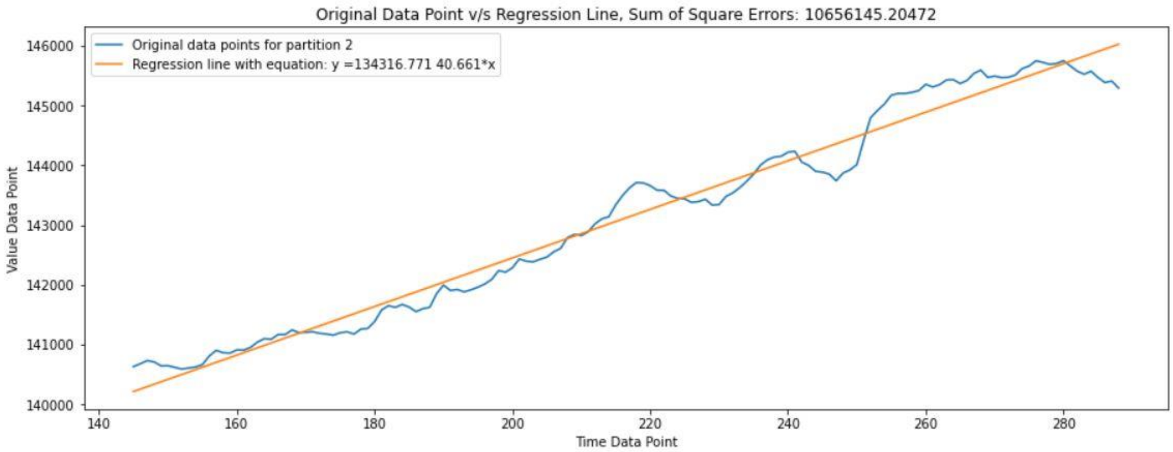
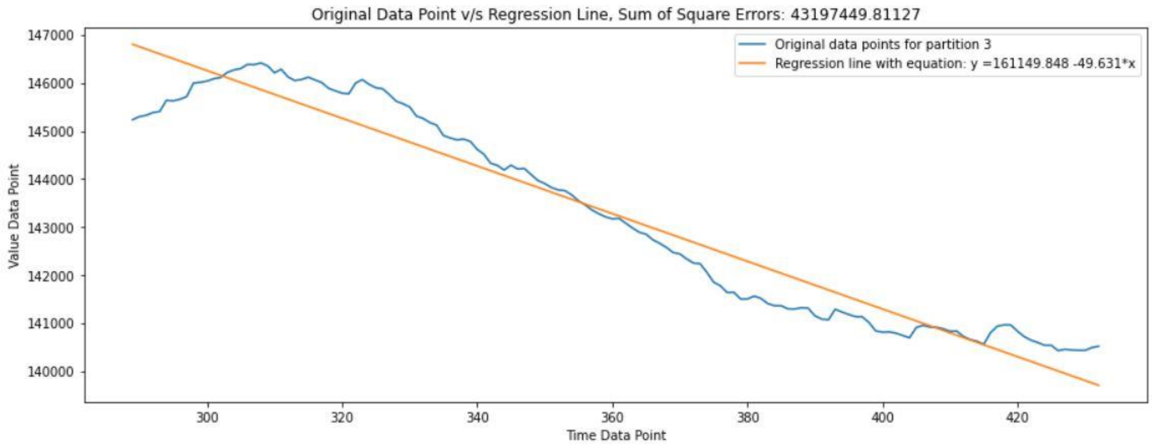
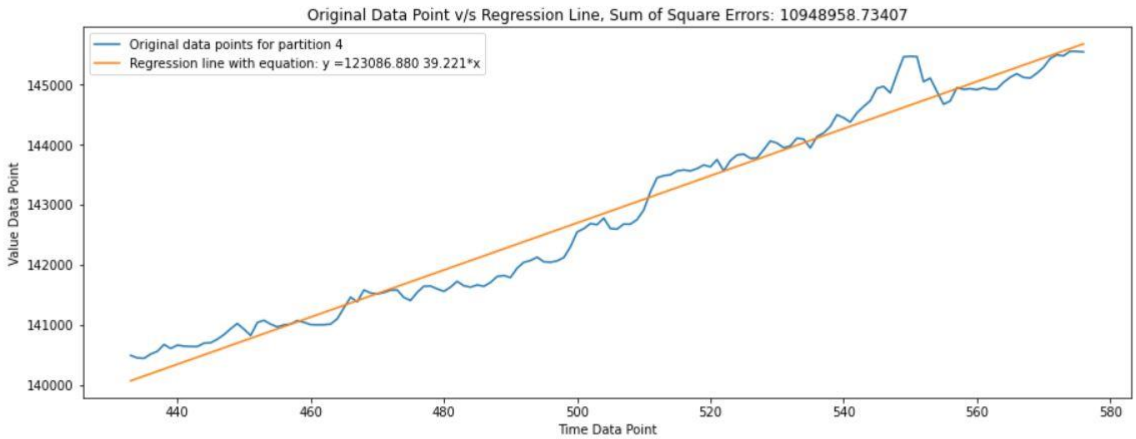





Original Data Point v/s Regression Line, Sum of Square Errors: 43197449.81127
Original data points for partition 3
Regression line with equation: y =161149.848 -49.631*x
Value Data Point
Time Data Point
147000
146000
145000
144000
143000
142000
141000
140000
300
320
340
360
380
400
420


Original Data Point v/s Regression Line, Sum of Square Errors: 10948958.73407
Original data points for partition 4
Regression line with equation: y =123086.880 39.221*x
Value Data Point
Time Data Point
145000
144000
143000
142000
141000
140000
440
460
480
500
520
540
560
580

5(a)

$C=0$ i.e. we choose $H=0$ (good soil) iff $P(H=0|\omega) \geq P(H=1|\omega)$

Now,

$$P(H=0|\omega) = \frac{P(\omega|H=0) \cdot P(H=0)}{P(\omega)} \quad \text{(by Bayes thm)}$$

$$P(H=1|\omega) = \frac{P(\omega|H=1) \cdot P(H=1)}{P(\omega)}$$

$C=0$ iff, $\cancel{P(H=0|\omega) \cdot P(H=0)} \cancel{\geq P(\omega|H=1) \cdot P(H=1)}$

$$P(H=0|\omega) \geq P(H=1|\omega)$$

$$\frac{P(\omega|H=0) \cdot P(H=0)}{P(\omega)} \geq \frac{P(\omega|H=1) \cdot P(H=1)}{P(\omega)}$$

$$P(\omega|H=0) \cdot P(H=0) \geq P(\omega|H=1) \cdot P(H=1) \quad [\because p(\omega) \text{ is +ve}]$$

$$P(\omega|H=0) \cdot p \geq p(\omega|H=1) \cdot (1-p)$$

Now, $\omega = \{\omega_1, \omega_2, \omega_3 \dots \omega_n\}$

$$P(\omega_1 \dots \omega_n | H=0) \cdot p \geq P(\omega_1 \dots \omega_n | H=1)(1-p)$$

Given that samples are conditionally independent given hypothesis

$$\therefore \prod_{i=1}^{n} P(\omega_i | H=0) \cdot p \geq \prod_{i=1}^{n} P(\omega_i | H=1)(1-p)$$

$$p \cdot \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\omega_i+\mu}{\sigma}\right)^2} \geq (1-p) \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\omega_i-\mu}{\sigma}\right)^2}$$

this is because $f_W(\omega|H=0) = N(\omega_i; -\mu, \sigma^2)$

& $f_W(\omega|H=1) = N(\omega_i; \mu, \sigma^2)$

$\therefore$ We have,

$$p \cdot e^{-\frac{1}{2} \sum_{i=1}^{n} \frac{(\omega_i+\mu)^2}{\sigma^2}} \geq (1-p) e^{-\frac{1}{2} \sum_{i=1}^{n} \frac{(\omega_i-\mu)^2}{\sigma^2}}$$

$$e^{\left(-\sum_{i=1}^{n}\frac{(\omega_i+\mu)^2}{2\sigma^2}+\frac{\sum(\omega_i-\mu)^2}{2\sigma^2}\right)} \geq \frac{(1-p)}{p}$$

$$e^{\frac{-\sum\cancel{\omega_i^2}-\sum\cancel{\mu^2}-\sum 2\omega_i\mu+\sum\cancel{\omega_i^2}+\sum\cancel{\mu^2}-\sum 2\omega_i\mu}{2\sigma^2}} \geq \frac{(1-p)}{p}$$

$$e^{\frac{-\cancel{4}^{2}\mu\sum_{i=1}^{n}\omega_i}{\cancel{2}\sigma^2}} \geq \frac{(1-p)}{p}$$

$$e^{\frac{2\mu\sum_{i=1}^{n}\omega_i}{\sigma^2}} \leq \left(\frac{p}{1-p}\right)$$

taking ln on both sides

$$\frac{2\mu\sum_{i=1}^{n}\omega_i}{\sigma^2} \leq \ln\left(\frac{p}{1-p}\right)$$

$$\boxed{\sum_{i=1}^{n}\omega_i \leq \frac{\sigma^2}{2\mu}\ln\left(\frac{p}{1-p}\right)}$$

this is the condition for choosing the hypothesis H=0

5(c) Average Error Probability (AEP) $= P(C=0|H=1)P(H=1) + P(C=1|H=0).P(H=0)$

$P(C=0|H=1) = P(\text{Choosing } H=0 \mid H=1)$

from 5(a), we know that we choose $H=0$ iff,

$$\sum_{i=1}^{n} \omega_i \leq \frac{\sigma^2}{2\mu} \ln\left(\frac{P}{1-P}\right)$$

$$\therefore P(C=0|H=1) = P\left(\sum_{i=1}^{n} \omega_i \leq \frac{\sigma^2}{2\mu} \ln\left(\frac{P}{1-P}\right) \Big| H=1\right)$$

As ω_i's are normally distributed given $H=1$ with μ, σ^2

$W' = \sum_{i=1}^{n} \omega_i$ is also normally distributed with mean $n\mu$ &

std. deviation $n\sigma^2$ (linear transformation of normal distribution)

$$\therefore P(C=0|H=1) = P\left(W' \leq \frac{\sigma^2}{2\mu} \ln\left(\frac{P}{1-P}\right) \Big| H=1\right)$$

let $\frac{\sigma^2}{2\mu} \ln\left(\frac{P}{1-P}\right)$ be "c"

$$\therefore P(C=0|H=1) = P(W' \leq c \mid H=1)$$

$$= \Phi\left(\frac{c - n\mu}{\sqrt{n}\,\sigma}\right)$$

$$\therefore P(C=0|H=1).P(H=1) = \Phi\left(\frac{c-n\mu}{\sqrt{n}\,\sigma}\right)(1-P) \quad \text{--- (1)}$$

Similarly, $P(C=1 \mid H=0) = P\left(\sum_{i=1}^{n} \omega_i > \frac{\sigma^2}{2\mu} \ln\left(\frac{p}{1-p}\right) \mid H=0\right)$

$$= P(W' > c \mid H=0)$$

where, $W' = \sum_{i=1}^{n} \omega_i$ & W'_i is normally distributed given $H=1$ with mean $-n\mu$ & std. deviation σ.

$$P(C=1 \mid H=0) = P(W' > c \mid H=0)$$

$$= 1 - \Phi\left(\frac{c+n\mu}{\sqrt{n}\,\sigma}\right)$$

$$\therefore P(C=1 \mid H=0) \cdot P(H=0) = \left(1 - \Phi\left(\frac{c+n\mu}{\sqrt{n}\,\sigma}\right)\right) \cdot (p) \quad \text{--- (2)}$$

from (1) & (2)

$$AEP = \Phi\left(\frac{c-n\mu}{\sqrt{n}\,\sigma}\right)(1-p) + \left[1 - \Phi\left(\frac{c+n\mu}{\sqrt{n}\,\sigma}\right)\right] \cdot p$$