Math 239

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Introduction to Graph Theory

4.1 Definitions

A graph is an ordered pair {V(G), E(G)} of finite sets V(G) and E(G) such that E is a subset of the set of unordered pairs of elements from V(G).

If {V(G), E(G)} is a graph then vertices v, w are adjacent in the graph G if {v,w} ϵ *E*.

The degree of a vertex, v, is the number of vertices that are adjacent to v, and is denoted by *d(v)* or *deg(v)*

A graph G is considered r-regular if every vertex has a degree r.

4.2 Isomorphism

Two graphs {V(G), E(G)} and {V́(G), E'(G)} are isomorphic if there is a bijection

$$\phi : \text{V} \to \text{V'}$$

such that E'(G) = {{ϕ(u), ϕ(v) | {u,v} ϵ E}.

Examples

- Empty Graph (V(G), ϕ)
- Complete Graph (V, $\binom{E}{2}$)
- Path Graph
- Null Graph {ϕ, ϕ}
- Cycle Graph, denoted by C_n:

Theorem: Handhake Lemma

Suppose {V(G), E(G)} is a graph, then $\sum_{v \epsilon V(G)}$ *deg(v)* = 2|E(G)|.

Proof: $\sum_{v \epsilon V(G)} deg(v) = \sum_{v \epsilon V(G)} |\{u \mid \{u, v\} \epsilon E(G)\}|$
$= \sum_{v \epsilon V(G)} | \{e \epsilon E(G) \mid v \epsilon e\}$
$= \sum_{e \epsilon E(G)} \sum_{v \epsilon V(G)} 1$
$= \sum_{e \epsilon E(G)} 2 = 2|E(G)|$

4.3 Degree

4.4 Bipartite Graphs

A graph *G* is bipartite if *V(G)* can be partitioned into two parts X, Y such that each edge has one vertex in X and one vertex in Y.

$$V = X \cup Y$$
$$\phi = X \cap Y$$

Empty Graphs are bipartite.

FACT: Even cycles are bipartite.

FACT: If H is a subgraph of a bipartite graph G, then H is bipartite.

Implication

If G has a subgraph that is an odd cycle, then G is not a bipartite graph.
Two More exmaples of Bipartite Graph
We already showed all the paths and all even cycles are bipartite.

$K_{p,q}$ = Complete bipartite graphs of p and q vertices.

How many vertices? p+q
What are the degrees? q p
How many edges? p.q

4.4.1 N - Dimensional Cube (Q_n)

V(G) = All binary strings of length n
E(G) = Two strings that differ in exactly one place.

Q_n has 2^n vertices. They have degrees, all n, and number of edges is $n2^{n-1}$

FACT: Let G be a bipartite graph with bipartition (X,Y), i.e, V(G) = X $\cup$ Y and X $\cap$ Y = ϕ, and every edge consists of vertex in X and a vertex in Y.

Then |E(G)| = $\sum_{v \epsilon X}$ *deg(v)* = $\sum_{v \epsilon Y}$ *deg(v)*.

X = Odd # 1's and Y = Even # 1's

4.5 Connectness

Last time, we agreed that K_n is connected for any empty graph $\geq$ 2 is disconnected.

- A graph G is connected if, for every two vertices v,u there is a path in G joining u & v.

- A graph is disconnected if it is not connected. That means there exists a pair of vertices, u,v such that no path in the graph joins u and v.

- How do we prove that Q_{758} is connected?
- How do we prove some path is disconnected?

4.6 Eulerian Circuits

4.7 Bridges

Chapter 5

Trees

5.1 Trees

5.2 Spanning Trees

5.3 Characterizing Bipartite Graphs

5.4 Minimum Spanning Trees

Chapter 6

Planar Graphs

6.1 Planarity

6.2 Euler's Formula

Theorem: **Proof:**

6.3 Platonic Solids

Iamges of Tetrahedron, octahedroon, hexahedron, icosahedron, dodecahedron

Theorem: Let d and d* be integers, noth at least 3. If there is a planar graph such that all the vertices have degree d & all faces of length d*, then (d, d*) $\in$ {(3, 3), (3, 4), (3, 5), (4, 3), (5,3)}

Proof: Handshake Lemma says d|V(G)| = 2|E(G)|. For faces, let r be the number of faces. Then Handshaking lemma for faces says d*r = 2|E(G)|. Also we have Eulers' formula:

$$p - q + r = 2$$

Therefore,

$$\frac{2}{d}|E(G)| - |E(G)| + \frac{2}{d*}|E(G)| = 2.$$

$$\frac{E(G)}{dd*}[2d^* - dd^* + 2d] = 2.$$

$$dd^* - 2d - 2d^* = \frac{-2dd*}{E(G)}$$

$$(d-2)(d^*-2) - 4 = \frac{-2dd*}{E(G)}$$

$$(d-2)(d^*-2) = 4 - \frac{2dd*}{E(G)}$$

What positive integer pairs a,b have a,b < 4? where a = (d-2) and b = (d-2)*

$\rightarrow${(1, 1), (1, 2), (1, 3), (2, 1), (3,1)}

So now when a,b $\in$ A, we get (d, d*) such that {(3, 3), (3, 4), (3, 5), (4, 3), (5,3)}

Let us take d = 3 and d* = 3;

$\frac{E(G)}{9}$(6-9+6) = 2, simplifying and solving for E(G) we get 6. Then 3V(G) = 2E(G) V(G) = 4 and hence G = K_4

Let us take d = 3 and d* = 5;
$\frac{E(G)}{15}$(6-15+10) = 2, simplifying and solving for E(G) we get 30. Then 3V(G) = 2E(G).
V(G) = 40, r = 12

Let us take d = 3 and d* = 4;
V(G) = 8, E(G) = 12 and r =6 Should be the same as hexahedron
$\triangle$ Joining the vertex of top triangle with bottom triangle
$\square\square$
$\triangle$

Colouring Planar Graphs Adjacent vertices get different colors

How many colours for K_n? n If G is connected planar graph then G has a vertex with degree ≤ 5

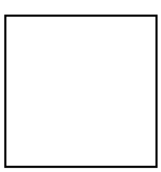
$$E(G) \leq 3V(G) - 6$$

Handshaking Lemma: $\sum$deg(V(G)) = 2E(G) $\leq$ 6V(G) - 12

6.4 Colouring and Planar Graphs

Colouring: assigns a colour to each vertex so that adjacent vertices get different colours.
1) Every bipartition in a bipartite graph can be coloured with two colours. Use different colour on each part of bipartition.
2)K_n can only be coloured with n colours.

Figures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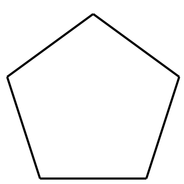
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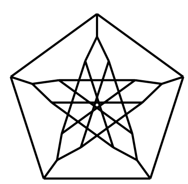
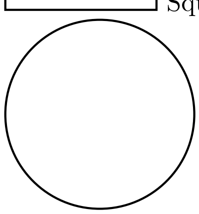
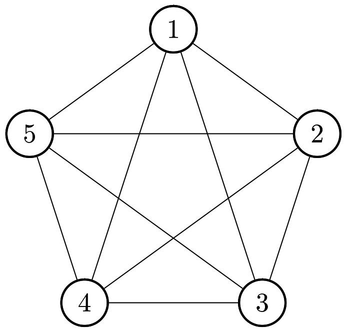

Some graph



K 100 K5  Peterson Graph Cross-Cover graph

6.4.1 Six Colour Theorem

Theorem: A planar graph can be coloured with six colours.

1) Every planar graph has a vertex with atmost 5.
Pf: Let C be a component of a planar graph G. If V(C) $\leq$ 6, then every vertex of G has a degree of at most 5. If V(C) $\geq$ 6, then because it is a connected planar graph , we know that E(C) $\leq$ 3V(G) - 6. There fore the average degree is $\sum_{v\epsilon V(C)} \frac{deg(v)}{V(C)} \leq \frac{6V(C)-12}{V(C)}$ = 6 - $\frac{12}{V(C)}$ < 6. There is a vertex v of C whose degree is at most the average. Thus deg(v) < 6 and so deg(v) $\leq$ 5.

2) To colour G, we proceed with induction on V(G)
Pf: If V(G) = 1, then G can be coloured with 1 (and therefore 6) colours.
For the induction step, choose a vertex V of G with degree at most 5. Colour G\V with at most 6 colours (using inductive assumption). The neighbours are coloured by the colouring of G\V

Proof: Because deg(v) $\leq$ 5, at most five of the 6 colours are used as colours for the neighbours of v. Thusm there is one of the six colours not used to colour a neighbour of v, we can use this colour to colour v. Hence proved.

6.4.2 Five Colour Theorem

Theorem: A planar graph can be coloured with five colours.

1) Every planar graph has a vertex with atmost 5.
Pf: Let C be a component of a planar graph G. If V(C) $\leq$ 6, then every vertex of G has a degree of at most 5. If V(C) $\geq$ 6, then because it is a connected planar graph , we know that E(C) $\leq$ 3V(G) - 6. There fore the average degree is $\sum_{v\epsilon V(C)} \frac{deg(v)}{V(C)} \leq \frac{6V(C)-12}{V(C)}$ = 6 - $\frac{12}{V(C)}$ < 6. There is a vertex v of C whose degree is at most the average. Thus deg(v) < 6 and so deg(v) $\leq$ 5.

2) To colour G, we proceed with induction on V(G)
Pf: If V(G) = 1, then G can be coloured with 1 (and therefore 6) colours.

For the induction step, choose a vertex V of G with degree at most 5. Colour G\V with at most 6 colours (using inductive assumption). The neighbours are coloured by the colouring of G\V

Problem: Deg(v) = 5. Because K_5 ia not planar, some two of the neighbours of v are not adjacent.
Solution: Let there be u and v. There two are not adjacent. Delete the edges from u to the other three neighbours of v & contract to v the edge uv and v as to create a smaller graph G'.

Proof: Continuing further from the solution, we see that by colouring G' with 5 colours by induction and colouring u and w, with the colour of the contracted vertex, every one else with their colour in G'. Since u, w have the same colour, neighbours of u and v have at most 4 colours in G\v. This leaves the fifth colour for v.

6.4.3 $K_{3,3}$ is not planar

Lemma: If a planar graph has no cycle and has atleast three vertices, then every face boundary has length ≥ 4.

Corollary: If G is a connected planar graph with ≥ 3 vertices and no 3 cycle, then $E(G) \leq 2V(G)$.

So by using this above corollary $K_{3,3}$ is not planar

6.5 Kuratowski's Theorem:

A graph is not planar if and only if it contains a subdivision of either K_5 or $K_{3,3}$
Fig 1 and Fig 2 to be embedded here.

Chapter 7

Matchings

7.1 Matching or Matching in Graphs

A *matching* in a graph G is a set M of edges of G, no two of which are incident with a common vertex.

Fig 3 and Fig 4

For any matching M in a graph, 2|M| $\leq$ |V(G)|.

A *perfect Matching* in a graph G is a matching M in G such that 2 = |V(G)|

The empty graph with at least one vertex has no perfect matching. Notice ϕ is a matching.

Note: Even cycles have perfect matching. If |V(G)| is odd then G does not have a perfect matching

In a $K_{p,q}$ graph we get the min{p, q} giving us the largest matching size.

7.1.1 Covers or Covering the edges of a graph

A cover of a graph G is a set C of vertices such that every edge is incident with at least one vertex in C.

Fact: If M is a matching and C is a cover in G, then |M| $\leq$ |C| .

In a *Cycle* of length n:
the size of a largest matching is $\lfloor \frac{n}{2} \rfloor$
the size of a smallest cover is $\lceil \frac{n}{2} \rceil$

In a K_n graph:
the size of the matching $\lfloor \frac{n}{2} \rfloor$
the cover for the cycle is: n-1

In a $K_{p,q}$ complete graph (with $p \leq q$):
the size of the matching is: p
the size of the cover is: p

Lemma: If M is a matching and C is a cover then $|M| < |C|$.

Proof: Let e = uv be an edge in M. By definition, either u or v is in C (or both), without loss of generality, assume it is u. Let e' be a different edge in M. One of the endpoints of e'(w) is in C. Can u = w? By definition of the matching, e and e' share no endpoints, so $u \neq$ to w. There are |M| edges un M. Each yields a unique vertex in C $\rightarrow |M| \leq |C|$

Maximum mathcing a alaregest matching G. (No matching exist with more edges)
If the size os a maximum matching is $\frac{n}{2}$, then it is a perfect

Minimum cover is a smallest cover in C.

Lemma: If M is a matching and C is a $|M| = |C|$ then, M is a maximum matching and C is the minimum cover.

Proof: Let M* be a maximum mathcing, then $| M^* | \leq | C| = | M |$. M has at least as many edges as M*(which is a maximum matching). So M is also a maximum matching. Let C* be a minimum cover. $| C^* | \geq | M| = | C |$. So C is also a minimum cover.

Lemma: If M has an augmenting path then i is not a maximum matching

Proof: Let P be the augmenting path. The edges alternate between being in M and not being in M. The endpoints are unsaturated. More edges of P are not in M than are in M (one more). Replace the edges in $M \cap P$ with the edges in P set difference M. New matching M'. If $v \epsilon P$ is saturated by M. We removing the edge in M is incident to v and replacing it with one in M'. If $v \epsilon P$ is unsaturated, we only add one edge incident to v to form the matching M'.. M' is a matching $|M'| = | M |$ is not a maximum matching.

7.1.2 Augmented Paths

An alternating Path is a path in G where the edges alternate between being in M and not being in M. An augmented Path is an alternating path where its endpoints are unsaturated.
Odd length

First + Last edge not in the matching
A vertex is a saturated by a matching M if it is incident to an edge in M. Otherwies it is unsaturated which is also called exposed. In a perfect matching, all vettices are saturated. If there's no perfect matching, at least one vertex is unsaturated in any matchings.

Lemma: Let G be a graph with matching M. If there is a M - augmented path in G, then M is not a maximum matching(swap the edges of the ones in M with the edges which are not in M)
Lemma: If M is not a max matching then it has an augmenting path. Will be done in tutorial
Let G be a graph with bipartition A, B and let M be a matching of G.
- Let X_0 be the set of vertices in A not saturated by M.
- Let Z be the set of vertices in G joined to a vertex in X_0 by an alternating path.
- If there is an augmenting path use it to update the matching.
- Otherwise: $X = A \cap Z$.
  $X = B \cap Z$
  Cover: $C = Y \cup (A \backslash X)$

Figure here
$X_0 = \{b, e\}$
Alternating Paths (Start from b\e) {b,e,4,5,a,d,1,2,3}
Figure 2 here
$X_0 = \{e\}$
$Z = \{e, 4, 5, b, d\}$
In order to have a augmenting path, Z must include an unsaturated vertex not in X_0. Since Z does not contain 3 there is no augmenting path
$X = A \cap Z = \{b, d, e\}$
$Y = B \cap Z = \{4, 5\}$
$C = Y \cup (A \setminus X) = \{4, 5\} \cup \{a, c\} = \{4, 5, a, c\}$ is a min cover.

Lemma: Let M be a matching of a bipartite graph with partition A, B. Then
a) There is no edge from X to $B\backslash Y$.
b) $C = Y \cup (A\backslash X)$ is a cover.
c) There is no edge of M from Y to $(A\backslash X)$.
d) If u is the set of unsaturated vertices in Y then $|M|= |C| - |U|$.
e)there is an augmenting path to each vertex in U.

Proof: a) Suppose there is an edge where $u \in X$, $v \in B \backslash Y$. There is an alternating oath P from X_0 to u P' = P + uv. uv is not an edge of M. Alternating path to v. So V should be in Z.
b) Edges from:
$X \to Y$ - one end in Y and one end in C
$X \to B\backslash Y$ - don't exist by part a of the proof
$A\backslash X \to Y$ - both end in C
$A\backslash X \to B\backslash Y$ - one end in C
Hence C is cover

c)Suppose the edge vw $\in$ M, v $\in$ A$\setminus$ X, w $\in$ Y, w $\in$ Z, so there is an alternating path from X_0 to w w$\in$B, so P has odd length (So to travel to the other set it should have an odd length length 1, 3, 5, ...). Starting from X_0, the first edge is not in M, so the last edge (incident edge to w) is also not in the matching. P' = P + vw is an alternating path and starts from X_0 to v.Thus v is in Z, hence it is in X. And since we said that v $\in$ A$\setminus$ X, it is a contradiction($\longrightarrow ! \longleftarrow$)

d) X $\to$ Y $*$
X $\to$ B$\setminus$Y - does not $\exists$
A$\setminus$X $\to$ Y - not in M by part c
A$\setminus$X $\to$ B $\setminus$Y $*$ - (2)
Number of edges X to Y in M. |U| unsaturated vertices in Y. |Y| - |U| saturated vertices in Y. One edge in M for each in M for each of these. Number of edges from X to Y in M = |Y| - |U|.From (2) now, 0 unsaturated vertices in A$\setminus$X. |A$\setminus$X| saturated vertices in A$\setminus$X. $X_0 \subseteq$ Z.So X_0 $\subseteq$ X. Number is equal to |A$\setminus$X|. So |M| = |Y| - |U|+|A$\setminus$X|. So by using part b of the proof, |M| = |C| - |U|.

e) Let v $\in$ U. V is unsaturated and v $\in$ Y and since Y $\in$ B $\cap$ Z then v $\in$ Z. Then there is an alternating path from X_0 to v then it is an augmenting path.
Hence proved the lemma.

7.2 Konig's Theorem

König's Theorem: In a bipartite graph, the maximum size of a mathcing is the minimum size of a cover.
Proof: Let M be a maximum matching. Then there is no augmenting path in G. Furthermore, |U| = 0. So by part b from the previous lemma, C = Y $\cup$ (A$\setminus$X) and then it is a cover. By part d) we get that M = C. $\square$.

D is a set of vertices.
Neighbour set, denoted by N(D), is the set of vertices adjacent to a vertex in D
Halls' Theorem: G is a bipartite graph with partition on A, B. G has a matching saturating every vertex in A if and only if every subset D of A satisfies

$$|N(D)| \geq |D|$$

Picture: D = {1, 2, 3}
N(D) = {a, c}
|N(D)| = 2 < 3 = |D|

7.3 Applications of Konig's Theorem

7.4 System of Distinct Represntatives