Unit-4: Inner Product Spaces

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Example-3

Find the angle between the matrices

$$A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Solution:

By definition of angle between vectors we have

$\theta = \cos^{-1}\left(\frac{\langle A, B\rangle}{\|A\|\|B\|}\right)$

Where $\langle A, B\rangle = Trace\left(AB^T\right)$

$$AB^T = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix} * \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 46 & 118 \\ 28 & 73 \end{bmatrix}$$

$\Rightarrow Trace\left(AB^T\right) = 46 + 73 = 119$

Next find
$\|A\| = \sqrt{Trace\,(AA^T)}$

$$AA^T = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix} * \begin{bmatrix} 9 & 6 \\ 8 & 5 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 194 & 122 \\ 122 & 77 \end{bmatrix}$$

$\|A\| = \sqrt{Trace\,(AA^T)} = \sqrt{194 + 77} = \sqrt{271}$
Similarly find $\|B\| = Trace\left(BB^T\right)$

$$BB^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} * \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 32 \\ 32 & 77 \end{bmatrix}$$

$\|B\| = \sqrt{Trace\,(BB^T)} = \sqrt{14 + 77} = \sqrt{91}$
Hence
$\theta = \cos^{-1}\left(\frac{\langle A, B\rangle}{\|A\|\|B\|}\right) = \cos^{-1}\left(\frac{119}{\sqrt{271}\sqrt{91}}\right)$ is the required angle between A and B.

Example-4:

Check whether $f(t) = \sin t$ and $g(t) = \cos t$ in $C[a, b]$ with $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t) * g(t)dt$ are orthogonal.

Solution:

Using given definition and functions, we get

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t) * g(t)dt = \int_{-\pi}^{\pi} \sin t \cos t dt$$

$$\Rightarrow \langle f, g\rangle = \int_{-\pi}^{\pi} \frac{\sin 2t}{2} dt = \left[-\frac{\cos 2t}{4}\right]_{-\pi}^{\pi} = -\left[\frac{\cos 2\pi - \cos 0}{4}\right] = 0$$

Therefore $\sin t$ and $\cos t$ are orthogonal each other.

Examples Continued.....

Example-5:

Find a non-zero vector that is orthogonal to $(1, 2, 1)$ and $(2, 5, 4)$ in R^3 .

Solution: Given $u = (1, 2, 1)$ and $v = (2, 5, 4)$
Let $w = (x, y, z)$ in R^3 is orthogonal to both u and v then by definition of orthogonality we have $\langle u, w \rangle = 0$ and $\langle v, w \rangle = 0$
From this we can write

$$\langle u, w \rangle = 0 \Rightarrow x + 2y + z = 0$$
$$\langle v, w \rangle = 0 \Rightarrow 2x + 5y + 4z = 0$$

This represents the system of homogeneous equation. The solution of this will give required orthogonal vector.
Consider coefficient matrix of the above system and reduce in echelon form,we get:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

There is no leading entry third column, therefore $z = k$ is free. Hence we get

$$Row - 2 \Rightarrow y + 2z = 0 \Rightarrow y = -2z = -2k$$
$$Row - 1 \Rightarrow x + 2y + z = 0 \Rightarrow x = -2y - z \Rightarrow x = 4k - k = 3k$$

Hence required orthogonal vector w which orthogonal to both u and v is given by

$$w = \begin{bmatrix} 3k \\ -2k \\ k \end{bmatrix}$$

Orthogonal and Orthonormal Sets:

Definition:

Let $(V, \langle , \rangle)$ be an Inner Product Space. Then a nonempty set $S \subset V$ of nonzero vectors is called an **orthogonal set** if all vectors in S are mutually orthogonal. That is, $0 \notin S$ and $\langle u, v \rangle = 0$ for any u, $v \in S$, $u \neq v$.

Definition:

An orthogonal set $S \subset V$ is called **orthonormal set** if $\|u\| = 1$ for any $u \in S$.

Note: Vectors $v_1, v_2, ..., v_k \in V$ form an orthonormal set if and only if

$$\langle v_i, v_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Orthogonal Compliment

Definition:

Let V be any inner product space and let $W \subset V$ be any subset then $W^{\perp} = \{u : \langle w, u\rangle = 0\}$ is the set of all vectors which are orthogonal to vectors in W is called Orthogonal compliment of W.

Note:

- If W and $W^{\perp}$ are orthogonal compliments then the basis of of $W^{\perp}$ are orthogonal to every vector of W.
- Finding the basis of W given vectors in $W^{\perp}$ we need just to solve system of homogeneous equations.
- The set of all orthogonal vectors are linearly independent.

Orthogonal Matrix:

A matrix P is said to orthogonal if and only if both columns and rows of P are mutually orthogonal to each other. Also for an orthogonal matrix $|P| = 1$ and $P^{-1} = P^{T}$.

Ex-1:

Let W be a subspace of R^5 spanned by $u = (1, 2, 3, -1, 2)$ and $v = (2, 4, 7, 2, -1)$. Find a basis of $W^{\perp}$.

Solution: Given vectors $u = (1, 2, 3, -1, 2)$ and $v = (2, 4, 7, 2, -1)$ are in W. Since $W^{\perp}$ is an orthogonal compliment of W for any $w = (x, y, z, p, q) \in W^{\perp}$ must satisfy
$u.w = 0$ and $v.w = 0$ (by definition of orthogonality).
$\Rightarrow u.w = x + 2y + 3z - p + 2q = 0.........(1)$
Similarly
$v.w = 2x + 4y + 7z + 2p - q = 0..........(2)$
Equation (1) and (2) represents homogeneous system

$$x + 2y + 3z - p + 2q = 0$$
$$2x + 4y + 7z + 2p - q = 0$$

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 2 & 4 & 7 & 2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 0 & 1 & 4 & -5 \end{bmatrix}$$

Is echelon form and hence the variables corresponding to column 2, 4 and 5 are free variables i.e.

$\Rightarrow y = k_1, p = k_2, q = k_3$

From echelon form of matrix we have

Row-2$\Rightarrow z + 4p - 5q = 0 \Rightarrow z = -4p + 5q \Rightarrow z = -4k_2 + 5k_3$

Row-1$\Rightarrow x + 2y + 3z - p + 2q = 0 \Rightarrow x = -2y - 3z + p - 2q$

$\Rightarrow x = -2k_1 - 3(-4k_2 + 5k_3) + k_2 - 2k_3 = 2k_1 + 13k_2 - 17k_3$

Thus the basis of $W^{\perp}$ are given by columns of rhs of below vector

$$\begin{bmatrix} x \\ y \\ z \\ p \\ q \end{bmatrix} = \begin{bmatrix} -2k_1 + 13k_2 - 17k_3 \\ k_1 \\ -4k_2 + 5k_3 \\ k_2 \\ k_3 \end{bmatrix} = k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 13 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} -17 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

Ex-2:

Find a basis of W of R^4 orthogonal to $u = (1, -2, 3, 4)$ and $v = (3, -5, 7, 8)$.

Solution: Given vectors $u = (1, -2, 3, 4)$ and $v = (3, -5, 7, 8)$ are in $W^{\perp}$. Since W is an orthogonal compliment of $W^{\perp}$ for any $w = (x, y, z, p, q) \in W$ must satisfy
$u.w = 0$ and $v.w = 0$ (by definition of orthogonality).
$\Rightarrow u.w = x - 2y + 3z + 4t = 0.........(1)$
Similarly
$v.w = 3x - 5y + 7z + 8t = 0.........(2)$
Equation (1) and (2) represents homogeneous system

$$x - 2y + 3z + 4t = 0$$
$$3x - 5y + 7z + 8t = 0$$

The coefficient matrix is

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 3 & -5 & 7 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -4 \end{bmatrix}$$

Is echelon form and hence the variables corresponding to columns 2 and 4 are free variables i.e.

$\Rightarrow z = k_1, t = k_2$

From echelon form of matrix we have

Row-2$\Rightarrow y - 2z - 4t = 0 \Rightarrow y = 2z + 4t \Rightarrow y = 2k_1 + 4k_2$

Row-1$\Rightarrow x - 2y + 3z + 4t = 0 \Rightarrow x = 2y - 3z - 4t$

$\Rightarrow x = 2(2k_1 + 4k_2) - 3k_1 - 4k_2 = k_1 + 4k_2$

Thus the basis of W are given by columns of rhs of below vector

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} k_1 + 4k_2 \\ 2k_1 + 4k_2 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 4 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

Therefore basis of W are $(1, 2, 1, 0)$ and $(4, 4, 0, 1)$.

Ex-3:

Find an orthogonal basis of $W^{\perp}$ where $W = (1, 2, 3, 1)$.

Solution: Let $w = (x, y, z, t)$ be orthogonal basis of $W^{\perp}$ then we must have $w.W = 0$
$\Rightarrow x + 2y + 3z + t = 0$
This show that $y = k_1, z = k_2$ and $t = k_3$ are free variables. Hence $x = -2k_1 - 3k_2 - k_3$
There fore

$$w = \begin{bmatrix} -2k_1 - 3k_2 - k_3 \\ k_1 \\ k_2 \\ k_3 \end{bmatrix} = k1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$ Clear

each of above columns is orthogonal to W but not orthogonal to each other.

Therefore let us choose some other vector $u = (x, y, z, t)$ which is orthogonal to W as well as these first column, we get
$x + 2y + 3z + t = 0, -2x + y = 0$
Writing its matrix
$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ -2 & 1 & 0 & 0 \end{bmatrix}$
Reducing it in to echelon form, we get
$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 5 & 6 & 2 \end{bmatrix}$
Hence the variables corresponding to columns 3 and 4 are free, i.e.
$z = k_1, t = k_2$
Row-2$\Rightarrow 5y = -6z - 2t \Rightarrow y = -\dfrac{6k_1}{5} - \dfrac{2k_2}{5}$
Row-1$\Rightarrow x = -2y - 3z - t = -2\left(-\dfrac{6k_1}{5} - \dfrac{2k_2}{5}\right) - 3k_1 - k_2$
$\Rightarrow x = -\dfrac{3k_1}{5} - \dfrac{k_2}{5}$

Second orthogonal vector is

$$v = \frac{k_1}{5}\begin{bmatrix}-3\\-6\\5\\0\end{bmatrix} + \frac{k_2}{5}\begin{bmatrix}-1\\-2\\0\\5\end{bmatrix}$$

Therefore required orthogonal basis are $(-2, 1, 0, 0)$ and $(-3, -6, 5, 0)$.

Similarly one can find other basis by constructing homogeneous system again.

Note:

The set of orthogonal vectors with number of vectors in that set equal to the dimension of the subspace or vector space forms a basis.

Ex-4:

Show that $S = \{u_1, u_2, u_3, u_4\}$ where $u_1 = (1, 1, 1, 1)$, $u_2 = (1, 1, -1, -1)$, $u_3 = (1, -1, 1, -1)$, $u_4 = (1, -1, -1, 1)$ is orthogonal and a basis of R^4. Express $v = (1, 3, -5, 6)$ as a linear combination of the vectors of S. Find the coordinates of the arbitrary vector $v = (a, b, c, d)$ in R^4 relative the basis S .

Solution: Clearly we see that

$u_1 \cdot u_2 = 1 + 1 - 1 - 1 = 0,\ u_1 \cdot u_4 = 1 - 1 - 1 + 1 = 0$
$u_1 \cdot u_3 = 1 - 1 + 1 - 1 = 0,\ u_2 \cdot u_3 = 1 - 1 - 1 + 1 = 0$
$u_2 \cdot u_4 = 1 - 1 + 1 - 1 = 0,\ u_3 \cdot u_4 = 1 + 1 - 1 - 1 = 0$

Hence u_1, u_2, u_3 and u_4 are orthogonal and hence S is orthogonal set and hence vectors of S are linearly independent. Further number vector in S is equal to dimension of R^4. Therefore S is basis set of R^4.

Note:

If u_1, u_2, u_3, u_4 are basis of some subspace S of inner product space then any $v \in S$ can be written as linear combination of u_1, u_2, u_3, u_4 and it is given by $v = c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4$ Where c_1, c_2, c_3, c_4 are scalar are called **Fourier coefficients** and given by $c_i = \dfrac{\langle u_i, v\rangle}{\langle u_i, u_i\rangle}$

Using the above note we get $v = c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4$ and $c_i = \dfrac{\langle u_i, v\rangle}{\langle u_i, u_i\rangle}$

$$c_1 = \frac{\langle u_1, v\rangle}{\langle u_1, u_1\rangle} = \frac{1+3-5+6}{1+1+1+1} = \frac{5}{4}$$

$$c_2 = \frac{\langle u_2, v\rangle}{\langle u_2, u_2\rangle} = \frac{1+3+5-6}{1+1+1+1} = \frac{3}{4}$$

$$c_3 = \frac{\langle u_3, v\rangle}{\langle u_3, u_3\rangle} = \frac{1-3-5-6}{1+1+1+1} = \frac{-13}{4}$$

$$c_4 = \frac{\langle u_4, v\rangle}{\langle u_4, u_4\rangle} = \frac{1-3+5+6}{1+1+1+1} = \frac{9}{4}$$

Hence $v = \frac{5}{4}u_1 + \frac{3}{4}u_2 - \frac{13}{4}u_3 + \frac{9}{4}u_4$.

With respect to $u = (a, b, c, d) = c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4$ where

$$c_1 = \frac{\langle u_1, u\rangle}{\langle u_1, u_1\rangle} = \frac{a+b+c+d}{1+1+1+1} = \frac{a+b+c+d}{4}$$

$$c_2 = \frac{\langle u_2, u\rangle}{\langle u_2, u_2\rangle} = \frac{a+b-c-d}{1+1+1+1} = \frac{a+b-c-d}{4}$$

$$c_3 = \frac{\langle u_3, u\rangle}{\langle u_3, u_3\rangle} = \frac{a-b+c-d}{1+1+1+1} = \frac{a-b+c-d}{4}$$

$$c_4 = \frac{\langle u_4, u\rangle}{\langle u_4, u_4\rangle} = \frac{a-b-c+d}{1+1+1+1} = \frac{a-b-c+d}{4}$$

Ex-5:

$P_2(t)$ is the vector space of polynomials of degree ≤ 2 with $\langle f, g\rangle = \int_0^1 f * g dt$. Find a basis of the subspace W orthogonal to $h(t) = 2t + 1$.

Solution: Let $f(t) = a_0 t^2 + a_1 t + a_2 \in P_2(t)$ is orthogonal to $h(t)$ then $\langle f, h\rangle = 0$

$\Rightarrow \langle f, h\rangle = \int_0^1 f * h dt = 0$

$\Rightarrow = \int_0^1 (2t+1) * (a_0 t^2 + a_1 t + a_2) dt = 0$

$\Rightarrow = \int_0^1 2a_0 t^3 + 2a_1 t^2 + a_0 t^2 + 2a_2 t + a_1 t + 2a_2 dt =$

$\left[2a_0 t^4/4 + (2a_1 + a_0)t^3/3 + (2a_2 + a_1)t^2/2 + 2a_2 t\right]_0^1$

$\Rightarrow = a_0/2 + (2a_1 + a_0)/2 + (2a_2 + a_1)/2 + 2a_2 = 0$

$\Rightarrow a_0 = 0, a_1 = 0, a_0 = 0$

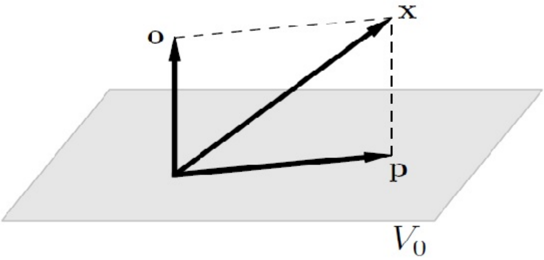
$\Rightarrow f(t) = 0$ is the only polynomial orthogonal to $h(t)$.

Orthogonal Projections:

Definition:

Let V be an inner product space and V_0 be a finite dimensional subspace of V. Then any vector $X \in V$ is uniquely represented as $X = P + O$, where $P \in V_0$ and $O \perp V_0$. Further P is called orthogonal projection of X on to V_0. In general the projection of X on to some vector $v_0 \in V$ is defined by

$$\hat{P} = Proj(X, v_0) = \frac{\langle X, v_0 \rangle}{\langle v_0, v_0 \rangle} v_0$$



Projection along the Spanning set:

Let $S = \{u_1, u_2, u_3, u_4\}$ be the spanning set of some subspace in inner product space then the projection of vector v along this subspace is given by $\hat{P} = c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4$, where c_i are Fourier coefficients given by $c_i = \dfrac{\langle u_i, v\rangle}{\langle u_i, u_i\rangle}$

Example-1:

Find the projection of the vector $v = (1, -2, 3, 4)$ along $w = (1, 2, 1, 2)$ in R^4.

Solution: By definition $\hat{P} = Proj_v(w) = \dfrac{\langle v, w\rangle}{\langle w, w\rangle} w$

$\langle v, w\rangle = v \cdot w = 1 - 4 + 3 + 8 = 8$

$\langle w, w\rangle = w \cdot w = 1 + 4 + 1 + 4 = 10$

$\hat{P} = Proj_v(w) = \dfrac{8}{10}\begin{bmatrix}1\\2\\1\\2\end{bmatrix}$ is the required projection of v along w.

Example-2:

Find the projection of the vector $v = t^2$ along $w = t + 1$ in $P(t)$, with respect to $\langle f, g \rangle = \int_0^1 f * g dt$.

Solution: By definition $\hat{P} = Proj_v(w) = \dfrac{\langle v, w \rangle}{\langle w, w \rangle} w$

$$\langle v, w \rangle = \int_0^1 v * w dt = \int_0^1 t^2 * (t+1) dt = \left[\frac{t^4}{4} + \frac{t^3}{3} \right]_0^1 = \frac{1}{4} + \frac{1}{3}$$

$$\langle v, w \rangle = \frac{7}{12}$$

$$\langle w, w \rangle = \int_0^1 w * w dt = \int_0^1 (t+1) * (t+1) dt =$$

$$\left[\frac{t^3}{3} + \frac{2t^2}{2} + t \right]_0^1 = \frac{1}{3} + 1 + 1 = \frac{7}{3}$$

$\hat{P} = Proj_v(w) = \dfrac{1}{4}(t+1)$ is the required projection of v along w.

Example-3:

Find the projection of the vector $v = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ along $w = \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}$ in $M_{2\times 2}$, with respect to $\langle v, w \rangle = Trace(vw^T)$.

Solution: By definition $\hat{P} = Proj_v(w) = \dfrac{\langle v, w \rangle}{\langle w, w \rangle} w$

$\langle v, w \rangle = trace(vw^T) = 1 + 2 + 15 + 20 = 38$

$\langle w, w \rangle = trace(ww^T) = 1 + 1 + 25 + 25 = 52$

$\hat{P} = Proj_v(w) = \dfrac{38}{52} \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}$ is the required projection of v along w.

Example-4:

Find the projection of the vector $v = (1,3,5,7)$ along $W = Span(w_i)$ in R^4. Where $S = \{(1,1,1,1),(1,-3,4,-2)\}$

Solution: By definition $\hat{P} = Proj_v(W) = \dfrac{\langle v, w_1\rangle}{\langle w_1, w_1\rangle} w_1 + \dfrac{\langle v, w_2\rangle}{\langle w_2, w_2\rangle} w_2$

$\langle v, w_1\rangle = v \cdot w_1 = 1+3+5+7 = 16$

$\langle w_1, w_1\rangle = w_1 \cdot w_1 = 1+1+1+1 = 4$

$\langle v, w_2\rangle = v \cdot w_2 = 1-9+20-14 = -2$

$\langle w_2, w_2\rangle = w_2 \cdot w_2 = 1+9+16+4 = 30$

$$\hat{P} = Proj_v(w) = \frac{16}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{-2}{30}\begin{bmatrix}1\\-3\\4\\-2\end{bmatrix} = \begin{bmatrix}\frac{59}{15}\\\frac{63}{15}\\\frac{56}{15}\\\frac{62}{15}\end{bmatrix}$$

is the required projection of v along W.

Gram-Schmidt orthogonalization:

Algorithm:

Given a set of vectors $v_1, v_2, v_3.....v_n$. Then to make this vectors mutually orthogonal we use the following procedure.

Step-1: Set $u_1 = v_1$.

Step-2: Compute $u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1$.

Step-3: Compute $u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2$.

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Step-n: Compute

$$u_n = v_n - \frac{\langle v_n, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_n, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \ldots\ldots - \frac{\langle v_n, u_{n-1} \rangle}{\langle u_{n-1}, u_{n-1} \rangle} u_{n-1}.$$

- To find orthonormal vectors divide every vector by its norm i.e. $\hat{u}_i = \frac{u_i}{\|u_i\|}$

Examples:

Ex-1:

Find an orthogonal basis and hence an orthonormal basis of the subspace W spanned by the following vectors, $v_1 = (1, 1, 1, 1)$, $v_2 = (1, 2, 4, 5)$ and $v_3 = (1, -3, -4, -2)$ of R^4 .

Solution: By Gram-Schmidt orthogonalization;

Step-1: Set $u_1 = v_1 = (1, 1, 1, 1)$

Step-2:Compute $u_2 = v_2 - \dfrac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1$

$\langle v_2, u_1 \rangle = 1 + 2 + 4 + 5 = 12$

$\langle u_1, u_1 \rangle = 1 + 1 + 1 + 1 = 4$

$$u_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix} - \frac{12}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

Step-3: Compute $u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2$

$\langle v_3, u_1 \rangle = 1 - 3 - 4 - 2 = -8$

$\langle v_3, u_2 \rangle = -2 + 3 - 4 - 4 = -7$

$\langle u_2, u_2 \rangle = 4 + 1 + 1 + 4 = 10$

$$u_3 = \begin{bmatrix} 1 \\ -3 \\ -4 \\ -2 \end{bmatrix} - \frac{-8}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-7}{10} \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -4 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} + \frac{7}{10} \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 1 + 2 - \frac{14}{10} \\ -3 + 2 - \frac{7}{10} \\ -4 + 2 + \frac{7}{10} \\ -2 + 2 + \frac{14}{10} \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 16 \\ -17 \\ -13 \\ 14 \end{bmatrix} \Rightarrow u_3 = \begin{bmatrix} 16 \\ -17 \\ -13 \\ 14 \end{bmatrix}$$

Hence the orthogonal basis of W are

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ u_2 = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \ u_3 = \begin{bmatrix} 16 \\ -17 \\ -13 \\ 14 \end{bmatrix}$$

Therefore the orthonormal basis of W are

$$\hat{u_1} = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{u_2} = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{10}} \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\hat{u_3} = \frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{910}} \begin{bmatrix} 16 \\ -17 \\ -13 \\ 14 \end{bmatrix}$$

Ex-2:

Find an orthogonal basis and hence an orthonormal basis of the subspace W spanned by the following vectors, $v_1 = (1, 1, 1, 1)$, $v_2 = (1, -1, 2, 2)$ and $v_3 = (1, 2, -3, -4)$ of R^4 .

Solution: By Gram-Schmidt orthogonalization;
Step-1: Set $u_1 = v_1 = (1, 1, 1, 1)$

Step-2:Compute $u_2 = v_2 - \dfrac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1$

$\langle v_2, u_1 \rangle = 1 - 1 + 2 + 2 = 4$
$\langle u_1, u_1 \rangle = 1 + 1 + 1 + 1 = 4$

$$u_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

Step-3: Compute $u_3 = v_3 - \frac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle} u_1 - \frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle} u_2$

$\langle v_3, u_1\rangle = 1 + 2 - 3 - 4 = -4$

$\langle v_3, u_2\rangle = 0 - 4 - 3 - 4 = -11$

$\langle u_2, u_2\rangle = 0 + 4 + 1 + 1 = 6$

$$u_3 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ -4 \end{bmatrix} - \frac{-4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-11}{6} \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ -4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{11}{6} \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 1 + 1 - 0 \\ 2 + 1 - \frac{22}{6} \\ -3 + 1 + \frac{11}{6} \\ -4 + 1 + \frac{11}{6} \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 12 \\ -4 \\ -1 \\ -7 \end{bmatrix} \Rightarrow u_3 = \begin{bmatrix} 12 \\ -4 \\ -1 \\ -7 \end{bmatrix}$$

Hence the orthogonal basis of W are

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; u_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \; u_3 = \begin{bmatrix} 12 \\ -4 \\ -1 \\ -7 \end{bmatrix}$$

Therefore the orthonormal basis of W are

$$\hat{u_1} = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{u_2} = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{u_3} = \frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{210}} \begin{bmatrix} 12 \\ -4 \\ -1 \\ -7 \end{bmatrix}$$

Ex-3:

Find an orthogonal basis and hence an orthonormal basis of the subspace W spanned by the following vectors, $S = \{1, t, t^2, t^3\}$ of $P_3(t)$ given $\langle f, g\rangle = \int_{-1}^{1} f * g dt$.

Solution: Let $f_1 = 1$, $f_2 = t, f_3 = t^2$ and $f_4 = t^3$.
By Gram-Schmidt orthogonalization
Set $g_1 = f_1 = 1$
Compute $g_2 = f_2 - \frac{\langle f_2, g_1\rangle}{\langle g_1, g_1\rangle} g_1$

$$\langle f_2, g_1\rangle = \int_{-1}^{1} f_2 * g_1 dt = \int_{-1}^{1} t dt = \left[\frac{t^2}{2}\right]_{-1}^{1} = 0$$

$$\langle g_1, g_1\rangle = \int_{-1}^{1} g_1 * g_1 dt = \int_{-1}^{1} 1 dt = [t]_{-1}^{1} = 2$$

$$\Rightarrow g_2 = t - \frac{0}{2} * 1 = t$$

Compute $g_3 = f_3 - \frac{\langle f_3, g_1 \rangle}{\langle g_1, g_1 \rangle} g_1 - \frac{\langle f_3, g_2 \rangle}{\langle g_2, g_2 \rangle} g_2$

$\langle f_3, g_1 \rangle = \int_{-1}^{1} f_3 * g_1 dt = \int_{-1}^{1} t^2 dt = \left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{2}{3}$

$\langle f_3, g_2 \rangle = \int_{-1}^{1} f_3 * g_2 dt = \int_{-1}^{1} t^3 dt = \left[\frac{t^4}{4}\right]_{-1}^{1} = 0$

$\langle g_2, g_2 \rangle = \int_{-1}^{1} g_2 * g_2 dt = \int_{-1}^{1} t^2 dt = \left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{2}{3}$

$\Rightarrow g_3 = t^2 - \frac{2/3}{2} 1 - \frac{0}{2/3} t = t^2 - \frac{1}{3} = \frac{1}{3}(3t^3 - 1) \Rightarrow g_3 = 3t^2 - 1$

Compute $g_4 = f_4 - \frac{\langle f_4, g_1 \rangle}{\langle g_1, g_1 \rangle} g_1 - \frac{\langle f_4, g_2 \rangle}{\langle g_2, g_2 \rangle} g_2 - \frac{\langle f_4, g_3 \rangle}{\langle g_3, g_3 \rangle} g_3$

$\langle f_4, g_1 \rangle = \int_{-1}^{1} f_4 * g_1 dt = \int_{-1}^{1} t^3 dt = \left[\frac{t^4}{4}\right]_{-1}^{1} = 0$

$\langle f_4, g_2 \rangle = \int_{-1}^{1} f_4 * g_2 dt = \int_{-1}^{1} t^4 dt = \left[\frac{t^5}{5}\right]_{-1}^{1} = \frac{2}{5}$

$$\langle f_4, g_3\rangle = \int_{-1}^{1} f_4 * g_3 dt = \int_{-1}^{1} t^3(3t^2-1)dt = \left[3\frac{t^6}{6} - \frac{t^4}{4}\right]_{-1}^{1} = 0$$

$$\langle g_3, g_3\rangle = \int_{-1}^{1} g_3 * g_3 dt = \int_{-1}^{1} (3t^2-1)^2 dt =$$

$$\left[9\frac{t^5}{5} + t - 6\frac{t^3}{3}\right]_{-1}^{1} = \frac{18}{5} + 2 - \frac{12}{3} = \frac{8}{5}$$

$$g_4 = t^3 - 0 - \frac{2/5}{2/3}t - 0 = \frac{1}{5}(5t^3-3t) \Rightarrow g_4 = 5t^3 - 3t$$

$$\langle g_4, g_4\rangle = \int_{-1}^{1} g_4 * g_4 dt = \int_{-1}^{1} (5t^3-3t)^2 dt =$$

$$\left[25\frac{t^7}{7} + 9\frac{t^3}{3} - 30\frac{t^5}{5}\right]_{-1}^{1} = \frac{50}{7} + \frac{18}{3} - \frac{60}{5} = \frac{8}{5}$$

Hence the orthogonal basis of W are $g_1 = 1$, $g_2 = t$, $g_3 = 3t^2 - 1$ and $g_4 = 5t^3 - 3t$.

Therefore the orthonormal basis of W are $\hat{g_1} = \dfrac{g_1}{\|g_1\|} = \dfrac{1}{\sqrt{2}}$,

$$\hat{g_2} = \frac{g_2}{\|g_2\|} = \frac{t}{\sqrt{\frac{2}{3}}},$$

$$\hat{g_3} = \frac{g_3}{\|g_3\|} = \frac{3t^2 - 1}{\sqrt{\frac{8}{5}}}$$

and

$$\hat{g_4} = \frac{g_4}{\|g_4\|} = \frac{5t^3 - 3t}{\sqrt{\frac{8}{5}}}.$$

Ex-4:

Find an orthogonal matrix P whose first row is $u = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ in R^3.

solution: Let $v = (x, y, z) \in R^3$ orthogonal to u, we get

$$u.v = 0 \Rightarrow \frac{x}{2} + \frac{2y}{3} + \frac{2z}{3} = 0$$

$$\Rightarrow x + 2y + 2z = 0 \Rightarrow y = k_1 \text{ and } z = k_2 \text{ are free}$$

$$\Rightarrow x = -2y - 2z = -2k_1 - 2k_2$$

$$\Rightarrow v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2k_1 - 2k_2 \\ k \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Here we see that clearly v_1 and v_2 are orthogonal to u, but not mutually orthogonal. Hence we make them orthogonal by Gram-Schmidt orthogonalizatin as follows;

Let $u_1 = v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \hat{u_1} = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Computing u_2 using

$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1$

$\langle v_2, u_1 \rangle = 4 + 0 + 0 = 4$

$\langle u_1, u_1 \rangle = 4 + 1 + 0 = 5$

$$\Rightarrow u_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 + \frac{8}{5} \\ -\frac{4}{5} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ -\frac{4}{5} \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix}$$

$$\Rightarrow u_2 = \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix} \Rightarrow \hat{u_2} = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{45}} \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix}$$

Ex-4 continued...

Hence the required orthogonal matrix is

$$P = \begin{bmatrix} \hat{u} \\ \hat{u_1} \\ \hat{u_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{\sqrt{45}} & -\frac{4}{\sqrt{45}} & \frac{5}{\sqrt{45}} \end{bmatrix}$$

Ex-5:

Find an orthogonal matrix P whose first two rows are linear combination of $u_1 = (1, 1, 1)$ and $u_2 = (1, -2, 3)$ in R^3.

Solution: Let $v = (x, y, z) \in R^3$ orthogonal to u_1 and u_2, we get

$v \cdot u_1 = 0 \Rightarrow x + y + z = 0$

$v \cdot u_2 = 0 \Rightarrow x - 2y + 3z = 0$

Hence the coefficient matrix is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \end{bmatrix} \Rightarrow z = k \text{ is free}$$

$$\Rightarrow -3y + 2z = 0 \Rightarrow y = \frac{2}{3}k$$

$$\Rightarrow x + y + z = 0 \Rightarrow x = -y - z = -\frac{2}{3}k - k = -\frac{5}{3}k$$

$$\Rightarrow v = \frac{k}{3}\begin{bmatrix} -5 \\ 2 \\ 3 \end{bmatrix} \Rightarrow v = \begin{bmatrix} -5 \\ 2 \\ 3 \end{bmatrix}$$

Ex-5 continued...

Clearly we see that $v$is orthogonal to both u_1 and u_2 but u_1 and u_2 are not orthogonal to each other. Hence we make them orthogonal by Gram-Schmidt orthogonalizatin as follows;

Let $v_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \hat{v_1} = \dfrac{v_1}{\|v_1\|} = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Computing v_2 using

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} u_1$$

$$\langle u_2, v_1 \rangle = 1 - 2 + 3 = 2$$

$$\langle v_1, v_1 \rangle = 1 + 1 + 1 = 3$$

$$\Rightarrow v_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{2}{3} \\ -2 - \frac{2}{3} \\ 3 - \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{8}{3} \\ \frac{7}{3} \end{bmatrix}$$

Ex-5 continued...

$$v_2 = \frac{1}{3}\begin{bmatrix} 1 \\ -8 \\ 7 \end{bmatrix}$$

$$\Rightarrow v_2 = \begin{bmatrix} 1 \\ -8 \\ 7 \end{bmatrix} \Rightarrow \hat{v_2} = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{114}}\begin{bmatrix} 1 \\ -8 \\ 7 \end{bmatrix}$$

$$\hat{v}\frac{v}{\|v\|} = \frac{1}{\sqrt{38}}\begin{bmatrix} -5 \\ 2 \\ 3 \end{bmatrix}$$

Hence the required orthogonal matrix is

$$P = \begin{bmatrix} \hat{v_1} \\ \hat{v_2} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{114}} & -\frac{8}{\sqrt{114}} & \frac{7}{\sqrt{114}} \\ -\frac{5}{\sqrt{38}} & \frac{2}{\sqrt{38}} & \frac{3}{\sqrt{38}} \end{bmatrix}$$

QR-Factorization:

Algorithm:

Given a inconsistent system $AX = b$ $QR-$factorization can be applied to get an approximate solution X^*. The following the procedure to find $A = Q * R$ where Q is orthogonal matrix and R is upper triangular matrix and this procedure is $QR-$factorization.

Step-1: Take the column of A as vectors $v_1, v_2, v_3.....$

Step-2: Using Gram-Schmidt orthogonaliztion procedure make the vectors in above as orthogonal and orthonormal vectors say $\hat{u}_1, \hat{u}_2, \hat{u}_3......$

Step-3: Write the orthogonal matrix Q by writing vectors $\hat{u}_1, \hat{u}_2, \hat{u}_3.....$ as columns in it. i.e $Q = [\hat{u}_1 \| \hat{u}_2 \| \hat{u}_3 \| .....]$.

Step-4: The construct upper triangular $R = [a_{ij}]$ by using $a_{ij} = \hat{u}_i \cdot v_j$.

Step-5: Thus we get $A = Q * R$ is QR factorization of A.

Step-6: To solve $AX = b$, put $A = QR$, we get $QRX = b \Rightarrow RX = Q^T b$ will give system normal to $AX = b$ and solving this we get an approximate solution X^*.

Examples:

Ex-1:

Find the $QR-$factorization of the matrix $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.

Solution:

Step-1: Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

We note that these vectors not orthogonal to each other.

Step-2: Make the vector in above step orthogonal using Gram-Schmidt orthogonalization procedure and also make them orthonormal.

Let $u_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \hat{u_1} = \dfrac{u_1}{\|u_1\|} = \begin{bmatrix} \dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{3}} \\ 0 \end{bmatrix}$

Compute $u_2 = v_2 - \dfrac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \dfrac{-1+0-1+0}{1+1+1+0} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix} \Rightarrow u_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix} \Rightarrow \hat{u_2} = \frac{u_2}{\|u_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{15}} \\ -\frac{1}{\sqrt{15}} \\ \frac{3}{\sqrt{15}} \end{bmatrix}$$

Compute

$$u_3 = v_3 - \frac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle} u_1 - \frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle} u_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{-1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{-2}{15}\begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{12}{15} \\ \frac{9}{15} \\ \frac{3}{15} \\ -\frac{9}{15} \end{bmatrix} = \frac{1}{15}\begin{bmatrix} -12 \\ 9 \\ 3 \\ -9 \end{bmatrix} \Rightarrow u_3 = \begin{bmatrix} -12 \\ 9 \\ 3 \\ -9 \end{bmatrix}$$

$$\Rightarrow \hat{u}_3 = \frac{u_3}{\|u_3\|} = \begin{bmatrix} -\frac{12}{\sqrt{315}} \\ \frac{9}{\sqrt{315}} \\ \frac{3}{\sqrt{315}} \\ -\frac{9}{\sqrt{315}} \end{bmatrix}$$

Step-3: Write an orthogonal matrix $Q = [\hat{u}_1 \| \hat{u}_2 \| \hat{u}_3]$, by writing orthonomal vectors obtained previous step as columns;

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{15}} & -\frac{12}{\sqrt{315}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{15}} & \frac{9}{\sqrt{315}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{15}} & \frac{3}{\sqrt{315}} \\ 0 & \frac{3}{\sqrt{15}} & -\frac{9}{\sqrt{315}} \end{bmatrix}$$

Step-4: Construct upper triangular matrix $R = [a_{ij}]$ using $a_{ij} = \hat{u}_i \cdot v_j$ with $a_{ij} = 0$ for $i > j$

$\Rightarrow a_{21} = 0, a_{31} = 0, a_{32} = 0$

$$\Rightarrow a_{11} = \hat{u}_1 \cdot v_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0 = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$a_{12} = \hat{u}_1 \cdot v_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = -\frac{1}{\sqrt{3}} + 0 - \frac{1}{\sqrt{3}} + 0 = -\frac{2}{\sqrt{3}}$$

$$a_{13} = \hat{u}_1 \cdot v_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = -\frac{1}{\sqrt{3}} + 0 + 0 + 0 = -\frac{1}{\sqrt{3}}$$

$$a_{22} = \hat{u}_2 \cdot v_2 = \begin{bmatrix} -\frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{15}} \\ -\frac{1}{\sqrt{15}} \\ \frac{3}{\sqrt{15}} \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{15}} + 0 + \frac{1}{\sqrt{15}} + \frac{3}{\sqrt{15}} = \frac{5}{\sqrt{15}}$$

$$a_{23} = \hat{u}_2 \cdot v_3 = \begin{bmatrix} -\frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{15}} \\ -\frac{1}{\sqrt{15}} \\ \frac{3}{\sqrt{15}} \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = -\frac{1}{\sqrt{15}} + 0 + 0 - \frac{3}{\sqrt{15}} = -\frac{4}{\sqrt{15}}$$

$$a_{33} = \hat{u}_3 \cdot v_3 = \begin{bmatrix} -\frac{12}{\sqrt{315}} \\ \frac{9}{\sqrt{315}} \\ \frac{3}{\sqrt{315}} \\ -\frac{9}{\sqrt{315}} \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$= \frac{12}{\sqrt{315}} + 0 + 0 + \frac{9}{\sqrt{315}} = -\frac{23}{\sqrt{315}}$$

Hence the upper triangular matrix

$$R = \begin{bmatrix} \sqrt{3} & -\frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{5}{\sqrt{15}} & -\frac{4}{\sqrt{15}} \\ 0 & 0 & -\frac{23}{\sqrt{315}} \end{bmatrix}$$

Therefore the required $QR-$factorization of given matrix A is

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{15}} & -\frac{12}{\sqrt{315}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{15}} & \frac{9}{\sqrt{315}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{15}} & \frac{3}{\sqrt{315}} \\ 0 & \frac{3}{\sqrt{15}} & -\frac{9}{\sqrt{315}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & -\frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{5}{\sqrt{15}} & -\frac{4}{\sqrt{15}} \\ 0 & 0 & -\frac{23}{\sqrt{315}} \end{bmatrix}$$

Ex-2:

Find the $QR-$factorization of the matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix}$.Hence

Solve $AX = b$, where $b = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Solution:

Let $v_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$

We note that these vectors not orthogonal to each other.
Make the vector in above step orthogonal using Gram-Schmidt orthogonalization procedure and also make them orthonormal.

Let $u_1 = v_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \hat{u_1} = \frac{u_1}{\|u_1\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}$

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} - \frac{-2-3+0}{4+1+0} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow$$

$$\hat{u_2} = \frac{u_2}{\|u_2\|} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Write an orthogonal matrix $Q = [\hat{u_1} \| \hat{u_2}] \Rightarrow Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{6}} \end{bmatrix}$

Construct upper triangular matrix $R = [a_{ij}]$ using $a_{ij} = \hat{u}_i \cdot v_j$ with $a_{ij} = 0$ for $i > j \Rightarrow a_{21} = 0$

$$\Rightarrow a_{11} = \hat{u}_1 \cdot v_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \frac{4}{\sqrt{5}} + \frac{1}{\sqrt{5}} + 0 = \frac{5}{\sqrt{5}} = \sqrt{5}$$

$$a_{12} = \hat{u}_1 \cdot v_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} = -\frac{2}{\sqrt{5}} - \frac{3}{\sqrt{5}} + 0 = -\frac{5}{\sqrt{5}} = -\sqrt{5}$$

$$a_{22} = \hat{u}_2 \cdot v_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} = -\frac{1}{\sqrt{6}} + \frac{6}{\sqrt{6}} + \frac{1}{\sqrt{6}} = \sqrt{6}$$

Hence the upper triangular matrix

$$R = \begin{bmatrix} \sqrt{5} & -\sqrt{5} \\ 0 & \sqrt{6} \end{bmatrix}$$

Therefore the required $QR-$factorization of given matrix A is

$$A = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & -\sqrt{5} \\ 0 & \sqrt{6} \end{bmatrix}$$

Next to find the solution of $AX = b$ put $A = QR$, we get $QRX = b$

$$\Rightarrow RX = Q^T b \Rightarrow \begin{bmatrix} \sqrt{5} & -\sqrt{5} \\ 0 & \sqrt{6} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{5} & -\sqrt{5} \\ 0 & \sqrt{6} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{4}{\sqrt{6}} \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{6} \\ \frac{4}{6} \end{bmatrix}$$

Least Square Method:

Working Rule:

Given a inconsistent system of equations $AX = b$, then to approximate solution X^* the following the procedure:

Step-1: Take A and find its transpose A^T.

Step-2: Compute A^TA and A^Tb.

Step-3: Construct normal equations $A^TAX = A^Tb$, we make the system consistent.

Step-4: Solve system of equation in above step-3 to get an approximate solution X^* called least square solution.

Step-5 Find the least square error in the solution using $E = \sqrt{\sum (AX^* - b)^2}$. Which measures how close the approximate solution to the required solution.

Ex-1:

Solve the following system of equations $AX = b$ by the method of least squares $2x + 1 = 3; x + 0y = 1; 0x - y = 2; -x + y = -1$. Hence find the least square error.

Solution: Writing the given system of equations in matrix form $AX = b$, we get

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

Next find $A^T \Rightarrow A^T = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{bmatrix}$

Then compute $A^T A$

$$\Rightarrow A^T A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 3 \end{bmatrix}$$

Compute $A^T b$

$$\Rightarrow A^T b = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

Write the normal equations $A^T AX = A^T b$

$$\Rightarrow \begin{bmatrix} 6 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

Solve this system of equations by any of the known methods
Writing augmented matrix

$$\Rightarrow \begin{bmatrix} 6 & 1 & : & 8 \\ 1 & 3 & : & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 1 & : & 8 \\ 0 & 17 & : & -8 \end{bmatrix} \Rightarrow 17y = -8, 6x + y = 8$$

$$\Rightarrow y = -\frac{8}{17} \text{ and } x = \frac{24}{17}$$

Hence approximate solution given system is $X = \begin{bmatrix} \frac{24}{17} \\ -\frac{8}{17} \end{bmatrix}$

Compute

$$E = AX^* - b = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{24}{17} \\ -\frac{8}{17} \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.647 \\ 0.4117 \\ -1.5294 \\ -0.8823 \end{bmatrix}$$

Hence the least square error is square root of sum of the square of components of E i.e.

$Er = \sqrt{\sum AX^* - b)^2}$

$\Rightarrow Er = \sqrt{(-0.647)^2 + (0.4117)^2 + (-1.5294)^2 + (-0.8823)^2}$

$\Rightarrow Er = \sqrt{3.7056} = 1.925$ is the least square error.

Ex-2:

Solve the following system of equations $AX = b$ by the method of least squares $x + 2y + z = -1; x + 3y + 2z = 2; 2x + 5y + 3z = 0;$ $2x + 0y + z = 1$ and $3x + y + z = -2$.

Solution: Writing the given system of equations in matrix form $AX = b$, we get

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

Next find $A^T \Rightarrow A^T = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 0 & 1 \\ 1 & 2 & 3 & 1 & 1 \end{bmatrix}$

Then compute $A^TA \Rightarrow A^TA = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 0 & 1 \\ 1 & 2 & 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$\Rightarrow A^TA = \begin{bmatrix} 19 & 18 & 14 \\ 18 & 39 & 24 \\ 14 & 24 & 16 \end{bmatrix}$

Compute A^Tb

$$\Rightarrow A^Tb = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 0 & 1 \\ 1 & 2 & 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$$

Ex-2 continued....

Write the normal equations $A^T AX = A^T b$

$$\begin{bmatrix} 19 & 18 & 14 \\ 18 & 39 & 24 \\ 14 & 24 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$$

Solve this system of equations by any of the known methods

Writing augmented matrix

$$\Rightarrow \begin{bmatrix} 19 & 18 & 14 : -3 \\ 18 & 39 & 24 : 2 \\ 14 & 24 & 16 : 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 : -1.5333 \\ 0 & 1 & 0 : -1.8667 \\ 0 & 0 & 1 : 4.2667 \end{bmatrix}$$

Hence approximate solution given system is

$$X = \begin{bmatrix} -1.5333 \\ -1.8667 \\ 4.2667 \end{bmatrix}$$

THANK YOU