

# Variational Inference and Generative Models

CS 330

# Course Reminders

Homework 3 due Monday next week.

Tutorial session on Thursday 4:30 pm

Be careful Azure usage — turning off machines when you are not using them!

# This Week

A Bayesian perspective on meta-learning

**Today:** Approximate Bayesian inference via variational inference



Bayes is back.

# Plan for Today

1. Latent variable models
2. Variational inference
3. Amortized variational inference
4. Example latent variables models

} Part of (optional) Homework 4

## Goals

- Understand latent variable models in deep learning
- Understand how to use (amortized) variational inference

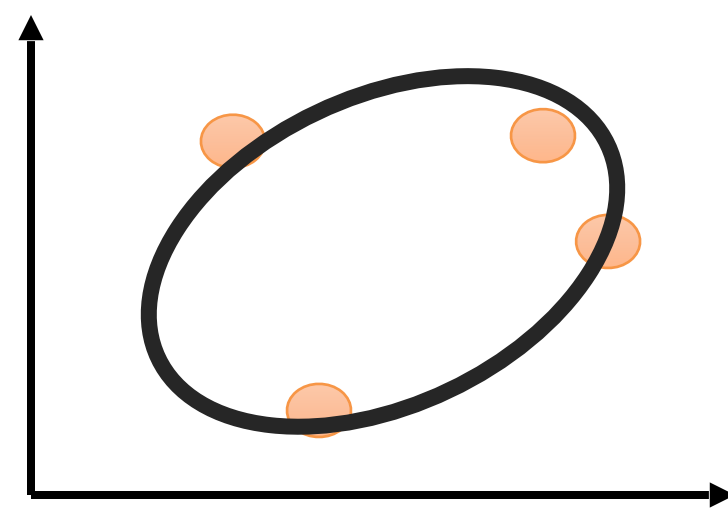
# Probabilistic models

These models can approximate the data distribution.  $P(x, y)$ . If we know  $p_c(x, y)$  then we can approximate  $P(y|x)$ .

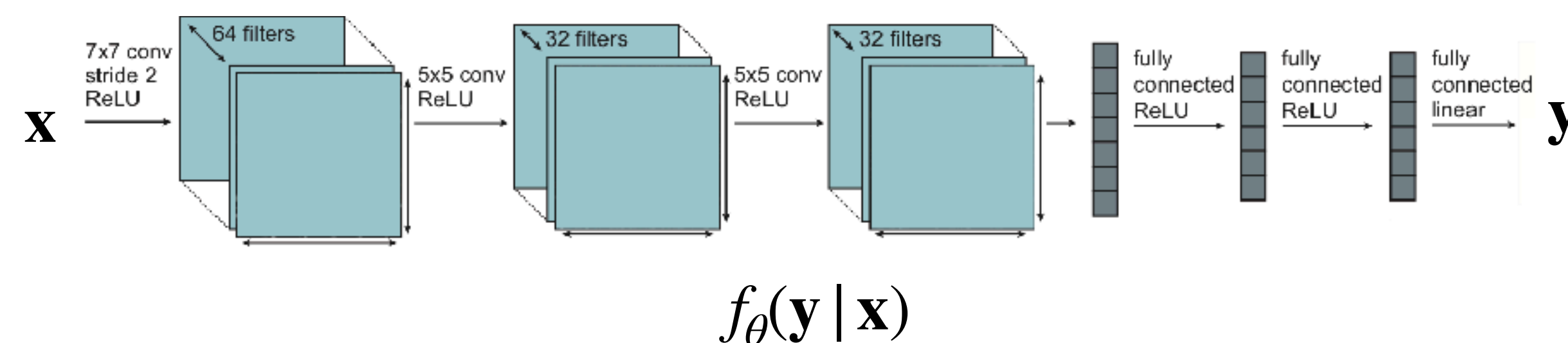
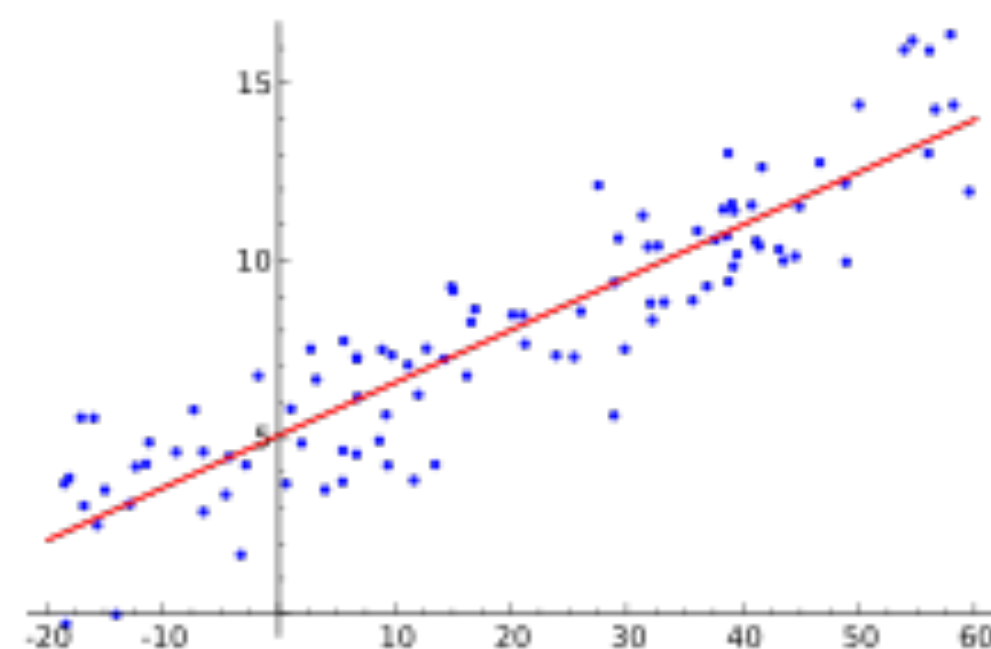
They are useful because:

1. They can fit model that can capture predictive uncertainty
2. Models are more principled by explicitly modelling data uncertainty

$p(x)$



$p(y|x)$



Most commonly:

- probability values of discrete categorical distribution
- mean and variance of a **Gaussian**

But it could be other distributions!

# How do we train probabilistic models?

the model:  $p_{\theta}(x)$

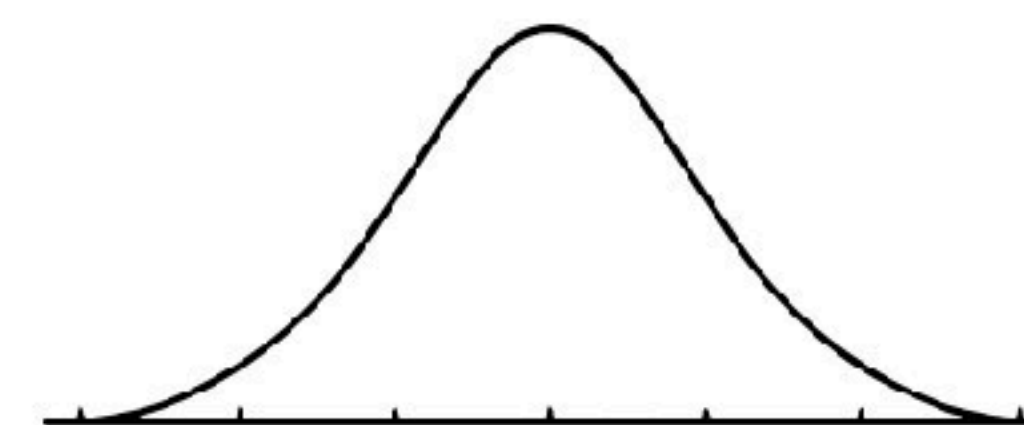
the data:  $\mathcal{D} = \{x_1, x_2, x_3, \dots, x_N\}$

maximum likelihood fit:

$$\theta \leftarrow \arg \max_{\theta} \frac{1}{N} \sum_i \log p_{\theta}(x_i)$$

Easy to evaluate & differentiate for categorical or Gaussian distributions.

i.e. cross-entropy, MSE losses



NORMAL DISTRIBUTION



PARANORMAL DISTRIBUTION

**Goal:** Can we model and train more complex distributions?

# When might we want more complex distributions?

- generative models of images, text, video, or other data
- represent uncertainty over labels (e.g. ambiguity arising from limited data, partial observability)
- represent uncertainty over *functions*

“HD Video: Riding a horse  
in the park at sunrise”





Meta-learning methods represent a deterministic  $p(\phi_i | \mathcal{D}_i^{\text{tr}}, \theta)$  (i.e. a point estimate)

Why/when is this a problem?

Few-shot learning problems may be *ambiguous*.  
(even with prior)

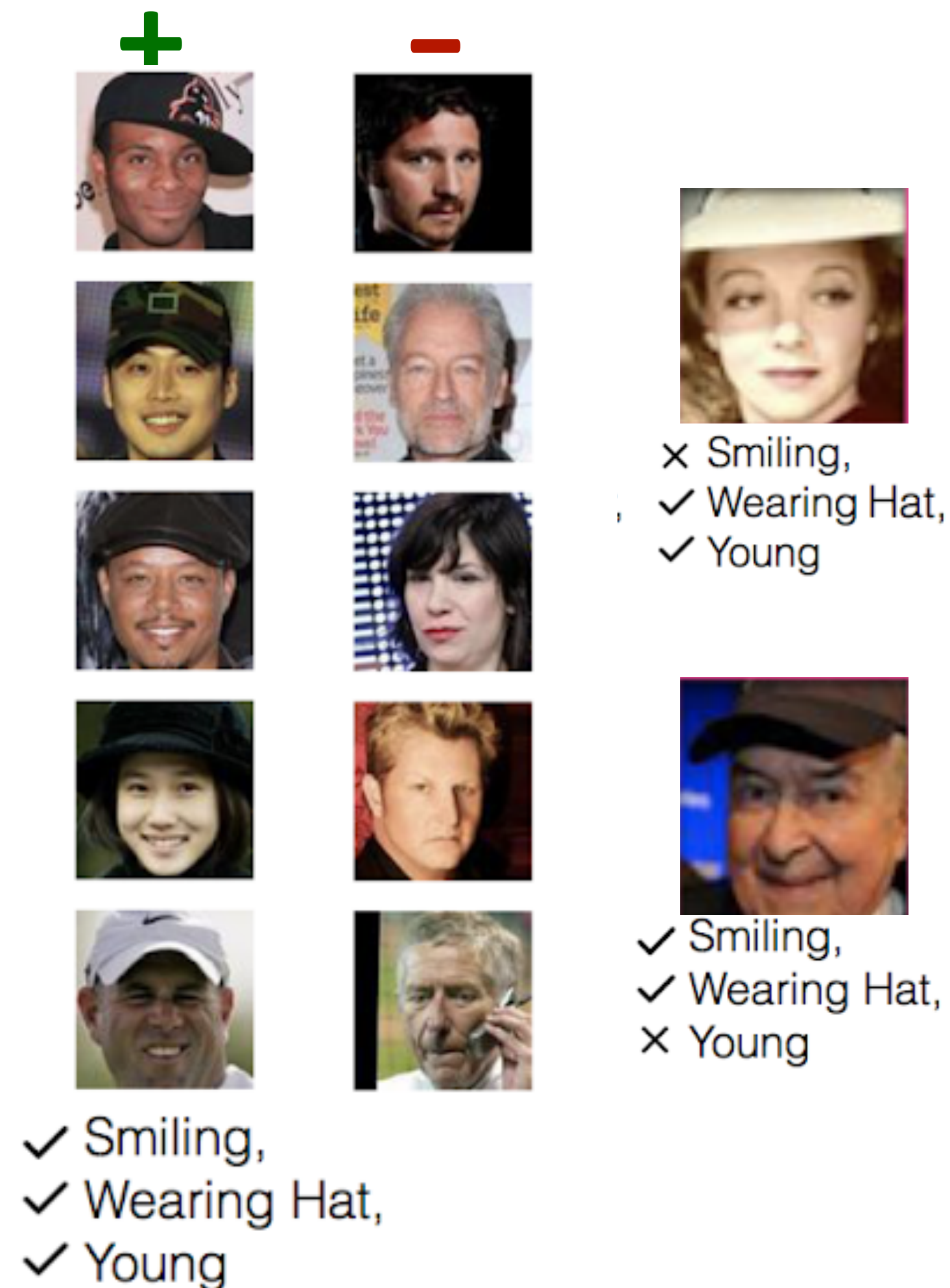
Can we learn to *generate hypotheses*  
about the underlying function?

i.e. sample from  $p(\phi_i | \mathcal{D}_i^{\text{tr}}, \theta)$

Important for:

- **safety-critical** few-shot learning  
(e.g. medical imaging)
- learning to **actively learn**
- learning to **explore** in meta-RL

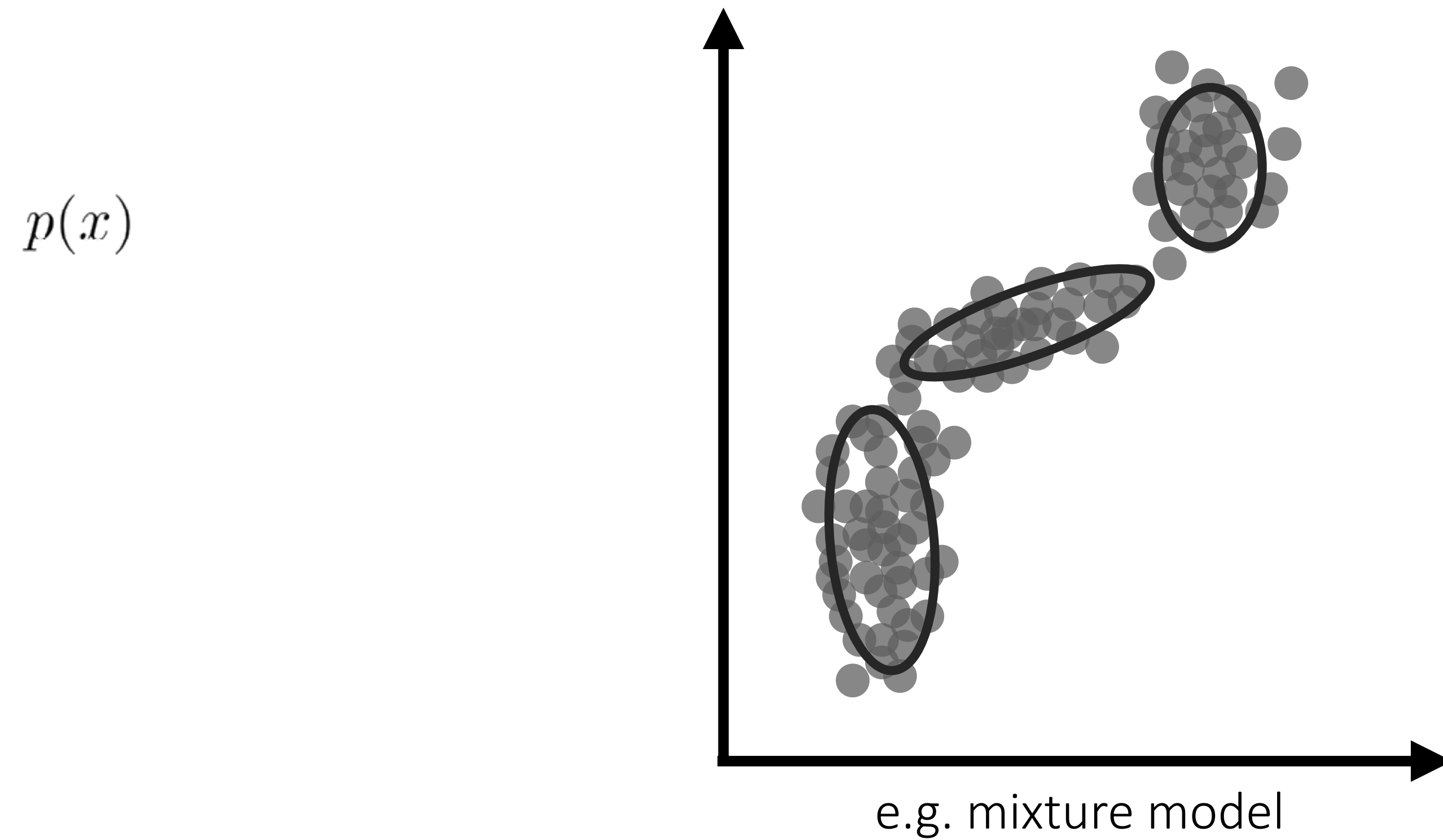
Active learning w/ meta-learning: Woodward & Finn '16,  
Konyushkova et al. '17, Bachman et al. '17



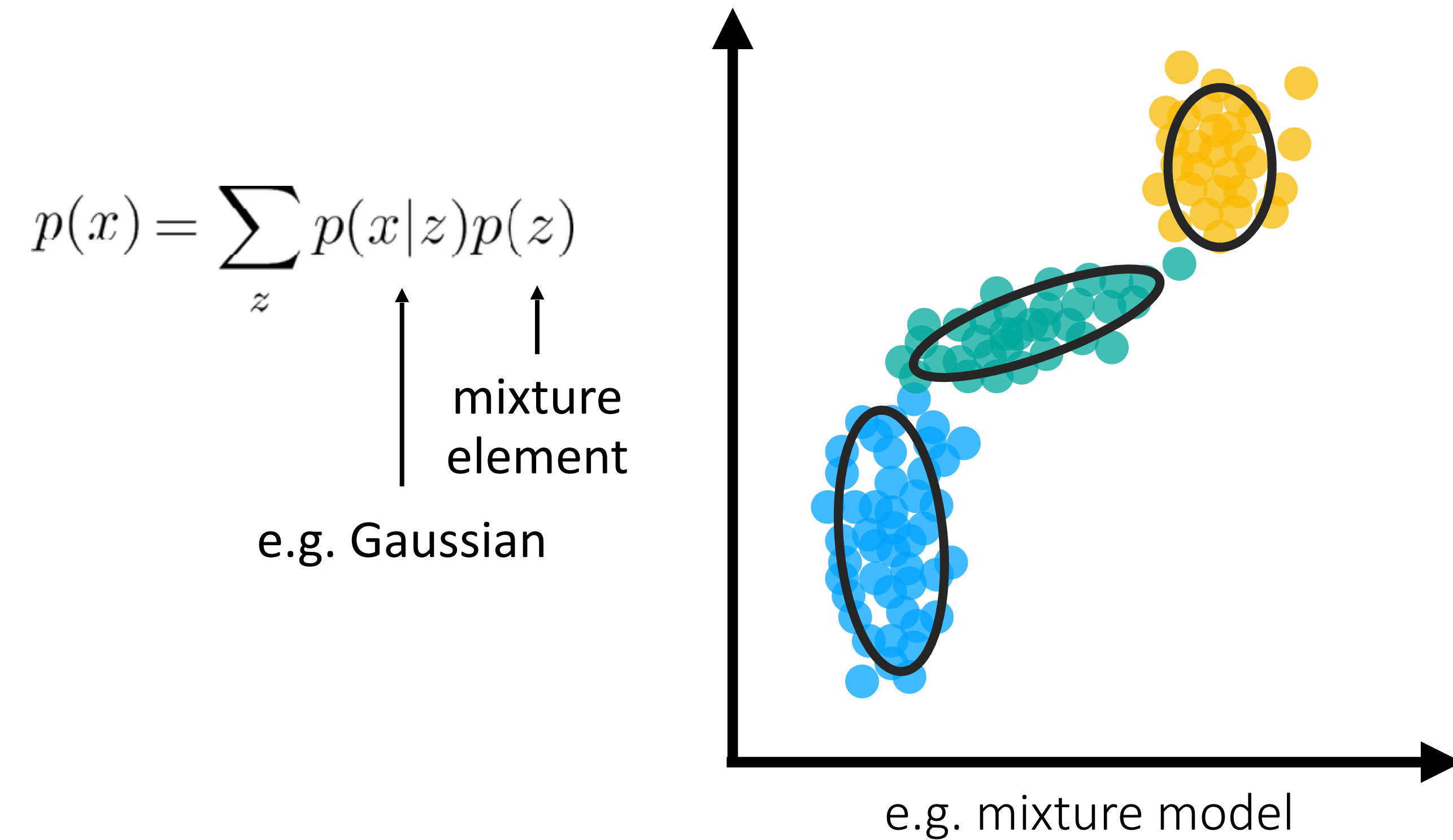
**Goal:** Can we model and train complex distributions?



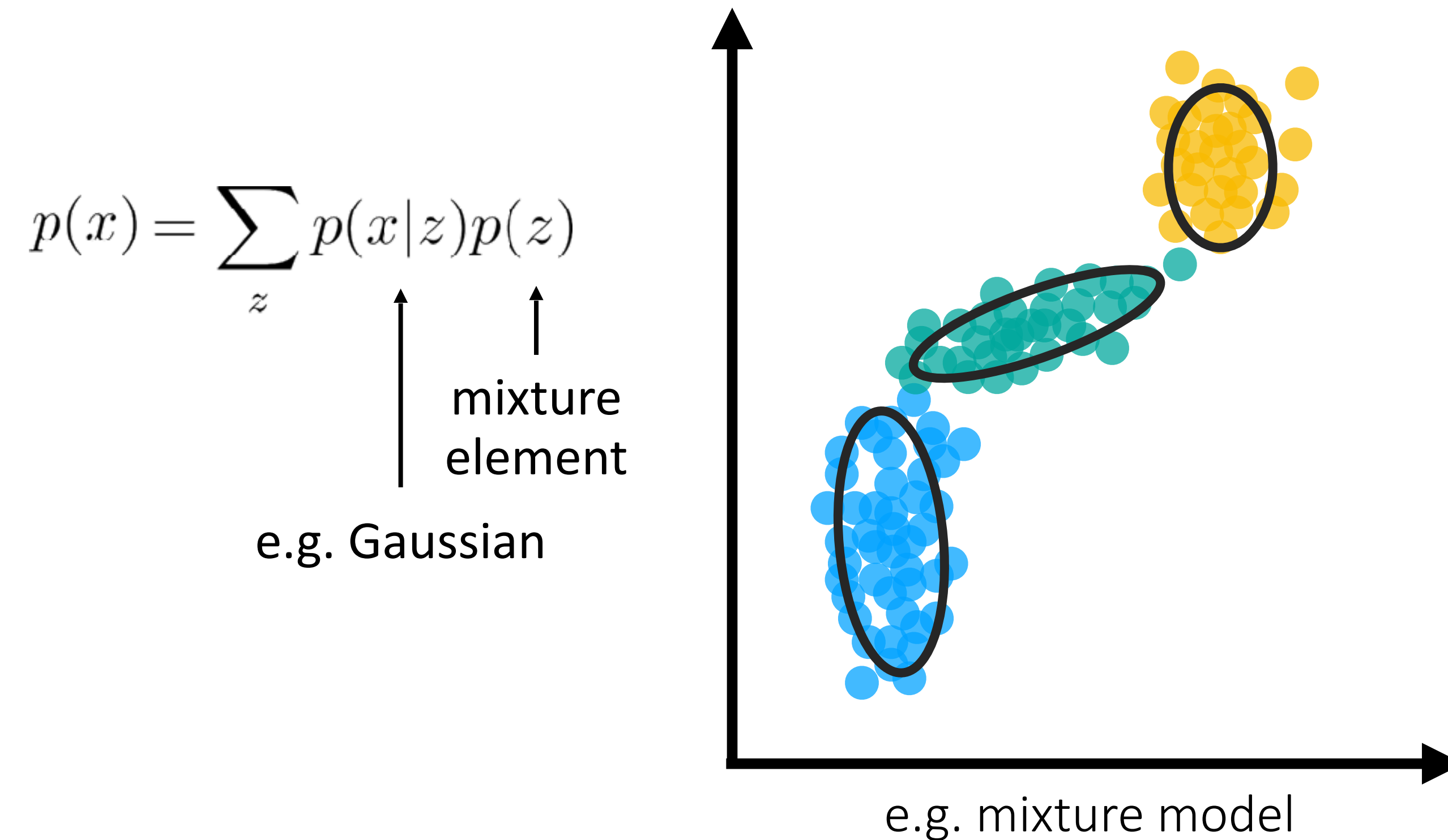
# Latent variable models: examples



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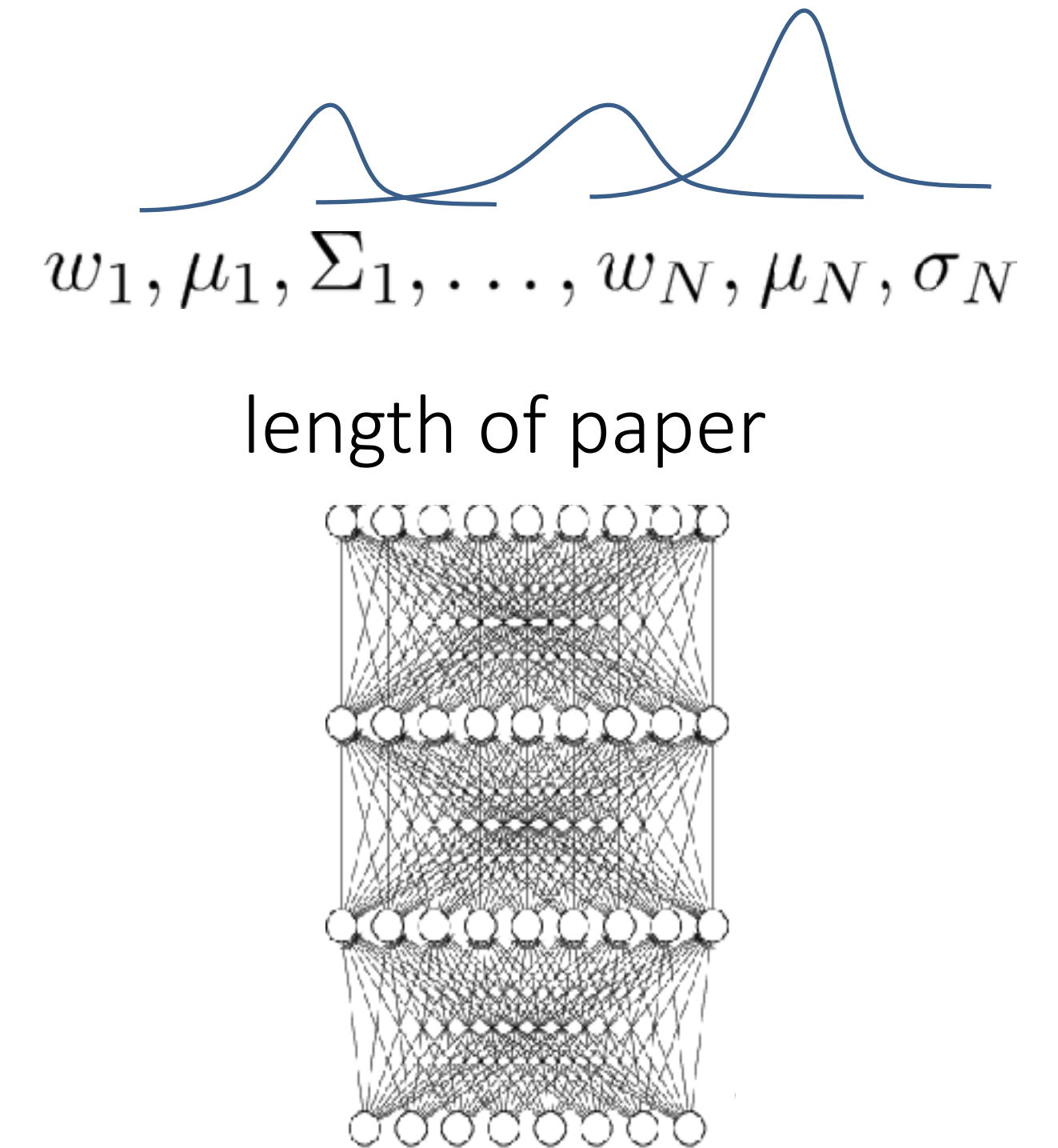


# Latent variable models: examples



$$p(y|x) = \sum_z p(y|x, z)p(z|x)$$

e.g. mixture density network




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**ImageNet Classification with Deep Convolutional  
Neural Networks**

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# Latent variable models in general

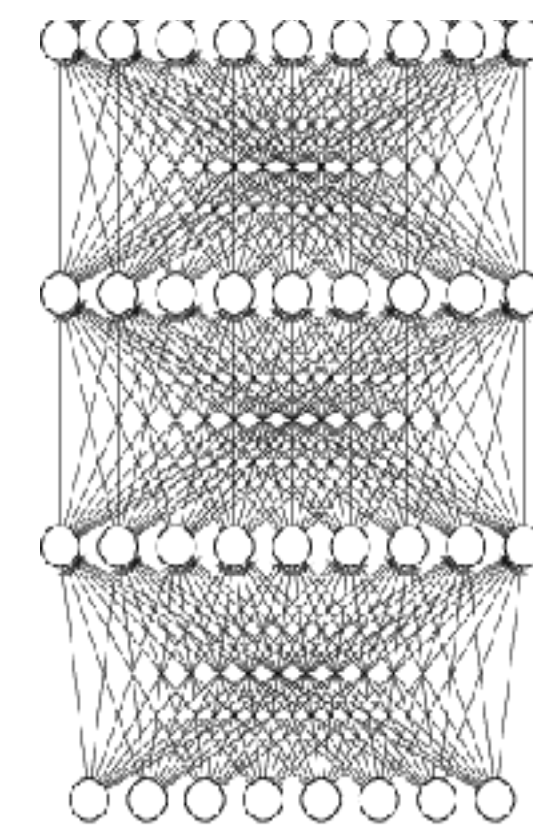
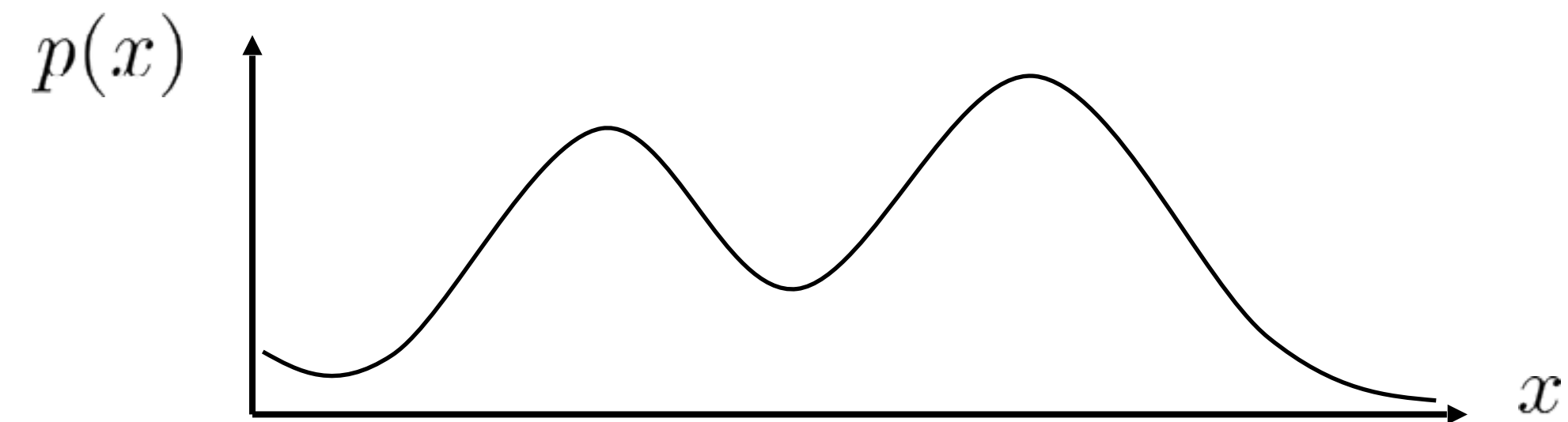
$$p(x) = \int p(x|z)p(z)dz$$

“easy” distribution  
(e.g., conditional Gaussian)

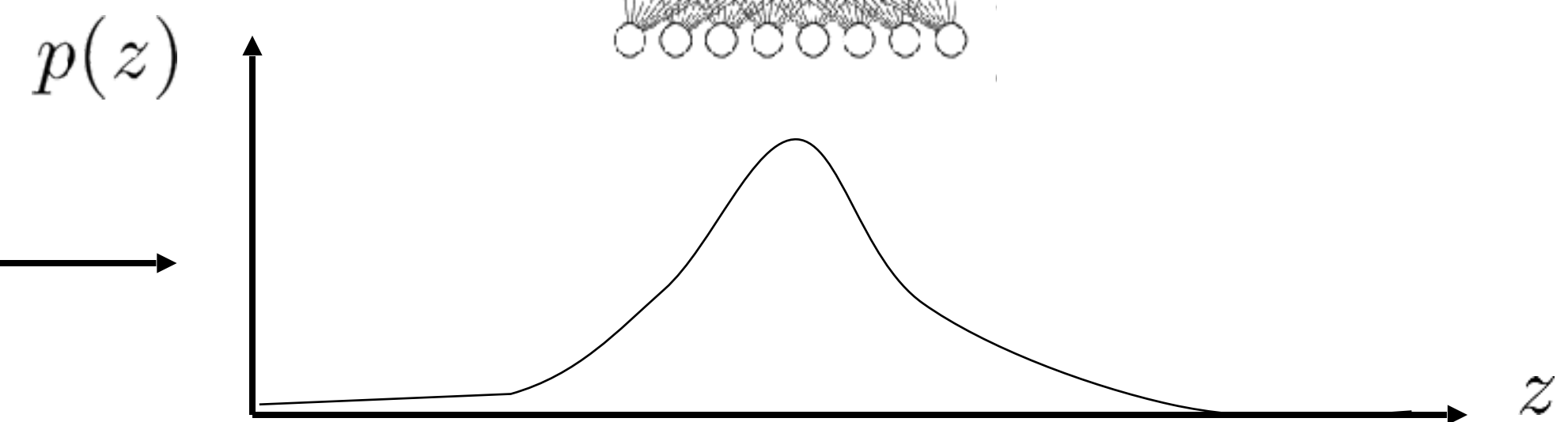
“easy” distribution  
(e.g., Gaussian)

“easy” distribution  
(e.g., Gaussian)

complicated distribution



$$p(x|z) = \mathcal{N}(\mu_{\text{nn}}(z), \sigma_{\text{nn}}(z))$$



💡 **Key idea:** represent complex distribution by composing two simple distributions

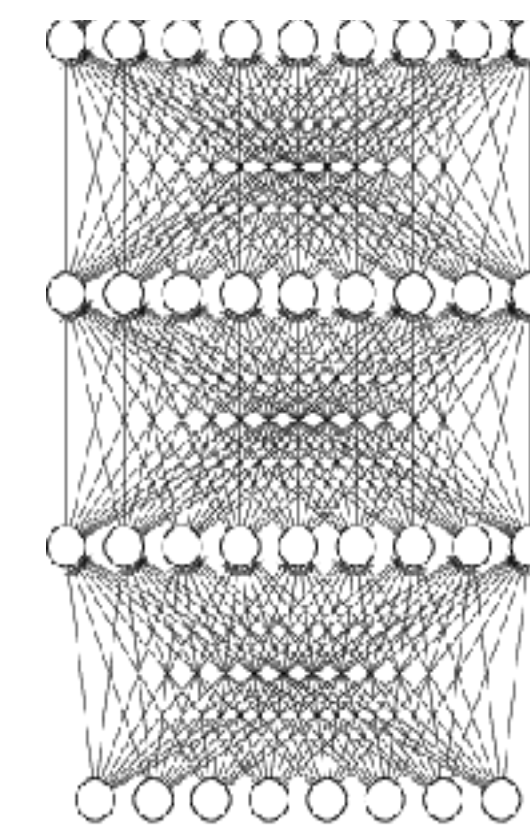
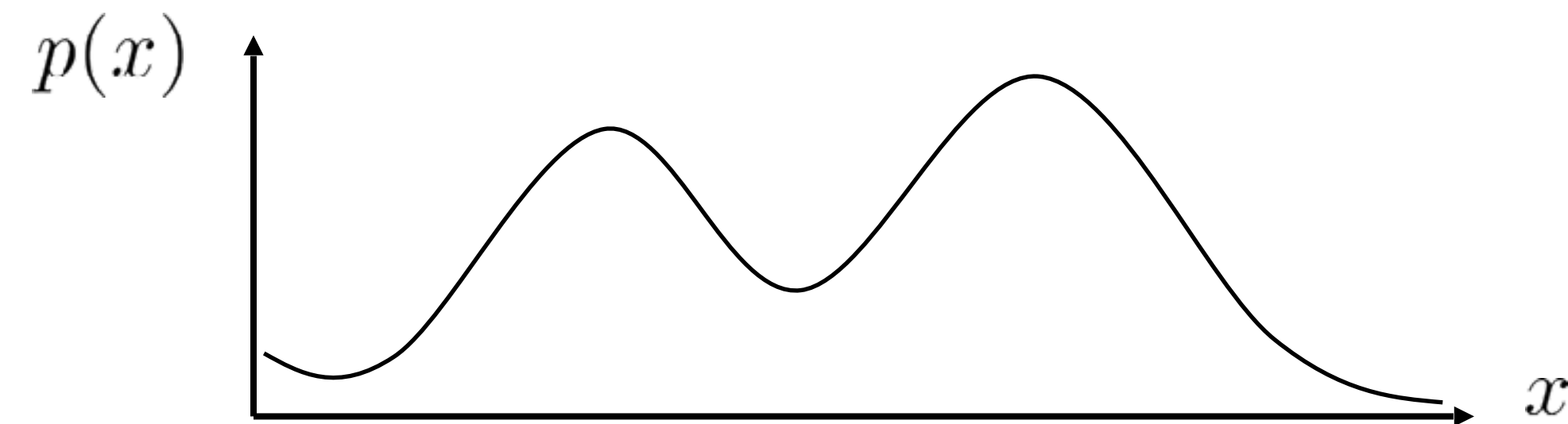
# Latent variable models in general

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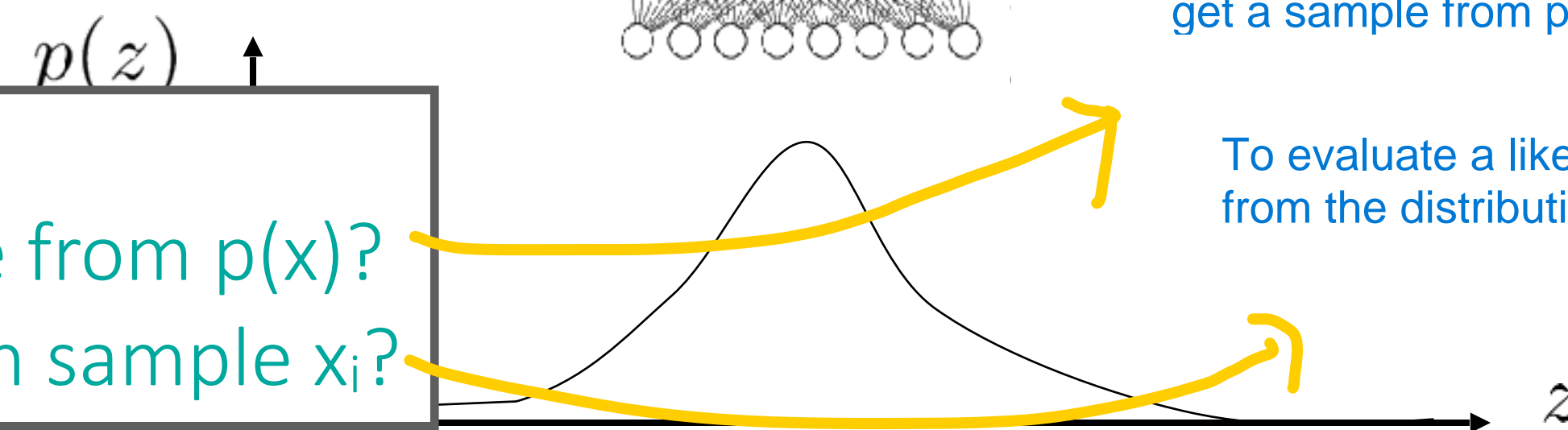


$$p(x|z) = \mathcal{N}(\mu_{nn}(z), \sigma_{nn}(z))$$

Sample a particular  $z$  and for that particular  $z$  you can get a sample from  $p(x|z=Z)$

## Questions:

1. Once trained, how do you generate a sample from  $p(x)$ ?
2. How do you evaluate the likelihood of a given sample  $x_i$ ?



To evaluate a likelihood sample out multiple  $z$ 's from the distribution, and then evaluate the integral

💡 **Key idea:** represent complex distribution by composing two simple distributions



# How do we train latent variable models?

the model:  $p_{\theta}(x)$

the data:  $\mathcal{D} = \{x_1, x_2, x_3, \dots, x_N\}$

maximum likelihood fit:

$$\theta \leftarrow \arg \max_{\theta} \frac{1}{N} \sum_i \log p_{\theta}(x_i)$$

$$p(x) = \int p(x|z)p(z)dz$$

$$\theta \leftarrow \arg \max_{\theta} \frac{1}{N} \sum_i \log \left( \int p_{\theta}(x_i|z)p(z)dz \right)$$



completely intractable



# Flavors of Deep Latent Variable Models

Use latent variables:

- generative adversarial networks (GANs)
- variational autoencoders (VAEs)
- normalizing flow models
- diffusion models

Do not use latent variables:

- autoregressive models
- (recall generative pre-training lecture)

All differ in how they are trained.

Let  $X$  be a random variable and  $g(X)$  be a function of that random variable then  
 $E(g(X)) = \text{summation } (g(X) \text{ pmf}(X))$   
and here in our case in the definition below the  $\text{pmf} = p(X)$

# Variational Inference

- A. Formulate a lower bound on the log likelihood objective.
- B. Check how tight the bound is.
- C. Variational inference -> *Amortized* variational inference
- D. How to optimize

# Estimating the log-likelihood

$$H(X) = -E(\log(P(X))) = - \int p(x) \log p(x) dx$$

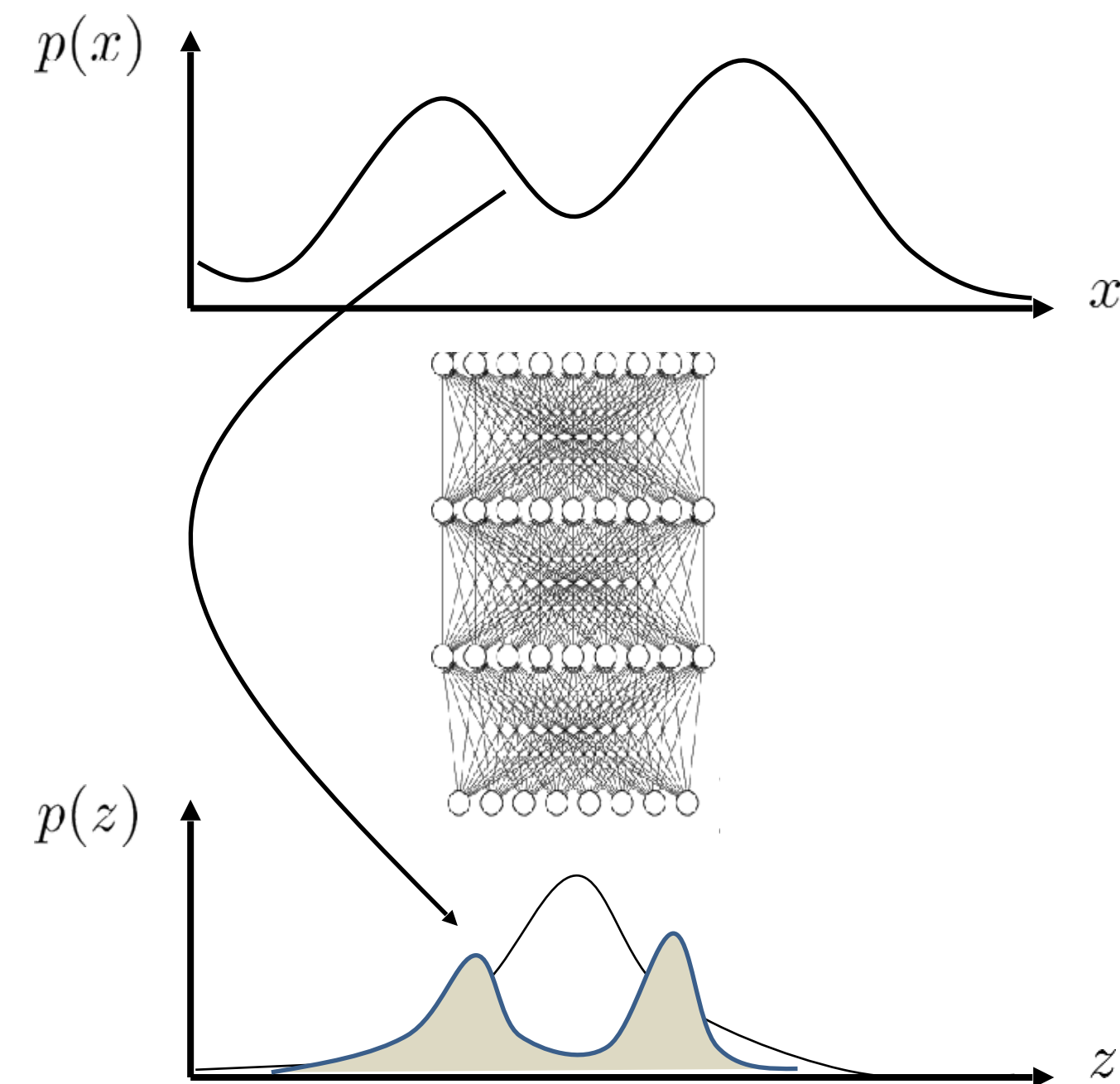
alternative: *expected* log-likelihood:

$$\theta \leftarrow \arg \max_{\theta} \frac{1}{N} \sum_i E_{z \sim p(z|x_i)} [\log p_{\theta}(x_i, z)]$$

but... how do we calculate  $p(z|x_i)$ ?

intuition: “guess” most likely  $z$  given  $x_i$ ,  
and pretend it’s the right one

...but there are many possible values of  $z$   
so use the distribution  $p(z|x_i)$



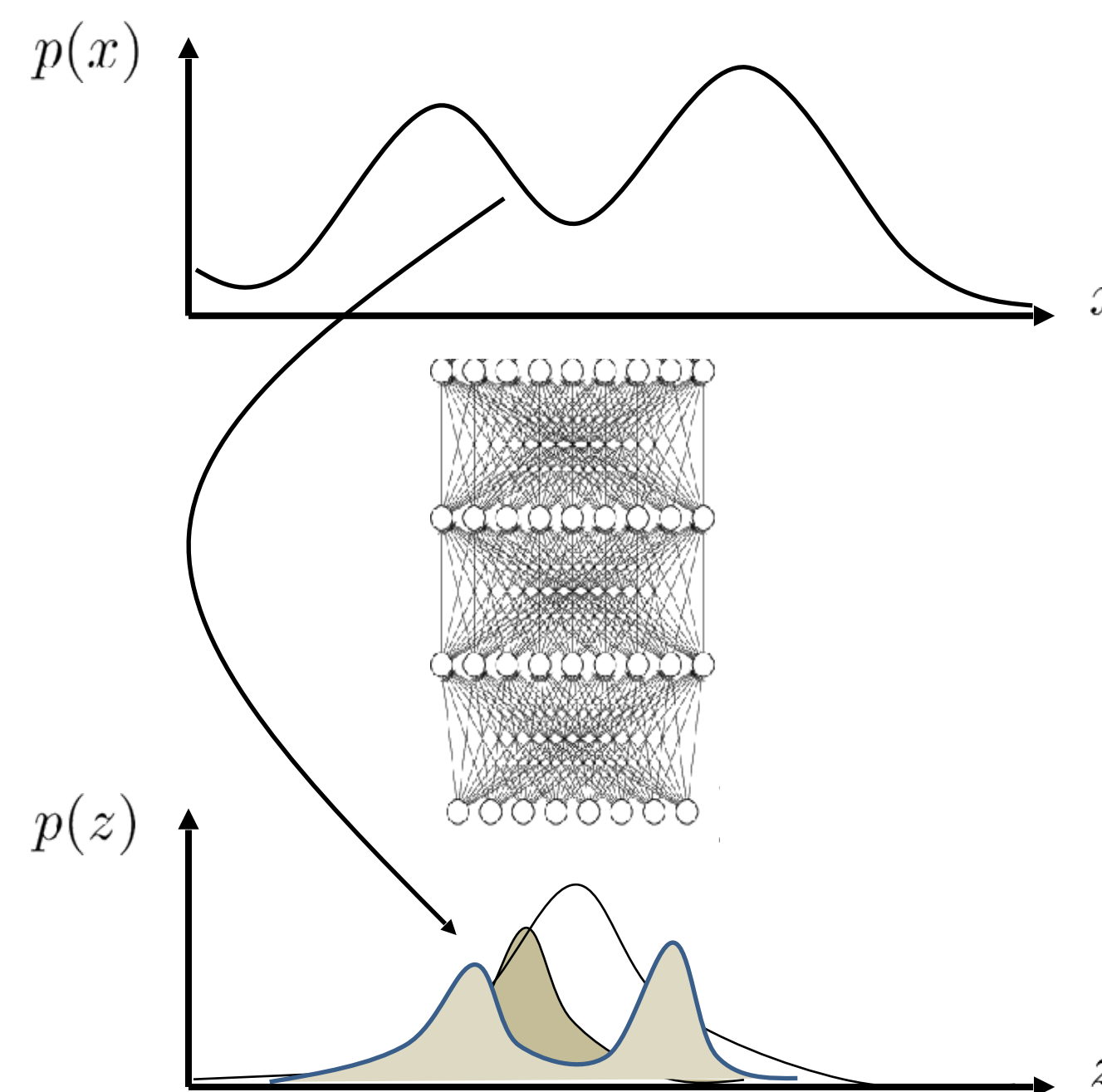
# The variational approximation

but... how do we calculate  $p(z|x_i)$ ?

what if we approximate with  $q_i(z) = \mathcal{N}(\mu_i, \sigma_i)$

can bound  $\log p(x_i)$ !

$$\begin{aligned}\log p(x_i) &= \log \int_z p(x_i|z)p(z) \\ &= \log \int_z p(x_i|z)p(z) \frac{q_i(z)}{q_i(z)} \\ &= \log E_{z \sim q_i(z)} \left[ \frac{p(x_i|z)p(z)}{q_i(z)} \right]\end{aligned}$$



# The variational approximation

but... how do we calculate  $p(z|x_i)$ ?

can bound  $\log p(x_i)$ !

Jensen's inequality

$$\log E[y] \geq E[\log y]$$

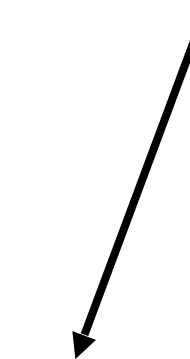
$$\log p(x_i) = \log \int_z p(x_i|z)p(z)$$

$$= \log \int_z p(x_i|z)p(z) \frac{q_i(z)}{q_i(z)}$$

$$= \log E_{z \sim q_i(z)} \left[ \frac{p(x_i|z)p(z)}{q_i(z)} \right]$$

$$\geq E_{z \sim q_i(z)} \left[ \log \frac{p(x_i|z)p(z)}{q_i(z)} \right] = E_{z \sim q_i(z)} [\log p(x_i|z) + \log p(z)] + \mathcal{H}_{q_i(z)}[q_i(z)]$$

maximizing this maximizes  $\log p(x_i)$

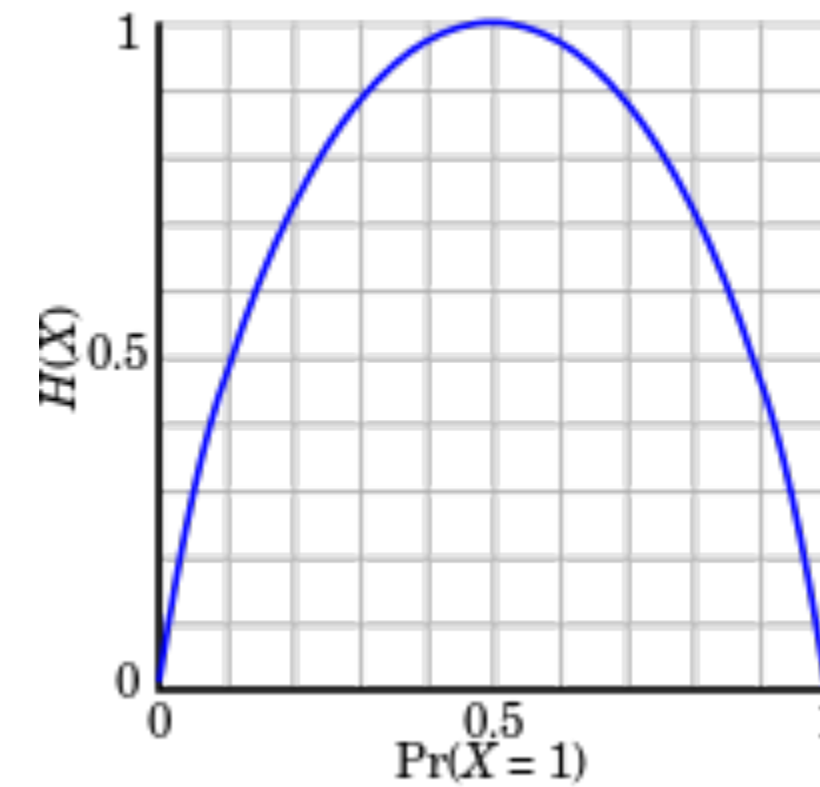


“evidence lower bound” (ELBO)



# A brief aside...

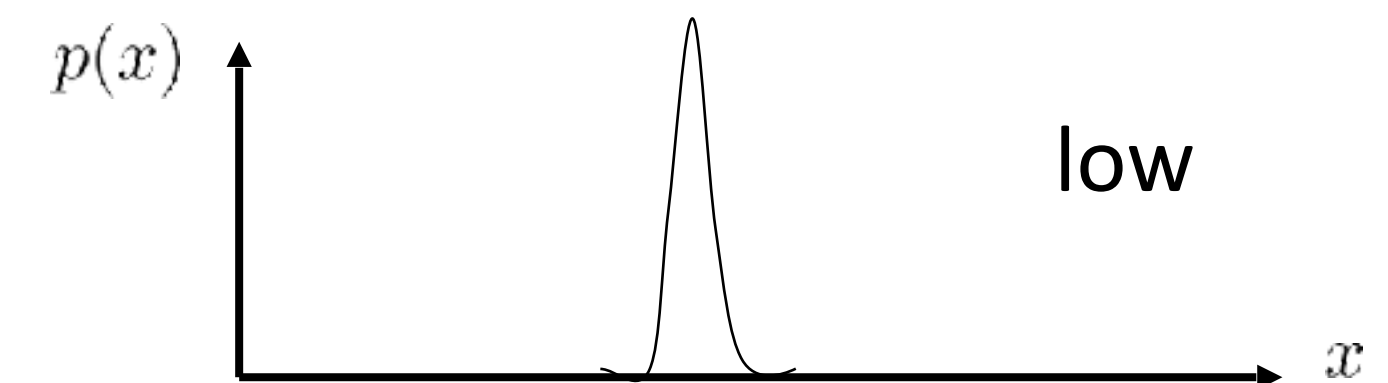
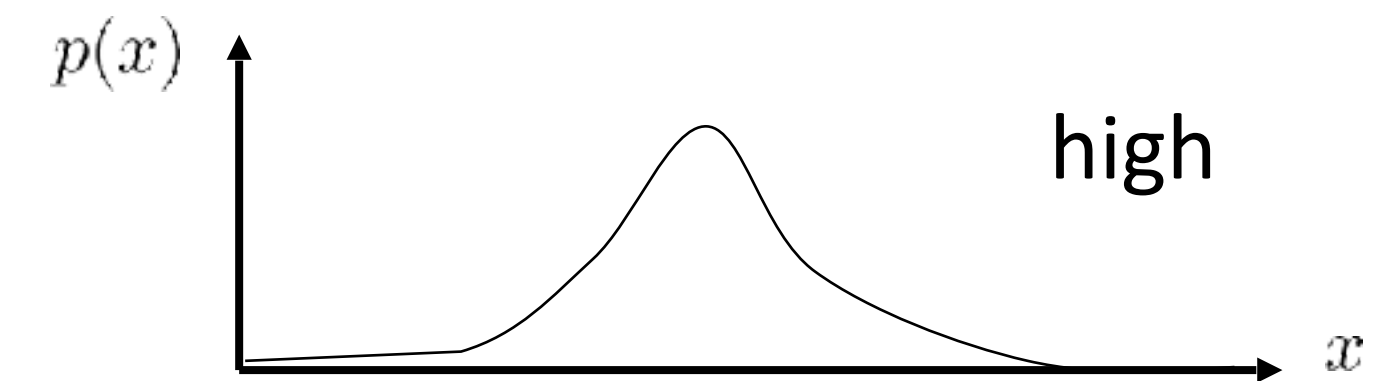
## Entropy:



$$\mathcal{H}(p) = -E_{x \sim p(x)}[\log p(x)] = - \int_x p(x) \log p(x) dx$$

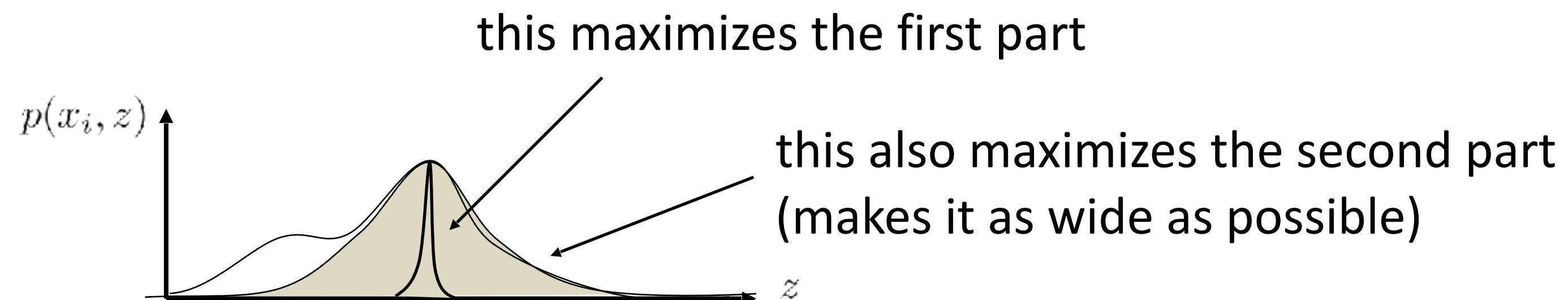
Intuition 1: how *random* is the random variable?

Intuition 2: how large is the log probability in expectation *under itself*



what do we expect this to do?

$$E_{z \sim q_i(z)}[\log p(x_i|z) + \log p(z)] + \mathcal{H}(q_i)$$





# A brief aside...

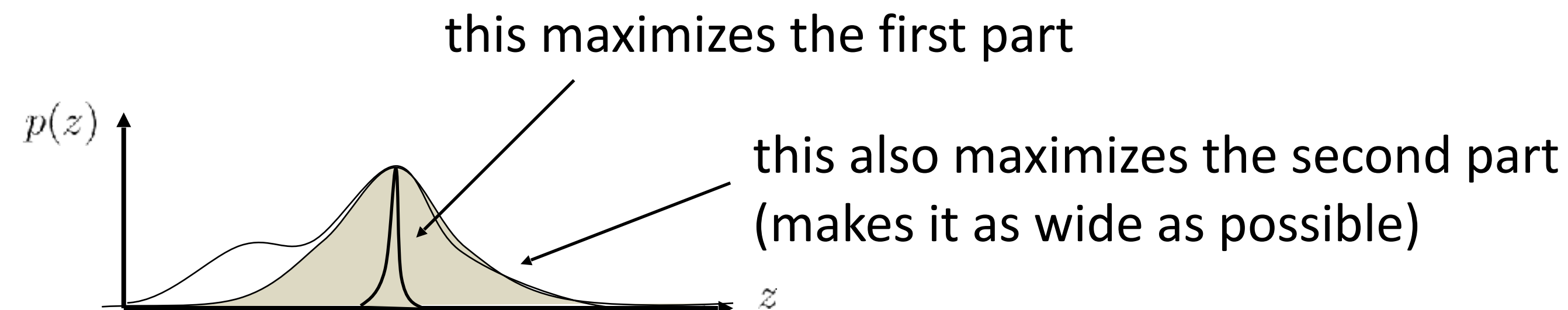
## KL-Divergence:

$$D_{\text{KL}}(q||p) = E_{x \sim q(x)} \left[ \log \frac{q(x)}{p(x)} \right] = E_{x \sim q(x)} [\log q(x)] - E_{x \sim q(x)} [\log p(x)] = -E_{x \sim q(x)} [\log p(x)] - \mathcal{H}(q)$$

Intuition 1: how *different* are two distributions? e.g. when  $q=p$ , KL divergence is 0

Intuition 2: how small is the expected log probability of one distribution under another, minus entropy?

why entropy?



# How tight is the lower bound?

$\mathcal{L}_i(p, q_i)$  “evidence lower bound” (ELBO)

$$\log p(x_i) \geq \overbrace{E_{z \sim q_i(z)} [\log p(x_i|z) + \log p(z)]}^{\mathcal{L}_i(p, q_i)} + \mathcal{H}(q_i)$$

what makes a good  $q_i(z)$ ?

intuition:  $q_i(z)$  should approximate  $p(z|x_i)$

approximate in what sense?

compare in terms of KL-divergence:  $D_{\text{KL}}(q_i(z) \| p(z|x))$

why?

$$\begin{aligned} D_{\text{KL}}(q_i(z) \| p(z|x_i)) &= E_{z \sim q_i(z)} \left[ \log \frac{q_i(z)}{p(z|x_i)} \right] = E_{z \sim q_i(z)} \left[ \log \frac{q_i(z)p(x_i)}{p(x_i, z)} \right] \\ &= -E_{z \sim q_i(z)} [\log p(x_i|z) + \log p(z)] + E_{z \sim q_i(z)} [\log q_i(z)] + E_{z \sim q_i(z)} [\log p(x_i)] \\ &= -E_{z \sim q_i(z)} [\log p(x_i|z) + \log p(z)] - \mathcal{H}(q_i) + \log p(x_i) \\ &= -\mathcal{L}_i(p, q_i) + \log p(x_i) \end{aligned}$$

$\log p(x_i) = D_{\text{KL}}(q_i(z) \| p(z|x_i)) + \mathcal{L}_i(p, q_i)$  **Note 1:** If KL divergence is 0, then bound is tight.

$$\log p(x_i) \geq \mathcal{L}_i(p, q_i)$$

# How tight is the lower bound?

$\mathcal{L}_i(p, q_i)$  “evidence lower bound” (ELBO)

$$\log p(x_i) \geq \overbrace{E_{z \sim q_i(z)} [\log p(x_i|z) + \log p(z)]}^{\mathcal{L}_i(p, q_i)} + \mathcal{H}(q_i)$$

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approximate in what sense?

compare in terms of KL-divergence:  $D_{\text{KL}}(q_i(z) \| p(z|x))$

why?

$$D_{\text{KL}}(q_i(z) \| p(z|x_i)) = -\mathcal{L}_i(p, q_i) + \log p(x_i)$$

**Note 2:** Maximizing  $L(p, q_i)$  w.r.t.  $q_i$  minimizes the KL divergence.

$$\log p(x_i) = D_{\text{KL}}(q_i(z) \| p(z|x_i)) + \mathcal{L}_i(p, q_i) \quad \text{Note 1: If KL divergence is 0, then bound is tight.}$$

$$\log p(x_i) \geq \mathcal{L}_i(p, q_i)$$

Optimization objective:

$$\max_{\theta, q_i} \frac{1}{N} \sum_i \mathcal{L}_i(p_\theta, q_i)$$

Imp (both  $\theta, q_i$ )

# Optimizing the ELBO

$\mathcal{L}_i(p, q_i)$  “evidence lower bound” (ELBO)

$$\log p(x_i) \geq \overbrace{E_{z \sim q_i(z)} [\log p_\theta(x_i|z) + \log p(z)]}^{\mathcal{L}_i(p, q_i)} + \mathcal{H}(q_i)$$

~~$$\theta \leftarrow \arg \max_{\theta} \frac{1}{N} \sum_i \log p_\theta(x_i)$$~~

$$\theta \leftarrow \arg \max_{\theta} \frac{1}{N} \sum_i \mathcal{L}_i(p, q_i)$$

for each  $x_i$  (or mini-batch):

let's say  $q_i(z) = \mathcal{N}(\mu_i, \sigma_i)$

calculate  $\nabla_{\theta} \mathcal{L}_i(p, q_i)$ :

use gradient  $\nabla_{\mu_i} \mathcal{L}_i(p, q_i)$  and  $\nabla_{\sigma_i} \mathcal{L}_i(p, q_i)$

sample  $z \sim q_i(z)$     Here they are sampling one  $z$  gradient ascent on  $\mu_i, \sigma_i$

$$\nabla_{\theta} \mathcal{L}_i(p, q_i) \approx \nabla_{\theta} \log p_\theta(x_i|z)$$

$$\theta \leftarrow \theta + \alpha \nabla_{\theta} \mathcal{L}_i(p, q_i)$$

update  $q_i$  to maximize  $\mathcal{L}_i(p, q_i)$  ← how?



# What's the **problem**?

for each  $x_i$  (or mini-batch):

calculate  $\nabla_{\theta} \mathcal{L}_i(p, q_i)$ :

sample  $z \sim q_i(z)$

$$\nabla_{\theta} \mathcal{L}_i(p, q_i) \approx \nabla_{\theta} \log p_{\theta}(x_i|z)$$

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let's say  $q_i(z) = \mathcal{N}(\mu_i, \sigma_i)$

use gradient  $\nabla_{\mu_i} \mathcal{L}_i(p, q_i)$  and  $\nabla_{\sigma_i} \mathcal{L}_i(p, q_i)$

gradient ascent on  $\mu_i, \sigma_i$

Question: How many parameters are there?

in terms of  $|\theta|, |z|, N$

$$|\theta| + (|\mu_i| + |\sigma_i|) \times N$$

# What's the **problem**?

for each  $x_i$  (or mini-batch):

calculate  $\nabla_{\theta} \mathcal{L}_i(p, q_i)$ :

sample  $z \sim q_i(z)$

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update  $q_i$  to maximize  $\mathcal{L}_i(p, q_i)$

let's say  $q_i(z) = \mathcal{N}(\mu_i, \sigma_i)$

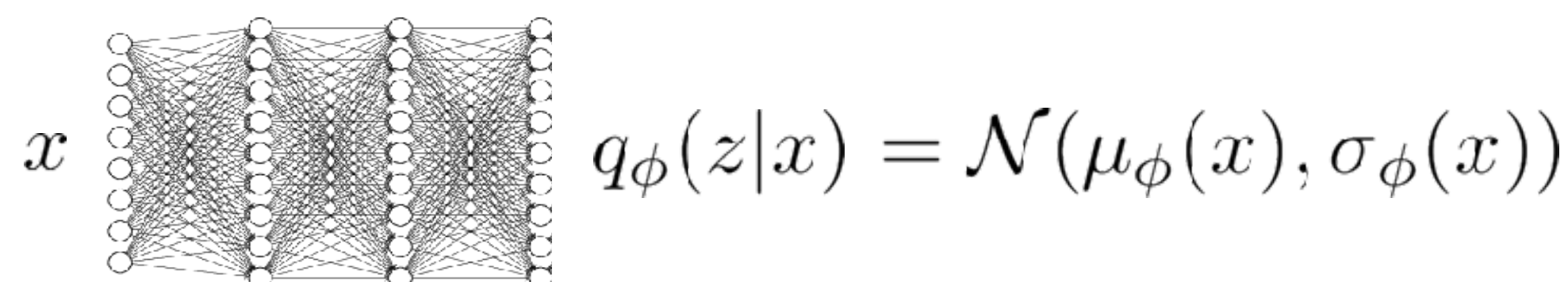
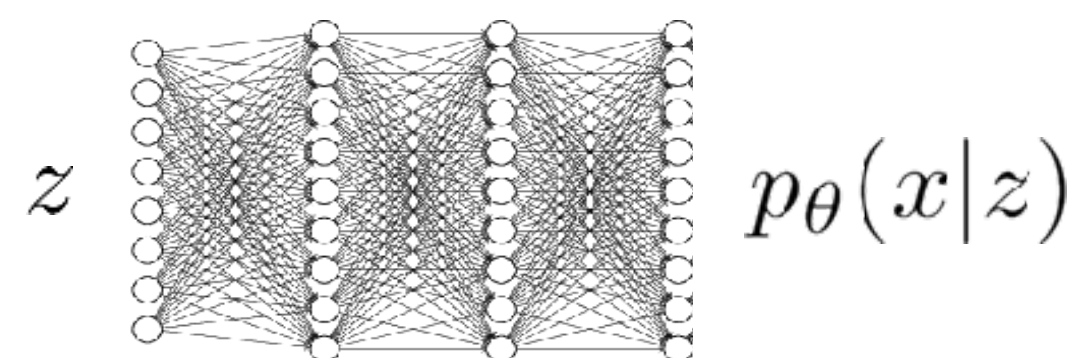
use gradient  $\nabla_{\mu_i} \mathcal{L}_i(p, q_i)$  and  $\nabla_{\sigma_i} \mathcal{L}_i(p, q_i)$

gradient ascent on  $\mu_i, \sigma_i$

**Question:** How many parameters are there?

intuition:  $q_i(z)$  should approximate  $p(z|x_i)$

what if we learn a *network*  $q_i(z) = q(z|x_i) \approx p(z|x_i)$ ?





# Amortized Variational Inference

- A. Formulate a lower bound on the log likelihood objective.
- B. Check how tight the bound is.
- C. Variational inference -> *Amortized* variational inference
- D. How to optimize

# What's the **problem**?

for each  $x_i$  (or mini-batch):

calculate  $\nabla_{\theta} \mathcal{L}_i(p, q_i)$ :

sample  $z \sim q_i(z)$

$$\nabla_{\theta} \mathcal{L}_i(p, q_i) \approx \nabla_{\theta} \log p_{\theta}(x_i|z)$$

$$\theta \leftarrow \theta + \alpha \nabla_{\theta} \mathcal{L}_i(p, q_i)$$

update  $q_i$  to maximize  $\mathcal{L}_i(p, q_i)$

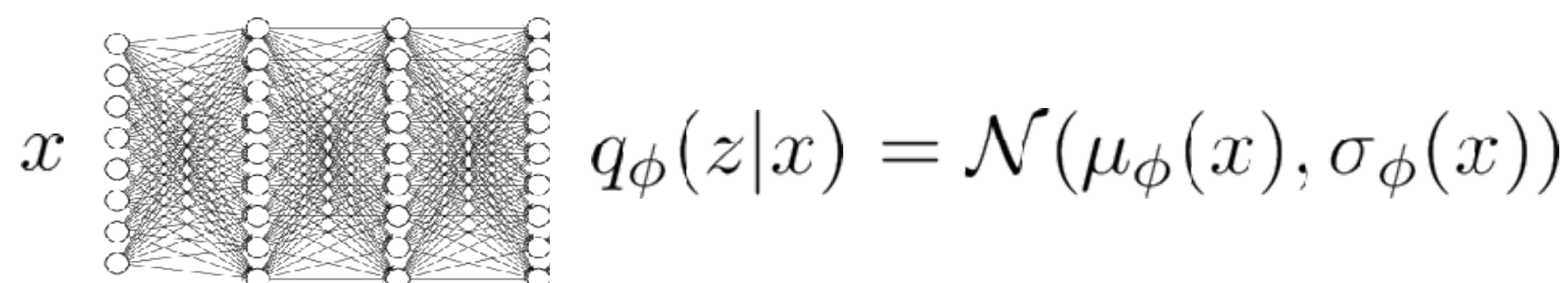
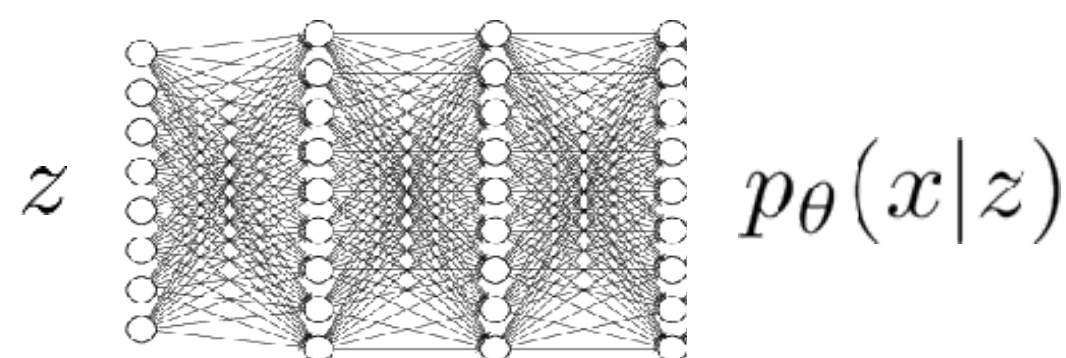
let's say  $q_i(z) = \mathcal{N}(\mu_i, \sigma_i)$

use gradient  $\nabla_{\mu_i} \mathcal{L}_i(p, q_i)$  and  $\nabla_{\sigma_i} \mathcal{L}_i(p, q_i)$

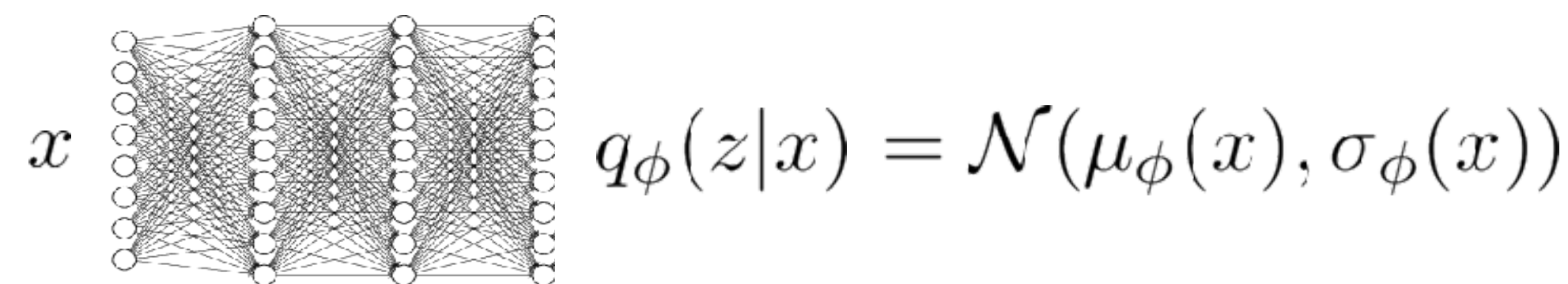
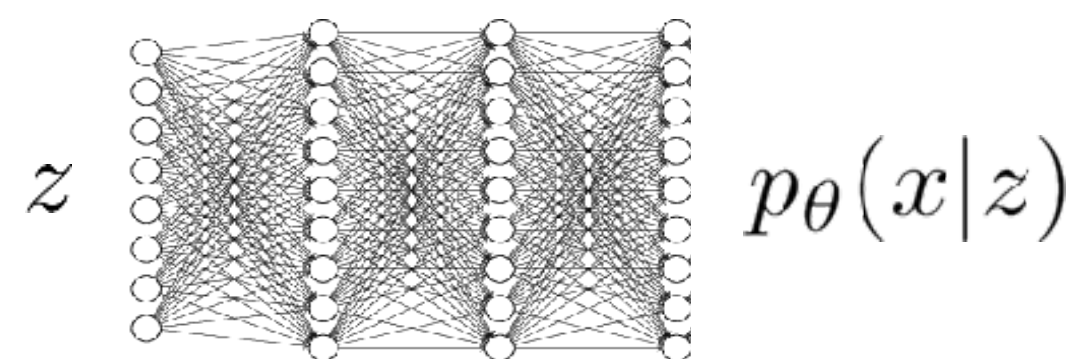
gradient ascent on  $\mu_i, \sigma_i$

**Question:** How many parameters are there?  $|\theta| + (|\mu_i| + |\sigma_i|) \times N$

intuition:  $q_i(z)$  should approximate  $p(z|x_i)$  what if we learn a *network*  $q_i(z) = q(z|x_i) \approx p(z|x_i)$ ?



# Amortized variational inference



for each  $x_i$  (or mini-batch):

calculate  $\nabla_\theta \mathcal{L}(p_\theta(x_i|z), q_\phi(z|x_i))$ :

sample  $z \sim q_\phi(z|x_i)$

$\nabla_\theta \mathcal{L} \approx \nabla_\theta \log p_\theta(x_i|z)$

$\theta \leftarrow \theta + \alpha \nabla_\theta \mathcal{L}$

$\phi \leftarrow \phi + \alpha \nabla_\phi \mathcal{L}$

how do we calculate this?

$$\log p(x_i) \geq \overbrace{E_{z \sim q_\phi(z|x_i)} [\log p_\theta(x_i|z) + \log p(z)]}^{\mathcal{L}(p_\theta(x_i|z), q_\phi(z|x_i))} + \mathcal{H}(q_\phi(z|x_i))$$

# Amortized variational inference

for each  $x_i$  (or mini-batch):

calculate  $\nabla_{\theta} \mathcal{L}(p_{\theta}(x_i|z), q_{\phi}(z|x_i))$ :

sample  $z \sim q_{\phi}(z|x_i)$

$$\nabla_{\theta} \mathcal{L} \approx \nabla_{\theta} \log p_{\theta}(x_i|z)$$

$$\theta \leftarrow \theta + \alpha \nabla_{\theta} \mathcal{L}$$

$$\phi \leftarrow \phi + \alpha \nabla_{\phi} \mathcal{L}$$

$$q_{\phi}(z|x) = \mathcal{N}(\mu_{\phi}(x), \sigma_{\phi}(x))$$

look up formula for  
entropy of a Gaussian

$$\mathcal{L}_i = \underbrace{E_{z \sim q_{\phi}(z|x_i)} [\log p_{\theta}(x_i|z) + \log p(z)]}_{J(\phi)} + \mathcal{H}(q_{\phi}(z|x_i))$$

$$J(\phi) = E_{z \sim q_{\phi}(z|x_i)} [r(x_i, z)]$$

# The reparameterization trick

$$\begin{aligned} J(\phi) &= E_{z \sim q_\phi(z|x_i)}[r(x_i, z)] \\ &= E_{\epsilon \sim \mathcal{N}(0,1)}[r(x_i, \mu_\phi(x_i) + \epsilon \sigma_\phi(x_i))] \end{aligned}$$

$$q_\phi(z|x) = \mathcal{N}(\mu_\phi(x), \sigma_\phi(x))$$

$$z = \mu_\phi(x) + \epsilon \sigma_\phi(x)$$

estimating  $\nabla_\phi J(\phi)$ :

sample  $\epsilon_1, \dots, \epsilon_M$  from  $\mathcal{N}(0, 1)$  (a single sample works well!)

$$\nabla_\phi J(\phi) \approx \frac{1}{M} \sum_j \nabla_\phi r(x_i, \mu_\phi(x_i) + \epsilon_j \sigma_\phi(x_i))$$

$$\epsilon \sim \mathcal{N}(0, 1)$$

independent of  $\phi$ !

- + Very simple to implement
- + Low variance
- Only continuous latent variables

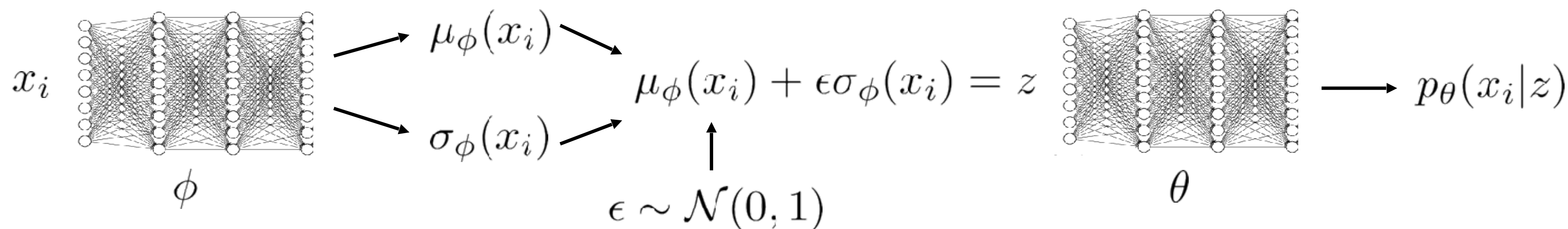
Discrete latent variables:

- vector quantization & straight-through estimator (“VQ-VAE”)
- policy gradients / “REINFORCE”



# Another way to look at everything...

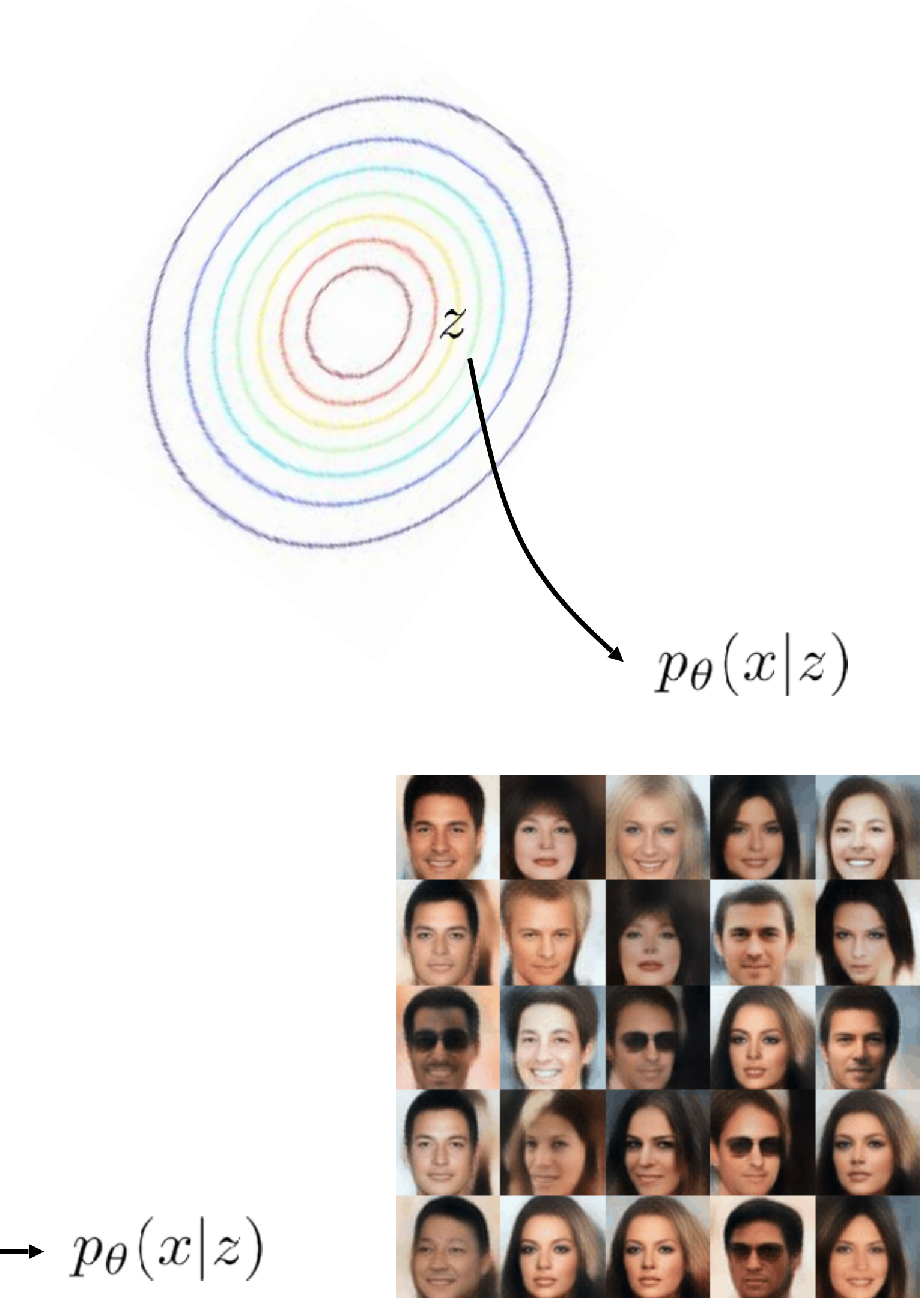
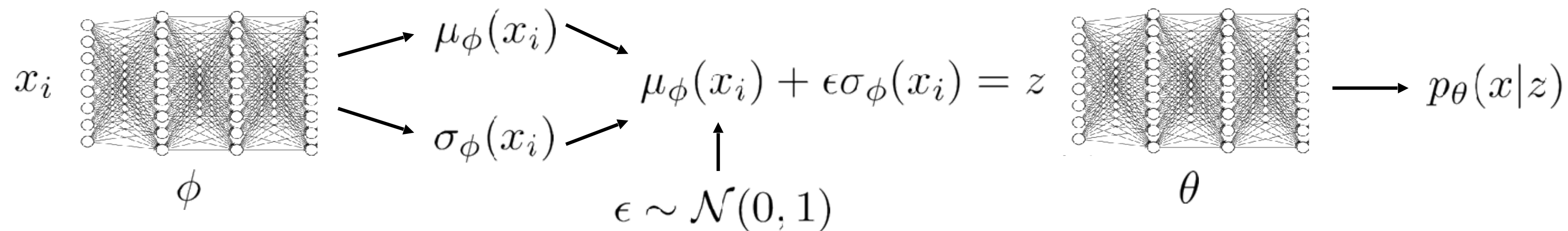
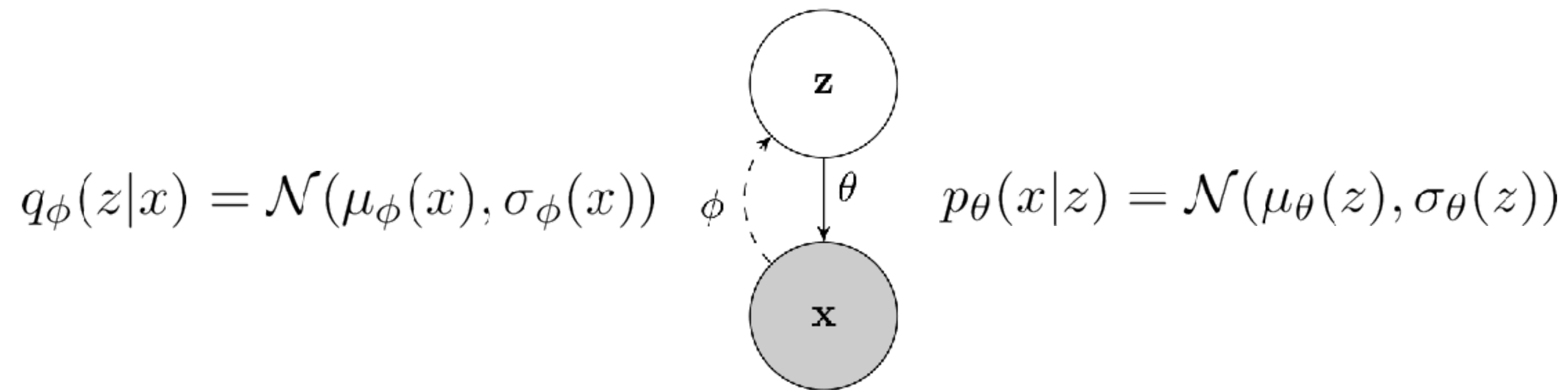
$$\begin{aligned}
 \mathcal{L}_i &= E_{z \sim q_\phi(z|x_i)} [\log p_\theta(x_i|z) + \log p(z)] + \mathcal{H}(q_\phi(z|x_i)) \\
 &= E_{z \sim q_\phi(z|x_i)} [\log p_\theta(x_i|z)] + \underbrace{E_{z \sim q_\phi(z|x_i)} [\log p(z)] + \mathcal{H}(q_\phi(z|x_i))}_{-D_{\text{KL}}(q_\phi(z|x_i) \| p(z))} \leftarrow \text{this has a convenient analytical form for Gaussians} \\
 &= E_{z \sim q_\phi(z|x_i)} [\log p_\theta(x_i|z)] - D_{\text{KL}}(q_\phi(z|x_i) \| p(z)) \\
 &= E_{\epsilon \sim \mathcal{N}(0,1)} [\log p_\theta(x_i | \mu_\phi(x_i) + \epsilon \sigma_\phi(x_i))] - D_{\text{KL}}(q_\phi(z|x_i) \| p(z)) \\
 &\approx \log p_\theta(x_i | \mu_\phi(x_i) + \epsilon \sigma_\phi(x_i)) - D_{\text{KL}}(q_\phi(z|x_i) \| p(z))
 \end{aligned}$$





# Example Models

# The variational autoencoder

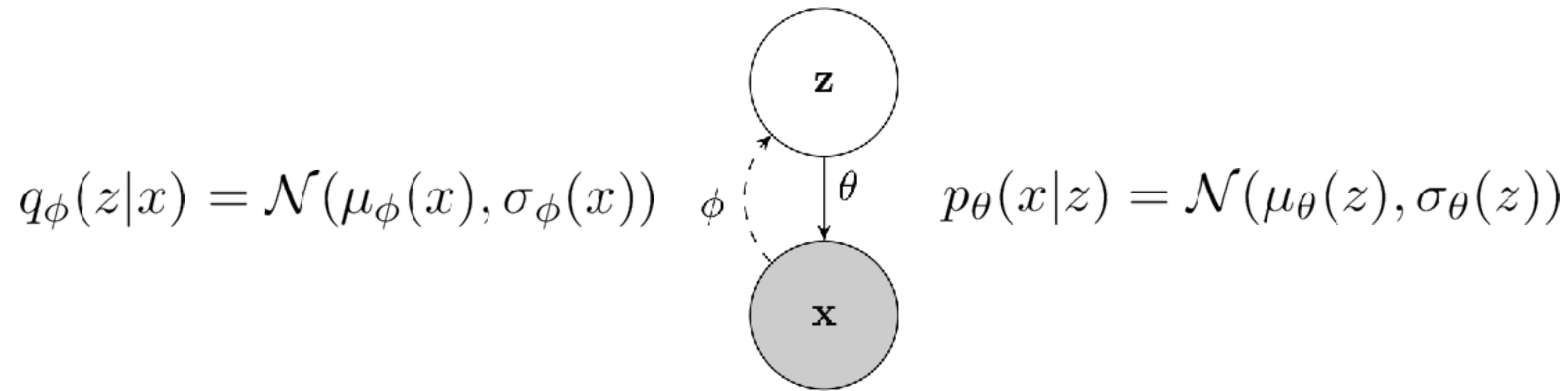


$$\max_{\theta, \phi} \frac{1}{N} \sum_i \log p_{\theta}(x_i | \mu_{\phi}(x_i) + \epsilon \sigma_{\phi}(x_i)) - D_{\text{KL}}(q_{\phi}(z|x_i) || p(z))$$

This KL Divergence term acts as a regularizer that ensures that the posterior distribution is closed to the prior ensuring that too much info is not from a single example



# Using the variational autoencoder



$$p(x) = \int p(x|z)p(z)dz$$

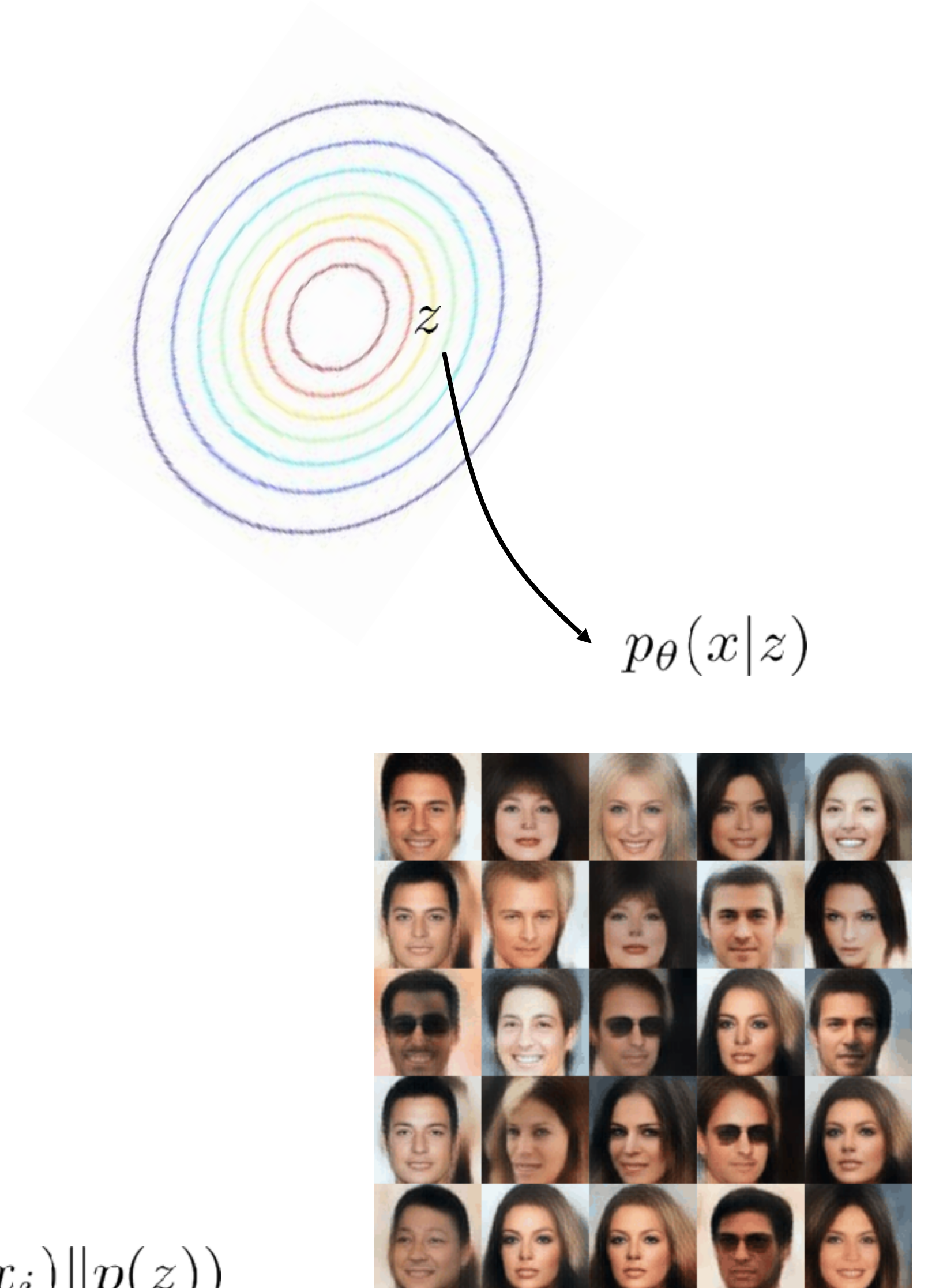
why does this work?

sampling:

$$z \sim p(z)$$

$$x \sim p(x|z)$$

$$\mathcal{L}_i = E_{z \sim q_{\phi}(z|x_i)}[\log p_{\theta}(x_i|z)] - D_{\text{KL}}(q_{\phi}(z|x_i) || p(z))$$







# Conditional models

$$\mathcal{L}_i = E_{z \sim q_\phi(z|x_i, y_i)} [\log p_\theta(y_i|x_i, z) + \log p(z|x_i)] + \mathcal{H}(q_\phi(z|x_i, y_i))$$

just like before, only now generating  $y_i$   
and *everything* is conditioned on  $x_i$

at test time:

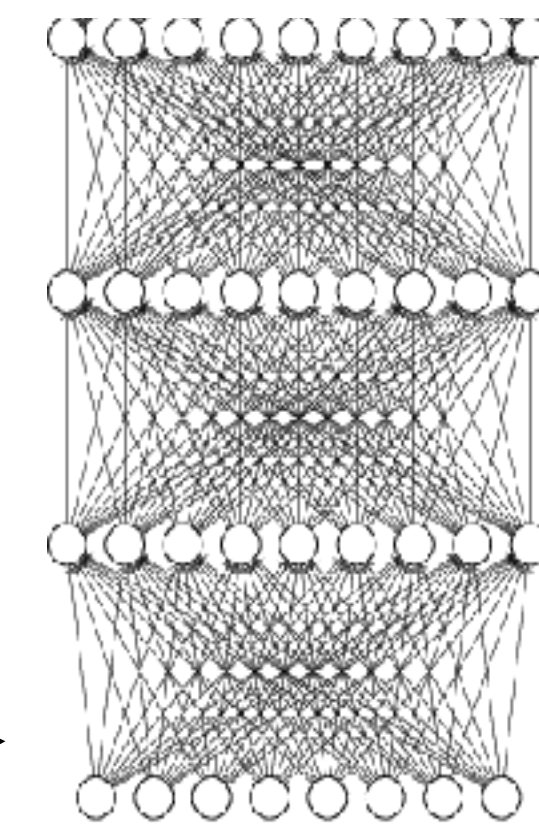
$$z \sim p(z|x_i)$$

$$y \sim p(y|x_i, z)$$

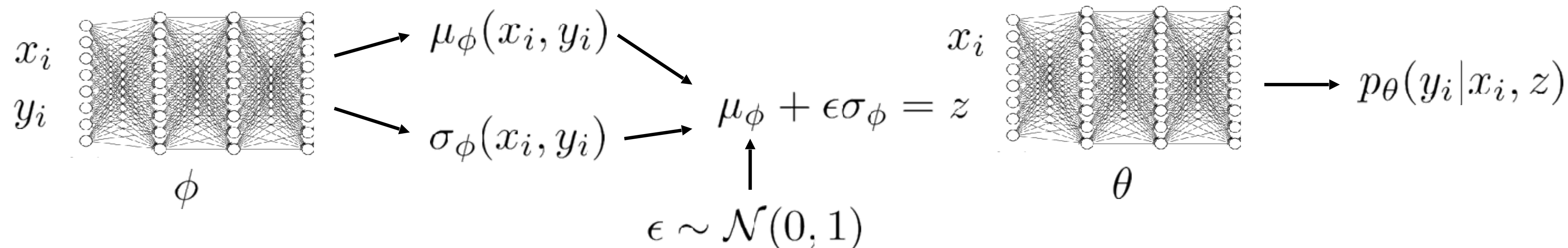
$$z \sim \mathcal{N}(0, \mathbf{I})$$

$$p(z)$$

can *optionally* depend on  $x$



class 109  
(brain coral)



# Plan for Today

1. Latent variable models
2. Variational inference
3. Amortized variational inference
4. Example latent variables models

} Part of (optional) Homework 4

## Goals

- Understand latent variable models in deep learning
- Understand how to use (amortized) variational inference



# Course Reminders

Homework 3 due Monday next week.

Tutorial session on Thursday 4:30 pm

**Next time:** Bayesian meta-learning

Be careful Azure usage — turning off machines when you are not using them!