MATHEMATICS-I

KAS-103T

Lecture No-1

Modul-I

Matrices

(By- Dr. Anuj Kumar)

Basices of Matrix Algebra and types of matrices

Introduction→ A matrix is a rectangular table of elements which may be numbers or any abstract quantity that can be added and multiplied.

Matrices are generally used in salving simultaneous equations, linear transformation, linear differential equation electrical circuit and robotics etc.

Matrix→ A set of mn elements (real or complex) arranged in a rectangular array of m rows and n columns is called matrix of order m by n, written as m×n.

A m×n matrix is usually written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{mn} \end{bmatrix}.$$

It is also denoted by

$A = [a_{ij}]_{m\times n}$ or

$A = [a_{ij}]$ where

$i = 1, 2, \ldots m$

$j = 1, 2, \ldots n$

and a_{ij} is the element in the ith row and jth column.

Types of Matrix→

① **Real Matrix→** A matrix is said to be real if all its elements are real numbers.

Ex→ $\begin{bmatrix} \sqrt{5} & -3 & 1 \\ 0 & -\sqrt{2} & 3 \end{bmatrix}$

② **Row Matrix→** A matrix has only one row and any number of columns is called a row matrix or row vector.

Ex→ i) $[1 \ \ 5 \ \ 6]_{1\times 3}$ (ii) $[4 \ \ 5]_{1\times 2}$.

③ **Column Matrix →** A matrix has only one column and any number of rows is called column matrix or column vector.

Ex → (i) $\begin{bmatrix} 2 \\ 5 \end{bmatrix}_{2\times 1}$ (ii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$

④ **Square matrix →** A matrix in which the number of rows equal to the number of columns is called a square matrix.

Ex → (i) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times 3}$.

⑤ **Null matrix or zero matrix →**

If all the elements of a matrix are zero is called null matrix.

⑥ **Diagonal matrix →** A square matrix in which all non-diagonal elements are zero is called a diagonal matrix.

Thus, $A = [a_{ij}]_{n\times n}$ is a diagonal matrix if $a_{ij} = 0$ for $i \neq j$.

Ex → $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Note →

$i<j$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$i>j$ $\quad$ $i=j$

⑦ **Scalar matrix →** A diagonal matrix in which all the diagonal elements are equal to a scalar (say K), is called a scalar matrix.

Thus $A = [a_{ij}]_{n\times n}$ is a scalar matrix if $a_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ K & \text{if } i = j. \end{cases}$

Ex → (i) $\begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{bmatrix}_{3\times 3}$ (ii) $\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}_{4\times 4}$.

⑧ **Unit matrix or identity matrix →**

A scalar matrix in which each diagonal element is unity (1) is called unit or identity matrix. Unit matrix of order n is denoted by I_n

Ex → $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

⑨ **Triangular matrix** → If all the elements below or above the principal diagonal are zero, is called triangular matrix.

(i) **Upper triangular matrix** → A square matrix in which all the elements below the principal diagonal are zero is called an upper triangular matrix.

Thus $A = [a_{ij}]_{n\times n}$ is an upper triangular matrix if $a_{ij} = 0$ for $i > j$.

Ex → (i) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

(ii) **Lower triangular matrix** → A square matrix in which all the elements above the principal diagonal are zero is called a lower triangular matrix.

Thus $A = [a_{ij}]_{n\times n}$ is a L.T.M if $a_{ij} = 0$ for $i < j$.

Ex → (i) $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$

⑩ **Singular and non-singular matrix** → A square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

Ex → (i) If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ then $|A| = 4 - 4 = 0 \Rightarrow$ A is singular.

(ii) If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ then $|A| = \cos^2\alpha + \sin^2\alpha = 1 \neq 0$

$\Rightarrow$ A is non-singular.

⑪ **Trace of matrix** → The sum of all the diagonal elements of a square matrix is called the trace of a matrix.

Ex → If $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 5 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ then $\boxed{\text{trace of } A = 2 + 5 + 3 = 10}$

⑫ **Inverse of matrix** → Let A is a non-singular square matrix then $A^{-1} = \frac{1}{|A|} \operatorname{adj} A$.

Note → Trick: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Ex: If $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$ then evaluate the value of the expression $A + 5I + 2A^{-1}$. (AKTU-2016)

Sol^n:

Given $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$ so $|A| = 0 + 2 = 2$.

then $A^{-1} = \frac{1}{|A|} \text{adj} A = \frac{1}{2} \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \Rightarrow 2A^{-1} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$

Now $A + 5I + 2A^{-1} = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2I.$

Hence $\boxed{A + 5I + 2A^{-1} = 2I}$

(13) **Nilpotent matrix** → A square matrix A is said to be nilpotent if $A^2 = 0$. Ex: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

It will be of index p if p is the least +ve integer such that

$$A^p = 0.$$

(14) **Idempotent matrix** → A square matrix A is called idempotent if $A^2 = A$. Ex: i) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(ii) Show that the matrix

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$ is idempotent.

(15) **Involutary matrix** → A square matrix A is said to be involutary if $A^2 = I$ where I is identity matrix.

Ex:

$$A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

MATHEMATICS-I
KAS-103T
Lecture No- 2

Module-I
Matrices
(By Dr. Anuj Kumar)

Types of matrices: Symmetric, Skew-symmetric and Orthogonal Matrices

(*) **Transpose of a matrix** → A matrix obtained by interchanging rows and columns of a matrix is called the transpose of a matrix. It is denoted by A^T or A'.

Ex→ If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Note→ $\boxed{\text{(i) } (A^T)^T = A \quad \text{(ii) } (A+B)^T = A^T + B^T \quad \text{(iii) } (AB)^T = B^T A^T.}$

(*) **Symmetric matrix**→ A square matrix $A = [a_{ij}]$ is called symmetric if $\boxed{a_{ij} = a_{ji} \ \forall\, i,j \text{ or } A^T = A}$

Ex→ $\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$.

Ex→ (i) Prove that $A = \begin{bmatrix} 1 & i & 0 \\ i & 2 & 4 \\ 0 & 4 & 3 \end{bmatrix}$ is symmetric.

Solⁿ→ Given $A = \begin{bmatrix} 1 & i & 0 \\ i & 2 & 4 \\ 0 & 4 & 3 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & i & 0 \\ i & 2 & 4 \\ 0 & 4 & 3 \end{bmatrix} = A$

$\Rightarrow A^T = A$. Hence A is symmetric matrix.

Note→ The diagonal elements of symmetric matrix are real.

(*) **Skew-symmetric matrix or anti-symmetric matrix**→

A square matrix $A = [a_{ij}]$ is called skew-symmetric if $\boxed{a_{ij} = -a_{ji} \ \forall\, i,j \text{ or } A^T = -A}$ Ex→ $\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$.

Note→ for diagonal elements, put $i=j$ then $a_{ii} = -a_{ii}$

$\Rightarrow 2a_{ii} = 0$

$\Rightarrow a_{ii} = 0 \ \forall\, i$

Therefore the diagonal elements of skew-symmetric matrix are zero

Ex: Prove that the matrix $A = \begin{bmatrix} 0 & -i & -4 \\ i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix}$ is skew-symmetric.

Solⁿ: Given

$$A = \begin{bmatrix} 0 & -i & -4 \\ i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix}$$

then $A^T = \begin{bmatrix} 0 & i & 4 \\ -i & 0 & -8 \\ -4 & 8 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -i & -4 \\ i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix} = -A$

$\Rightarrow A^T = -A$ then A is skew-symmetric matrix.

Note: Every square matrix can be uniquely expressed as the sum of symmetric matrix and a skew-symmetric matrix.

$$A = \frac{1}{2}A + \frac{1}{2}A$$

$$= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$= P + Q$$

Symmetric matrix   Skew-symmetric matrix.

Ex: Express the following matrix $A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$ as the sum of a ~~square~~ symmetric matrix and a skew-symmetric matrix.

Solⁿ: Given $A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$ then $A^T = \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

Now Symmetric matrix

$$P = \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 9/2 & 3 \\ 9/2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix}$$

Skew-symmetric matrix

$$Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 5/2 & -2 \\ -5/2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

Hence $A = P + Q$ where P is symmetric and Q is skew-symmetric.

Ex→ Express each of the following matrices as the sum of a a symmetric and a skew-symmetric matrix :

i) $\begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$ ii) $\begin{bmatrix} a & a & b \\ c & b & b \\ c & a & c \end{bmatrix}$.

⊛ **Orthogonal Matrix**→ A square matrix A is called orthogonal matrix if $\boxed{AA^T = A^TA = I \quad \text{or} \quad A^T = A^{-1}}$

Note→ (i) we have $AA^T = I$ then $|AA^T| = |I|$

$\Rightarrow |A| \cdot |A^T| = 1$

$\Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1 \neq 0 \Rightarrow$ A is non-singular.

(ii) $A^T = A^{-1}$

then $AA^T = AA^{-1}$

$\Rightarrow AA^T = I \quad [\because AA^{-1} = I]$.

Ex→ Prove that $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is orthogonal.

Solⁿ→ Given $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ then $A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

Now $AA^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} + \frac{1}{2} & -\frac{1}{2} + \frac{1}{2} \\ -\frac{1}{2} + \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\Rightarrow AA^T = I$. Hence A is orthogonal matrix and $\boxed{A^{-1} = A^T}$

Ex-1→ Prove that the following matrices are othogonal and find A^{-1}.

i) $\frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$ ii) $\frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$ iii) $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

Ex-2→ Find a, b, c if $A = \frac{1}{9}\begin{bmatrix} -8 & 4 & a \\ 4 & 4 & 7 \\ a & b & c \end{bmatrix}$ is orthogonal.

Sol^n: Given $A = \frac{1}{9}\begin{bmatrix} -8 & 4 & a \\ 1 & 4 & b \\ 4 & 7 & c \end{bmatrix}$ then $A^T = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ a & b & c \end{bmatrix}$.

Since A is orthogonal then $AA^T = I$

$$\Rightarrow \frac{1}{9}\cdot\frac{1}{9}\begin{bmatrix} -8 & 4 & a \\ 1 & 4 & b \\ 4 & 7 & c \end{bmatrix}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{81}\begin{bmatrix} 64+16+a^2 & -8+16+ab & -32+28+ac \\ -8+16+ab & 1+16+b^2 & 4+28+bc \\ -32+28+ac & 4+28+bc & 16+49+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{81}\begin{bmatrix} 80+a^2 & 8+ab & -4+ac \\ 8+ab & 17+b^2 & 32+bc \\ -4+ac & 32+bc & 65+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On compairing both side,

* $\frac{1}{81}(80+a^2) = 1 \Rightarrow 80+a^2 = 81 \Rightarrow a^2 = 1 \Rightarrow \boxed{a=1}$
* $8+ab = 0 \Rightarrow 8+b=0 \Rightarrow \boxed{b=-8}$
* $-4+ac = 0 \Rightarrow -4+c=0 \Rightarrow \boxed{c=4}$

MATHEMATICS - I
KAS-103T
Lecture No. - 3

Module - I
Matrices
(By Dr. Anuj Kumar)

Complex and Unitary Matrices

Complex matrix → A matrix is called complex matrix if at least one element of matrix is a complex number $a+ib$ where a, b are real.

Conjugate of matrix → The matrix $\overline{A}$ formed by replacing the elements of a matrix A by their respective conjugate number is called the conjugate of A.

Thus, if $A = [a_{ij}]_{m \times n}$ then $\overline{A} = [\overline{a}_{ij}]_{m \times n}$.

Ex → $A = \begin{bmatrix} 2+3i & -7i \\ 5 & -1-i \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 2-3i & 7i \\ 5 & -1+i \end{bmatrix}$

Note → (i) $\overline{\overline{A}} = A$ (ii) $(\overline{A+B}) = \overline{A} + \overline{B}$ (iii) $(\overline{kA}) = \overline{k}\,\overline{A}$ (iv) $(\overline{AB}) = \overline{A}\,\overline{B}$

Transpose of conjugate matrix → Let A is a matrix and $\overline{A}$ its conjugate. Then transpose of $\overline{A}$ is $A^* \text{ or } A^\theta = (\overline{A})^T$

Note → (i) $(A^*)^* = A$ (ii) $(A+B)^* = A^* + B^*$ (iii) $(kA)^* = \overline{k}A^*$ (iv) $(AB)^* = B^* A^*$

Hermitian Matrix → A square matrix $A = [a_{ij}]_{m \times n}$ is said to be Hermitian if $A^* = A$ or $a_{ij} = \overline{a_{ji}} \;\forall\, i, j$

Note → for diagonal elements put $i = j$, then

$a_{ii} = \overline{a_{ii}} \Rightarrow a+ib = a-ib$ [Let $a_{ii} = a+ib$]

$\Rightarrow 2ib = 0 \Rightarrow b = 0$ [$\because 2i \neq 0$]

$\Rightarrow a_{ii} = a$ (real number)

$\therefore$ Diagonal elements of Hermitian matrix are real.

Ex → Prove that $A = \begin{bmatrix} 5 & 2+i & -3i \\ 2-i & -3 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$ is a Hermitian matrix.

Sol Given $A = \begin{bmatrix} 5 & 2+i & -3i \\ 2-i & -3 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 5 & 2-i & 3i \\ 2+i & -3 & 1+i \\ -3i & 1-i & 0 \end{bmatrix}$

and $A^* = (\overline{A})^T = \begin{bmatrix} 5 & 2+i & -3i \\ 2-i & -3 & 1-i \\ 3i & 1+i & 0 \end{bmatrix} = A.$

$\Rightarrow A^* = A \Rightarrow A$ is Hermitian Matrix.

Skew-Hermitian matrix → A square matrix $A = [a_{ij}]$ is called skew-Hermitian if $\boxed{A^* = -A \text{ or } a_{ij} = -\overline{a_{ji}} \ \forall i,j}$

Note → for diagonal elements, put $i=j$ then

$$a_{ii} = -\overline{a_{ii}} \Rightarrow a_{ii} + \overline{a_{ii}} = 0$$

$$\Rightarrow (x+iy) + (x-iy) = 0 \qquad [\text{Take } a_{ii} = x+iy]$$

$$\Rightarrow 2x = 0 \Rightarrow x = 0.$$

$$\therefore \boxed{a_{ii} = iy} \Rightarrow a_{ii} \text{ is imaginary or } 0.$$

Hence the diagonal elements of a skew-Hermitian matrix is either 0 or purely imaginary.

Ex → Show that $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is skew-Hermitian matrix.

Sol Given $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$

& $A^* = (\overline{A})^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = -\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} = -A.$

$\Rightarrow A^* = -A \Rightarrow A$ is skew-Hermitian matrix.

Ex → Check the matrices for Hermitian and skew-Hermitian.

(i) $\begin{bmatrix} 3i & 1+i & 7 \\ -1+i & 0 & -2-i \\ -7 & 2-i & -i \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$

Note → (i) If A is a Hermitian matrix, then iA is Skew-Hermitian.
(ii) If A is a skew-Hermitian matrix, then iA is Hermitian.

Ex: Show that the matrix A is Hermitian and iA is skew-Hermitian where A is

(i) $\begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix}$ (AKTU-2012) (ii) $\begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$ (AKTU-2015)

Sol^n: (i) Given $A = \begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix}$ then $\bar{A} = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$

& $A^* = (\bar{A})^T = \begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix} = A \Rightarrow A^* = A$

Hence A is Hermitian.

Let $B = iA = i\begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix} = \begin{bmatrix} 2i & 4+3i \\ -4+3i & 2i \end{bmatrix}$

then $\bar{B} = \begin{bmatrix} -2i & 4-3i \\ -4-3i & -2i \end{bmatrix}$ & $B^* = (\bar{B})^T = \begin{bmatrix} -2i & -4-3i \\ 4-3i & -2i \end{bmatrix}$

$$= -\begin{bmatrix} 2i & 4+3i \\ -4+3i & 2i \end{bmatrix}$$

$$= -B.$$

$\Rightarrow B^* = -B$

Hence $B = iA$ is skew-Hermitian matrix.

Note: Every square matrix can be uniquely expressed as the sum of a Hermitian matrix and a skew-Hermitian matrix.

Let $A = \frac{1}{2}(A + A)$

$= \frac{1}{2}[(A + A^*) + (A - A^*)]$

$= \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)$

$= P + Q$

P: Hermitian Matrix

Q: Skew-Hermitian Matrix.

Ex: Express the matrix $A = \begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}$ as the sum of a Hermitian and a skew-Hermitian matrix. (AKTU-2016)

Sol^n → Given $A = \begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}$ then $\bar{A} = \begin{bmatrix} 2-3i & 0 & -4i \\ 5 & -i & 8 \\ 1+i & -3-i & 6 \end{bmatrix}$

$\therefore A^* = (\bar{A})^T = \begin{bmatrix} 2-3i & 5 & 1+i \\ 0 & -i & -3-i \\ -4i & 8 & 6 \end{bmatrix}$

Now Hermitian matrix

$$P = \frac{1}{2}(A + A^*) = \frac{1}{2}\begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2-3i & 5 & 1+i \\ 0 & -i & -3-i \\ -4i & 8 & 6 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 5 & 1+5i \\ 5 & 0 & 5-i \\ 1-5i & 5+i & 12 \end{bmatrix}$$

& Skew-Hermitian matrix

$$Q = \frac{1}{2}(A - A^*) = \frac{1}{2}\begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2-3i & 5 & 1+i \\ 0 & -i & -3-i \\ -4i & 8 & 6 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 6i & -5 & -1+3i \\ 5 & 2i & 11+i \\ 1+3i & -11+i & 0 \end{bmatrix}$$

Hence $A = P + Q$ where P is Hermitian and Q is skew-Hermitian matrix.

Ex+2 → Express the matrix $A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$ as the sum of Hermitian matrix and skew-Hermitian matrix. (AKTU-2010)

③ **Unitary matrix →** A square matrix A is called unitary if $\boxed{AA^* = A^*A = I. \quad \text{or} \quad A^{-1} = A^*}$.

Ex+1 → Prove that the matrix $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ is unitary, and hence find A^{-1}.

(AKTU-2011).

Sol^n: Given

$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

then $\bar{A} = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix}$ & $A^* = (\bar{A})^T = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$.

Now $A \cdot A^* = \frac{1}{4}\begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix} \cdot \begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix} 1-i^2-i^2+1 & 1-i^2+i^2-1 \\ 1-i^2-1+i^2 & 1-i^2+1-i^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $AA^* = I$ Hence A is unitary matrix.

For unitary matrix, $A^{-1} = A^* = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$.

Ex: Prove that following matrices are unitary

(i) $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ (AKTU-2013) (ii) $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$.

(iii) Show that the matrix $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$ is a unitary matrix, where ω is cube root of unity. (AKTU-2010, 2017).

Sol^n: (iii) Given $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$

then $\bar{A} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$.

& $(A^*) = (\bar{A})^T = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$.

Now $AA^* = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$.

Note:

$x = 1^{1/3}$

$\Rightarrow x^3 = 1$

$\Rightarrow x^3 - 1 = 0$

$\Rightarrow (x-1)(x^2+x+1) = 0$

$\therefore x = 1,\ x = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

Let

$1,\ \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

then $\omega^3 = 1$

$1 + \omega + \omega^2 = 0$

$\bar{\omega} = \omega^2$

$\bar{\omega}^2 = \omega$

$$\Rightarrow AA^* = \frac{1}{3}\begin{bmatrix} 3 & 1+\omega^2+\omega & 1+\omega+\omega^2 \\ 1+\omega+\omega^2 & 1+\omega^3+\omega^3 & 1+\omega^2+\omega^4 \\ 1+\omega^2+\omega & 1+\omega^4+\omega^2 & 1+\omega^3+\omega^3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \quad [\text{as } \omega^3 = 1 \ \& \ 1+\omega+\omega^2 = 0]$$

$\Rightarrow AA^* = I$. Hence A is unitary.

Ex: Show that the matrix $\begin{bmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{bmatrix}$ is unitary if $\alpha^2+\beta^2+\gamma^2+\delta^2=1$. (AKTU-2018).

Sol^n: Let $A = \begin{bmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{bmatrix}$ then $\bar{A} = \begin{bmatrix} \alpha-i\gamma & -\beta-i\delta \\ \beta-i\delta & \alpha+i\gamma \end{bmatrix}$

& $A^* = (\bar{A})^T = \begin{bmatrix} \alpha-i\gamma & \beta-i\delta \\ -\beta-i\delta & \alpha+i\gamma \end{bmatrix}$.

For a square matrix A to be unitary,

$$AA^* = I = A^*A \longrightarrow (1)$$

Now $AA^* = \begin{bmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{bmatrix} \cdot \begin{bmatrix} \alpha-i\gamma & \beta-i\delta \\ -\beta-i\delta & \alpha+i\gamma \end{bmatrix}$

$$= \begin{bmatrix} \alpha^2-i^2\gamma^2+\beta^2-i^2\delta^2 & \alpha\beta-i\alpha\delta+i\beta\gamma-i^2\gamma\delta-\alpha\beta-i\beta\gamma+i\alpha\delta+i^2\delta\gamma \\ \alpha\beta-i\beta\gamma+i\alpha\delta-i^2\delta\gamma-\beta\alpha-i\alpha\delta+i\beta\gamma+i^2\gamma\delta & \beta^2-i^2\delta^2+\alpha^2-i^2\gamma^2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2+\beta^2+\delta^2+\gamma^2 & 0 \\ 0 & \alpha^2+\beta^2+\delta^2+\gamma^2 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Eq^n (1) satisfied only when

$$\alpha^2+\beta^2+\delta^2+\gamma^2=1.$$

Hence A is unitary if $\alpha^2+\beta^2+\gamma^2+\delta^2=1$.

MATHEMATICS-I
KAS-103T
Lecture No-4

Module-I
Matrices
(By Dr. Anuj Kumar)

Inverse using elementary transformation

Elementary transformation or E-operation →

The operations on a matrix such as

i) adding or subtracting two rows or columns.

ii) multiplying or divide any row or column by a non-zero scalar quantity.

iii) Interchanging any two rows or columns.

These operations on a matrix are called elementary transformation.

Inverse using elementary transformation →

The elementary row transformation which reduce square matrix A to the unit matrix, when applied to the unit matrix, gives the inverse matrix A^{-1}.

Working Rule →

① Write the given matrix A as $A = IA$ $\boxed{\text{Note: } AA^{-1} = I}$

② Reduce the matrix A on L.H.S to identity matrix using only row or only column transformation.

③ We get $I = A^{-1}A$, then A^{-1} is called inverse of matrix A and vice-versa.

Trick to reduce matrix A into identity matrix I →

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Non-zero element or 1

* Use only a_{11} for making zero of a_{21} & a_{31}.
* Use only a_{22} for making zero of a_{12} & a_{32}.
* Use only a_{33} for making zero of a_{13} & a_{23}.

Ex-1: Using elementary Row-transformation, find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ (AKTU-2018)

Sol^n: Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

we have $A = I_3 A$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

$R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} . A$$

$R_3 \to \frac{R_3}{2}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} . A$$

$R_2 \to R_2 - 2R_3$, $R_1 \to R_1 - 3R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{15}{2} & \frac{11}{2} & -\frac{3}{2} \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

$\Rightarrow I = A^{-1} A$

Hence

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

Ex-21 Using elementary column transformation. find matrix inverse. where $A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$ (AKTU-2010)

Sol^n: Let

$$A = I_3 A$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . A$$

$C_2 \to C_2 - 3C_1$, $C_3 \to C_3 - 3C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 4 \\ 3 & -1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$C_3 \to C_3 + 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & -1 & -7 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -9 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} A$$

$C_3 \to -\frac{1}{7} C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9/7 \\ 0 & 1 & -2/7 \\ 0 & 0 & -1/7 \end{bmatrix} . A$$

$C_1 \to C_1 - 3C_3$, $C_2 \to C_2 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{20}{7} & -\frac{18}{7} & \frac{9}{7} \\ 6/7 & 5/7 & -2/7 \\ 3/7 & -1/7 & -\frac{1}{7} \end{bmatrix} . A$$

$C_2 \to -\frac{1}{2} C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{20}{7} & \frac{18}{14} & 9/7 \\ 6/7 & 5/7 & -2/7 \\ 3/7 & 1/14 & -1/7 \end{bmatrix} . A$$

$C_1 \to C_1 - 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -38/7 & \frac{18}{14} & 9/7 \\ 11/7 & -5/14 & -2/7 \\ 2/7 & \frac{1}{14} & -\frac{1}{7} \end{bmatrix}$$

$\Rightarrow \boxed{I = A^{-1} A}$

Hence $A^{-1} = \begin{bmatrix} -\frac{38}{7} & \frac{18}{14} & \frac{9}{7} \\ \frac{11}{7} & -\frac{5}{14} & -\frac{2}{7} \\ 2/7 & \frac{1}{14} & -\frac{1}{7} \end{bmatrix}$.

Ex 1: Find inverse using elementary row transformation.

i) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ (AKTU-2012) ii) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ (AKTU-2014)

iii) $\begin{bmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ (AKTU-2013) iv) $\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$

(AKTU-2003)

Ex 2: Find inverse using elementary column transformation.

i) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ (AKTU-2008)

ii) $\begin{bmatrix} \frac{1}{3} & \frac{1}{5} & \frac{1}{7} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{11} \\ \frac{1}{7} & \frac{1}{11} & \frac{1}{13} \end{bmatrix}$ (AKTU-2009)

Ex-3: Transform $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$ into a unit matrix

by using elementary transformation. (AKTU-2011).

Note: If row transformation or column transformation is not given in question, then you can solve by row transformation only or only column transformation.

Ex-4: Reduce the matrix $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ to the upper

triangular form by using e-transformation.

MATHEMATICS-I
KAS-103T
Lecture No- 5

Module-I
Matrices
(By- Dr. Anuj Kumar)

Rank of matrix using elementary transformation by Echelon Form

Any matrix A is said to be in Echelon form if

① The first non-zero element from the left of a non-zero row is 1 and called leading entry or pivot element.

② If a column contains a leading entry then all entries below that leading entry are zero.

③ Every zero row of the matrix occurs below a non-zero row.

④ For each non-zero row, the leading entry in the lower row occurs to the right of the leading entry in the above row.

Tricks for Echelon form →

For Echelon form of matrix

① Columnwise zero elements may be decreasing by at least one zero.

② Row-wise zero elements may be increasing by at least one zero.

③ Extra zero element in rows and columns does not effect Echelon form.

Note → The Echelon form of a square matrix is upper triangular matrix.
Imp

Rank of matrix by Echelon matrix or Rank of matrix by Echlen form →

The rank of matrix in Echelon form is equal to the number of non-zero rows of the matrix.
It is denoted by Rank(A) or $\rho(A)$.

Note → (i) Rank (Null matrix) $= 0$ | (ii) $\rho(A) \leq$ order A.

Ex-1 Find the rank of the matrix.

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$ (AKTU-2018-19)

Solⁿ Let

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \text{ by } R_1 \to \frac{R_1}{2}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ by } \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$\therefore \rho(A) =$ no. of non-zero row $= 1$

Ex-2 Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ by Echelon-form. (Ans-1)

Ex-3 Find the rank of the following matrices by reducing in Echelon-form.

(i) $\begin{bmatrix} 3 & 2 & -1 \\ 4 & 2 & 6 \\ 7 & 4 & 5 \end{bmatrix}$ (AKTU-2016)

(ii) $\begin{bmatrix} -2 & -1 & 3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ (AKTU-2018)

Solⁿ (i) Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 2 & 6 \\ 7 & 4 & 5 \end{bmatrix}$

$$\sim \begin{bmatrix} 3 & 2 & -1 \\ 1 & 0 & 7 \\ 7 & 4 & 5 \end{bmatrix} \text{ by } R_2 \to R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 7 \\ 3 & 2 & -1 \\ 7 & 4 & 5 \end{bmatrix} \text{ by } R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 2 & -22 \\ 0 & 4 & -44 \end{bmatrix} \text{ by } \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 2 & -22 \\ 0 & 0 & 0 \end{bmatrix} \text{ by } R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -11 \\ 0 & 0 & 0 \end{bmatrix} \text{ by } R_2 \to \frac{R_2}{2}$$

$$\boxed{\therefore \rho(A) = 2}$$

(ii) Let $A = \begin{bmatrix} -2 & -1 & 3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & -3 & -1 \\ -2 & -1 & 3 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \quad \text{by } R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -4 & -2 \\ 0 & -1 & 5 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \quad \text{by } R_2 \to R_2 - R_1,\; R_3 \to R_3 + 2R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & -1 & 5 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \quad \text{by } R_2 \to \frac{R_2}{2}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \quad \text{by } R_3 \to R_3 + R_2,\; R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \quad \text{by } R_3 \to \frac{R_3}{3}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{by } R_4 \to R_4 - 3R_3$$

$$\boxed{\therefore\ \rho(A) = \text{no. of non-zero rows} = 3}$$

Ex:- find the rank of matrices by using elementary transformation

(i) $\begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$ (AKTU-2010)

(ii) $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$ (AKTU-2011) (Ans-3)

(iii) $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \end{bmatrix}$ (AKTU-2014) (Ans-3)

Solⁿ (i) Let

$$A = \begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 13 & -2 & -8 \end{bmatrix} \text{ by } R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -28 & -14 \end{bmatrix} \text{ by } R_3 \to R_3 - 13R_2$$

$$\boxed{\therefore \rho(A) = 3}$$

Ex:- Find rank

(i) $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -3 \end{bmatrix}$ (Ans-4)

(ii) $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ (AKTU-2012) (Ans-4)

(iii) $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ (Ans-2)

Rank of matrix is
Number of linearly independent rows or columns

Rank and Nullity theorem →

If A is a matrix then

Rank of A + Nullity of A = Number of columns.

Ex: Find the rank and nullity of the matrix & verify rank & nullity theorem.

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

Sol: Given

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} \text{ by } \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_2 \to \frac{R_2}{(-1)}$$

$$\boxed{\therefore \text{ Rank of } A = \rho(A) = 2}$$

Since Rank of A + Nullity of A = No. of columns.

$\Rightarrow$ Nullity of A = No. of columns − Rank of A

$= 4 - 2$

$= 2.$

$\Rightarrow$ $\boxed{\text{Nullity of } A = 2}$

Hence also, Rank A + Nullity of A = $2 + 2 = 4$
= No. of columns.

Ex: Find the rank and nullity of the following matrices

① $\begin{bmatrix} 8 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 2 \\ 0 & 4 & 2 \end{bmatrix}$ ② $\begin{bmatrix} 1 & -2 \\ 0 & 0 \\ -3 & 6 \end{bmatrix}$

③ $\begin{bmatrix} 1 & -2 & 3 & -4 \\ 2 & -3 & 4 & -1 \\ 3 & -4 & 1 & -2 \\ 6 & -1 & 2 & -3 \end{bmatrix}$

---

* Special Examples *

Ex: Find the value of p for which the matrix

$$A = \begin{bmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{bmatrix}$$ is of rank 1. (AKTU-2012)

Sol: Given

$$A = \begin{bmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{p}{3} & \frac{p}{3} \\ p & 3 & p \\ p & p & 3 \end{bmatrix} \text{ by } R_1 \to \frac{R_1}{3}$$

$$\sim \begin{bmatrix} 1 & \frac{p}{3} & \frac{p}{3} \\ 0 & 3-\frac{p^2}{3} & p-\frac{p^2}{3} \\ 0 & p-\frac{p^2}{3} & 3-\frac{p^2}{3} \end{bmatrix} \text{ by } \begin{matrix} R_2 \to R_2 - pR_1 \\ R_3 \to R_3 - pR_1 \end{matrix}$$

Since Rank of $A = 1$

Then Echelon form will contain only one non-zero row

So IInd & IIIrd row must be zero.

$\Rightarrow 3 - \frac{p^2}{3} = 0$ & $p - \frac{p^2}{3} = 0$

$\Rightarrow 9 - p^2 = 0$ & $3p - p^2 = 0$

$\Rightarrow p^2 = 9$ & $p(p-3) = 0$

$\Rightarrow p = \pm 3$ & $p = 0, 3$.

$\boxed{\text{Hence } p = 3 \text{ if } \rho(A) = 1}$

Ex 2) Find all values of μ for which rank of the matrix

$$A = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$$ is equal to 3.

Ans $\boxed{\mu = 1, 2, 3}$

Rank of the matrix by determinant method →

The rank of a matrix is an order of highest non-zero determinant obtained from the matrix.

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

then $D_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$D_2 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$, $D_3 = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$, $D_4 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$

$D_5 = \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$, $D_6 = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix}$, $D_7 = \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}$

$D_8 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$, $D_9 = \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$, $D_{10} = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

$D_{11} = |a_1|$, $D_{12} = |a_2|$, ... , $D_{19} = |c_3|$.

Ex) $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & -1 & 1 \end{bmatrix}$

then $D_1 = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & -1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}$

$= 1(1-2) - 2(-1+4) - 1(1-2)$

$= -1 - 6 + 1 = -6 \neq 0$

$\Rightarrow D_1 \neq 0$.

Hence $\rho(A) = $ order of $D_1 = 3$.

Ex-2: find the rank of the matrix

Sol:
$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 5 \\ 1 & 4 & 4 \end{bmatrix}.$$

Now $D_1 = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 2 & 5 \\ 1 & 4 & 4 \end{vmatrix}$

$= 1(8-20) - 2(0-5) - 1(0-2)$

$= -12 + 10 + 2$

$\Rightarrow D_1 = -12 + 12 = 0$

$\Rightarrow D_1 = 0.$

$D_2 = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2 \neq 0$

$\Rightarrow D_2 \neq 0$

$\therefore \boxed{\rho(A) = \text{order of } D_2 = 2}$

Ex-3: for what value of b the rank of the matrix is 2.

Sol: Let $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix}$

Since $\rho(A) = 2$ (given)

then $D_1 = \begin{vmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{vmatrix} = 0$

$\Rightarrow 1 \begin{vmatrix} 3 & 2 \\ 13 & 10 \end{vmatrix} - 5 \begin{vmatrix} 0 & 2 \\ b & 10 \end{vmatrix} + 4 \begin{vmatrix} 0 & 3 \\ b & 13 \end{vmatrix} = 0$

$\Rightarrow 1(30-26) - 5(0-2b) + 4(0-3b) = 0$

$\Rightarrow 4 + 10b - 12b = 0$

$\Rightarrow 4 - 2b = 0$

$\Rightarrow \boxed{b = 2}.$

Ex: Find the value of 'K' such that the rank of matrix is 3

where $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & K \end{bmatrix}$ Ans $\boxed{K = 1}$

MATHEMATICS-I.
KAS-103T
Lecture No- 6

Module-I
Matrices
(By Dr. Anuj Kumar)

Rank of matrix by Normal form →

Rank of matrix by Canonical form or Normal form →

By using both row and column elementary transformation matrix A can be reduced one of the following forms known as canonical or normal form.

(i) I_r (ii) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_r & 0 \end{bmatrix}$.

The number r is known as rank of matrix A

i.e $\boxed{\rho(A) = r}$

Here for (i) & (ii) for square matrix
and (iii) & (iv) for non-square matrix.

Note:- (i) Use row operation for the elements lies below the ~~first~~ diagonal.
(ii) Use column operation for the elements lies above the diagonal.

Ex:1 Find the rank of the matrix

$$A = \begin{bmatrix} -2 & -1 & 3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$ by using normal form.

(AKTU-2018)

Solⁿ: Given

$$A = \begin{bmatrix} -2 & -1 & 3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & -3 & -1 \\ -2 & -1 & 3 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \text{ by } R_3 \leftrightarrow R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -4 & -2 \\ 0 & -1 & 5 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \text{ by } R_2 \to R_2 - R_1,\; R_3 \to R_3 + 2R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 5 & 1 \\ 0 & 2 & -4 & -2 \end{bmatrix} \text{ by } R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix} \text{ by } R_3 \to R_3 + R_2,\; R_4 \to R_4 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix} \quad R_3 \to \frac{R_3}{6}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_4 \to R_4 + 6R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } C_3 \to C_3 - C_1,\; C_4 \to C_4 - C_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \text{ by } C_3 \to C_3 - C_2,\; C_4 \to C_4 + C_2$$

$$\sim \left[\begin{array}{c|c} I_3 & 0 \\ \hline 0 & 0 \end{array}\right]$$

Here $\boxed{\rho(A) = 3}$

Ex 24 find the rank of a matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$ by normal form.

(AKTU-2015)

Sol^n Given

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix} \text{ by } R_2 \to R_2 + 2R_1, \; R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix} \text{ by } R_2 \to \frac{R_2}{8}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 9/4 & -8 \end{bmatrix} \text{ by } R_3 \to R_3 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 1 & -\frac{32}{9} \end{bmatrix} \text{ by } R_3 \to \frac{4}{9} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 1 & -\frac{32}{9} \end{bmatrix} \text{ by } C_2 \to C_2 - 2C_1, \; C_3 \to C_3 - C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{32}{9} \end{bmatrix} \text{ by } C_3 \to C_3 - \frac{5}{8} C_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \text{ by } C_4 \to C_4 + \frac{32}{9} C_3$$

$$\sim [I_3 \,\vdots\, 0]$$

$$\boxed{\text{Hence } \rho(A) = 3}$$

(*) Ex: Find the rank by normal form

i) $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$ (Ans-3) (AKTU-2012, 2010)

ii) $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \end{bmatrix}$ (AKTU-2014)

iii) $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$ (AKTU-2009).

Determination of non-singular matrices P & Q such that PAQ is in normal form→

Working Rule→ Let A be any matrix of order $m \times n$ i.e $A_{m\times n}$.

① Write $A_{m\times n} = I_{m\times n} \, A \, I_{n\times n}$.

② Reduce the matrix A on L.H.S in normal form by using Row & column transformation.

③ i) If row-transformation is applied on L.H.S then it must be applied on pre-factor of A on R.H.S.

ii) If column transformation is applied on L.H.S then it must be applied on post-factor of A on R.H.S.

Ex: Find then non-singular matrices P & Q such that PAQ is in normal form & hence find rank of A.

where $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$ (AKTU-2015)

Soln: Write

$$A = I_{3\times3} \, A \, I_{4\times4} \quad (\text{as } A_{3\times4})$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

by $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & -6 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ by } R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

by $C_2 \to C_2 - 2C_1$
$C_3 \to C_3 - 3C_1$
$C_4 \to C_4 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & +7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & -3 & 2 \\ 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ by } C_2 \to \frac{C_2}{-6}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

by $C_3 \to C_3 + 5C_2$
$C_4 \to C_4 - 7C_2$

$$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

where P & Q are non-singular.

& $\rho(A) = 2$.

Imp. Note → P & Q depends on operations
so P & Q not fixed and may be in other form

Ex1 Find the non-singular matrices P & Q such that PAQ are normal form.

(i) $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ (AKTU-2002)

(ii) $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

MATHEMATICS - I
KAS-103T
Lecture No - 7

(By Dr. Anuj Kumar)

System of linear equation →

Consider the system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

This system can be reduced in matrix form as $AX=B$.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

where A is called coefficient matrix, B is called constant matrix & X is variable matrix.

Non-Homogeneous system of linear equation →

The system of linear-equations $AX=B$ is called non-homogeneous if $B \neq 0$.

Homogeneous system of linear equation →

The system of linear-equations $AX=B$ is called homogeneous if $B=0$. i.e $\boxed{AX=0}$.

Solution of Homogeneous system of linear equation →

Working Rule → ① Convert the given system of linear equation in matrix form as $AX=0$

② find rank of A i.e $\rho(A)$.

③ (i) If $\rho(A)$ = no. of unknowns then system has zero solⁿ or unique solⁿ or trivial solⁿ. $(x_1=0, x_2=0, \cdots, x_n=0)$.

(ii) If $\rho(A)<$ no. of unknowns, then system has infinite number of non-zero solution. (non-trivial)

v.imp → **Note →** The Homogeneous system has non-trivial solⁿ if $|A|=0$

Ex-1: Test the consistency and solve the following system of linear equations $2x-y+3z=0$, $-x+2y+z=0$, $3x+y-4z=0$.

Sol^n: Given

$$\left.\begin{aligned} 2x-y+3z&=0 \\ -x+2y+z&=0 \\ 3x+y-4z&=0 \end{aligned}\right\} \rightarrow (1)$$

Convert the given system of equations into matrix form as

$AX=0$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (2)$$

Imp:

Note: Use only row operations in linear symultaneous equations.

Now

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 4 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix} \text{ by } R_1 \to R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & 5 \\ 0 & -2 & -16 \end{bmatrix} \text{ by } R_2 \to R_2 + R_1,\ R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -11 \\ 0 & -2 & -16 \end{bmatrix} \text{ by } R_2 \to R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -11 \\ 0 & 0 & -38 \end{bmatrix} \text{ by } R_3 \to R_2 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -11 \\ 0 & 0 & 1 \end{bmatrix} \text{ by } R_3 \to \frac{R_3}{-38}.$$

$\therefore$ $\rho(A) = 3 =$ no. of unknowns. Then the system has trivial sol^n or unique sol^n.

$$\boxed{x=0,\ y=0,\ z=0}$$

Ex-2.1 Salve the following system of equations
$4x+3y-z=0$, $3x+4y+z=0$, $x-y-2z=0$, $5x+y-4z=0$

Sol^n Given

$$\left.\begin{aligned} 4x+3y-z&=0\\ 3x+4y+z&=0\\ x-y-2z&=0\\ 5x+y-4z&=0 \end{aligned}\right\} \rightarrow (1)$$

Convert the given system of linear equations into matrix form as

$$AX=0$$

$$\Rightarrow \begin{bmatrix} 4 & 3 & -1\\ 3 & 4 & 1\\ 1 & -1 & -2\\ 5 & 1 & -4 \end{bmatrix}\begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} 0\\0\\0\end{bmatrix} \rightarrow (2)$$

Now
$$A = \begin{bmatrix} 4 & 3 & -1\\ 3 & 4 & 1\\ 1 & -1 & -2\\ 5 & 1 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2\\ 3 & 4 & 1\\ 4 & 3 & -1\\ 5 & 1 & -4 \end{bmatrix} \text{ by } R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -1 & -2\\ 0 & 7 & 7\\ 0 & 7 & 7\\ 0 & 6 & 6 \end{bmatrix} \text{ by } \begin{array}{l} R_2 \to R_2 - 3R_1\\ R_3 \to R_3 - 4R_1\\ R_4 \to R_4 - 5R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & -2\\ 0 & 1 & 1\\ 0 & 1 & 1\\ 0 & 1 & 1 \end{bmatrix} \text{ by } \begin{array}{l} R_2 \to \frac{R_1}{7}\\ R_3 \to \frac{R_3}{7}\\ R_4 \to \frac{R_4}{6} \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & -2\\ 0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix} \text{ by } \begin{array}{l} R_3 \to R_3 - R_2\\ R_4 \to R_4 - R_2 \end{array}$$

$\therefore$ $\rho(A) = 2 <$ no. of unknowns

Hence the system has infinite number of non-trivial sol^n.

Now system (2) reduces to

$$x - y - 2z = 0$$
$$y + z = 0$$

[Free variable = No. of unknowns − Rank]. $= 3 - 2 = 1$.

Let $z = K$ (say)

then $y = -K$

& $x = K$.

Hence

$$X = \begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} K\\ -K\\ K\end{bmatrix} = K\begin{bmatrix} 1\\ -1\\ 1\end{bmatrix}, K \in R.$$

Ex-3: Find solⁿ of the system of equation
$x+3y-2z=0$, $2x-y+4z=0$, $x-11y+14z=0$.

Ex-4: Solve
$$2x+y+2z=0$$
$$x+y+3z=0$$
$$4x+3y+8z=0$$
(AKTU-2011).

Ex-5: Solve the equations using matrix method :
$x_1+3x_2+2x_3=0$, $2x_1-x_2+3x_3=0$, $3x_1-5x_2+4x_3=0$
$x_1+17x_2+4x_3=0$.

(Ist type) (Special Examples)

Ex-1: Find the values of λ for which the equations
$$x+(\lambda+4)y+(4\lambda+2)z=0$$
$$x+2(\lambda+1)y+(3\lambda+4)z=0$$
$$2x+3\lambda y+(3\lambda+4)z=0$$
(AKTU-2015)

have a non-trivial solⁿ. Also find the solⁿ in each case.

Solⁿ: Convert the given system of equation into matrix form as

$AX=0$

$$\Rightarrow \begin{bmatrix} 1 & \lambda+4 & 4\lambda+2 \\ 1 & 2\lambda+2 & 3\lambda+4 \\ 2 & 3\lambda & 3\lambda+4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

Now $A = \begin{bmatrix} 1 & \lambda+4 & 4\lambda+2 \\ 1 & 2\lambda+2 & 3\lambda+4 \\ 2 & 3\lambda & 3\lambda+4 \end{bmatrix}$

$R_2 \to R_2-R_1$, $R_3 \to R_3-2R_1$.

$$\sim \begin{bmatrix} 1 & \lambda+4 & 4\lambda+2 \\ 0 & \lambda-2 & -\lambda+2 \\ 0 & \lambda-8 & -5\lambda \end{bmatrix}$$

$R_3 \to R_3-R_2$

$$\sim \begin{bmatrix} 1 & \lambda+4 & 4\lambda+2 \\ 0 & \lambda-2 & -\lambda+2 \\ 0 & -6 & -4\lambda-2 \end{bmatrix}$$

$R_3 \to (\lambda-2)R_3+6R_2$

$$\sim \begin{bmatrix} 1 & \lambda+4 & 4\lambda+2 \\ 0 & \lambda-2 & -\lambda+2 \\ 0 & 0 & -4\lambda^2+16 \end{bmatrix}$$

Since the system have a non-trivial solⁿ then $\rho(A) <$ no. of unknown.

$\Rightarrow$ we must have $-4\lambda^2+16=0$

$\Rightarrow \boxed{\lambda=\pm 2}$

For $\lambda=-2$ → Eqⁿ (1) gives

$$\begin{bmatrix} 1 & 2 & -6 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x+2y-6z=0$

$\quad -4y+4z=0$

[free variable = no. of unknowns − Rank $= 3-2 = 1$]

Let $z=K$ (say)

then $y=K$

& $x=6z-2y=6K-2K=4K$.

$$\Rightarrow X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 4K \\ K \\ K \end{bmatrix}=K\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \quad K\in R.$$

For $\lambda=2$ → Eqⁿ (1) gives

$$\begin{bmatrix} 1 & 6 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x+6y+10z=0$ [free variable $=3-1=2$]

Let $z=K_1, \; y=K_2$

then $x=-6K_2-10K_1$

Hence
$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -10K_1-6K_2 \\ K_2 \\ K_1 \end{bmatrix}, \quad K_1, K_2\in R.$$

Ex-2 → Find the values of λ for which the equations

$(11-\lambda)x-4y-7z=0$

$7x-(\lambda+2)y-5z=0$

$10x-4y-(6+\lambda)z=0$. passes a non-trivial solⁿ.

For these values of λ, find the solⁿ also.

II type Ex-1 → Find the values of K for which the system of equations

$(3K-8)x+3y+3z=0$

$3x+(3K-8)y+3z=0$

$3x+3y+(3K-8)z=0$

has a non-trivial solⁿ. [Ans $K=\frac{2}{3}, \frac{11}{3}, \frac{11}{3}$]

Ex-2:- Find the value of K sothat the equations

$x+y+3z=0$

$4x+3y+kz=0$

$2x+y+2z=0$ have a non-trivial solution.

Solⁿ:- Convert the given system of equation into matrix form as

$AX=O$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 3 \\ 4 & 3 & K \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the system have a non-trivial solⁿ then

$\rho(A) < 3$ i.e $|A|=0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 3 \\ 4 & 3 & K \\ 2 & 1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow 1 \begin{vmatrix} 3 & K \\ 1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 4 & K \\ 2 & 2 \end{vmatrix} + 3 \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (6-K) - (8-2K) + 3(4-6) = 0$$

$$\Rightarrow 6 - K - 8 + 2K - 6 = 0$$

$$\Rightarrow K - 8 = 0 \Rightarrow \boxed{K=8}.$$

Note:- If solⁿ is not asked in the question, then use the above method.

Non-Homogeneous system of linear equations →

The system of linear equations $AX=B$ is called non-homogeneous if $B \neq 0$.

Working Rule →

① Convert the given system of linear equations into matrix form as $AX=B$.

② find the augmented matrix $[A:B]$.

③ Find $\rho[A:B]$ and $\rho(A)$.

④ Case-I → If $\boxed{\rho[A:B] = \rho(A) = \text{no. of unknowns}}$.
Then the system is consistent and have unique solution.

Case-II → If $\boxed{\rho[A:B] = \rho(A) < \text{No. of unknowns}}$ Then the system is consistent and have infinite solution.

Case-III → If $\boxed{\rho[A:B] \neq \rho(A)}$. Then the system is inconsistent and have no solution.

Note:- The system will have a unique solution if coefficient matrix A is non-singular i.e $|A| \neq 0$.

Ex-1 → Test the consistency of the system of equations
$x+y+z=-3$, $3x+y-2z=-2$ & $2x+4y+7z=7$.

Soln → Given
$$\left.\begin{aligned} x+y+z&=-3 \\ 3x+y-2z&=-2 \\ 2x+4y+7z&=7 \end{aligned}\right\} \longrightarrow ①$$

Convert the given system of equation into matrix form as
$AX=B$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix} \longrightarrow ②$$

Now augmented matrix is

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{array}\right]$$

Note→ Use only row-operation

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{array}\right] \quad \text{by } R_2 \to R_2 - 3R_1, \; R_3 \to R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{array}\right] \quad \text{by } R_3 \to R_3 + R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & 1 & 5/2 & -7/2 \\ 0 & 0 & 0 & 20 \end{array}\right] \quad \text{by } R_2 \to \frac{R_2}{-2}$$

$\therefore \; \rho[A:B] \neq \rho(A)$ $\quad$ [As $\rho[A:B] = 3$ & $\rho(A) = 2$].

Hence the given system of equation is inconsistent & have no solution.

Ex-2→ Test the consistency of linear equations

$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0$$

(AKTU-2008, 2011).

Sol^n→

Convert the given system of equations into matrix form as

$$AX = B$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix} \longrightarrow (1)$$

Now augmented matrix is

$$[A:B] = \left[\begin{array}{ccc:c} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 4 & 12 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{array}\right] \quad \text{by } R_1 \to R_1 + R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 4 & 12 \\ 0 & 3 & 5 & 16 \\ 0 & -2 & -16 & -36 \end{array}\right] \text{ by } \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 4 & 12 \\ 0 & 1 & -11 & -20 \\ 0 & -2 & -16 & -36 \end{array}\right] \text{ by } R_2 \to R_2 + R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 4 & 12 \\ 0 & 1 & -11 & -20 \\ 0 & 0 & -38 & -76 \end{array}\right] \text{ by } R_3 \to R_3 + 2R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 4 & 12 \\ 0 & 1 & -11 & -20 \\ 0 & 0 & 1 & 2 \end{array}\right] \text{ by } R_3 \to \frac{R_3}{-38}$$

$\therefore\ \rho[A:B] = 3 = \rho(A) =$ no. of unknowns

Hence the system is consistent & have unique sol^n.

Now eq^n ① reduces to,

$$\begin{aligned} x + y + 4z &= 12 \\ y - 11z &= -20 \\ z &= 2 \end{aligned}$$

then $y = 2$

$x = 12 - y - 4z = 12 - 2 - 8 = 2.$

Hence $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$

Ex:- Test the consistency for the system.

or

Find the sol^n for the system of equations

$x + y + z = 6$, $x + 2y + 3z = 14$, $x + 4y + 7z = 30$. (AKTU-2012)

Sol^n: Given

$$\left.\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 14 \\ x + 4y + 7z &= 30 \end{aligned}\right\} \to ①$$

Convert the given system of equations into matrix form as $AX = B$.

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix} \longrightarrow ②$$

Now augmented matrix is

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 14 \\ 1 & 4 & 7 & : & 30 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 8 \\ 0 & 3 & 6 & : & 24 \end{bmatrix} \text{ by } R_2 \to R_2 - R_1, \; R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 8 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \text{ by } R_3 \to R_3 - 3R_2$$

$\because \rho[A:B] = 2 = \rho(A) <$ no. of unknowns (3).

Hence the system is consistent & have infinite solⁿ.

Now ② reduces to,

$$x + y + z = 6$$
$$y + 2z = 8$$

[free variable = No. of unknowns − rank A = 3 − 2 = 1]

$\therefore$ Take $z = K$.

$y = 8 - 2K$

& $x = 6 - y - z$

$= 6 - (8 - 2K) - K$

$= K - 2$.

Hence $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} K-2 \\ 8-2K \\ K \end{bmatrix}$, $K \in R$.

Ex-4: Show that the equations $2x + 6y + 11 = 0$, $6x + 20y - 6z + 3 = 0$ & $6y - 18z + 1 = 0$ are not consistent. (AKTU-2011).

Ex-5: Test the consistency and hence, solve the following set of equations:

$$10y + 3z = 0$$
$$3x + 3y + z = 1$$
$$2x - 3y - z = 5$$
$$x + 2y = 4$$

(AKTU- 2018).

Ex-6: Apply the matrix method to solve the system of eqⁿ

$x + 2y - z = 3$, $3x - y + 2z = 1$, $2x - 2y + 3z = 2$, $x - y + z = -1$

Ans - $x = -1$, $y = 4$, $z = 4$. (AKTU-2014)

Ex-7: Apply rank test to examine if the following system of equations is consistent, solve them.

$$2x+4y-z=9\ ,\ 3x-y+5z=5\ ,\ 8x+2y+9z=19.$$

Ans: $x=-\frac{19}{14}k+\frac{29}{14}$

$y=\frac{13}{14}k+\frac{17}{14}\ ,\ z=k.$

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* Special Examples *

Ex-1: Investigate, for what values of λ and μ do the system of equations

$$x+y+z=6\ ,\quad x+2y+3z=10\ ,\quad x+2y+\lambda z=\mu$$

have (i) no soln (ii) unique soln (iii) infinite solutions? (AKTU-2016, 2018)

Soln: Given

$$\left.\begin{aligned} x+y+z&=6\\ x+2y+3z&=10\\ x+2y+\lambda z&=\mu\end{aligned}\right\}\rightarrow (1)$$

Convert the given system of equations into matrix form as

$$AX=B$$

$$\Rightarrow \begin{bmatrix}1&1&1\\1&2&3\\1&2&\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\\mu\end{bmatrix}\rightarrow (2)$$

Now Augmented matrix

$$[A:B]=\left[\begin{array}{ccc:c}1&1&1&6\\1&2&3&10\\1&2&\lambda&\mu\end{array}\right]$$

$$\sim\left[\begin{array}{ccc:c}1&1&1&6\\0&1&2&4\\0&1&\lambda-1&\mu-6\end{array}\right] \text{ by } R_2\rightarrow R_2-R_1,\ R_3\rightarrow R_3-R_1$$

$$\sim\left[\begin{array}{ccc:c}1&1&1&6\\0&1&2&4\\0&0&\lambda-3&\mu-10\end{array}\right] \text{ by } R_3\rightarrow R_3-R_2$$

Case-I: If $\lambda=3,\ \mu\neq 10$

$\rho(A)=2\ ,\ \rho[A:B]=3$

$\Rightarrow \rho[A:B]\neq\rho(A)$

Hence the system has no soln

Case-II: If $\lambda\neq 3$, μ have any value

$\rho[A:B]=\rho(A)=3=$ no. of unknowns

Hence the system has unique soln.

Case-III:

If $\lambda=3,\ \mu=10$

$\rho[A:B]=\rho(A)=2$

$<$ no. of unknowns

Hence the system have infinite no. of soln.

Ex-2: Investigate the values of λ & μ so that the equations

$2x+3y+5z=9$, $7x+3y-2z=8$, $2x+3y+\lambda z=\mu$

have (a) no sol^n (b) a unique sol^n & (c) an infinite number of sol^n.

Ans (a) $\lambda=5, \mu\neq 9$ (b) $\lambda\neq 5$, μ arbitrary (c) $\lambda=5, \mu=9$

(AKTU-2009)

Ex-3: For what value of λ & μ, the system of linear equations

$x+y+z=6$, $x+2y+5z=10$, $2x+3y+\lambda z=\mu$

has no solution and also find the solution when $\lambda=2$ & $\mu=10$.

(AKTU-2011, 2013).

Ex-4: Show that the system of equations

$3x+4y+5z=a$, $4x+5y+6z=b$, $5x+6y+7z=c$

does not have a sol^n unless $a+c=2b$.

(AKTU-2008, 2012).

Ex-5: What value of 'K' the equations

$x+y+z=1$, $2x+y+4z=K$ & $4x+y+10z=K^2$

has a sol^n. Also find the sol^n in each case. (AKTU-2015).

Sol^n 5: Given
$$\left.\begin{aligned} x+y+z&=1\\ 2x+y+4z&=K\\ 4x+y+10z&=K^2 \end{aligned}\right\} \rightarrow (1)$$

Convert the given system of equations into matrix form as

$$AX=B$$

$$\begin{bmatrix} 1 & 1 & 1\\ 2 & 1 & 4\\ 4 & 1 & 10 \end{bmatrix}\begin{bmatrix} x\\ y\\ z \end{bmatrix}=\begin{bmatrix} 1\\ K\\ K^2 \end{bmatrix} \rightarrow (2)$$

Now Augmented matrix is

$$[A:B]=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 2 & 1 & 4 & K\\ 4 & 1 & 10 & K^2 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 0 & -1 & 2 & K-2\\ 0 & -3 & 6 & K^2-4 \end{array}\right] \quad \text{by } R_2\to R_2-2R_1,\ R_3\to R_3-4R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 0 & -1 & 2 & K-2\\ 0 & 0 & 0 & K^2-3K+2 \end{array}\right] \quad \text{by } R_3\to R_3-3R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 0 & 1 & -2 & -K+2\\ 0 & 0 & 0 & K^2-3K+2 \end{array}\right] \quad \text{by } R_2\to R_2(-1) \longrightarrow (3)$$

The system of equation will have a soln if

$\rho[A \vdots B] = \rho(A)$

$\Rightarrow$ IIIrd row must be zero row.

$\Rightarrow \quad k^2 - 3k + 2 = 0 \quad \Rightarrow k^2 - k - 2k + 2 = 0 \Rightarrow k(k-1) - 2(k-1) = 0$

$\Rightarrow \quad (k-1)(k-2) = 0 \quad \Rightarrow \boxed{k = 1,\ k = 2}.$

Case-I: If $k = 2$, Then ③ becomes.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

then ② gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x + y + z = 1$

$\qquad y - 2z = 0$

[free variable = no. of unknowns $- \rho(A)$ $= 3 - 2 = 1$]

Let $z = k$

$\therefore \ y = 2k$

$x = 1 - y - z = 1 - k - 2k = 1 - 3k.$

Hence $\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-3k \\ 2k \\ k \end{bmatrix}, \ k \in R.$

Case-II: If $k = 1$, Then ③ becomes

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now ② gives $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow x + y + z = 1$

$\qquad y - 2z = 1$

[Free variable $= 3 - 2 = 1$]

Take $z = k_1$

$y = 1 + 2k_1$

$x = 1 - y - z = 1 - (1 + 2k_1) - k_1 = -3k_1$

Hence $\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3k_1 \\ 1+2k_1 \\ k_1 \end{bmatrix}, \ k_1 \in R.$

Ex: Find the value of λ such that the following equations have unique soln:

$$\lambda x+2y-2z-1=0, \quad 4x+2\lambda y-z-2=0, \quad 6x+6y+\lambda z-3=0$$

and use matrix method to solve these equations when $\lambda=2$.

(AKTU-2013)

Soln: Given $AX=B$

$$\Rightarrow \begin{matrix} \lambda x+2y-2z=1 \\ 4x+2\lambda y-z=2 \\ 6x+6y+\lambda z=3 \end{matrix} \Rightarrow \begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \rightarrow (1)$$

The system will have a unique soln if A is non-singular

$$\Rightarrow |A| \neq 0$$

$$\Rightarrow \begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix} \neq 0 \Rightarrow (\lambda-2)(\lambda^2+2\lambda+15) \neq 0$$

$$\Rightarrow \boxed{\lambda \neq 2}.$$

Now for $\lambda=2$, (1) gives

$$\begin{bmatrix} 2 & 2 & -2 \\ 4 & 4 & -1 \\ 6 & 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \rightarrow (2)$$

(Solve your self).

MATHEMATICS-I

KAS-103T

Lecture-No. 9

Module-I

Matrices

By Dr. Anuj Kumar

Characteristic equation, Cayley-Hamilton Theorem

Cayley Hamilton Theorem →

Every square matrix A satisfies its own characteristic eqn.

i.e for a square matrix A, the ch. eqn is

$$|A-\lambda I|=0$$

$$\Rightarrow \quad a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0 \longrightarrow (1)$$

Then A satisfies eqn (1),

$$\boxed{a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_nI = 0.}$$

Inverse by Cayley Hamilton theorem

Multiplying by A^{-1} in above eqn we get

$$a_0A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \cdots + a_nA^{-1} = 0.$$

$$\Rightarrow \quad A^{-1} = -\frac{1}{a_n}\left[a_0A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I\right]$$

Note: v.imp

(i) For 2x2 matrix A characteristic eqn is $|A-\lambda I|=0$

or $\boxed{\lambda^2 - (\text{Trace of } A)\lambda + |A| = 0}$

(ii) For 3x3 matrix A, the ch. eqn i.e $|A-\lambda I|=0$

or $\boxed{\lambda^3 - (\text{Trace of } A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A| = 0}$.

Ex-14 Verify Cayley-Hamilton theorem for the matrix

$$A=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$ Hence compute A^{-1}. Also evaluate

$$A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 + 23A - 9I.$$

(AKTU-2016, 2018).

Sol: Here ch. eqn is $|A-\lambda I|=0$

or $\lambda^3 - (\text{Trace of } A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A| = 0. \longrightarrow (1)$

Now Trace of $A = 2+2+2 = 6$.

$A_{11} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$, $A_{22} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$, $A_{33} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$ then $A_{11}+A_{22}+A_{33} = 9$

& $|A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$

$= 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}$

$= 2\times 3 + 1\times -1 + (-1) = 6-1-1=4.$

Using these values in (1), we get

$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \longrightarrow (2)$

To verify Cayley - Hamilton theorem, we have to show that

$A^3 - 6A^2 + 9A - 4I = 0. \longrightarrow (3)$

Now $A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & +21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$

$\therefore A^3 - 6A^2 + 9A - 4I$

$= \begin{bmatrix} 22 & -21 & +21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 22 & -21 & +21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$

$\Rightarrow A^3 - 6A^2 + 9A - 4I = 0.$ Hence Cayley-Hamilton Theorem Verified.

Now multiplying eqn (3) by A^{-1}, we get

$A^2 - 6A + 9I - 4A^{-1} = 0$

$\Rightarrow 4A^{-1} = A^2 - 6A + 9I$

$$\Rightarrow 4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow 4A^{-1} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \Rightarrow \boxed{A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}}.$$

Also, $A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 + 23A - 9I$

$= A^3(A^3 - 6A^2 + 9A - 2I) - 12A^2 + 23A - 9I$

$= A^3(A^3 - 6A^2 + 9A - 4I + 2I) - 12A^2 + 23A - 9I$

$= A^3(0 + 2I) - 12A^2 + 23A - 9I$

$= 2A^3 - 12A^2 + 23A - 9I$

$= (2A^3 - 12A^2 + 18A - 8I) + 5A - I$

$= 2(A^3 - 6A^2 + 9A - 4I) + 5A - I$

$= 2\cdot 0 + 5A - I$

$= 5A - I$

$$= 5\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -5 & 5 \\ -5 & 9 & -5 \\ 5 & -5 & 9 \end{bmatrix}.$$

Ex-2 → Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and verify Cayley-Hamilton Theorem. Compute A^{-1}. Also find the matrix representation by

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$$

(AKTU - 2010, 2013).

Ans → $\begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$

Ex-3: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find A^{-1} and A^4 using Cayley-Hamilton theorem. Also show that for every integer $n \geq 3$,

$$A^n = A^{n-2} + A^2 - I. \quad \text{(AKTU-2015)}$$

Hint: By C.H. eq^n

$$A^3 - A^2 - A + I = 0$$

$$\Rightarrow A^3 - A^2 = A - I$$

Premultiplying both sides successively by A, we get

$$A^4 - A^3 = A^2 - A$$

$$A^5 - A^4 = A^3 - A^2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$A^{n-1} - A^{n-2} = A^{n-3} - A^{n-4}$$

$$A^n - A^{n-1} = A^{n-2} - A^{n-3}.$$

Adding these equations,

$$A^n - A^2 = A^{n-2} - I \Rightarrow \boxed{A^n = A^{n-2} + A^2 - I}, \ n \geq 3$$

Ex-4: Verify C.H.T & find A^{-1}.

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ (AKTU-2014, 2015)

(ii) $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ (AKTU 2012)

(iii) $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ (AKTU-2014)

Ex-5: If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, Use Cayley-Hamilton Theorem to express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a linear polynomial in A.

(Ans → $-4A + 5I$). (2010)

Ex-6: Express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A where $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. (AKTU-2017).

(Ans → $138A - 403I$)

$\underline{Sol^n \to 6)}$ Given $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

then ch. eq^n is $|A - \lambda I| = 0$

or

$\lambda^2 - (\text{Trace } A)\lambda + |A| = 0$

$\Rightarrow \lambda^2 - 5\lambda + 7 = 0 \to (1)$

By C.H.T, we have

$A^2 - 5A + 7I = 0$

$\Rightarrow A^2 = 5A - 7I$

then $A^3 = 5A^2 - 7A$

$= 5(5A - 7I) - 7A = 25A - 35I - 7A$

$\Rightarrow A^3 = 18A - 35I$

$\therefore A^4 = 18A^2 - 35A$

$= 18(5A - 7I) - 35A = 90A - 126I - 35A$

$= 55A - 126I$

& $A^5 = 55A^2 - 126A$

$= 55(5A - 7I) - 126A = 275A - 385I - 126A$

$\Rightarrow A^5 = 149A - 385I$

Now $2A^5 - 3A^4 + A^2 - 4I$

$= 2(149A - 385I) - 3(55A - 126I) + 5A - 7I - 4I$

$= 298A - 770I - 165A + 378I + 5A - 11I$

$= 138A - 403I.$

MATHEMATICS-I
KAS-103T
Lecture No- 10

Module-I
Matrices
By Dr. Anuj Kumar

Eigen Values & Eigen Vector or Characteristic roots & Characteristic Vector or Latent roots & Latent Vectors.

Let A be any square matrix then

$\boxed{\phi(\lambda) = |A - \lambda I|}$ $\longrightarrow$ Characteristic Polynomial

$\boxed{|A-\lambda I| = 0}$ $\rightarrow$ Characteristic Equation.

On solving ch. eq^n $|A-\lambda I|=0$, we get characteristic roots or eigen values. $\lambda = \lambda_1, \lambda_2, \lambda_3 \ldots$.

If $AX = \lambda X$

or $(A-\lambda I)X = 0$

then X is called eigen vector or ch. vector corresponding to the eigen value λ.

Note:- ① There are infinite eigen vector corresponding to the eigen value λ.

② The set of all eigen values of A is called spectrum of A.

Ex-1) Find the eigen values and corresponding eigen vectors of the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. (CAKTU-2008, 2018).

Sol^n: Given $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

then ch. eq^n is $|A - \lambda I| = 0$

or

$$\lambda^3 - (\text{Trace of } A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A| = 0 \longrightarrow ①$$

Trace of $A = 2+2+2 = 6$, $A_{11} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$, $A_{22} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$, $A_{33} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$

& $|A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 4$.

Using these values in ①, we get $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \longrightarrow ②$.

On solving eqn ②, we get $(\lambda-1)(\lambda-1)(\lambda-4)=0$

$$\Rightarrow \boxed{\lambda = 1, 1, 4} \rightarrow \text{Eigen values of } A.$$

Eigen vector corresponding to $\lambda = 1, 1, 4 \rightarrow$

Let X be the eigen vector for λ then

$$(A-\lambda I)X = 0$$

$$\Rightarrow \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow ③$$

For $\lambda = 1, 1 \rightarrow$ Eqn ③ gives

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{by } R_2 \to R_2 + R_1, \; R_3 \to R_3 - R_1$$

$$\Rightarrow x_1 - x_2 + x_3 = 0$$

[free variable = no. of unknowns − rank = 3 − 1 = 2]

Take $x_3 = K_1$ & $x_2 = K_2$

$$\therefore x_1 = x_2 - x_3 = K_2 - K_1$$

Hence $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K_2 - K_1 \\ K_2 \\ K_1 \end{bmatrix}$, $K_1, K_2 \in R$

are the eigen vectors corresponding to the eigenvalue $\lambda = 1, 1$.

For $\lambda = 4 \rightarrow$ Equation ③ gives

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -2 \\ -1 & -2 & -1 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{by } R_1 \leftrightarrow R_3$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{by } R_2 \to R_2 + R_1, \; R_3 \to R_3 + 2R_1$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{by } R_3 \to R_3 - R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ by } R_2 \to \frac{R_2}{(-3)}$$

$\Rightarrow \quad x_1 - x_2 - 2x_3 = 0$ [Free Variable $= 3-2 = 1$]

$x_2 + x_3 = 0$

Let $x_3 = p$

$\therefore \quad x_2 = -p$

$x_1 = x_2 + 2x_3 = -p + 2p = p$.

Hence

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} p \\ -p \\ p \end{bmatrix} = p \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \; p \in R.$$

i.e the eigen vectors corresponding to $\lambda = 4$.

Ex: Find the characteristic roots & characteristic vector for the matrices

(i) $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ (AKTU-2015) $[\lambda = -2, 6, 3]$

(ii) $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$ (AKTU-2011) $[\lambda = 0, 1, -2]$.

(iii) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ (AKTU 2010, 2011) $[\lambda = -3, -3, 5]$

(iv) $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ $[\lambda = 0, 3, 15]$ (AKTU-2011)

(v) $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ (AKTU-2010) $[\lambda = 3, 2, 5 \to$ diagonal elements$]$.

(vi) $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ (AKTU-2014).

Solⁿ (vi): Given $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

then ch. eqⁿ is $|A - \lambda I| = 0$

or $\lambda^2 - (\text{Trace of } A)\lambda + |A| = 0$

$\Rightarrow \lambda^2 - 0 \cdot \lambda + 1 = 0$

$\Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda^2 = i^2$

$\Rightarrow \boxed{\lambda = \pm i}$

Let X be eigen vector for eigen value λ then

$$(A - \lambda I) X = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & -1 \\ 2 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow (1)$$

For $\lambda = i \rightarrow$ Eqn (1) gives

$$\begin{bmatrix} 1-i & -1 \\ 2 & -1-i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -1-i \\ 1-i & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{by } R_1 \leftrightarrow R_2$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{-1-i}{2} \\ 1-i & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{by } R_1 \rightarrow \frac{R_1}{2}$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1-i}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{by } R_2 \rightarrow R_2 - (1-i) R_1$$

$$\Rightarrow x_1 + \frac{(-1-i)}{2} x_2 = 0 \qquad [\text{free variable} = 2-1 = 1]$$

$$\therefore x_1 = \frac{(1+i)}{2} K. \qquad \text{Let } x_2 = K.$$

Hence $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{(1+i)K}{2} \\ K \end{bmatrix} = K \begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix}, K \in R$

is the eigen vector corresponding to eigenvalue $\lambda = i$.

For $\lambda = -i \rightarrow$ Eqn (1) gives

$$\begin{bmatrix} +1+i & -1 \\ 2 & -1+i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -1+i \\ +1+i & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{by } R_1 \leftrightarrow R_2$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{-1+i}{2} \\ +1+i & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{by } R_1 \rightarrow \frac{R_1}{2}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{-1+i}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{by } R_2 \rightarrow R_2 - (i-1).R_1$$

$$\Rightarrow x_1 + \frac{(-1+i)}{2} x_2 = 0 \qquad [\text{free Variable} = 2-1 = 1]$$

$$\Rightarrow x_1 = \frac{(1-i)}{2} p, \qquad \text{Let } x_2 = p.$$

Hence $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = p \begin{bmatrix} \frac{1-i}{2} \\ 1 \end{bmatrix}, p \in R$ is the eigen vector for $\lambda = -i$.

Ex-1 <- Shortcut Method for short question ->

Find the eigen value of the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ corresponding to the eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$. (AKTU-2017)

Sol^n → Given $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$. Let λ be the eigen value for the eigenvector $X = \begin{bmatrix} 101 \\ 101 \end{bmatrix}$. Then

$$(A - \lambda I) X = 0 \quad \text{or} \quad AX = \lambda X.$$

$$\Rightarrow \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 101 \\ 101 \end{bmatrix} = \lambda \begin{bmatrix} 101 \\ 101 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 404+202 \\ 202+404 \end{bmatrix} = \lambda \begin{bmatrix} 101 \\ 101 \end{bmatrix}$$

$$\Rightarrow \lambda \begin{bmatrix} 101 \\ 101 \end{bmatrix} = \begin{bmatrix} 606 \\ 606 \end{bmatrix}$$

$$\Rightarrow \lambda \begin{bmatrix} 101 \\ 101 \end{bmatrix} = 6 \begin{bmatrix} 101 \\ 101 \end{bmatrix} \Rightarrow \boxed{\lambda = 6}.$$

Properties → ① A & A^T have same eigen values.

② If $\lambda_1, \lambda_2, \lambda_3$ are eigen values of a matrix A, Then

(i) $k\lambda_1, k\lambda_2, k\lambda_3$ " " " " " " " " " KA.

(ii) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ " " " " " " " " " A^{-1}.

(iii) $\lambda_1^2, \lambda_2^2, \lambda_3^2$ " " " " " " " " " A^2.

(iv) $\lambda_1^3, \lambda_2^3, \lambda_3^3$ " " " " " " " " " A^3.

Imp ③ The eigen values of diagonal matrix or triangular matrix are its diagonal elements.

Ex-2 Find the eigen values of $3A^3 + 5A^2 - 6A + 2I$ where $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$. Also find eigen values of A^{-1}.

Sol^n → Since A is triangular matrix, then its diagonal elements are eigen values of A.

Hence eigen values of A are $1, 3, -2$.

Then eigen values of A^2 are $1, 9, 4$

eigen values of A^3 are $1, 27, -8$

eigen values of A^{-1} are $1, \frac{1}{3}, -\frac{1}{2}$.

Also the eigen values of I are 1, 1, 1 (As it is diagonal matrix)

Now the eigen values of

$3A^3+5A^2-6A+2I$ are

$3(1, 27, -8)+5(1, 9, 4)-6(1, 3, -2)+2(1,1,1)$

$=(3, 81, -24)+(5, 45, 20)-(6, 18, -12)+(2,2,2)$

$=(4, 110, 10)$ i.e 4, 110, 10.

Ex-2: find the eigen values of A^2 & A^{-1} if

(i) $A=\begin{bmatrix}1&0&0\\3&6&0\\2&3&-2\end{bmatrix}$ (ii) $A=\begin{bmatrix}3&0&0\\0&-3&0\\0&0&5\end{bmatrix}$ (iii) $A=\begin{bmatrix}1&2\\0&3\end{bmatrix}$.

Ex-3: If the eigen values of A are 1, 1, 1 the find the eigen values of $A^2+3A+5I$.

Properties: ① The sum of all eigen values = Trace of matrix

② Eigen values are always unique.

③ The product of all eigen values $=|A|$.

Note: ① The ch. roots of unitary or orthogonal matrices are of unit modulus.

② The ch. roots of a Hermitian matrix are all real.

③ The ch. roots of a skew-Hermitian matrix are all zero or purely imaginary.

④ Eigen values of (i) Nilpotent matrix → 0, 0

(ii) Idempotent matrix → 0, 1

(iii) Involutary matrix → −1, 1.

⑤ 0 is the eigen value of A iff $|A|=0$.

Ex-1: Two eigen values of the matrix $A=\begin{bmatrix}2&2&1\\1&3&1\\1&2&2\end{bmatrix}$ are equal to 1, then find the third eigen value.

Sol^n: Let $\lambda_1=1$, $\lambda_2=1$, $\lambda_3=?$. Since Sum of eigen values = Trace of A.

$\Rightarrow \lambda_1+\lambda_2+\lambda_3=2+3+2 \Rightarrow 1+1+\lambda_3=7 \Rightarrow \boxed{\lambda_3=5}$

Ex-2: If 2 & 3 are two eigen values of matrix A & $|A|=24$. Find third eigen value.

Sol^n: Let $\lambda_1=2$, $\lambda_2=3$, $\lambda_3=?$.

We have $\lambda_1\lambda_2\lambda_3=|A| \Rightarrow 2\cdot 3\cdot\lambda_3=24 \Rightarrow \boxed{\lambda_3=4}$

Ex: For what value of 'x', the eigen values of the given matrix A are real

$$A = \begin{bmatrix} 10 & 5+i & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$ (AKTU-2017)

Solⁿ:

The eigen values of A are real if A is Hermitian.

$\Rightarrow \quad A^* = A$

$$\Rightarrow \begin{bmatrix} 10 & \bar{x} & 4 \\ 5-i & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix} = \begin{bmatrix} 10 & 5+i & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

$\Rightarrow \quad \boxed{x = 5-i}$ On comparing.

MATHEMATICS-I

KAS-103T

Lecture No - 11

Module-I

Matrices

By. Dr. Anuj Kumar

Linearly Independent Vectors→

The vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called L.I vectors, if there exists scalar $a_1, a_2, \ldots, a_n$ such that

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0$$

$$\Rightarrow a_1 = 0, \quad a_2 = 0, \quad \ldots, \quad a_n = 0.$$

Ex→ The vectors $\alpha_1 = (1,0,0)$, $\alpha_2 = (0,1,0)$, $\alpha_3 = (0,0,1)$ are L.I.

As $\exists$ a_1, a_2, a_3 scalars such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$$

$$\Rightarrow a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1) = (0,0,0)$$

$$\Rightarrow (a_1,0,0) + (0,a_2,0) + (0,0,a_3) = (0,0,0)$$

$$\Rightarrow (a_1, a_2, a_3) = (0,0,0)$$

$$\Rightarrow \boxed{a_1 = 0, \ a_2 = 0, \ a_3 = 0}$$

Note: ① $\alpha_1, \alpha_2, \alpha_3$ are L.I if $|A| = |\alpha_1\ \alpha_2\ \alpha_3| \neq 0$

② $\alpha_1, \alpha_2, \alpha_3$ are L.D if $|A| = |\alpha_1\ \alpha_2\ \alpha_3| = 0$.

Ex-1 Prove that the vectors $\alpha_1 = (1,0,0)$, $\alpha_2 = (0,1,0)$, $\alpha_3 = (0,0,1)$ are L.I.

Solⁿ We have

$$|A| = |\alpha_1\ \alpha_2\ \alpha_3| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\Rightarrow |A| \neq 0$

$\Rightarrow \alpha_1, \alpha_2, \alpha_3$ are L.I.

Ex-2 Prove that the vectors $(1,6,4)$, $(0,2,3)$ & $(0,1,2)$ are linearly Independent. (AKTU-2019-20)

Solⁿ We have

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 6 & 2 & 1 \\ 4 & 3 & 2 \end{vmatrix} = 1(4-3) = +1 \neq 0$$

$\Rightarrow |A| \neq 0 \Rightarrow$ The given vectors are L.I.

Ex-3 Prove that the vectors $(1,-1,1)$, $(2,1,1)$, $(3,0,2)$ are linearly dependent. (AKTU-2017)

Solⁿ Let $\alpha_1 = (1,-1,1)$, $\alpha_2 = (2,1,1)$, $\alpha_3 = (3,0,2)$

then $$|A| = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 1(2-0) - 2(-2-0) + 3(-1-1)$$
$$= 2 + 4 - 6 = 0$$

$\Rightarrow |A| = 0$

$\Rightarrow \alpha_1, \alpha_2, \alpha_3$ are L.D.

Ex-4 Prove that the vectors $\alpha_1 = [a_1, b_1]$, $\alpha_2 = [a_2, b_2]$ are L.D iff $a_1 b_2 - a_2 b_1 = 0$.

Solⁿ The vectors α_1, α_2 are L.D iff $|A| = 0$ iff $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$

iff $a_1 b_2 - a_2 b_1 = 0$.

Ex: If the vectors $(0, 1, K)$, $(1, K, 1)$ & $(K, 1, 0)$ are linearly dependent, then find the value of a. (AKTU-2016)

Sol: Since the vectors

$$\alpha_1 = (0, 1, K),\ \alpha_2 = (1, K, 1)\ \&\ \alpha_3 = (K, 1, 0)$$

are L.D

then $|A| = 0$

$$\Rightarrow \begin{vmatrix} 0 & 1 & K \\ 1 & K & 1 \\ K & 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow 0(0-1) - 1(0-K) + K(1-K^2) = 0$$

$$\Rightarrow K + K - K^3 = 0$$

$$\Rightarrow K^3 - 2K = 0$$

$$\Rightarrow K(K^2 - 2) = 0$$

$$\Rightarrow K = 0,\ K = \pm\sqrt{2}.$$

Ex: Show that the row vectors of the matrix

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$ are L.I.

Ex: Find the value of λ for which the vectors $(1, -2, \lambda)$, $(2, -1, 5)$ & $(3, -5, 7\lambda)$ are L.D. [Ans $\lambda = \frac{5}{14}$].

Similarity Transformation →

Let A and B be two square matrices of order n. Then B is said to be similar to A if there exists a non-singular matrix P such that

$$B = P^{-1} A P \rightarrow (1)$$

Eqn (1) is called a similarity transformation.

Diagonalisation of a matrix →

Diagonalisation of a matrix A is the process of reduction of A to a diagonal form.

Reduction of matrix to diagonal form →

If a square matrix A of order n has n linearly independent eigenvectors, then a matrix P can be found such that

$D = P^{-1} A P$ is a diagonal matrix.

where P is called modal matrix & D is spectral matrix of A.

Working rule → Given matrix A:

① Find eigen values & eigen vectors of A.
② Write modal matrix 'P' whose columns are eigen vectors.
③ find P^{-1} by shortcut method.
④ find $P^{-1}AP = D$ where D is required diagonal form of A.

Note→ ① If A has $\lambda_1, \lambda_2, \lambda_3$ eigen values then its diagonal form is $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$.

② If A is a matrix of order n then it can be diagonalize only when it has n L.I eigen vectors otherwise not.

Ex-1→ Write the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ in diagonal form by similarity transformation. (AKTU-2007)

Sol^n→ Given $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

Here Ch. eq^n is $|A - \lambda I| = 0$

or $\lambda^2 - (\text{Trace of } A)\lambda + |A| = 0$

$\Rightarrow \lambda^2 - 7\lambda + 10 = 0$

$\Rightarrow (\lambda - 2)(\lambda - 5) = 0$

$\Rightarrow \boxed{\lambda = 2, 5}$ → Eigen values of A.

Eigen vector for $\lambda = 2$→

Let X be an eigen vector for $\lambda = 2$.

Then $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow ①$$

For $\lambda = 2$→ $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by $R_2 \to R_2 - R_1$

[free variable $= 2 - 1 = 1$]

$\Rightarrow 2x_1 + x_2 = 0$

Let $x_1 = K$ then $x_2 = -2K$. Hence $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K \\ -2K \end{bmatrix} = K\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $K \in R$.

Eigen vector for $\lambda = 5$:-

Let X be eigen vector for $\lambda = 5$

Then $(A - \lambda I) X = 0$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{from } ①)$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ by } R_2 \to R_2 + 2R_1$$

$\Rightarrow -x_1 + x_2 = 0$ [free variable $= 2-1=1$]

Let $x_2 = K_1$ then $x_1 = K_1$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K_1 \\ K_1 \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; K_1 \in R.$$

Now modal matrix is

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \quad \text{then } P^{-1} = \frac{1}{|P|} \operatorname{adj} P = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}.$$

Then
$$P^{-1}AP = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -4 & 5 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 6 & 0 \\ 0 & 15 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D.$$

which is required diagonal form.

Ex-2 :- Find the eigen values and corresponding eigen vectors of the following matrix and hence diagonalise it.

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$ (AKTU-2011, 2014)

Soln :- Here ch. eqn is $|A - \lambda I| = 0$ or
$$\lambda^3 - (\text{Trace of } A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - |A| = 0 \longrightarrow ①$$

Now Trace of $A = -2+1+0 = -1$

$A_{11} = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} = -12$, $A_{22} = \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} = -3$, $A_{33} = \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = -2-4 = -6$

$\therefore A_{11} + A_{22} + A_{33} = -12 - 3 - 6 = -21$

& $|A| = (0+12+12) - (3 - 24 + 0) = 24 + 21 = 45$.

Using in ①, $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0 \Rightarrow (\lambda-5)(\lambda^2 + 6\lambda + 9) = 0$

$\Rightarrow (\lambda-5)(\lambda+3)^2 = 0$

$$\Rightarrow \boxed{\lambda = 5, -3, -3}$$ Eigen values of A.

Let X be eigen vector of eigen value $\lambda \rightarrow$

Then

$$(A - \lambda I) X = 0$$

$$\Rightarrow \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow (2)$$

For $\lambda = 5 \rightarrow$ Eqn (2) gives

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ by } R_1 \leftrightarrow -R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ by } \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 7R_1 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ by } R_3 \to R_3 - 2R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ by } R_2 \to \frac{R_2}{-8}$$

$$\Rightarrow x_1 + 2x_2 + 5x_3 = 0$$
$$x_2 + 2x_3 = 0$$

[free variable $= 3 - 2 = 1$]

Let $x_3 = K$,

$\therefore x_2 = -2K$ & $x_1 = -2x_2 - 5x_3 = 4K - 5K = -K$.

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -K \\ -2K \\ K \end{bmatrix} = K \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, K \in R.$$

is the eigen vector for $\lambda = 5$.

For $\lambda = -3 \rightarrow$ Eqn (2) gives

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ by } \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_2 \end{matrix}$$

$\Rightarrow x_1 + 2x_2 - 3x_3 = 0$ [Free variable $= 3-1 = 2$]

Let $x_3 = K_1$, $x_2 = K_2$

$\therefore x_1 = 3x_3 - 2x_2 = 3K_1 - 2K_2$.

$$\text{Hence } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3K_1 - 2K_2 \\ K_2 \\ K_1 \end{bmatrix} = \begin{bmatrix} 3K_1 - 2K_2 \\ 0K_1 + 1K_2 \\ 1K_1 + 0K_2 \end{bmatrix}$$

$$= K_1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + K_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad K_1, K_2 \in R.$$

are eigenvectors for $\lambda = -3, -3$.

Now modal matrix is

$$P = \begin{bmatrix} -1 & 3 & -2 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\text{then } P^{-1} = \frac{1}{|P|} \operatorname{adj} P = \frac{1}{8} \begin{bmatrix} -1 & -2 & 3 \\ 1 & 2 & 5 \\ -2 & 4 & 6 \end{bmatrix}$$

Then

$$P^{-1}AP = \frac{1}{8} \begin{bmatrix} -1 & -2 & 3 \\ 1 & 2 & 5 \\ -2 & 4 & 6 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 & -2 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} -5 & 10 & 15 \\ -3 & -6 & -15 \\ 6 & -12 & -18 \end{bmatrix} \begin{bmatrix} -1 & 3 & -2 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 40 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & -24 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = D$$

which is required diagonal form.

Ex-3: Reduce the matrix $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ to the diagonal form.

V.Imp

(AKTU-2011, 2017). 2008,

Hint:

- Ch. eq^n $\lambda^3 - \lambda^2 - 5\lambda + 5 = 0$
- $\lambda = 1, \pm\sqrt{5}$.
- For $\lambda = 1$, $X = k_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$
- for $\lambda = \sqrt{5}$, $X = k_2 \begin{bmatrix} \sqrt{5}-1 \\ 1 \\ -1 \end{bmatrix}$
- for $\lambda = -\sqrt{5}$, $X = k_3 \begin{bmatrix} \sqrt{5}+1 \\ -1 \\ 1 \end{bmatrix}$
- Modal matrix $P = \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$.

& $P^{-1} = \dfrac{1}{2\sqrt{5}} \begin{bmatrix} 0 & -2\sqrt{5} & -2\sqrt{5} \\ 1 & \sqrt{5}+2 & 1 \\ 1 & -\sqrt{5}+2 & 1 \end{bmatrix}$

& $D = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$.

Ex-4: Reduce in diagonal form

(i) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (2011) (ii) $\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$ (2015) (iii) $\dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ (2014).

Ex-5: Show that the matrix $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ has less than three linearly independent eigen vector. Is it possible to obtain a similarity transformation that will diagonalise this matrix.

V.Imp

(AKTU-2014, 2006, 2019 Sessional Exam)

Ans: $\boxed{\text{Not diagonalisable}}$