AKTU : Engineering Mathematics-1

Vector Calculus

Lec-1

By Gulshan sir

Today's Target

- ✓ Introduction to Vectors

Vector Calculus : Syllabus

Vector differentiation:

- Gradient, Curl and Divergence and their Physical interpretation,
- Directional derivatives.

Vector Integration:

- Line integral
- Surface integral
- Volume integral
- Gauss's Divergence theorem
- Green's theorem and Stoke's theorem (without proof) and their applications.

Introduction to Vectors

Vector: A physical quantity is said to be a vector quantity if it has

1) Magnitude
2) Direction
3) Obey vector law of addition

Examples : Displacement , velocity, acceleration, force etc.

Scalar : A physical quantity that has magnitude only is known as scalar.

Examples : Distance, time, speed area, volume etc.

Representation of Vector

$$A \xrightarrow{\quad a \quad} B$$

$\overrightarrow{AB}$ or $\overrightarrow{a}$

$A \rightarrow$ initial point

$B \rightarrow$ Terminal point

Magnitude

$|\overrightarrow{AB}|$ or $|\overrightarrow{a}|$ or a

Unit Vector

The unit vector in the direction of a vector $\vec{a}$ is denoted by $\hat{a}$

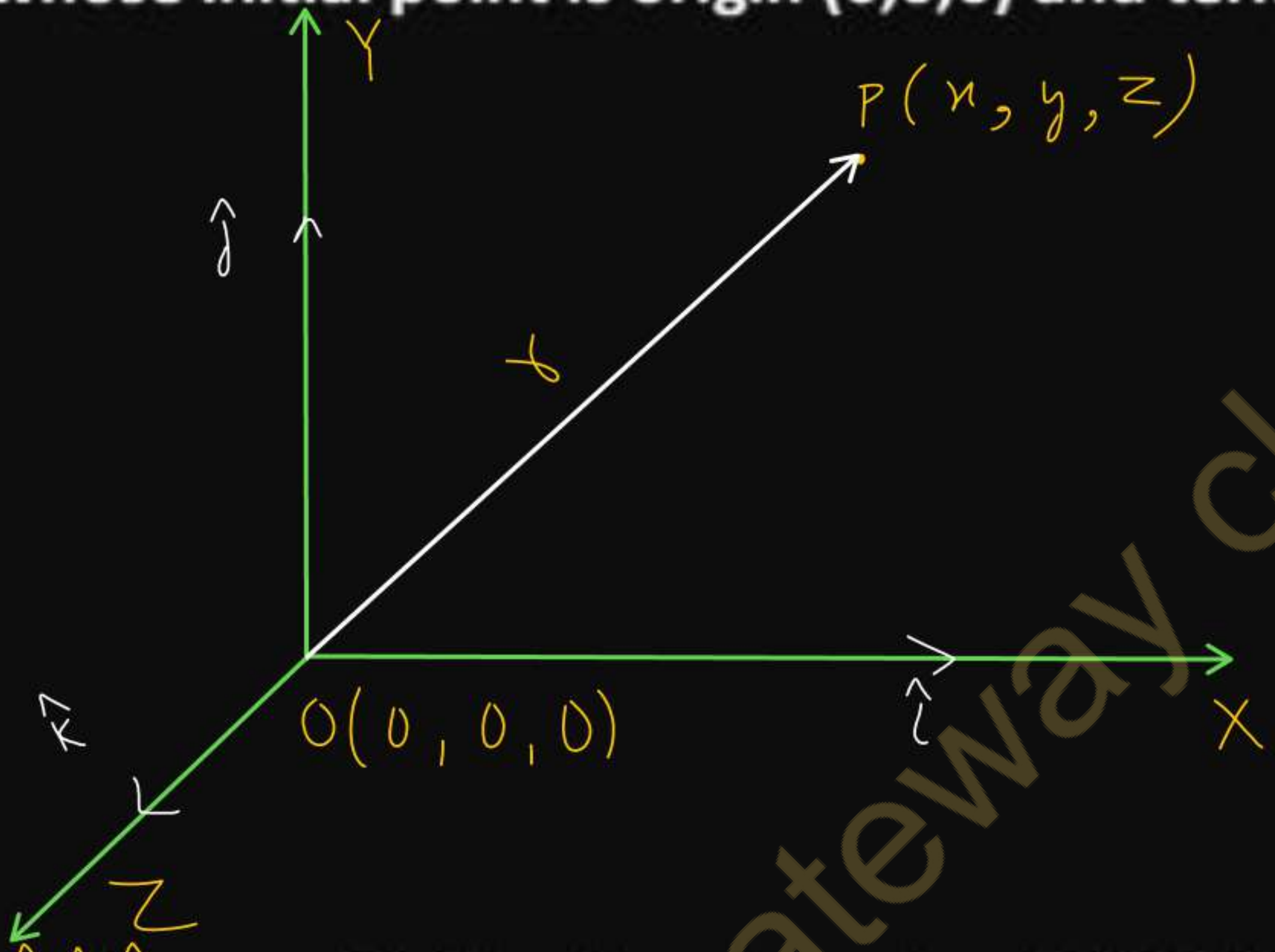
$$\boxed{\hat{a} = \frac{\vec{a}}{|\vec{a}|}}$$

Note : The magnitude of unit vector is unity

$$\boxed{|\hat{a}| = 1}$$

Position Vector

A vector whose initial point is origin (0,0,0) and terminal point is P(x,y,z)



Position vector of P

$$\boxed{\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}}$$

Magnitude

$$\boxed{|\vec{OP}| = |\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}}$$

Note : 1. $\hat{i}, \hat{j}, \hat{k}$ are called unit vectors along OX, OY, OZ axes

2. This form of vector is called component form of vector

a) x,y,z are called the scalar component of $\vec{r}$

b) $x\hat{i}, y\hat{j}, z\hat{k}$ are called the vector component of $\vec{r}$

Vector joining two points

$A(x_1, y_1, z_1)$ $\qquad$ $B(x_2, y_2, z_2)$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{AB} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k} - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$\boxed{\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}}$$

$$\boxed{|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

Important Points : If $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

(i) $\vec{a} + \vec{b} = (a_1 + a_2)\hat{i} + (b_1 + b_2)\hat{j} + (c_1 + c_2)\hat{k}$

(ii) $\vec{a} - \vec{b} = (a_1 - a_2)\hat{i} + (b_1 - b_2)\hat{j} + (c_1 - c_2)\hat{k}$

(iii) $\vec{a} = \vec{b}$ if and only if

$a_1 = a_2$

$b_1 = b_2$

$c_1 = c_2$

Multiplication of vector by scalar

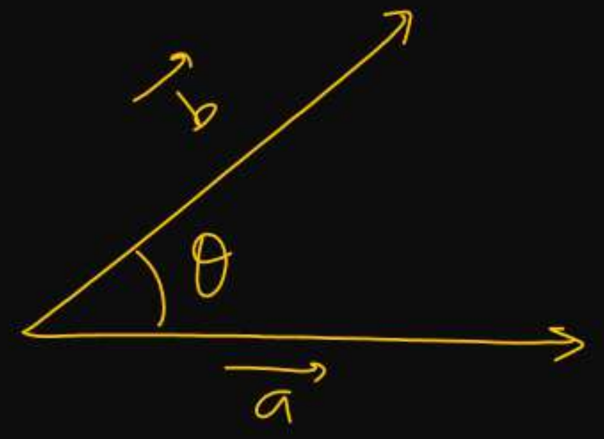
Let $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

$$\lambda\vec{a} = \lambda a_1\hat{i} + \lambda b_1\hat{j} + \lambda c_1\hat{k}$$

$$\boxed{|\lambda\vec{a}| = \lambda|\vec{a}|}$$

Scalar product (or DOT Product) of two Vectors

The scalar product of two non-zero vectors $\vec{a}$ and $\vec{b}$ is defined as



$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta}$$

Where θ is the angle between $\vec{a}$ and $\vec{b}$

$$0 \le \theta \le \pi$$

$$\boxed{\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}}$$

Results :

1. $\vec{a} \cdot \vec{b}$ is a real number
2. Two non-zero vectors $\vec{a}$ and $\vec{b}$ are perpendicular/orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$

3. For unit vector i, j, k

$\hat{i}.\hat{i} = 1$

$\hat{j}.\hat{j} = 1$

$\hat{k}.\hat{k} = 1$

$\hat{i}.\hat{j} = 0$

$\hat{j}.\hat{k} = 0$

$\hat{k}.\hat{i} = 0$

4. Scalar product in terms of component

Let $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

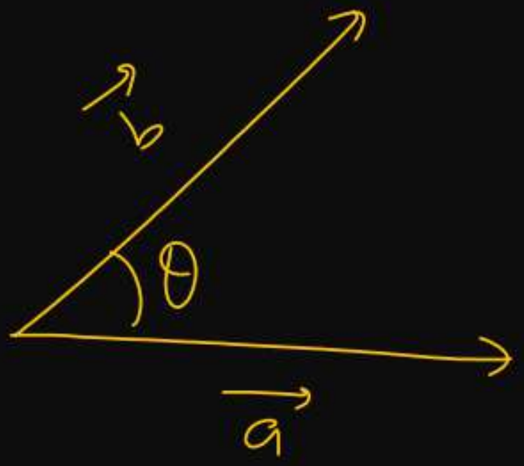
$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

5. $\boxed{\vec{a}.\vec{b} = a_1a_2 + b_1b_2 + c_1c_2}$

(5) $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

Vector product (or Cross Product) of two Vectors

The vector product of two non-zero vectors $\vec{a}$ and $\vec{b}$ is defined as



$$\boxed{\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}}$$

Note : $\vec{a} \times \vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$

Where

θ is the angle between $\vec{a}$ and $\vec{b}$

$0 \leq \theta \leq \pi$

$$\boxed{\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|\sin\theta}}$$

Results :

(i) $\vec{a} \times \vec{b}$ is a vector

(ii) Two non-zero vectors $\vec{a}$ and $\vec{b}$ are parallel (or collinear) if and only if $\vec{a} \times \vec{b} = 0$

$$\boxed{\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})}$$

Note $\vec{a} \times \vec{a} = 0$

$\vec{a} \times (-\vec{a}) = 0$

(iii) For unit vectors

$\hat{i} \times \hat{i} = 0$

$\hat{j} \times \hat{j} = 0$

$\hat{k} \times \hat{k} = 0$

$\hat{i} \times \hat{j} = \hat{k}$

$\hat{j} \times \hat{k} = \hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$

$\hat{j} \times \hat{i} = -\hat{k}$

$\hat{k} \times \hat{j} = -\hat{i}$

$\hat{i} \times \hat{k} = -\hat{j}$

(iv) Vectors product in terms of component

Let $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

$$\boxed{\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}$$

Scalar triple product

$\vec{a}, \vec{b}, \vec{c}$

It represent the volume of parallelepiped

Important Point

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

(i) Scalar Triple product in terms of component

Let $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

$\vec{c} = a_3\hat{i} + b_3\hat{j} + c_3\hat{k}$

$$\boxed{[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

(ii) Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if $\boxed{[\vec{a}\ \vec{b}\ \vec{c}] = 0}$

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Vector Calculus

Lec-2

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Vector Calculus

Vector Differentiation:

$\nabla \to Del$

or

$Nabla\ Vector\ operator$

$$\boxed{\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$$

Scalar function
(Ø)

Vector function
(Ø)

Gradient
Grad Ø = ∇Ø

(i) Divergence
Div Ø = ∇.Ø

$(ii)\ \cancel{curl}\ curl\emptyset = \nabla \times \emptyset$

Gradient

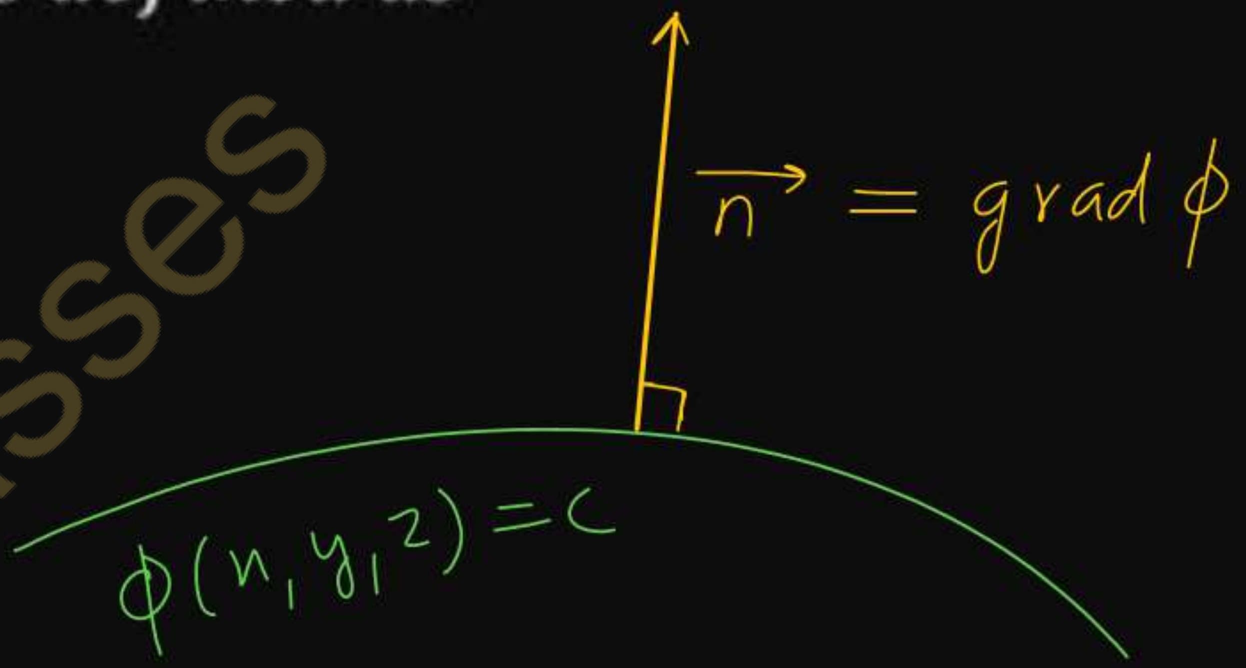
The Gradient of scalar point function $\boxed{\phi(x, y, z)}$ *is defined as*

$$grad\phi = \nabla\phi$$

$$grad\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$\boxed{grad\phi = \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)}$$

Where $\boxed{\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$ is called Delta or Nabla vector operator denoted by ∇



Geometrical Interpretation

$\nabla\phi$ *is normal to the surface* $\phi(x, y, z) = c$

Properties of gradient

(a) If $\emptyset$ is a constant scalar point function, then $\nabla\emptyset = \vec{0}$

(b) If $\emptyset_1$ and $\emptyset_2$ are two scalar point functions, then

(i) $\nabla(\emptyset_1 \pm \emptyset_2) = \nabla\emptyset_1 \pm \nabla\emptyset_2$

(ii) $\nabla(c_1\emptyset_1 \pm c_2\emptyset_2) = c_1\nabla\emptyset_1 \pm c_2\nabla\emptyset_2$ where c_1 and c_1 are constant

(iii) $\nabla(\emptyset_1\emptyset_2) = \emptyset_1\nabla\emptyset_2 + \emptyset_2\nabla\emptyset_1$

(iv) $\nabla\left(\dfrac{\emptyset_1}{\emptyset_2}\right) = \dfrac{\emptyset_2\nabla\emptyset_1 - \emptyset_1\nabla\emptyset_2}{{\emptyset_2}^2}$

Q.1 *If* $\phi = 3x^2y - y^3z^2$, *find the grad* ϕ *at point* $(2, 0, -2)$.

$$\phi = 3x^2y - y^3z^2$$

$$\text{grad}\,\phi = \nabla\phi$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{i}(6xy - 0) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(0 - 2y^3z)$$

At $(2, 0, -2)$

$$\text{grad}\,\phi = \hat{i}(0-0) + \hat{j}(12-0) + \hat{k}(0-0)$$

$$\boxed{\text{grad}\,\phi = 12\hat{j}}$$

Practice Q.1 $If\ \emptyset = 3x^2y - y^3z^2, find\ the\ grad\ \emptyset\ at\ point\ (1, -2, -1)$

$Ans. -12\hat{i} - 9\hat{j}$-$16\hat{k}$

Q.2 Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$)

$$x^3 + y^3 + 3xyz = 3$$

$$x^3 + y^3 + 3xyz - 3 = 0$$

Let $\phi = x^3 + y^3 + 3xyz - 3$

$$\text{grad}\,\phi = \nabla \phi$$

$$= \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(x^3+y^3+3xyz-3) + \hat{j}\frac{\partial}{\partial y}(x^3+y^3+3xyz-3) + \hat{k}\frac{\partial}{\partial z}(x^3+y^3+3xyz-3)$$

$$= \hat{i}(3x^2+3yz) + \hat{j}(3y^2+3xz) + \hat{k}(3xy)$$

At $(1, 2, -1)$

$$\text{grad}\,\phi = \hat{i}(3-6) + \hat{j}(12-3) + \hat{k}(6)$$

$$\text{grad}\,\phi = -3\hat{i} + 9\hat{j} + 6\hat{k}$$

Normal vector

$$\vec{n} = \text{grad}\,\phi$$

$$\vec{n} = -3\hat{i} + 9\hat{j} + 6\hat{k}$$

Unit normal vector

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|}$$

$$\hat{n} = \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{(-3)^2 + 9^2 + 6^2}}$$

$$\hat{n} = \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{3\sqrt{14}}$$

$$\hat{n} = \frac{\cancel{3}(-\hat{i} + 3\hat{j} + 2\hat{k})}{\cancel{3}\sqrt{14}}$$

$$\boxed{\hat{n} = \frac{1}{\sqrt{14}}(-\hat{i} + 3\hat{j} + 2\hat{k})}$$

Practice Q.2 Find a unit vector normal to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$)

Ans. $\dfrac{-\hat{\imath} + 2\hat{\jmath} + \hat{k}}{3}$

Q.3 If $u = x + y + z \quad v = x^2 + y^2 + z^2, \qquad w = yz + zx + xy$ prove that grad u, gradv and gradw are coplanar vectors.

$u = x + y + z$

$v = x^2 + y^2 + z^2$

$w = yz + zx + xy$

$\text{grad}\, u = \nabla u$

$= \hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}$

$= \hat{i} \times 1 + \hat{j} \times 1 + \hat{k} \times 1$

$$\boxed{\text{grad}\, u = \hat{i} + \hat{j} + \hat{k}}$$

$\text{grad}\, v = \nabla v$

$= \hat{i}\frac{\partial v}{\partial x} + \hat{j}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial v}{\partial z}$

$$\boxed{\text{grad}\, v = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}}$$

$\text{grad}\, w = \nabla w$

$= \hat{i}\frac{\partial w}{\partial x} + \hat{j}\frac{\partial w}{\partial y} + \hat{k}\frac{\partial w}{\partial z}$

$$\boxed{\text{grad}\, w = (z+y)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}}$$

Scalar triple product

$$[\text{grad}\,u \;\; \text{grad}\,v \;\; \text{grad}\,w] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{vmatrix}$$

$$R_2 \longrightarrow R_2 + R_3$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ y+z & x+z & x+y \end{vmatrix} = 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & x+z & x+y \end{vmatrix}$$

since R_1 and R_2 are same

$$\boxed{[\text{grad}\,u \;\; \text{grad}\,v \;\; \text{grad}\,w] = 0}$$

Hence grad u, grad v, grad w are coplanar vectors

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Vector Calculus : Vector Differentiation

Lec-3

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Gradient

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$grad\phi = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

Geometrical Interpretation

$\nabla\phi$ *is normal to the surface* $\phi(x, y, z) = c$

$$\vec{n} = \nabla\phi$$

Q.1 ***Find the angle between the surfaces*** $\boxed{x^2+y^2+z^2=9}$ ***and*** $\boxed{z=x^2+y^2-3}$ ***at the point*** $(2,-1,2)$.

$x^2+y^2+z^2=9$

$x^2+y^2+z^2-9=0$

Let $\phi_1 = x^2+y^2+z^2-9$

$z = x^2+y^2-3$

$x^2+y^2-z-3=0$

Let $\phi_2 = x^2+y^2-z-3=0$

Angle between ϕ_1 and ϕ_2 = Angle between $\vec{n_1}$ and $\vec{n_2}$

Where

$\vec{n_1} = \text{grad}\,\phi_1 = \nabla\phi_1$

$\vec{n_1} = \left(\hat{i}\frac{\partial}{\partial x}\phi_1 + \hat{j}\frac{\partial}{\partial y}\phi_1 + \hat{k}\frac{\partial}{\partial z}\phi_1\right)$

$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2-9) + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2-9) + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2-9)$

$\vec{n_1} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$$\vec{n_2} = \text{grad}\,\phi_2 = \nabla \phi_2$$

$$= \hat{i}\frac{\partial \phi_2}{\partial x} + \hat{j}\frac{\partial \phi_2}{\partial y} + \hat{k}\frac{\partial \phi_2}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2-z-3) + \hat{j}\frac{\partial}{\partial y}(x^2+y^2-z-3) + \hat{k}\frac{\partial}{\partial z}(x^2+y^2-z-3)$$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$\vec{n_1}$ and $\vec{n_2}$ at $(2,-1,2)$

$$\vec{n_1} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\vec{n_1} = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

$$\vec{n_2} = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\vec{n_2} = 4\hat{i} - 2\hat{j} - \hat{k}$$

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}$$

$$= \frac{(4\hat{i} - 2\hat{j} + 4\hat{k}) \cdot (4\hat{i} - 2\hat{j} - \hat{k})}{\sqrt{4^2+(-2)^2+4^2}\,\sqrt{4^2+(-2)^2+(-1)^2}}$$

$$\cos\theta = \frac{16 + \cancel{4} - \cancel{4}}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\boxed{\theta = \cos^{-1}\frac{8}{3\sqrt{21}}}$$

Q.2 Calculate the angle between the normals to the surface $\boxed{xy = z^2}$ at the points $(4, 1, 2)$ and $(3, 3, -3)$.

$$xy = z^2$$

$$xy - z^2 = 0$$

Let $\phi = xy - z^2$

$$\text{grad}\,\phi = \nabla \phi$$

$$= \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(xy - z^2) + \hat{j}\frac{\partial}{\partial y}(xy - z^2) + \hat{k}\frac{\partial}{\partial z}(xy - z^2)$$

$$= y\hat{i} + x\hat{j} - 2z\hat{k}$$

Normal vector at $(4,1,2)$

$$\vec{n_1} = \text{grad}\,\phi$$

$$= y\hat{i} + x\hat{j} + 2z\hat{k}$$

$$\vec{n_1} = \hat{i} + 4\hat{j} - 4\hat{k}$$

Normal vector at $(3,3,-3)$

$$\vec{n_2} = \text{grad}\,\phi$$

$$\vec{n_2} = y\hat{i} + x\hat{j} + 2z\hat{k}$$

$$\vec{n_2} = 3\hat{i} + 3\hat{j} + 6\hat{k}$$

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}$$

$$= \frac{(\hat{i}+4\hat{j}-4\hat{k})\cdot(3\hat{i}+3\hat{j}+6\hat{k})}{\sqrt{1^2+4^2+(-4)^2}\,\sqrt{3^2+3^2+6^2}}$$

$$= \frac{3+12-24}{\sqrt{33}\,\sqrt{54}}$$

$$= \frac{-9}{\sqrt{33\times 54}} = \frac{-9}{\sqrt{3\times 11\times 3\times 3\times 3\times 2}}$$

$$\cos\theta = \frac{-\cancel{9}}{\cancel{9}\sqrt{22}}$$

$$\cos\theta = \frac{-1}{\sqrt{22}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{-1}{\sqrt{22}}\right)}$$

Practice Q.1 *If θ is the acute angle between the surfaces* $xy^2z = 3x + z^2$ *and* $3x^2 - y^2 + 2z = 1$ *at the point* $(1, -2, 1)$ *show that* $\cos\theta = \dfrac{3}{7\sqrt{6}}$

Q.3 Prove that $\nabla \log r = \dfrac{\vec{r}}{r^2}$

We know that

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

LHS

$$\nabla \log r = \hat{i}\frac{\partial}{\partial x}(\log r) + \hat{j}\frac{\partial}{\partial y}(\log r) + \hat{k}\frac{\partial}{\partial z}(\log r)$$

$$= \frac{1}{r} \times \frac{\partial r}{\partial x}\hat{i} + \frac{1}{r} \times \frac{\partial r}{\partial y}\hat{j} + \frac{1}{r}\frac{\partial r}{\partial z}\hat{k}$$

$$= \frac{1}{r} \times \frac{x}{r}\hat{i} + \frac{1}{r} \times \frac{y}{r}\hat{j} + \frac{1}{r}\frac{z}{r}\hat{k}$$

$$= \frac{1}{r^2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\boxed{\nabla \log r = \frac{\vec{r}}{r^2}}$$

Hence proved

Q.4 Show that $\nabla r^n = n r^{n-2}\vec{r}$ and hence evaluate $\nabla \frac{1}{r}$ whre $\vec{r} = x\hat{j} + y\hat{i} + z\hat{k}$.

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

$\frac{\partial r}{\partial x} = \frac{x}{r}$

$\frac{\partial r}{\partial y} = \frac{y}{r}$

$\frac{\partial r}{\partial z} = \frac{z}{r}$

LHS

$\nabla r^n = \hat{i}\frac{\partial(r^n)}{\partial x} + \hat{j}\frac{\partial(r^n)}{\partial y} + \hat{k}\frac{\partial(r^n)}{\partial z}$

$= n r^{n-1}\frac{\partial r}{\partial x}\hat{i} + n r^{n-1}\frac{\partial r}{\partial y}\hat{j} + n r^{n-1}\frac{\partial r}{\partial z}\hat{k}$

$= n r^{n-1}\times\frac{x}{r}\hat{i} + n r^{n-1}\times\frac{y}{r}\hat{j} + n r^{n-1}\times\frac{z}{r}\hat{k}$

$= \frac{n r^{n-1}}{r}(x\hat{i} + y\hat{j} + z\hat{k})$

$= n r^{n-2}\times\vec{r}$

$\boxed{\nabla r^n = n r^{n-2}\vec{r}}$

Hence proved

Put $n = -1$

$\nabla r^{-1} = -1\times r^{-1-2}\vec{r}$

$\nabla\frac{1}{r} = -1\times r^{-3}\times\vec{r}$

$\boxed{\nabla\frac{1}{r} = \frac{-\vec{r}}{r^3}}$

Practice Q.2 If $\vec{r} = x\hat{j} + y\hat{i} + z\hat{k}$, then show that grad r = $\frac{\vec{r}}{r}$.

Thank You

AKTU : Engineering Mathematics-1

Vector Calculus : Vector Differentiation

Lec-4

Today's Target

- ✓ Directional Derivative
- ✓ Univ. Questions
- ✓ Practice Questions

By Gulshan sir

Directional Derivative

The directional derivative of a scalar point function f at a point $P\,(x, y, z)$ in the direction unit vector $\hat{a}$ is given by

$$\boxed{\frac{df}{ds} = (\text{grad} f) \cdot \hat{a}}$$

Note : *The maximum/greatest rate of increase of $f = |\nabla f|$*

Or

Maximum/Greatest value of directional derivative = $|\nabla f|$

Q.1 Find the directional derivative of $\boxed{\phi(x,y,z) = x^2yz + 4xz^2}$ at $\boxed{(1,-2,1)}$ in the direction of $\boxed{2\hat{i} - \hat{j} - 2\hat{k}}$. Find also the greatest rate of increase of ϕ.

$$\phi(x,y,z) = x^2yz + 4xz^2$$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^2yz + 4xz^2) + \hat{j}\frac{\partial}{\partial y}(x^2yz + 4xz^2) + \hat{k}\frac{\partial}{\partial z}(x^2yz + 4xz^2)$$

$$\nabla\phi = \hat{i}(2xyz + 4z^2) + \hat{j}(x^2z + 0) + \hat{k}(x^2y + 8xz)$$

At $(1,-2,1)$

$$\nabla\phi = \hat{i}(\cancel{-4} + \cancel{4}) + \hat{j}(1+0) + \hat{k}(-2+8)$$

$$\boxed{\nabla\phi = \hat{j} + 6\hat{k}}$$

Let $\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$

Unit vector

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

$$= \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{9}}$$

$$\hat{a} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3}$$

$$\hat{a} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

Now

Directional derivative

$$\frac{d\phi}{ds} = (\text{grad}\,\phi)\cdot\hat{a}$$

$$= (0\hat{i} + \hat{j} + 6\hat{k})\cdot\left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

$$= 0 - \frac{1}{3} + \cancel{6}^{\,2}\left(\frac{-2}{\cancel{3}}\right)$$

$$= -\frac{1}{3} - 4$$

$$\frac{d\phi}{ds} = \frac{-1-12}{3}$$

$$\boxed{\frac{d\phi}{ds} = -\frac{13}{3}}$$

Greatest rate of increase of $\phi = |\nabla\phi|$

$$= |\hat{j} + 6\hat{k}|$$

$$= \sqrt{1^2 + 6^2}$$

$$\boxed{= \sqrt{37}}$$

Q.2 *Find the directional derivative of* $\boxed{\emptyset = (x^2 + y^2 + z^2)^{-\frac{1}{2}}}$ *at the point* $\boxed{(3,1,2)}$ *in the direction of vector* $\boxed{yz\hat{i} + zx\hat{j} + xy\hat{k}}$.

$$\phi = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}} + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-\frac{1}{2}} + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$\nabla\phi = \hat{i}\left[-\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{3}{2}}\cancel{2}x\right] + \hat{j}\left[-\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{3}{2}}\cancel{2}y\right] + \hat{k}\left[-\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{3}{2}}\cancel{2}z\right]$$

$$\nabla\phi = -(x^2+y^2+z^2)^{-\frac{3}{2}}(x\hat{i} + y\hat{j} + z\hat{k})$$

At $(3,1,2)$

$$\nabla\phi = \frac{-1}{(x^2+y^2+z^2)^{3/2}}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$\nabla\phi = \frac{-1}{(9+1+4)^{3/2}}(3\hat{i}+\hat{j}+2\hat{k})$$

$$\nabla\phi = \frac{-1(3\hat{i}+\hat{j}+2\hat{k})}{(14)^{3/2}}$$

$$\text{grad}\,\phi = \nabla\phi = \frac{-(3\hat{i}+\hat{j}+2\hat{k})}{14\sqrt{14}}$$

Let $\vec{a} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

Unit vector

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{yz\hat{i}+zx\hat{j}+xy\hat{k}}{\sqrt{y^2z^2+z^2x^2+x^2y^2}}$$

At $(3,1,2)$

$$\hat{a} = \frac{2\hat{i}+6\hat{j}+3\hat{k}}{\sqrt{4+36+9}} = \frac{2\hat{i}+6\hat{j}+3\hat{k}}{7}$$

Directional Derivative

$$\frac{d\phi}{ds} = (\text{grad}\,\phi)\cdot\hat{a} = \frac{(-3\hat{i}-\hat{j}-2\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})}{14\sqrt{14}\times 7}$$

$$\boxed{\frac{d\phi}{ds} = \frac{-9}{49\sqrt{14}}}$$

Q.3 *Find the directional derivative of the function* $\boxed{f = x^2 - y^2 + 2z^2}$ *at the point* $\boxed{P\,(1,2,3)}$ *in the direction of line PQ where Q is the point* $\boxed{(5,0,4).}$

in what direction it will be maximum? Find also the magnitude of this maximum.

$$f = x^2 - y^2 + 2z^2$$

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$\nabla f = \hat{i}\frac{\partial}{\partial x}(x^2 - y^2 + 2z^2) + \hat{j}\frac{\partial}{\partial y}(x^2 - y^2 + 2z^2) + \hat{k}\frac{\partial}{\partial z}(x^2 - y^2 + 2z^2)$$

$$\nabla f = 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$

At $(1,2,3)$

$$\nabla f = 2\hat{i} - 4\hat{j} + 12\hat{k}$$

Now

$P(1,2,3)$ and $Q(5,0,4)$

$$\overrightarrow{PQ} = (5-1)\hat{i} + (0-2)\hat{j} + (4-3)\hat{k}$$

$$\overrightarrow{PQ} = 4\hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{a} = 4\hat{i} - 2\hat{j} + \hat{k}$$

Unit vector

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16+4+1}}$$

$$\hat{a} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

Directional derivative

$$\frac{df}{ds} = (\nabla f) \cdot \hat{a}$$

$$\frac{df}{ds} = (2\hat{i} - 4\hat{j} + 12\hat{k}) \frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}}$$

$$= \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}} = \frac{28}{\sqrt{21}} \times \frac{\sqrt{21}}{\sqrt{21}}$$

$$= \frac{\overset{4}{\cancel{28}}\sqrt{21}}{\underset{3}{\cancel{21}}} = \frac{4}{3}\sqrt{21}$$

$$\boxed{\frac{df}{ds} = \frac{4}{3}\sqrt{21}}$$

It will be maximum in the direction of ∇f

$$\text{Maximum value} = |\nabla f| = 2\hat{i} - 4\hat{j} + 12\hat{k}$$

$$= \sqrt{4+16+144} = \sqrt{164} = 2\sqrt{41}$$

Q.4 ***Find the directional derivative of*** $\boxed{\emptyset = 5x^2y - 5y^2z + \left(\frac{5}{2}\right)z^2x}$ ***at the point*** $\boxed{(1,1,1)}$ ***in the direction of line*** $\boxed{\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}}$.

$$\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}\left(5x^2y - 5y^2z + \frac{5}{2}z^2x\right) + \hat{j}\frac{\partial}{\partial y}\left(5x^2y - 5y^2z + \frac{5}{2}z^2x\right) + \hat{k}\frac{\partial}{\partial z}\left(5x^2y - 5y^2z + \frac{5}{2}z^2x\right)$$

$$\nabla\phi = \hat{i}\left(10xy + \frac{5}{2}z^2\right) + \hat{j}\left(5x^2 - 10yz\right) + \hat{k}\left(-5y^2 + 5xz\right)$$

$$\nabla\phi = \frac{25}{2}\hat{i} - 5\hat{j} + 0\hat{k}$$

Given line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$$

$$\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$$

Unit vector

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

$$\hat{a} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{4+4+1}}$$

$$\hat{a} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$$

Directional Derivative

$$\frac{d\phi}{ds} = (\nabla\phi) \cdot \hat{a} = \left(\frac{25}{2}\hat{i} - 5\hat{j} + 0\hat{k}\right) \cdot \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$$

$$= \left(\cancel{\frac{50}{2}}^{25} + 10\right) \times \frac{1}{3}$$

$$\boxed{\frac{d\phi}{ds} = \frac{35}{3}}$$

Q.5 ***Find the directional derivative of the function*** $\boxed{\emptyset = xy^2 + yz^3}$ ***at the point*** $\boxed{(2,-1,1)}$ ***in the direction of the normal to the surface*** $\boxed{x\log z - y^2 + 4 = 0}$ ***at*** $\boxed{(2,-1,1)}$.

$$\phi = xy^2 + yz^2$$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy^2 + yz^3) + \hat{j}\frac{\partial}{\partial y}(xy^2 + yz^3) + \hat{k}\frac{\partial}{\partial z}(xy^2 + yz^3)$$

$$\nabla\phi = \hat{i}(y^2) + \hat{j}(2xy + z^3) + \hat{k}(3yz^2)$$

At $(2,-1,1)$

$$\nabla\phi = \hat{i} + (-4+1)\hat{j} + \hat{k}(-3)$$

$$\nabla\phi = \hat{i} - 3\hat{j} - 3\hat{k}$$

Let $f = x\log z - y^2 + 4$

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$\nabla f = \hat{i}\frac{\partial}{\partial x}(x\log z - y^2 + 4) + \hat{j}\frac{\partial}{\partial y}(x\log z - y^2 + 4) + \hat{k}\frac{\partial}{\partial z}(x\log z - y^2 + 4)$$

$$\nabla f = (\log z)\hat{i} - 2y\hat{j} + \frac{x}{z}\hat{k}$$

At $(2, -1, 1)$

$$\nabla f = 0\hat{i} + 2\hat{j} + 2\hat{k}$$

where

$$\vec{a} = \nabla f$$

$$\vec{a} = 0\hat{i} + 2\hat{j} + 2\hat{k}$$

Unit vector

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{0\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{4+4}} = \frac{0\hat{i} + 2\hat{j} + 2\hat{k}}{2\sqrt{2}}$$

Directional Derivative

$$\frac{d\phi}{ds} = (\nabla\phi)\cdot\hat{a} = (\hat{i} - 3\hat{j} - 3\hat{k})\cdot\frac{(0\hat{i} + 2\hat{j} + 2\hat{k})}{2\sqrt{2}}$$

$$= \frac{0 - 6 - 6}{2\sqrt{2}} = \frac{-\cancel{12}\,6}{\cancel{2}\sqrt{2}} = \frac{-6}{\sqrt{2}}$$

$$\boxed{\frac{d\phi}{ds} = -3\sqrt{2}}$$

Q.6 Find the directional derivative of $\frac{1}{r}$ in the direction of $\vec{r}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$\cancel{2} r \frac{\partial r}{\partial x} = \cancel{2} x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

Let $\phi = \frac{1}{r}$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} + \hat{j}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + \hat{k}\left(\frac{-1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= -\frac{1}{r^2}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$\nabla\phi = \frac{-1}{r^3}(x\hat{i} + y\hat{j} + z\hat{k})$$

Unit vector

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$\hat{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}$$

Directional Derivative

$$\frac{d\phi}{ds} = \nabla\phi \cdot \hat{r} = \frac{-1}{r^3}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) \cdot \frac{\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}{r}$$

$$= \frac{-1}{r^4}\left(x^2 + y^2 + z^2\right)$$

$$= -\frac{1}{r^{4-2}} \times \cancel{r^2}$$

$$\boxed{\frac{d\phi}{ds} = -\frac{1}{r^2}}$$

Q.7 *Find the directional derivative of $\frac{1}{r^2}$ in the direction of $\vec{r}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.*

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AKTU : Engineering Mathematics-1

Vector Calculus : Vector Differentiation

Lec-5

Today's Target

- ✓ Divergence and Curl of a Vector Point Function
- ✓ Univ. Questions
- ✓ Practice Questions

By Gulshan sir

Divergence of a Vector Point Function

The divergence of a diffrentiable vector point function $\vec{V}$ *is denoted by* $div\,\vec{V}$ and is defined as

$$\boxed{div\,\vec{V} = \nabla.\vec{V}}$$

where $\boxed{\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$

$$\boxed{\vec{V} = V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k}}$$

$$div\vec{V} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k}\right)$$

$$\boxed{div\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}}$$

Note : *Divergence of a vector point function is a scalar point function.*

Q.1 If $\vec{V} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$, then find div V at (1, 1, 1).

Univ. Exam

GW

$$\vec{V} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$$

$$\text{div}\,\vec{V} = \nabla \cdot \vec{V}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) - \frac{\partial}{\partial z}(3yz^2)$$

$$= y^2 + 2x^2z - 6yz$$

At (1, 1, 1)

$$\text{div}\,\vec{V} = y^2 + 2x^2z - 6yz$$

$$\text{div}\,\vec{V} = 1^2 + 2\times1\times - 6\times1\times1$$

$$= 1 + 2 - 6$$

$$= 3 - 6$$

$$\boxed{\text{div}\,\vec{V} = -3}$$

Q. 2 *If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that $div\,\vec{r} = 3$.*

Univ. Exam

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{div}\,\vec{r} = \nabla \cdot \vec{r}$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$\boxed{\text{div}\,\vec{r} = 3}$$

Solenoidal Vector :

If $\boxed{div\,\vec{V} = 0}$, then $\vec{V}$ *is called solenoidal vector.*

Univ. Exam

Q.3 *Show that vector* $\boxed{\vec{V} = (x + 3y)\hat{i} + (y - 3z)\hat{j} + (x - 2z)\hat{k}}$ *is solenoidal.*

$\vec{V} = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$

$div\,\vec{V} = \nabla \cdot \vec{V}$

$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$

$= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-3z) + \frac{\partial}{\partial z}(x-2z)$

$div\,\vec{V} = 1 + 1 + (-2)$

$= \cancel{2} - \cancel{2}$

$\boxed{div\,\vec{V} = 0}$

Hence, $\vec{V}$ is a solenoidal vector

Curl of a Vector Point Function

The curl (or rotation) of a differentiable vector point function $\vec{V}$ is denoted by $\boxed{\text{curl}\,\vec{V}}$ and is defined as

$$\boxed{\text{curl}\,\vec{V} = \nabla \times \vec{V}}$$

where

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$$

$$\boxed{\text{curl}\,V = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}}$$

Note : *Curl of a vector point function is also a vector point function.*

$Q.4$ *If* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, *show that curl* $\vec{r} = \vec{0}$.

Univ. Exam

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\text{curl } \vec{r} = \nabla \times \vec{r}$

$$\text{curl } \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right) - \hat{j}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$= \vec{0}$$

Irrotational Vector :

If curl $\vec{V} = \vec{0}$, then $\vec{V}$ *is called irrotational vector.*

Univ. Exam

Q.5 ***Show that vector field*** $\boxed{\vec{V} = (\sin y + z)\hat{i} + (x\cos y - z)\hat{j} + (x - y)\hat{k}}$ ***is irrotational.***

$$\vec{V} = (\sin y + z)\hat{i} + (x\cos y - z)\hat{j} + (x-y)\hat{k}$$

$$\text{curl } \vec{V} = \nabla \times \vec{V}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & x\cos y - z & x-y \end{vmatrix}$$

$$\text{curl } \vec{V} = \hat{i}\left[\frac{\partial}{\partial y}(x-y) - \frac{\partial}{\partial z}(x\cos y - z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(x-y) - \frac{\partial}{\partial z}(\sin y + z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x\cos y - z) - \frac{\partial}{\partial y}(\sin y + z)\right]$$

$$= \hat{i}(\cancel{-1} + \cancel{1}) - \hat{j}(\cancel{1} - \cancel{1}) + \hat{k}(\cancel{\cos y} - \cancel{\cos y})$$

$\boxed{\text{curl } \vec{V} = 0}$ Hence, $\vec{V}$ is irrotational

Q.6 Find the divergence and curl of the vector Univ. Exam

$$\vec{R} = (x^2 + yz)\hat{\imath} + (y^2 + zx)\hat{\jmath} + (z^2 + xy)\hat{k}.$$

$$\operatorname{div} \vec{R} = \nabla \cdot \vec{R}$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(x^2+yz) + \frac{\partial}{\partial y}(y^2+zx) + \frac{\partial}{\partial z}(z^2+xy)$$

$$= 2x + 2y + 2z$$

$$\boxed{\operatorname{div} \vec{R} = 2(x+y+z)}$$

$$\operatorname{curl} \vec{R} = \nabla \times \vec{R}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+yz & y^2+zx & z^2+xy \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(z^2+xy) - \frac{\partial}{\partial z}(y^2+zx)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}(z^2+xy) - \frac{\partial}{\partial z}(x^2+yz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y^2+zx) - \frac{\partial}{\partial y}(x^2+yz)\right]$$

$$\operatorname{curl} \vec{R} = \hat{\imath}(x-x) - \hat{\jmath}(y-y) + \hat{k}(z-z)$$

$$\boxed{\operatorname{curl} \vec{R} = 0}$$

Hence $\vec{R}$ is an irrotational vector

Practice Q.1 *Find the div* $\vec{F}$ *and curl* $\vec{F}$ *where* $\boxed{\vec{F} = grad(x^3 + y^3 + z^3 - 3xyz).}$

Univ. Exam

Ans div = 6(x+y+z)

Curl = 0

Practice Q.2 ***Find the the value of K if the vector***

$$\vec{V} = (x + ky)\hat{\imath} + (ky - 3z)\hat{\jmath} + (x - 2z)\hat{k}$$

is a solenoidal vector

Univ. Exam

Ans. K = 1

AKTU : Engineering Mathematics-1

Vector Calculus : Vector Differentiation

Lec-6

Today's Target

- ✓ Divergence and Curl (Part-2)
- ✓ Univ. Questions
- ✓ Practice Questions

By Gulshan sir

Physical Interpretation of DIVERGENCE

- It gives the rate of flow originating per unit volume at any point in vector field .
- If the fluid is incompressible

$$\boxed{div\,\vec{V} = 0}$$

- If $\boxed{div\,\vec{V} = 0}$

  $\vec{V}$ is solenoidal vector function

Physical Interpretation of curl

- It gives angular velocity at any point in vector field .

- If $\boxed{curl\,\vec{V} = 0}$

  then $\vec{V}$ is irrotational vector

Note : for a constant vector $\vec{a}$

$$\boxed{div\,\vec{a} = 0} \qquad \boxed{\cancel{div\,\vec{a} = 0}} \qquad curl\,\vec{a} = 0$$

$Q.1$ If $\vec{A} = xz^2\hat{\imath} + 2y\hat{\jmath} - 3xz\hat{k}$ and $\vec{B} = 3xz\hat{\imath} + 2yz\hat{\jmath} - z^2\hat{k}$, find the value of $\vec{A} \times (\nabla \times \vec{B})$

Univ. Exam

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xz & 2yz & -z^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-z^2) - \frac{\partial}{\partial z}(2yz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-z^2) - \frac{\partial}{\partial z}(3xz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(3xz)\right]$$

$$= \hat{i}(0 - 2y) - \hat{j}(0 - 3x) + \hat{k}(0 - 0)$$

$$\boxed{\nabla \times \vec{B} = -2y\hat{i} + 3x\hat{j}}$$

$$\vec{A}\times(\nabla\times\vec{B}) = (xz^2\hat{i} + 2y\hat{j} - 3xz\hat{k})\times(-2y\hat{i} + 3x\hat{j})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ xz^2 & 2y & -3xz \\ -2y & 3x & 0 \end{vmatrix}$$

$$= \hat{i}\left[0 + 9x^2z\right] - \hat{j}\left[0 - 6xyz\right] + \hat{k}\left[3x^2z^2 + 4y^2\right]$$

$$\boxed{\vec{A}\times(\nabla\times\vec{B}) = 9x^2z\,\hat{i} + 6xyz\,\hat{j} + (3x^2z^2 + 4y^2)\hat{k}}$$

Velocity Potential or Scalar potential

If vector $\vec{V}$ is an irrotational vector then

$$\boxed{\text{curl}\ \vec{V} = 0}$$

Let ϕ is a scalar poteintial.

Let $\vec{V} = \nabla\phi$

$$\vec{V}\cdot\vec{dr} = \nabla\phi\cdot\vec{dr}$$

$$\vec{V}\cdot\vec{dr} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$\vec{V}\cdot\vec{dr} = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$\boxed{\vec{V}\cdot\vec{dr} = d\phi}$$

$$\boxed{d\phi = \vec{V}\cdot\vec{dr}}$$

$$\int d\phi = \int\vec{V}\cdot\vec{dr}$$

$$\boxed{\phi = \int\vec{V}\cdot\vec{dr}}$$

Q.2 *A fluid motion is given by* $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$.

(i) *Is this motion irrotational? If so, find the velocity potential.*

(ii) *Is the motion possible for an incompressible fluid?*

$$\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$$

The motion is irrotational if

$$\text{curl}\,\vec{V} = 0$$

$$\text{curl}\,\vec{V} = \nabla \times \vec{V}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(y+z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right]$$

$$= \hat{i}(\cancel{1}-\cancel{1}) - \hat{j}(\cancel{1}-\cancel{1}) + \hat{k}(\cancel{1}-\cancel{1})$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k}$$

$$= \vec{0}$$

Hence, motion is irrotational

We know that

$$d\phi = \vec{v} \cdot \vec{dr}$$

$$d\phi = \left[(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}\right] \cdot \left[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right]$$

$$d\phi = (y+z)dx + (z+x)dy + (x+y)dz$$

$$d\phi = y\,dx + z\,dx + z\,dy + x\,dy + x\,dz + y\,dz$$

$$= \boxed{x\,dy + y\,dx} + \boxed{x\,dz + z\,dx} + \boxed{y\,dz + z\,dy}$$

$$d\phi = d(xy) + d(xz) + d(yz)$$

Integrate both side

$$\boxed{\phi = xy + xz + yz + C}$$

Motion is possible for incompressible fluid if

$$\text{div}\,\vec{v} = 0$$

$$\text{div}\,\vec{v} = \nabla \cdot \vec{v}$$

$$\text{div}\,\vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}\right]$$

$$\text{div}\,\vec{v} = \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y}(z+x) + \frac{\partial}{\partial z}(x+y)$$

$$\text{div}\,\vec{v} = 0 + 0 + 0 = 0$$

Q.3 *A fluid motion is given by* $\boxed{\vec{v} = (y\sin z - \sin x)\hat{\imath} + (x\sin z + 2yz)\hat{\jmath} + (xy\cos z + y^2)\hat{k}.}$

Univ. Exam

Is the motion irrotational? If so, find the velocity potential.

We know that

$$d\phi = \vec{V} \cdot \vec{dr}$$

$$d\phi = \left[(y\sin z - \sin x)\hat{\imath} + (x\sin z + 2yz)\hat{\jmath} + (xy\cos z + y^2)\hat{k}\right] \cdot \left[dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}\right]$$

$$d\phi = (y\sin z - \sin x)dx + (x\sin z + 2yz)dy + (xy\cos z + y^2)dz$$

$$d\phi = y\sin z\,dx - \underline{\sin x\,dx} + x\sin z\,dy + 2yz\,dy + xy\cos z\,dz + y^2 dz$$

$$d\phi = -\sin x\,dx + (y\sin z\,dx + x\sin z\,dy + xy\cos z\,dz) + y^2 dz + 2yz\,dy$$

$$d\phi = d(\cos x) + d(xy\sin z) + d(y^2 z)$$

Integrate both side

$$\boxed{\phi = \cos x + xy\sin z + y^2 z + C}$$

Q.4 *Find the constants a, b, c so that*

$$\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}. \text{ is irrotational}$$

$$\text{If } \vec{F} = \text{grad}\emptyset, \quad \text{show that } \emptyset = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz.$$

Since $\vec{F}$ is an irrotational

$$\text{curl}\,\vec{F} = 0$$

$$\nabla \times \vec{F} = 0$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$\hat{i}\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$= 0$$

$$\hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2) = 0$$

$$(c+1)\hat{i} + (a-4)\hat{j} + (b-2)\hat{k} = \vec{0}$$

$$(c+1)\hat{i} + (a-4)\hat{j} + (b-2)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\Rightarrow \quad c+1 = 0 \qquad a-4 = 0 \qquad b-2 = 0$$

$$\boxed{c = -1} \qquad \boxed{a = 4} \qquad \boxed{b = 2}$$

We know that

$$d\phi = \vec{F} \cdot \vec{dr}$$

Practice Question.1

Univ. Exam

A vector field is given by $\vec{A} = (x^2 + xy^2)\hat{\imath} + (y^2 + x^2y)\hat{\jmath}$ *. Show that the field is irrotational and find the scalar potential.*

$$Ans.\ \emptyset = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + C$$

Practice Question.2

Univ. Exam

Show that $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ *is irrotational*.
Find the velocity potential $\emptyset$ *such that* $\vec{A} = \nabla\emptyset$.

Practice Question.3

Univ. Exam

Prove that $\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$. *is both solenoidal and irrotational*

Also *find the scalar potential* $\emptyset$ *such that* $\vec{F} = grad\emptyset$.

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Thank You

AKTU : Engineering Mathematics-1

Vector Calculus : Vector Integration

Lec-7

Today's Target

- ✓ Line Integral
- ✓ Univ. Questions
- ✓ Practice Questions

By Gulshan sir

Line Integral

Line integral of a vector function $\vec{F}(x, y, z)$ *along the curve C is defined as*

$$\boxed{\text{Line Integral} = \int_C \vec{F} \cdot \vec{dr}}$$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\boxed{\vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}}$$

Note:

1. ***Work***

If $\vec{F}$ *represent the variable force acting on a particle along a curve C, then work (W) is*

$$\boxed{W = \int_C \vec{F} \cdot \vec{dr}}$$

2. *Circulation*

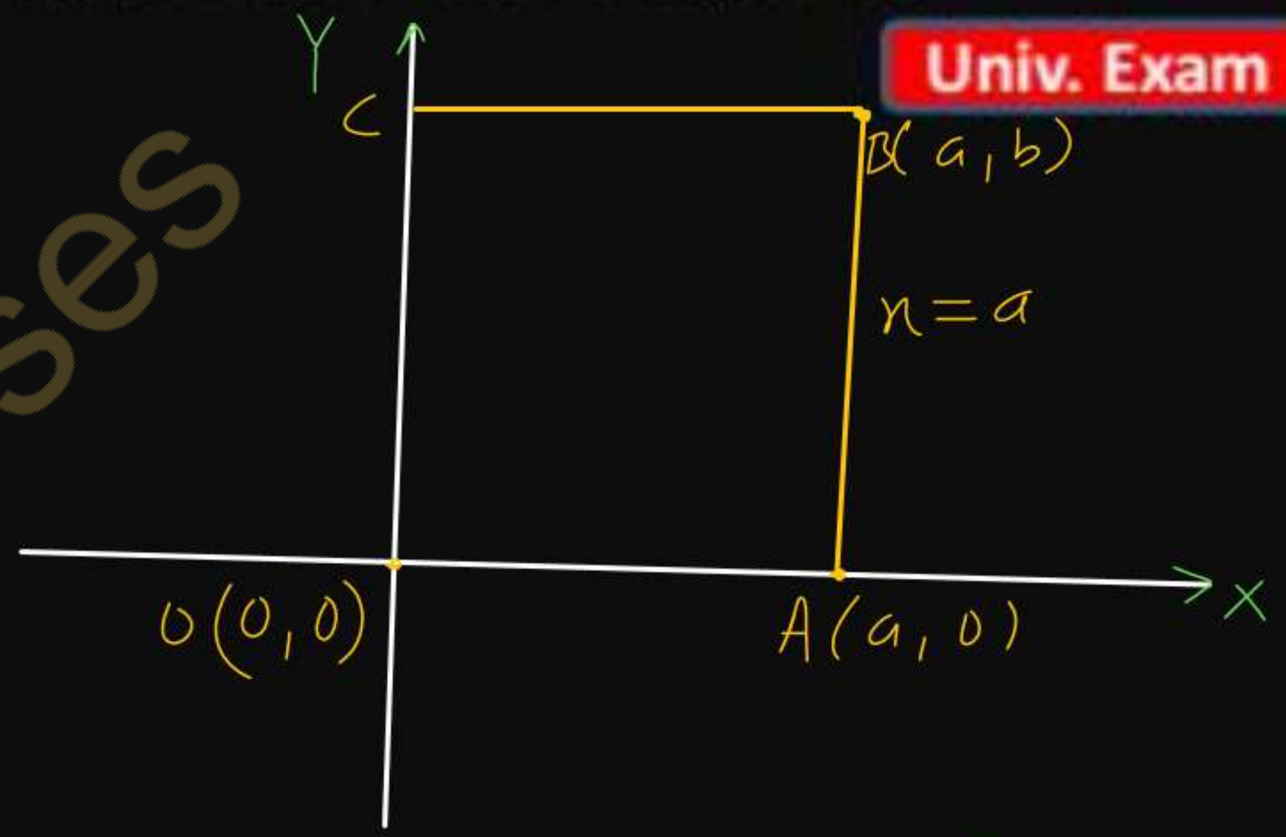
If $\vec{F}$ *represent the velocity of a fluid particle and C is closed curve*

$$\boxed{\textit{circulation} = \oint_C \vec{F}\cdot\vec{dr}}$$

3. If $\oint \vec{F}.\overrightarrow{dr} = 0$, then $\vec{F}$ *is said to be irrotational .*

Q. 1 *Find the total work done by a force* $\boxed{\vec{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}}$ *in moving a point from* $(0, 0)$ *to* (a, b) *along the rectangle bounded by the lines* $\boxed{x = 0, x = a, y = 0 \text{ and } y = b.}$

Univ. Exam



$$\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$$

$$\vec{dr} = dx\hat{i} + dy\hat{j}$$

$$\vec{F}\cdot\vec{dr} = \left[(x^2+y^2)\hat{i} - 2xy\hat{j}\right]\cdot\left(dx\hat{i} + dy\hat{j}\right)$$

$$\vec{F}\cdot\vec{dr} = (x^2+y^2)dx - 2xy\,dy \quad \text{—— (1)}$$

$$W = \int_C \vec{F}\cdot\vec{dr}$$

$$W = \int_{OA} \vec{F}\cdot\vec{dr} + \int_{AB} \vec{F}\cdot\vec{dr}$$

<u>Along OA</u>

$y = 0$

$dy = 0$

$x = 0$ to $x = a$

<u>Along AB</u>

$x = a$

$dx = 0$

$y = 0$ to $y = b$

$$W = \int_{OA} \vec{F} \cdot \vec{dr} + \int_{AB} \vec{F} \cdot \vec{dr}$$

$$= \int_0^a x^2 dx + \int_0^b (-2ay) dy$$

$$= \int_0^a x^2 dx - 2a \int_0^b y \, dy$$

$$= \left(\frac{x^3}{3} \right)_0^a - 2a \left(\frac{y^2}{2} \right)_0^b$$

$$= \frac{a^3}{3} - \cancel{2} a \times \frac{b^2}{\cancel{2}}$$

$$\boxed{W = \frac{a^3}{3} - ab^2}$$

Univ. Exam

Q. 2 *Suppose* $\boxed{\vec{F}(x,y,z) = x^3\hat{\imath} + y\hat{\jmath} + z\hat{k}}$ *is the force field. Find the work done by* $\vec{F}$ *along the line from* $\boxed{(1,2,3) \text{ to } (3,5,7)}$.

$$\vec{F} = x^3\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\vec{F}\cdot\vec{dr} = x^3dx + y\,dy + z\,dz \quad \text{---} \; (1)$$

Equation of line passing through $(1,2,3)$ and $(3,5,7)$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = t \text{ (say)}$$

$x = 2t+1 \qquad dx = 2dt \qquad$ At $(1,2,3)$

$y = 3t+2 \qquad dy = 3\,dt \qquad t = 0$

$z = 4t+3 \qquad dz = 4dt \qquad$ At $(3,5,7)$

$t = 1$

$$W = \int_C \vec{F}\cdot\vec{dr} = \int_C x^3dx + y\,dy + z\,dz$$

$$W = \int_0^1 (2t+1)^3 \times 2dt + (3t+2)\times 3dt + (4t+3)\times 4dt$$

$$W = \int_0^1 2(2t+1)^3dt + 3(3t+2)dt + 4(4t+3)dt$$

$$W = \int_0^1 \left[2(8t^3+12t^2+6t+1) + 9t+6 + 16t+12\right] dt$$

$$= \int_0^1 (16t^3 + 24t^2 + 12t + 2 + 9t + 6 + 16t + 12)\, dt$$

$$= \int_0^1 (16t^3 + 24t^2 + 37t + 20)\, dt$$

$$= \left[\frac{16t^4}{4} + \frac{24t^3}{3} + \frac{37t^2}{2} + 20t\right]_0^1$$

$$= \cancel{\frac{16}{4}}^{\,4} + \cancel{\frac{24}{3}}^{\,8} + \frac{37}{2} + 20$$

$$= 12 + 20 + \frac{37}{2}$$

$$= 32 + \frac{37}{2}$$

$$= \frac{64+37}{2}$$

$$= \frac{101}{2}$$

$$\boxed{W = \frac{101}{2} \text{ units}}$$

2nd Method

$$W = \int_c \vec{F} \cdot \vec{dr}$$

$$W = \int_c x^3 dx + y\,dy + z\,dz$$

$$W = \int_c x^3 dx + \int_c y\,dy + \int_c z\,dz$$

$(1, 2, 3)$ to $(3, 5, 7)$

$$W = \int_1^3 x^3 dx + \int_2^5 y\,dy + \int_3^7 z\,dz$$

Q.3 *If $\vec{A} = (x-y)\hat{i} + (x+y)\hat{j}$, evaluate $\oint \vec{A}.\vec{dr}$ around the curve C consisting of $y = x^2$ and $x = y^2$.*

Univ. Exam

$\vec{A} = (x-y)\hat{i} + (x+y)\hat{j}$

$\vec{dr} = dx\hat{i} + dy\hat{j}$

$\vec{A}.\vec{dr} = (x-y)dx + (x+y)dy$ — (1)

Now

$y = x^2$ — (2)

$x = y^2$ — (3)

Solve (2) and (3)

$y = (y^2)^2$

$y = y^4$

$y^4 - y = 0$

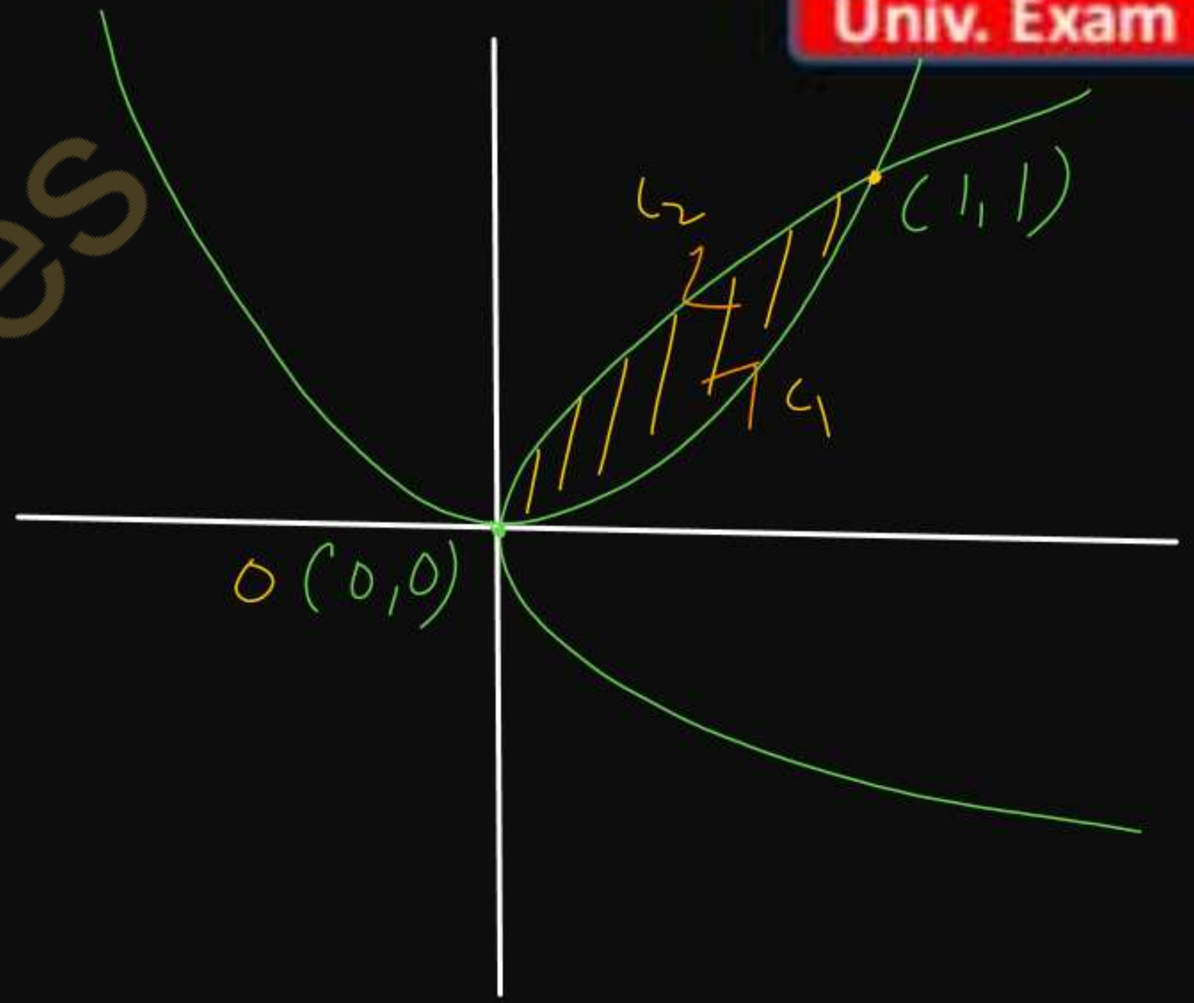
$y(y^3-1) = 0$

$y = 0$

$y = 1$

When $y = 0$, $x = 0$

When $y = 1$, $x = 1$



$$\boxed{\oint \vec{A}.\vec{dr} = \int_{C_1} \vec{A}.\vec{dr} + \int_{C_2} \vec{A}.\vec{dr}}$$

For C_1

$y = x^2$

$dy = 2x\,dx$

$x = 0$ to $x = 1$

For C_2

$x = y^2$

$dx = 2y\,dy$

$y = 1$ to $y = 0$

$$\oint \vec{A}\cdot d\vec{r} = \int_{C_1} (x-y)\,dx + (x+y)\,dy + \int_{C_2} (x-y)\,dx + (x+y)\,dy$$

$$= \int_0^1 (x - x^2)\,dx + (x + x^2)\,2x\,dx + \int_1^0 (y^2 - y)\,2y\,dy + (y^2 + y)\,dy$$

$$= \int_0^1 (x - x^2 + 2x^2 + 2x^3)\,dx + \int_1^0 (2y^3 - 2y^2 + y^2 + y)\,dy$$

$$= \int_0^1 (x + x^2 + 2x^3)\,dx + \int_1^0 (2y^3 - y^2 + y)\,dy$$

$$= \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{\cancel{2}x^4}{\cancel{4}\,2}\right)_0^1 + \left[\frac{\cancel{2}y^4}{\cancel{4}\,2} - \frac{y^3}{3} + \frac{y^2}{2}\right]_1^0$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + 0 - \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{2}\right)$$

$$\cancel{\frac{1}{2}} + \frac{1}{3} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \frac{1}{3} - \cancel{\frac{1}{2}}$$

$$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\boxed{\oint \vec{A}\cdot d\vec{r} = \frac{2}{3}}$$

Q.4 ***Find the work done in moving a particle in the force field***

$$\boxed{\vec{F} = 3x^2\hat{\imath} + (2xz - y)\hat{\jmath} + z\hat{k}}$$

along the curve defined by $x^2 = 4y$ ***and*** $3x^3 = 8z$ ***from*** $x = 0$ ***to*** $x = 2$.

Univ. Exam

$\vec{F} = 3x^2\hat{\imath} + (2xz - y)\hat{\jmath} + z\hat{k}$

$\vec{dr} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$

$\vec{F}\cdot\vec{dr} = 3x^2dx + (2xz - y)dy + z\,dz$

Let $\boxed{x = t}$

$4y = x^2$

$4y = t^2$

$\boxed{y = \frac{t^2}{4}}$

$8z = 3x^3$

$8z = 3t^3$

$\boxed{z = \frac{3}{8}t^3}$

$x = t$

$dx = dt$

$y = \frac{t^2}{4}$

$dy = \frac{\cancel{2}t}{\cancel{4}2} = \frac{t}{2}$

$z = \frac{3}{8}t^3$

$dz = \frac{9}{8}t^2$

when $x = 0$

$t = 0$

when $x = 2$

$t = 2$

We know that

$W = \int_C \vec{F}\cdot\vec{dr}$

$$W = \int \vec{F} \cdot \vec{dr}$$

$$= \int 3x^2\,dx + (2xz - y)\,dy + z\,dz$$

$$= \int_0^2 3t^2\,dt + \left(\cancel{2}t \times \frac{3}{\cancel{8}_4}t^3 - \frac{t^2}{4}\right)\frac{t}{2}dt + \frac{3}{8}t^3 \times \frac{9t^2}{8}\,dt$$

$$= \int_0^2 \left(3t^2 + \left(\frac{3t^4}{4} - \frac{t^2}{4}\right) \times \frac{t}{2} + \frac{27t^5}{64}\right)dt$$

$$= \int_0^2 \left(3t^2 + \frac{3t^5}{8} - \frac{t^3}{8} + \frac{27}{64}t^5\right)dt$$

$$= \left(\frac{\cancel{3}t^3}{\cancel{3}} + \frac{\cancel{3}t^6}{\cancel{48}_{16}} - \frac{t^4}{32} + \frac{\cancel{27}^{9}}{64} \times \frac{t^6}{\cancel{6}_2}\right)_0^2$$

$$= \left(t^3 + \frac{t^6}{16} - \frac{t^4}{32} + \frac{9t^6}{128}\right)_0^2$$

$$= 8 + \frac{\cancel{64}^{4}}{\cancel{16}} - \frac{16}{32} + \frac{9 \times \cancel{64}}{\cancel{128}_2}$$

$$= 8 + 4 - \frac{1}{2} + \frac{9}{2}$$

$$= 12 + 4 = 16$$

$$\boxed{W = 16 \text{ units}}$$

Prac.Q. *Compute the work done by the force* $\vec{F} = (2y+3)\hat{\imath} + xz\hat{\jmath} + (yz - x)\hat{k}$ *when it moves a particle from the point* $(0,0,0,)$*to the point* $(2,1,1)$*along the crve*

$x = 2t^2, y = t, z = t^3$.

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AKTU : Engineering Mathematics-1

Vector Calculus : Vector Integration

Lec-8

Today's Target

- ✓ Green's Theorem
- ✓ Univ. Questions
- ✓ Practice Questions

By Gulshan sir

Green's Theorem

If C is the regular closed curve in xy – plane and R be the region bounded by C, then

$$\boxed{\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

Where

M and N are differentiable functions inside and on c .

Univ. Exam

Q.1 *Use Green's theorem to evaluate* $\boxed{\int_C (x^2+xy)dx + (x^2+y^2)dy}$ *where C is the square formed by the lines* $\boxed{y=\pm 1, x=\pm 1.}$

$$\int_C (x^2+xy)dx + (x^2+y^2)dy$$

$M = x^2+xy \qquad N = x^2+y^2$

$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 2x$

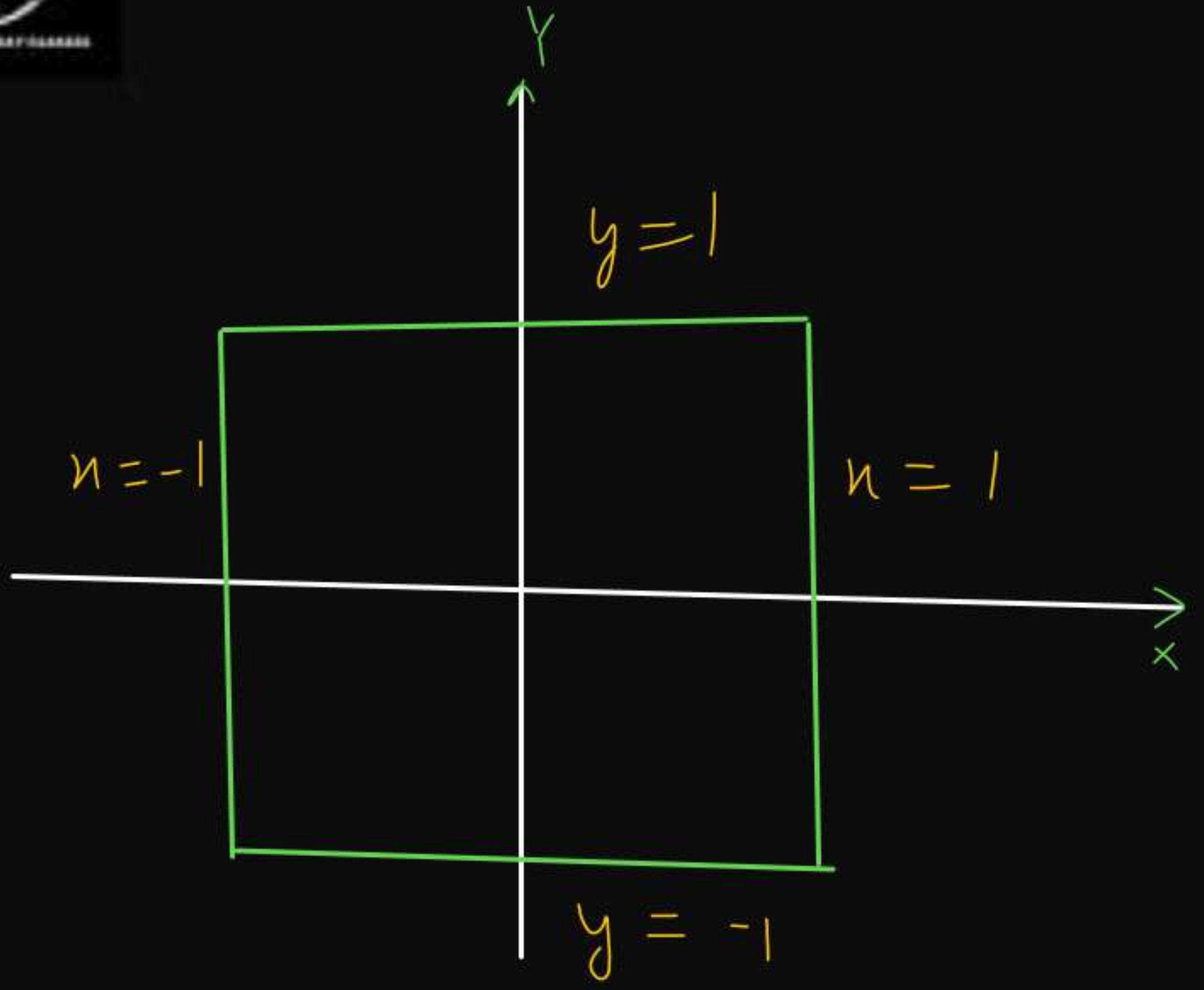
$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x = x$

By Green's theorem

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\int_C (x^2+xy)dx + (x^2+y^2)dy = \iint_R x\,dx\,dy$$

$$= \int_{x=-1}^{1} \int_{y=-1}^{1} x\,dy\,dx$$



$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad (\text{even})$$
$$= 0 \quad (\text{odd})$$

$$= \int_{x=-1}^{1} x\,dx \int_{y=-1}^{1} dy = \int_{-1}^{1} x\,dx\,(y)_{-1}^{1}$$

$$= \int_{-1}^{1} x\,dx\,(1+1) = 2\int_{-1}^{1} x\,dx$$

$$= 2\int_{-1}^{1} x\,dx$$

$$= 2 \times 0$$

$$= 0$$

Q.2 ***Using Green's theorem to evaluate*** $\boxed{\int_C x^2y\,dx + x^2dy \text{ where}}$ **Univ. Exam**

C is the boundary described anti − clockwise of the trianle with vertices $(0,0); (1,0); (1,1)$.

$$\int_C x^2y\,dx + x^2dy$$

$M = x^2y$ | $N = x^2$

$\dfrac{\partial M}{\partial y} = x^2$ | $\dfrac{\partial N}{\partial x} = 2x$

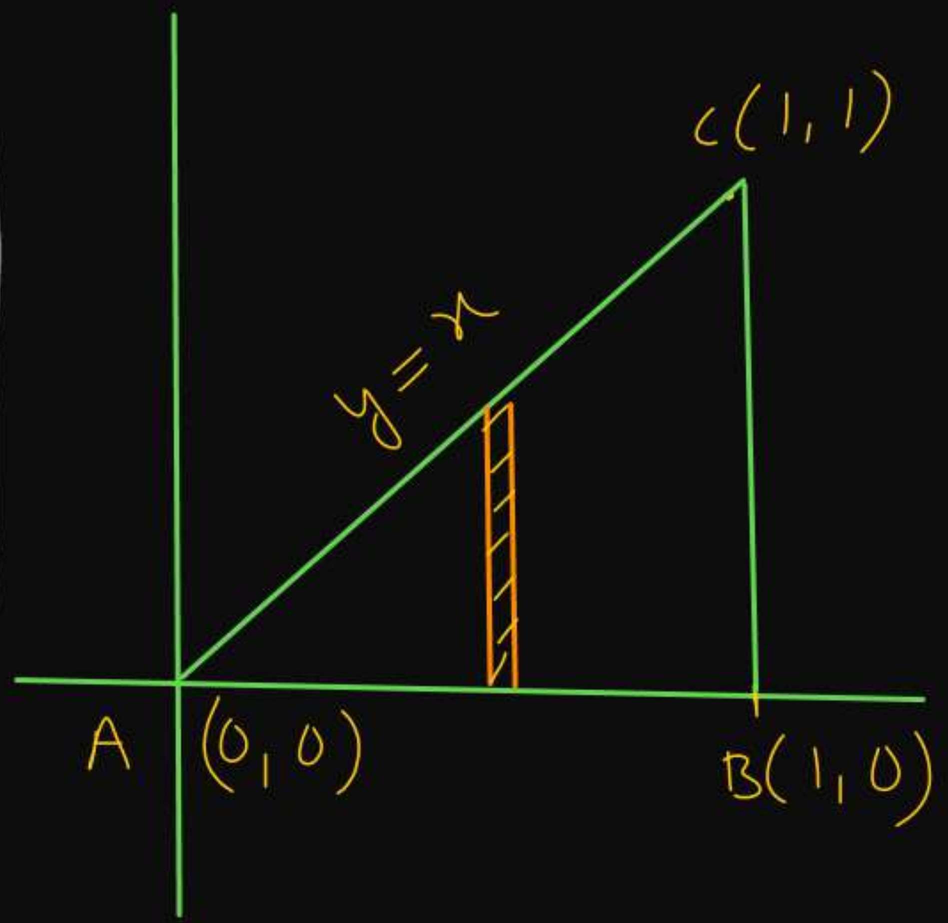
$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x^2$$

By Green's theorem

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\int_C x^2y\,dx + x^2dy = \iint_R (2x - x^2)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{x} (2x - x^2)\,dx\,dy$$



$y = mx$

$y = x$

$$= \int_0^1 \int_0^x (2x - x^2)\, dx\, dy$$

$$= \int_0^1 (2x - x^2)\, dx \int_0^x dy$$

$$= \int_0^1 (2x - x^2)\, (y)_0^x$$

$$= \int_0^1 (2x - x^2)\, x$$

$$= \int_0^1 (2x^2 - x^3)\, dx$$

$$= \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_0^1$$

$$= \frac{2}{3} - \frac{1}{4}$$

$$= \frac{8 - 3}{12}$$

$$\boxed{= \frac{5}{12}}$$

Q.3 *Apply Green's theorem to evaluate* $\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$ **Univ. Exam**

where C is the boundary of the area enclosed by the x − axis and the upper half of the circle $x^2 + y^2 = a^2$.

$$\int_C (2x^2-y^2)dx + (x^2+y^2)dy$$

$M = 2x^2 - y^2$ | $N = x^2 + y^2$

$\frac{\partial M}{\partial y} = -2y$ | $\frac{\partial N}{\partial x} = 2x$

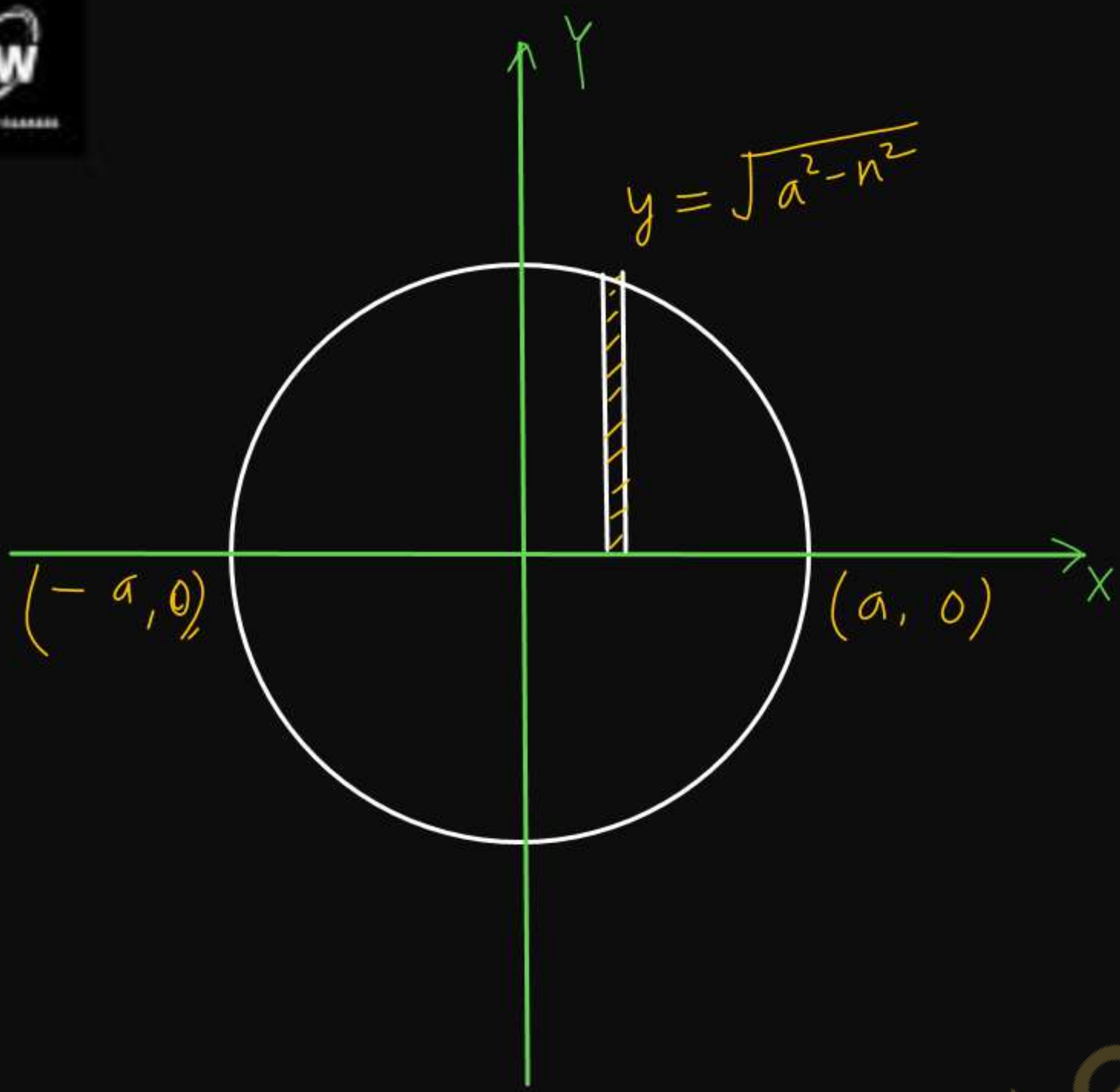
$$\frac{\partial N}{\partial x} - \frac{\partial M}{y} = 2x + 2y$$

By Green's theorem

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$\int_C (2x^2-y^2)dx + (x^2+y^2)dy = \iint_R 2(x+y)\, dx\, dy$$

$$= \int_{x=-a}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} 2(x+y)\, dx dy$$



$$x^2+y^2=a^2$$

$$y=\sqrt{a^2-x^2}$$

$$=\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} 2(x+y)\,dx\,dy$$

$$=2\int_{-a}^{a}\left(xy+\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}}$$

$$=2\int_{-a}^{a}\left(x\sqrt{a^2-x^2}+\frac{a^2-x^2}{2}\right)dx$$

$$=2\int_{-a}^{a}x\sqrt{a^2-x^2}+\int_{-a}^{a}(a^2-x^2)\,dx$$

$$=0+2\int_{0}^{a}(a^2-x^2)\,dx$$

$$\boxed{\begin{aligned}\int_{-a}^{a} f(x) &= 2\int_{0}^{a} f(x)\,dx \quad \text{(even)}\\ &= 0 \quad \text{(odd)}\end{aligned}}$$

$$=2\left(a^2x-\frac{x^3}{3}\right)_0^a$$

$$=2\left(a^3-\frac{a^3}{3}\right)$$

$$=2\times\frac{2a^3}{3}$$

$$\boxed{=\frac{4a^3}{3}}$$

Q. 4 Using Green's theorem to evaluate $\int_C 2y^2dx + 3xdy$ where C is the boundary of the closed region bounded by $y = x$, and $y = x^2$.

Univ. Exam

$$\int_C 2y^2dx + 3xdy$$

$M = 2y^2$

$\frac{\partial M}{\partial y} = 4y$

$N = 3x$

$\frac{\partial N}{\partial x} = 3$

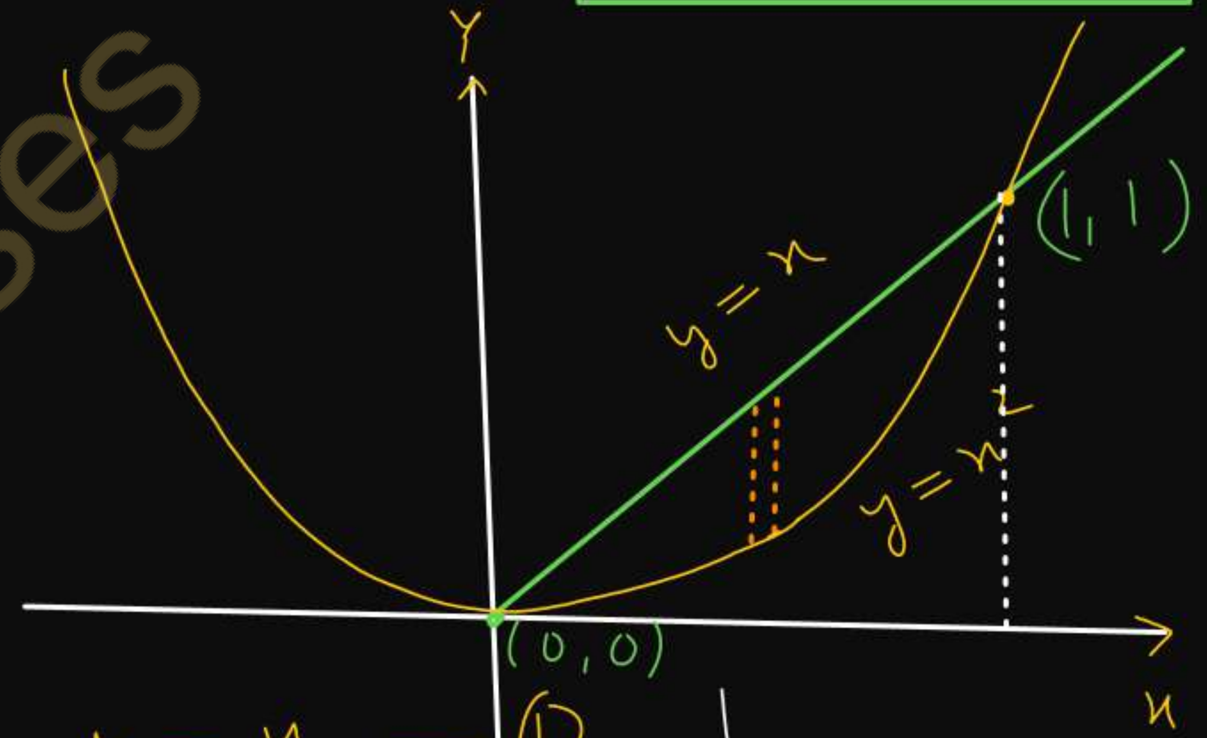
$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3 - 4y$$

By Green's theorem

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$\int_C 2y^2dx + 3xdy = \iint_R (3-4y)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (3-4y)\,dxdy$$



$y = x$ — ①

$y = x^2$ — ②

solve ① and ②

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0$

$x = 1$

$y = 0$

$y = 1$

$$= \int_0^1 \int_{x^2}^{x} (3-4y)\, dy\, dx$$

$$= \int_0^1 \left(3y - 2y^2\right)_{x^2}^{x} dx$$

$$= \int_0^1 \left(3x - 2x^2 - 3x^2 + 2x^4\right) dx$$

$$= \int_0^1 \left(3x - 5x^2 + 2x^4\right) dx$$

$$= \left(\frac{3x^2}{2} - \frac{5x^3}{3} + \frac{2x^5}{5}\right)_0^1$$

$$= \frac{3}{2} - \frac{5}{3} + \frac{2}{5}$$

$$= \frac{45 - 50 + 12}{30}$$

$$\boxed{= \frac{7}{30}}$$

Q.5 *Verify Green's theorem in the plane for* $\boxed{\int_C (3x^2 - 8y^2)dx + (4x - 6xy)dy}$ *where*

C is the region bounded by the parabolas $\boxed{y = \sqrt{x}, \text{ and } y = x^2}$.

$$\int_C (3x^2 - 8y^2)dx + (4x - 6xy)dy$$

$$M = 3x^2 - 8y^2$$

$$\frac{\partial M}{\partial y} = -16y$$

$$N = 4x - 6xy$$

$$\frac{\partial N}{\partial x} = 4 - 6y$$

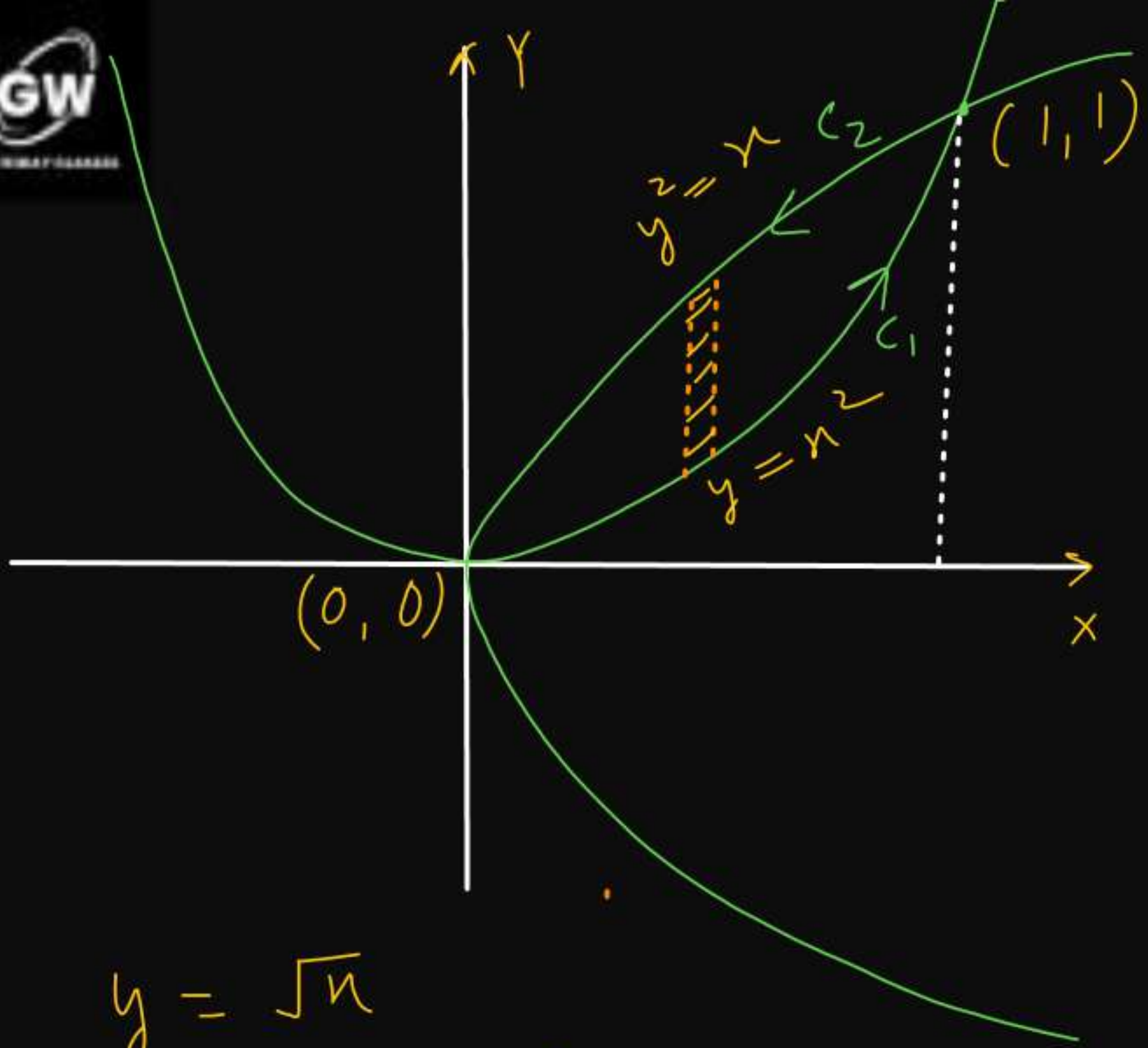
$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 4 - 6y - (-16y)$$

$$= 10y + 4$$

By Green's theorem

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

RHS



$y = \sqrt{x}$

$y^2 = x$ — (1)

$x^2 = y$ — (2)

$$= \iint_R (10y+4)\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (10y+4)\,dx\,dy$$

$$= \int_0^1 \left(5y^2+4y\right)_{x^2}^{\sqrt{x}}$$

$$= \int_0^1 \left(5x + 4\sqrt{x} - 5x^4 - 4x^2\right)dx$$

$$= \int_0^1 \left(5x + 4x^{1/2} - 5x^4 - 4x^2\right)dx$$

$$= \left(\frac{5x^2}{2} + \frac{8}{3}x^{3/2} - x^5 - \frac{4x^3}{3}\right)_0^1$$

$$= \frac{5}{2} + \frac{8}{3} - 1 - \frac{4}{3}$$

$$= \frac{15+16-6-8}{6}$$

$$\boxed{= \frac{17}{6}}$$

LHS

$$\int M\,dx + N\,dy = \int_C (3x^2-8y^2)dx + (4x-6xy)dy$$

$$= \int_{C_1+C_2} (3x^2-8y^2)dx + (4x-6xy)dy$$

$$= \int_{C_1} (3x^2-8y^2)dx + (4x-6xy)dy + \int_{C_2} (3x^2-8y^2)dx + (4x-6xy)dy$$

$$= \int_0^1 (3x^2-8x^4)dx + (4x-6x^3)2x\,dx +$$

$$\int_1^0 (3y^4-8y^2)2y\,dy + (4y^2-6y^3)\,dy$$

For C_1

$y = x^2$

$dy = 2x\,dx$

$x = 0$ to $x = 1$

For C_2

$y^2 = x$

$2y\,dy = dx$

$y = 0$ to $y = 1$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^2 - 12x^4)\,dx + \int_1^0 (6y^5 - 16y^3 + 4y^2 - 6y^3)\,dy$$

$$= \int_0^1 (11x^2 - 20x^4)\,dx + \int_1^0 (6y^5 - 22y^3 + 4y^2)\,dy$$

$$= \left(\frac{11x^3}{3} - 4x^5\right)_0^1 + \left(y^6 - \frac{22y^4}{4} + \frac{4y^3}{3}\right)_1^0$$

$$= \left(\frac{11}{3} - 4\right) - \left(1 - \frac{22}{4} + \frac{4}{3}\right)$$

$$= \frac{11}{3} - 4 - 1 + \frac{11}{2} - \frac{4}{3}$$

$$= \frac{22 - 24 - 6 + 33 - 8}{6}$$

$$= \frac{55 - 38}{6}$$

$$\boxed{= \frac{17}{6}}$$

LHS = RHS

Hence

Green's theorem is verified

Univ. Exam

Prac. Q. 1 Evaluate by Green's theorem $\int (Cosx\, siny - xy)dx + sinx\, cosy\, dy$ where C is the circle $x^2 + y^2 = 1$

Ans. 0

Univ. Exam

Prac. Q. 2 Verify Green's theorem in the plane for $\int (xy + y^2)dx + x^2\,dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

AKTU : Engineering Mathematics-1

Vector Calculus : Vector Integration

Lec-9

By Gulshan sir

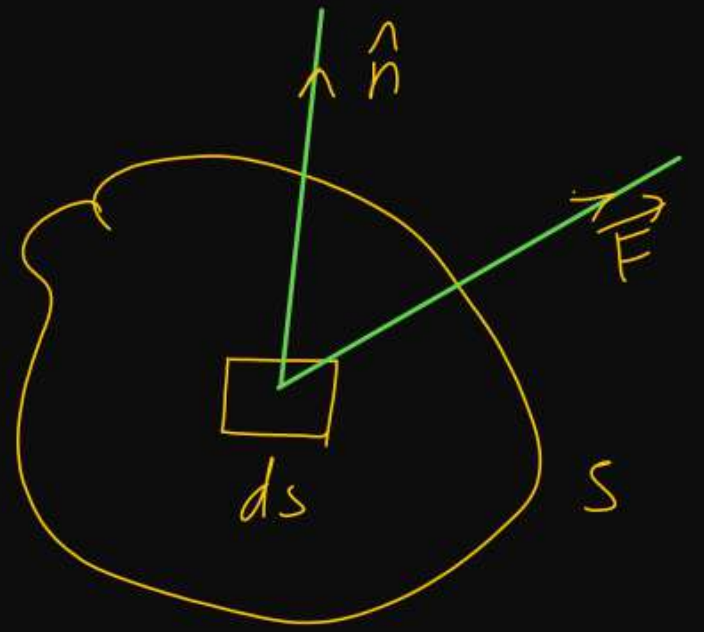
Today's Target

- ✓ Stoke's Theorem
- ✓ Univ. Questions
- ✓ Practice Questions

Surface Integral

Surface Integral of $\vec{F}$ over surface S is given as

$$\boxed{\text{surface integral} = \iint_R \vec{F}\cdot\hat{n}\,ds}$$



where

(i) $\boxed{\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}}$

(ii) *If R be the projection of S on xy – plane then* $\boxed{ds = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}}$

(iii) *If R be the projection of S on yz – plane then* $\boxed{ds = \dfrac{dz\,dy}{|\hat{n}\cdot\hat{i}|}}$

(iv) *If R be the projection of S on zx – plane then* $\boxed{ds = \dfrac{dx\,dz}{|\hat{n}\cdot\hat{j}|}}$

Stoke's Theorem (Relation between line and surface integral)

If S is an open surface bounded by a closed curve C and $\boxed{\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}}$ *is any vector point function having continous first order partial derivatives then*

$$\boxed{\oint_C \vec{F}\cdot\vec{dr} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds}$$

Where $\hat{n}$ *is a unit normal vector at any point of S*

Q.1 Evaluate $\oint_C \vec{F}\cdot\vec{dr}$ by Stoke's theorem where $\boxed{\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}}$ and C is the boundary of the triangle with vertices at $\boxed{(0,0,0), (1,0,0) \text{ and } (1,1,0).}$

Univ. Exam

$$\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$$

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -x-z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-x-z) - \frac{\partial}{\partial z}(x^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-x-z) - \frac{\partial}{\partial z}(y^2)\right]$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)\right]$$

$$= \hat{i}(0-0) - \hat{j}(-1-0) + \hat{k}(2x-2y)$$

$$= 0\hat{i} + \hat{j} + 2(x-y)\hat{k}$$

$$\boxed{\text{curl } \vec{F} = 0\hat{i} + \hat{j} + 2(x-y)\hat{k}}$$

since z coordinates are zero

$\therefore$ Triangle formed in X-Y plane

$$\boxed{\hat{n} = \hat{k}}$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \frac{dx\,dy}{|\hat{k}\cdot\hat{k}|}$$

$$= \frac{dx\,dy}{|\hat{k}|^2}$$

$$\boxed{ds = dx\,dy}$$

By stoke's theorem

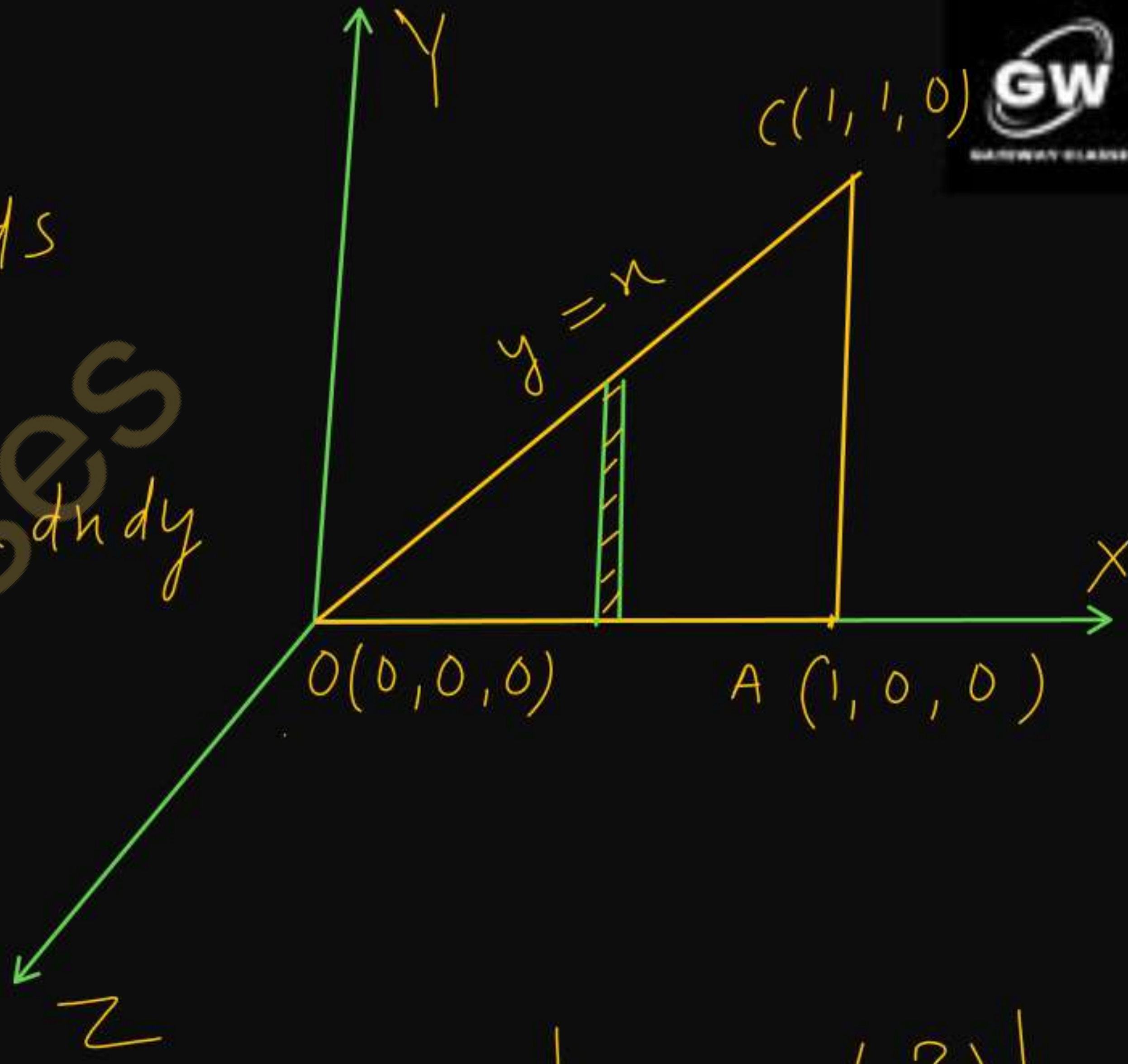
$$\oint \vec{F}\cdot\vec{dr} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$= \iint \{\hat{j} + 2(x-y)\hat{k}\}\cdot\hat{k}\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{x} 2(x-y)\,dx\,dy$$

$$= 2\int_{x=0}^{1}\left(xy - \frac{y^2}{2}\right)_0^x dx = 2\int_{x=0}^{1}\left(x^2 - \frac{x^2}{2}\right)dx = \cancel{2}\int_0^1 \frac{x^2}{\cancel{2}}\,dx = \left(\frac{x^3}{3}\right)_0^1$$

$$\boxed{\oint_C \vec{F}\cdot\vec{dr} = \frac{1}{3}}$$



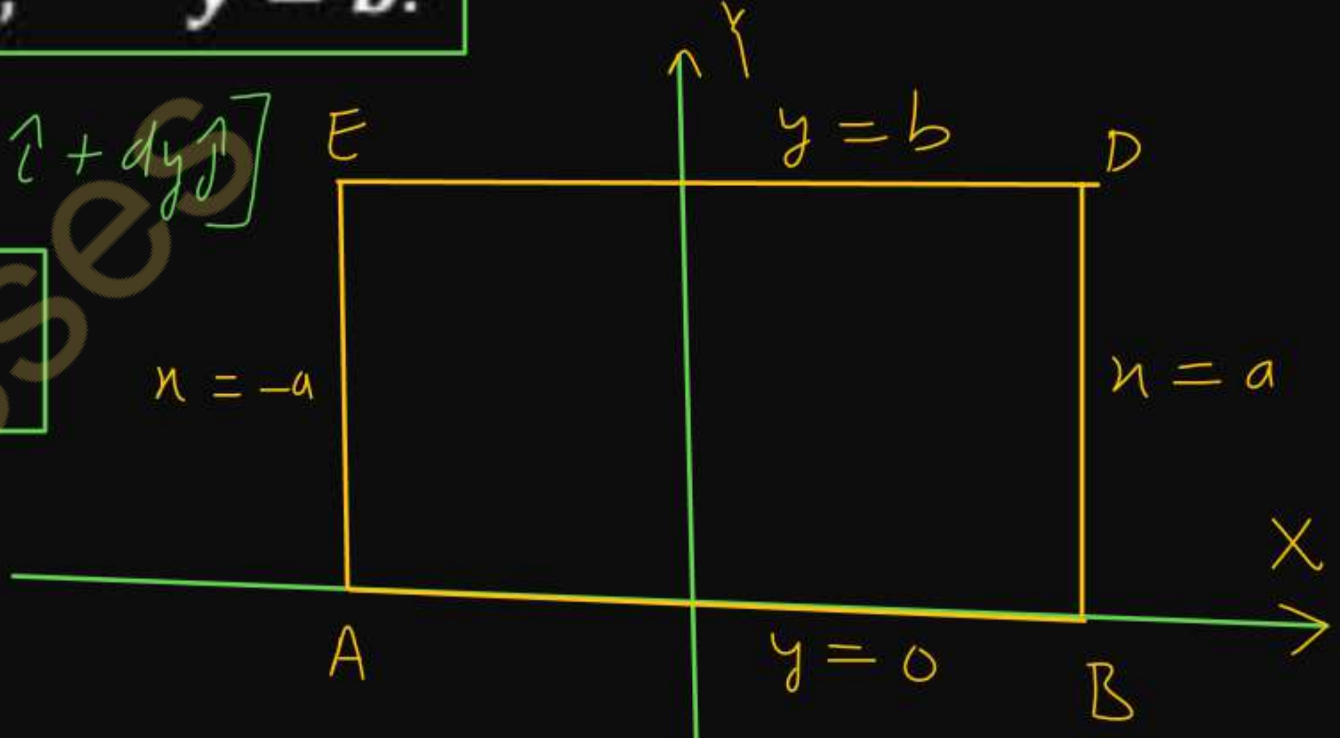
Q.2 ***Verify Stoke's theorem for*** $\boxed{\vec{F} = (x^2+y^2)\hat{\imath} - 2xy\hat{\jmath}}$ ***taken round the rectangle bounded by the lines*** $\boxed{x = \pm a, \quad y = 0, \quad y = b.}$

Univ. Exam

$\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$

$\vec{F}\cdot d\vec{r} = [(x^2+y^2)\hat{i} - 2xy\hat{j}]\cdot[dx\,\hat{i} + dy\,\hat{j}]$

$$\boxed{\vec{F}\cdot d\vec{r} = (x^2+y^2)dx - 2xy\,dy}$$



Stoke's theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

LHS

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_{AB} \vec{F}\cdot d\vec{r} + \oint_{BD} \vec{F}\cdot d\vec{r} + \oint_{DE} \vec{F}\cdot d\vec{r} + \oint_{EA} \vec{F}\cdot d\vec{r}$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{-a}^{a} x^2\,dx + \int_0^b (-2ay)\,dy + \int_a^{-a} (x^2+b^2)\,dx + \int_b^0 2ay\,dy$$

AB	BD	DE	EA
$y = 0$	$x = a$	$y = b$	$x = -a$
$dy = 0$	$dx = 0$	$dy = 0$	$dx = 0$
$x = -a$ to $x = a$	$y = 0$ to $y = b$	$x = a$ to $x = -a$	$y = b$ to $y = 0$

$$\oint \vec{F}\cdot d\vec{r} = \left(\frac{x^3}{3}\right)_{-a}^{a} - \cancel{2}a\left(\frac{y^2}{\cancel{2}}\right)_0^b + \left(\frac{x^3}{3}+b^2x\right)_a^{-a} + \cancel{2}a\left(\frac{y^2}{\cancel{2}}\right)_b^0$$

$$= \cancel{\frac{a^3}{3}} + \cancel{\frac{a^3}{3}} - ab^2 - \cancel{\frac{a^3}{3}} - ab^2 - \cancel{\frac{a^3}{3}} - ab^2 - ab^2$$

$$= -4ab^2$$

RHS

$$\iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$\operatorname{curl}\vec{F} = \hat{i}(0+0) - \hat{j}(0-0) + \hat{k}(-2y-2y)$$

$$= 0\hat{i} + 0\hat{j} - 4y\hat{k} = -4y\hat{k}$$

$$\hat{n} = \hat{k}$$

$$\operatorname{curl}\vec{F}\cdot\hat{n} = (-4y\hat{k})\cdot\hat{k} = -4y$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$ds = dx\,dy$$

$$\iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds = \int_{x=-a}^{a}\int_{y=0}^{b}(-4y)\,dx\,dy$$

$$= -4\int_{-a}^{a}\left(\frac{y^2}{2}\right)_0^b dx$$

$$= -\cancel{4}^{2}\int_{-a}^{a}\frac{b^2}{\cancel{2}}\,dx$$

$$= -2b^2\int_{-a}^{a}dx$$

$$= -2b^2(x)_{-a}^{a} = -2b^2(a+a)$$

$$\iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds = -4ab^2$$

LHS = RHS

Hence stoke's theorem is verified

Q.3 Verify Stoke's theorem for $\boxed{\vec{F} = (y - z + 2)\hat{\imath} + (yz + 4)\hat{\jmath} - xz\hat{k}}$ over the surface of cube $\boxed{x = 0, y = 0, z = 0, x = 2, y = 2, z = 2}$ above the XOY – plane(open the bottom).

Univ. Exam

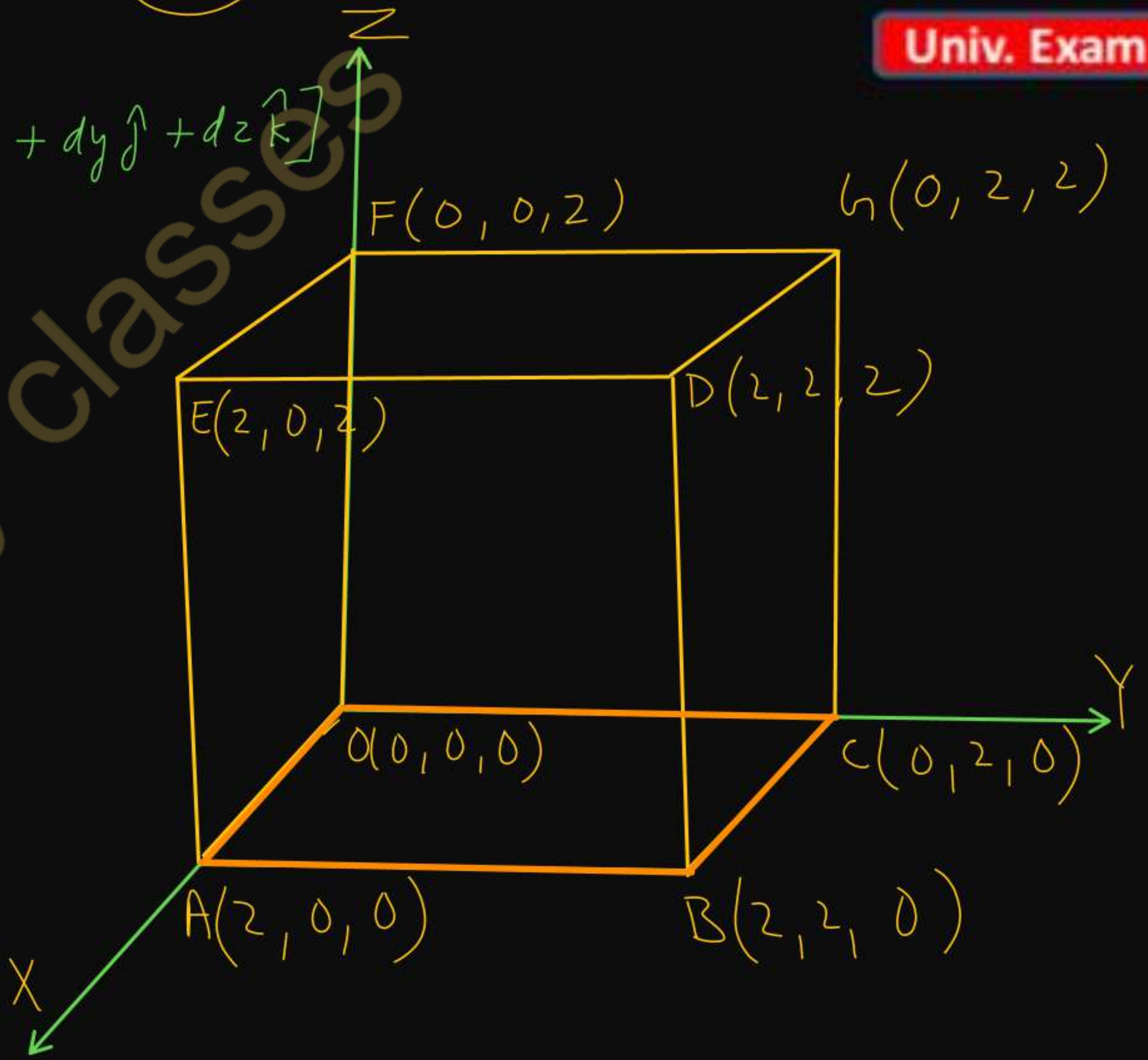
$\vec{F} = (y-z+2)\hat{i} + (yz+4)\hat{j} - xz\hat{k}$

$\vec{F}\cdot\vec{dr} = [(y-z+2)\hat{i} + (yz+4)\hat{j} - xz\hat{k}]\cdot[dx\hat{i} + dy\hat{j} + dz\hat{k}]$

$$\boxed{\vec{F}\cdot\vec{dr} = (y-z+2)dx + (yz+4)dy - xz\,dz}$$

Stoke's theorem

$$\oint_C \vec{F}\cdot\vec{dr} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds$$



LHS

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r}$$

For OA	For AB	For BC	For CO
$y = 0$	$z = 0$	$z = 0$	$x = 0$
$z = 0$	$x = 2$	$y = 2$	$z = 0$
$dy = 0$	$dz = 0$	$dz = 0$	$dx = 0$
$dz = 0$	$dx = 0$	$dy = 0$	$dz = 0$
$x = 0$ to $x = 2$	$y = 0$ to $y = 2$	$x = 2$ to $x = 0$	$y = 2$ to $y = 0$

$$\vec{F}\cdot d\vec{r} = (y - z + 2)\,dx + (yz + 4)\,dy - xz\,dz$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^2 2\,dx + \int_0^2 4\,dy + \int_2^0 4\,dx + \int_2^0 4\,dy$$

$$= 2(x)_0^2 + 4(y)_0^2 + 4(x)_2^0 + 4(y)_2^0$$

$$= 2(2-0) + 4(2-0) + 4(0-2) + 4(0-2)$$

$$= 4 + \cancel{8} - \cancel{8} - 8$$

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = -4}$$

RHS

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$\text{curl}\,\vec{F} = \nabla\times\vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y-z+2) & (yz+4) & -xz \end{vmatrix}$$

$$= \hat{i}(0-y) - \hat{j}(-z+1) + \hat{k}(0-1)$$

$$\boxed{\text{curl}\,\vec{F} = -y\hat{i} + (z-1)\hat{j} - \hat{k}}$$

For ABDE

$\hat{n} = \hat{i}$, $x = 2$

$$ds = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = \frac{dy\,dz}{|\hat{i}\cdot\hat{i}|} = dy\,dz$$

$$\int_{z=0}^{2}\int_{y=0}^{2}\left[-y\hat{i} + (z-1)\hat{j} - \hat{k}\right]\cdot(\hat{i})\,dy\,dz$$

$$= \int_0^2\int_0^2 (-y)\,dy\,dz$$

$$= -\int_0^2 \left(\frac{y^2}{2}\right)_0^2 dz$$

$$= -\int_0^2 2\,dz = -2(z)_0^2$$

$$= -2(2-0)$$

$$= -4$$

For COFG

$\hat{n} = -\hat{i}$

$ds = \dfrac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$

$= \dfrac{dy\,dz}{|-\hat{i}\cdot\hat{i}|}$

$= dy\,dz$

$x = 0$

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \int\int [y\hat{i} + (z-1)\hat{j} - \hat{k}]\cdot(-\hat{i})\,dy\,dz$$

$$= \int_{z=0}^{2}\int_{y=0}^{2} y\,dy\,dz = \int_0^2 \left(\frac{y^2}{2}\right)_0^2 dz = \int_0^2 2\,dz$$

$$= 2(z)_0^2 = 2(2-0) = 4$$

Q.4 *Apply Stoke's theorem to evaluate* $\boxed{\int_C [(x+y)dx + (2x-z)dy + (y+z)dz]}$ *where C is the boundary of the triangle with vertices at* $\boxed{(2,0,0), (0,3,0) \text{ and } (0,0,6).}$

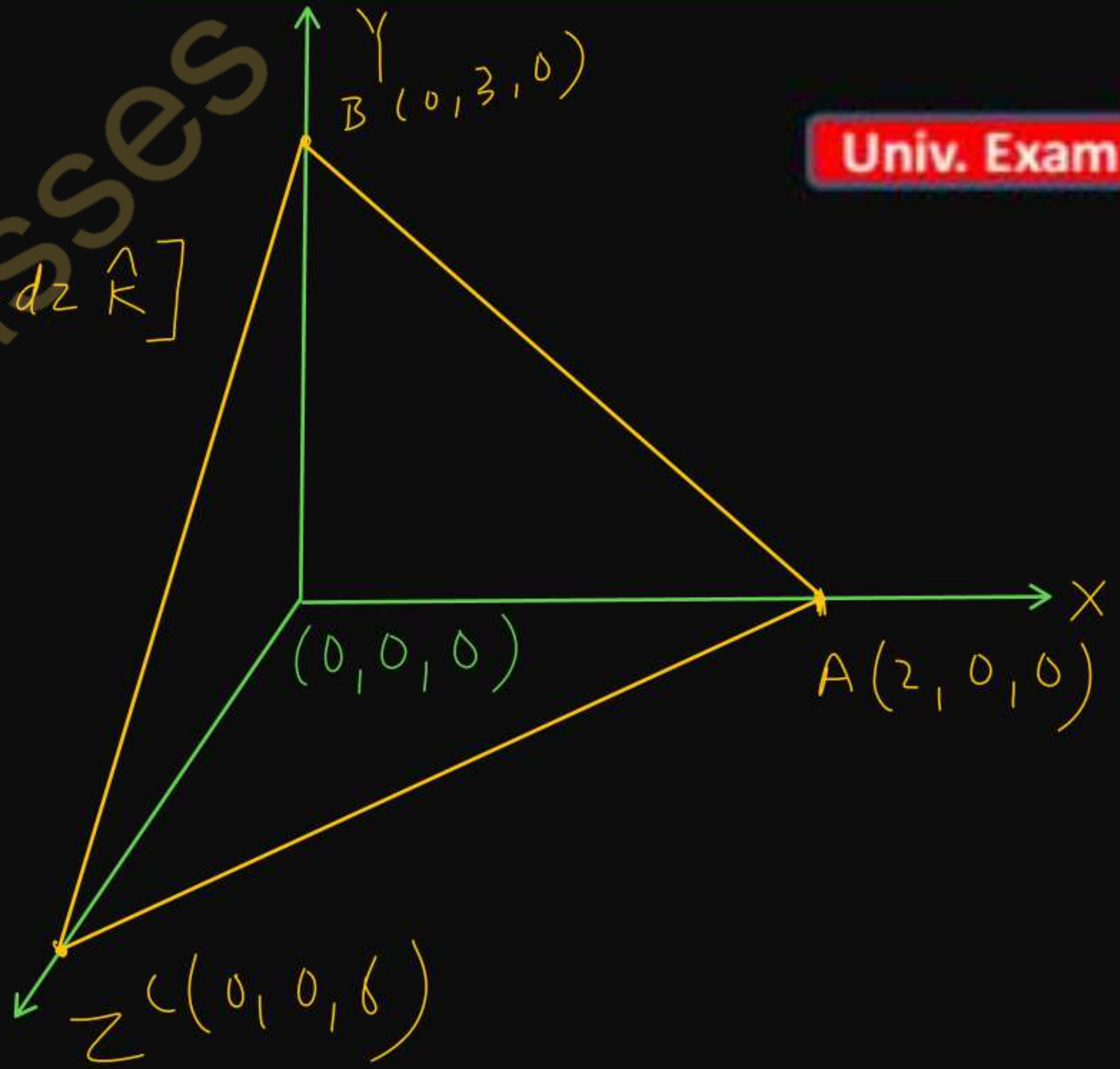
Univ. Exam

$$\int_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$$

$$= \int_C \left[(x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}\right]\left[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right]$$

$$= \int_C \vec{F} \cdot \vec{dr}$$

$$\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$$



Stoke's theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\, ds$$

Equation of surface (Plane) ABC

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$$

Let $\phi = \frac{x}{2}+\frac{y}{3}+\frac{z}{6}-1$

Ans 21

P.Q.1 Verify Stoke's theorem for $\vec{F} = x^2\hat{\imath} + xy\hat{\jmath}$ integrated round the square whose sides are $x = 0, y = 0, x = a, y = a$ in the plane $z = 0$. Univ. Exam

P.Q.2 Verify Stoke's theorem for $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and surfaces S is the portion of the sphere $x^2 + y^2 + z^2 = 1$ above the xy – plane.

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P.Q.3 *Verify Stoke's theorem for the vector field* $\vec{F} = (x^2 - y^2)\hat{\imath} + 2xy\hat{\jmath}$ *integrated round the rectangle in the plane* $z = 0$ *and bounded by the lines* $x = 0, y = 0, x = a, y = b$.

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P.Q.4 Verify Stoke's theorem for the vector field $\vec{F} = (2y + z, x - z, y - x)$ *taken over the triange ABC cut from the plane* $x + y + z = 1$ *by the coordinates.*

Univ. Exam

AKTU : Engineering Mathematics-1

Vector Calculus : Vector Integration

Lec-10

Today's Target

- ✓ Gauss Divergence Theorem
- ✓ Univ. Questions
- ✓ Practice Questions

By Gulshan sir

Gauss Divergence Theorem(Relation between Surface and Volume Integral)

If $\vec{F}$ is a vector point function having continous first order partial derivatives in the region V bounded by a closed surface S then

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV}$$

surface integral volume integral

Where $\hat{n}$ is the outward drawn unit normal vector to the surface S

Univ. Exam

Q.1 *Find* $\iint_S \vec{F}.\hat{n}ds$ *where* $\vec{F} = (2x + 3Z)\hat{\imath} - (xz + y)\hat{\jmath} + (y^2 + 2z)\hat{k}$ *and S is the surface of sphere having centre at* $\boxed{(3, -1, 2)}$ *and* $\boxed{\text{radius } 3.}$

$$\vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$$

$$\operatorname{div}\vec{F} = \nabla.\vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$$

$$= \frac{\partial}{\partial x}(2x+3z) - \frac{\partial}{\partial y}(xz+y) + \frac{\partial}{\partial z}(y^2+2z)$$

$$\operatorname{div}\vec{F} = 2 - 1 + 2$$

$$\boxed{\operatorname{div}\vec{F} = 3}$$

By Gauss Divergence theorem

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV$$

$$= \iiint_V 3\,dV = 3\iiint_V dV$$

$$= 3V$$

Here V is the volume of sphere

$$V = \frac{4}{3}\pi r^3$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \cancel{3} \times \frac{4}{\cancel{3}} \pi r^3$$

$$= 4 \times \pi \times 3^3$$

$$= 108\pi$$

Univ. Exam

Q.2 Evaluate $\iint_S \vec{r}.\hat{n}ds$ **where S is a closed surface and** $\vec{r} = x\hat{i} + \hat{j} + z\hat{k}$.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{div}\,\vec{r} = \nabla.\vec{r}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= 1 + 1 + 1$$

$$\boxed{\text{div}\,\vec{r} = 3}$$

By Gauss Divergen theorem

$$\iint_S \vec{F}.\hat{n}ds = \iiint_V \text{div}\,\vec{F}\,dV$$

put $\vec{F} = \vec{r}$

$$\iint_S \vec{r}.\hat{n}ds = \iiint_V \text{div}\,\vec{r}\,dV$$

$$= \iiint_V 3\,dV = 3\iiint_V dV$$

$$\boxed{\iint_S \vec{r}.\hat{n}ds = 3V}$$

Q.3 Evaluate $\boxed{\iint_S (y^2z^2\vec{i} + z^2x^2\hat{j} + z^2y^2\vec{k}).\hat{n}ds}$, where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy plane and bounded by this plane.

Univ. Exam

$$\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}).\hat{n}\,ds$$

where

$$\vec{F} = y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}$$

$$\text{div}\vec{F} = \nabla.\vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(z^2y^2)$$

$$\text{div}\vec{F} = 0 + 0 + 2zy^2$$

$$\text{div}\vec{F} = 2zy^2$$

By Gauss Divergence theorem

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \text{div}\vec{F}\,dV$$

$$= \iiint_V 2zy^2\,dV$$

Change into spherical coordinates

Put $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$dV = r^2\sin\theta\, dr\, d\theta\, d\phi$

$r = 0$ to $r = 1$

$\theta = 0$ to $\theta = \frac{\pi}{2}$

$\phi = 0$ to $= 2\pi$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = 2\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r\cos\theta \times r^2\sin^2\theta\sin^2\phi \times r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 r^5\sin^3\theta\cos\theta\sin^2\phi\, dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2}\left(\frac{r^6}{6}\right)_0^1 \sin^3\theta\cos\theta\sin^2\phi\, dr\, d\theta\, d\phi$$

$$= \frac{\cancel{2}}{3\cancel{6}}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left(\sin^3\theta\cos\theta\, d\theta\right)\sin^2\phi\, d\phi$$

Put $\sin\theta = t$

$\cos\theta\, d\theta = dt$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2}\left(t^3\, dt\right)\sin^2\phi\, d\phi$$

$$= \frac{1}{3}\int_0^{2\pi}\left(\frac{t^4}{4}\right)_0^{\pi/2}\sin^2\phi\, d\phi$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{1}{3\times 4}\int_0^{2\pi}\left(\sin^4\theta\right)_0^{\pi/2}\sin^2\phi\,d\phi$$

$$= \frac{1}{12}\int_0^{2\pi}(1-0)\sin^2\phi\,d\phi$$

$$= \frac{1}{12}\int_0^{2\pi}\sin^2\phi\,d\phi$$

$$= \frac{1}{12}\int_0^{2\pi}\left(\frac{1-\cos 2\phi}{2}\right)d\phi$$

$$= \frac{1}{24}\int_0^{2\pi}(1-\cos 2\phi)\,d\phi$$

$$= \frac{1}{24}\left(\phi - \frac{\sin 2\phi}{2}\right)_0^{2\pi}$$

$$= \frac{1}{24}(2\pi - 0)$$

$$= \frac{\cancel{2}\pi}{\cancel{24}_{12}}$$

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{\pi}{12}}$$

P.Q. 1 Evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot d\bar{S}$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant.

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Ans. 0

P.Q.2 *Verify divergence theorem for* $\vec{F} = x^3\hat{\imath} - y^3\hat{\jmath} + z^3\hat{k}$ *taken over the surface of sphere* $x^2 + y^2 + z^2 = a^2$.

Univ. Exam

Q.4 Use divergence theorem to evaluate the surface integral

$\iint_S (xdydz + ydzdx + zdxdy)$ **where s is the portion of the plane** $\boxed{x + 2y + 3z = 6}$

Univ. Exam

$$\iint_S x\,dydz + y\,dzdx + z\,dxdy$$

$$= \iint_S (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (dydz\,\hat{i} + dzdx\,\hat{j} + dxdy\,\hat{k})$$

where

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\mathrm{DIV}\,\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$$

$$\mathrm{DIV}\,\vec{F} = 1 + 1 + 1$$

$$\boxed{\mathrm{div}\,\vec{F} = 3}$$

By Gauss Divergence theorem

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \mathrm{div}\,\vec{F}\, dV$$

$$\iint_S x\,dydz + y\,dzdx + z\,dxdy =$$

$$\iiint_V 3\, dV$$

$$= 3\int_{x=0}^{6}\int_{y=0}^{\frac{6-x}{2}}\int_{z=0}^{\frac{6-x-2y}{3}} dz\, dy\, dx$$

$$= 3\int_{0}^{6}\int_{0}^{\frac{6-x}{2}} \left(z\right)_{0}^{\frac{6-x-2y}{3}} dy\, dx$$

$$= \cancel{3}\int_{0}^{6}\int_{0}^{\frac{6-x}{2}} \left(\frac{6-x-2y}{\cancel{3}}\right) dy\, dx$$

$$= \int_{0}^{6}\int_{0}^{\frac{6-x}{2}} \left(6-x-2y\right) dy\, dx$$

$$= \int_{0}^{6} \left(6y - xy - y^2\right)_{0}^{\frac{6-x}{2}} dx$$

$$= \int_{0}^{6} 6\left(\frac{6-x}{2}\right) - x\left(\frac{6-x}{2}\right) - \left(\frac{6-x}{2}\right)^2 dx$$

$$= \int_{0}^{6} \frac{72 - 12x - 2x(6-x) - (36 + x^2 - 12x)}{4} dx$$

$$= \frac{1}{4}\int_{0}^{6} \left(72 - \cancel{12x} - 12x + 2x^2 - 36 - x^2 + \cancel{12x}\right) dx$$

$$= \frac{1}{4}\int_{0}^{6} \left(x^2 - 12x + 36\right) dx$$

$$= \frac{1}{4}\left(\frac{x^3}{3} - 6x^2 + 36x\right)_0^6$$

$$= \frac{1}{4}\left(\frac{6^3}{3} - 6\times 6^2 + 36\times 6\right)$$

$$= \frac{1}{4}\left(\frac{216}{3} - \cancel{216} + \cancel{216}\right)$$

$$= \frac{\cancel{216}^{\cancel{54}\ 18}}{\cancel{4}\times\cancel{3}}$$

$$\boxed{= 18}$$

Q.5 ***Verify divergence theorem for*** $\boxed{\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}}$ ***taken over the cube bounded by the planes*** $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

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$$\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$$

$$\operatorname{div}\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz)$$

$$\operatorname{div}\vec{F} = 4z - 2y + y$$

$$\boxed{\operatorname{div}\vec{F} = 4z - y}$$

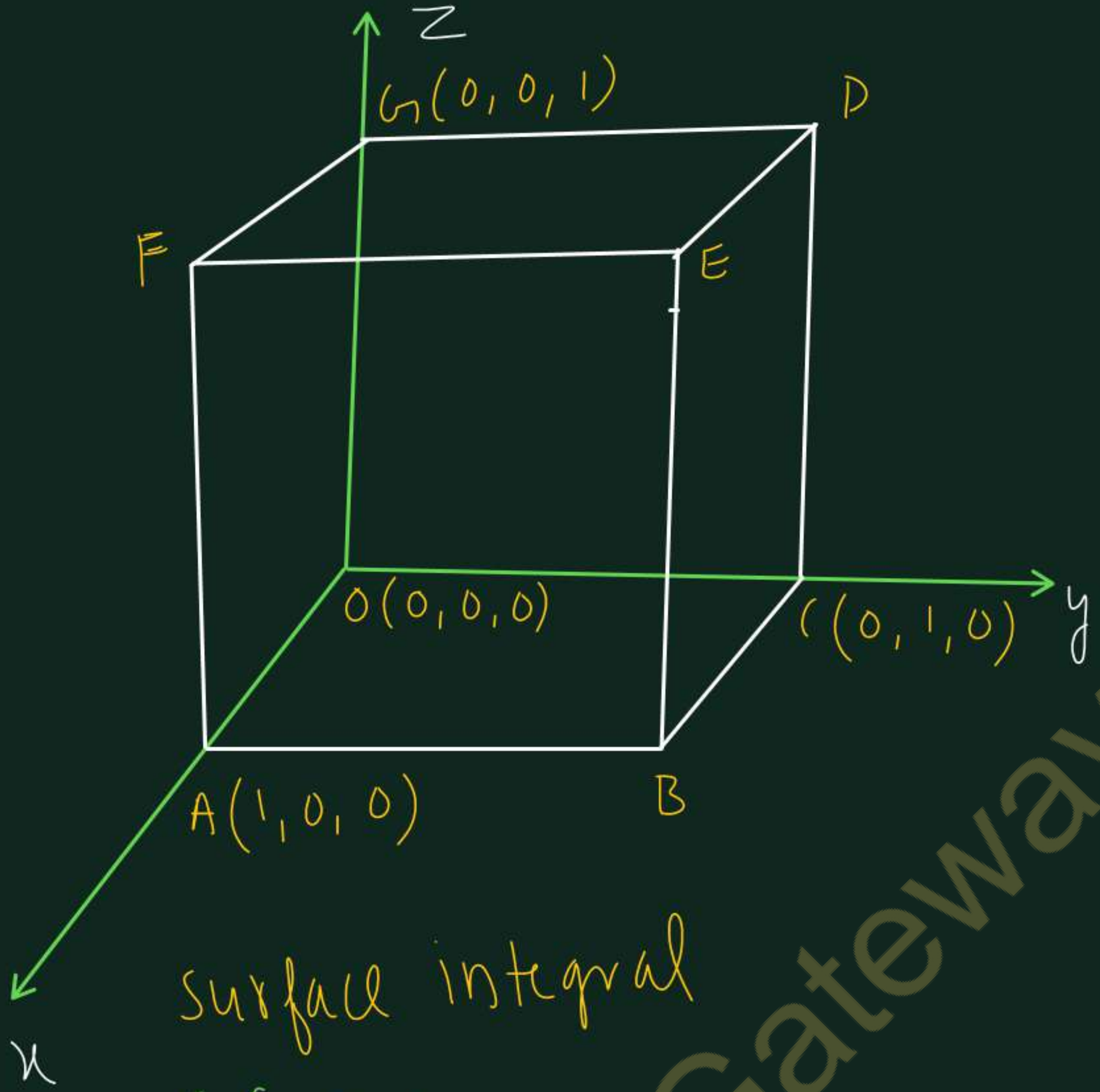
volume integral

$$\iiint_V \operatorname{div}\vec{F}\, dV = \int_0^1\int_0^1\int_0^1 (4z - y)\, dz\, dy\, dx$$

$$= \int_0^1\int_0^1 \left(2z^2 - yz\right)_0^1 dy\, dx$$

$$= \int_0^1\int_0^1 (2 - y)\, dy\, dx = \int_0^1 \left(2y - \frac{y^2}{2}\right)_0^1 dx$$

$$= \int_0^1 \frac{3}{2}\, dx = \frac{3}{2}(x)_0^1 \quad \boxed{= \frac{3}{2}} \quad \text{—(1)}$$



surface integral

$$= \iint_S \vec{F}\cdot\hat{n}\,ds$$

where, S consist 6 surfaces of cube

For OABC

$\hat{n} = -\hat{k}$ | $dz = 0$

$z = 0$ | $ds = dx\,dy$

$$\iint \vec{F}\cdot\hat{n}\,ds = \iint (4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k})\cdot(-\hat{k})\,dx\,dy$$

$$= \iint (-yz)\,dx\,dy$$

$$= 0$$

For DEFG

$\hat{n} = \hat{k}$ $\quad dz = 0$

$z = 1$ $\quad ds = dx\,dy$

$$\iint \vec{F}\cdot\hat{n}\,ds = \iint (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(\hat{k})\,dx\,dy$$

$$= \iint yz\,dx\,dy = \int_0^1\int_0^1 y\,dy\,dx$$

$$= \int_0^1 \left(\frac{y^2}{2}\right)_0^1 dx = \frac{1}{2}\int_0^1 dx$$

$$= \frac{1}{2}$$

For BCDE

$\hat{n} = \hat{j}$ $\quad dy = 0$

$y = 1$ $\quad ds = dx\,dz$

$$\iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(\hat{j})\,dx\,dz$$

$$= \int_0^1\int_0^1 -y^2\,dx\,dz = -\int_0^1\int_0^1 dz\,dx$$

$$= -\int_0^1 (z)_0^1 = -\int_0^1 dx$$

$$= -1$$

For AOGF

$\hat{n} = -\hat{j}$ $\quad dy = 0$

$ds = dx\,dz$

$y = 0$

$$\iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xz\hat{i} - y^2\hat{j} + yz\hat{k})(-\hat{j})\,dx\,dz$$

$$= \int_0^1\int_0^1 y^2\,dx\,dz$$

$$= \int_0^1\int_0^1 0\,dx\,dz$$

$$= 0$$

For ABEF

$\hat{n} = \hat{i}$ $\quad dx = 0$

$ds = dy\,dz$

$x = 1$

$$\iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(\hat{i})\,dy\,dz$$

$$= \int_0^1\int_0^1 4xz\,dy\,dz = \int_0^1\int_0^1 4z\,dy\,dz$$

$$= 4\int_0^1\left(\frac{z^2}{2}\right)_0^1 dy = \frac{4}{2}\int_0^1 dy$$

$$= 2(y)_0^1 = 2$$

For OCDG

$\hat{n} = -\hat{i}$ $dx = 0$

$x = 0$ $ds = dy\,dz$

$$\iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xz\,\hat{i} - y^2\hat{j} + yz\hat{k})(-\hat{i})\,dy\,dz$$

$$= \int_0^1\int_0^1 (-4xz)\,dy\,dz$$

$$= \int_0^1\int_0^1 0\,dy\,dz = 0$$

Hence

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 0 + \tfrac{1}{2} - 1 + 0 + 2 + 0$$

$$\boxed{\iint \vec{F}\cdot\hat{n}\,ds = \frac{3}{2}} \quad \text{— (2)}$$

From (1) and (2)

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV}$$

Hence divergen theorem is verified

P.Q.3 Verify divergence theorem for $\vec{F} = x^2\hat{\imath} + z\hat{\jmath} + yz\hat{k}$ taken over the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.$

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P.Q.4 *Verify divergence theorem for* $\vec{F} = (x^3 - yz)\hat{\imath} + (y^3 - zx)\hat{\jmath} + (z^3 - xy)\hat{k}$ *taken over the cube bounded by the planes* $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.$

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P.Q.5 Evaluate $\iint (a^2x^2 + b^2y^2 + c^2z^2)^{-\frac{1}{2}}, d\bar{S}$ where S is the surface of the ellipsoid $ax^2 + by^2 + cz^2 = 1$.

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Ans. $\dfrac{4\pi}{\sqrt{abc}}$

Syllabus Completed

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Thank You