Lec- No- 14

Laplace Transform and its formulae

Transformation → A transformation is a mathematical device which converts one func^n into another.

Ex→ **Differential Transform →**

Let $f(x) = \sin x$

then $Df(x) = \frac{d}{dx}\sin x = \cos x = g(x)$

∴ Differential operator D gives a new func^n $g(x)$ when apply on the func^n $f(x)$.

Laplace Transform →

or

Laplace Transformation →

Let $F(t)$ be a func^n of t defined $\forall\, t \geq 0$.

Then the Laplace transform of $F(t)$, denoted by $L\{F(t)\}$, is defined by

$$\boxed{L\{F(t)\} = \int_0^{\infty} e^{-pt} F(t)\, dt = f(p)}$$

provided that the integral exists where p is a parameter may be real or complex.

Note-1→ We can use parameter s also for p.

i.e
$$\boxed{L\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\, dt = f(s)}$$

Note-2→ Sometime we use the notation $L\{F(t)\} = \overline{F}(p)$ or $L\{y(t)\} = \overline{y}(p)$ or $L\{F(t)\} = \phi(p)$ etc.

Use of Laplace Transform →

Laplace transform is widely used by scientists and engineers in solving linear differential eqⁿ - ordinary as well as partial.

It reduces an ordinary differential eqⁿ into an algebraic eqⁿ.

~~Linear~~ Linear Property of Laplace Transformation →

Laplace transformation is a Linear transformation.

i.e Let $F_1(t)$ & $F_2(t)$ are funcⁿ of t, then

$$\boxed{L\{C_1F_1(t)+C_2F_2(t)\} = C_1L\{f_1(t)\}+C_2L\{f_2(t)\}}$$

where C_1 & C_2 are constants.

Laplace Transform of some elementary funcⁿ →

① Laplace Transform of $F(t)=1$ →

Solⁿ We have $L\{F(t)\} = \int_0^\infty e^{-pt}F(t)\,dt$

$$\therefore L\{1\} = \int_0^\infty e^{-pt}.1\,dt = \int_0^\infty e^{-pt}dt$$

$$= \left[\frac{e^{-pt}}{(-p)}\right]_0^\infty$$

$$= \left[\frac{e^{-\infty}}{(-p)} - \frac{e^0}{(-p)}\right]$$

$$= -\frac{1}{(-p)} = \frac{1}{p}. \qquad [\because e^{-\infty}=0 \text{ \& } e^0=1]$$

$$\Rightarrow \boxed{L\{1\} = \frac{1}{p},\ p>0}$$

② Laplace transform of the func^n $F(t)=t^n$, n is any real number.

Sol^n: we have $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)dt$

$$\therefore L\{t^n\} = \int_0^\infty e^{-pt} t^n dt$$

$$= \int_0^\infty e^{-pt} t^{(n+1)-1} dt \qquad \left[\because \int_0^\infty e^{-at} t^{m-1} dt = \frac{\Gamma m}{a^m} \text{ by Gamma func}^n\right]$$

$$= \frac{\Gamma n+1}{p^{n+1}}$$

$$\Rightarrow \boxed{L\{t^n\} = \frac{\Gamma n+1}{p^{n+1}}}$$ where n is any real number >-1 & $p>0$.

Note-1: If n is +ve integer then $\Gamma n+1 = \lfloor n$

$$\therefore L\{t^n\} = \frac{\lfloor n}{p^{n+1}}, \; p>0$$

Note-2: If $n=1$ then

$$\boxed{L\{t\} = \frac{\lfloor 1}{p^{1+1}} = \frac{1}{p^2}}$$

③ Prove that $L\{e^{at}\} = \frac{1}{p-a}$, $p>a$.

Sol^n: we have $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)dt$

$$\therefore L\{e^{at}\} = \int_0^\infty e^{-pt} e^{at} dt$$

$$= \int_0^\infty e^{-(p-a)t} dt$$

$$= \left[\frac{e^{-(p-a)t}}{-(p-a)}\right]_0^\infty$$

$$= \left[\frac{e^{-\infty}}{-(p-a)} - \frac{e^0}{-(p-a)}\right] = \left[0 - \frac{1}{-(p-a)}\right]$$

$$= \frac{1}{p-a} \text{ if } p>a$$

$$\Rightarrow \boxed{L\{e^{at}\} = \frac{1}{p-a} \text{ if } p>a}$$

④ Prove $L\{\sin at\} = \frac{a}{p^2+a^2}$, $p>0$

Proof:- we have $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$

$$\therefore L\{\sin at\} = \int_0^\infty e^{-pt} \sin at\,dt$$

$$= \left[\frac{e^{-pt}}{(-p)^2+a^2}\left\{-p\sin at - a\cos at\right\}\right]_0^\infty$$

$\left[\text{By } \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]\right]$

$$= \left[\frac{e^{-\infty}}{p^2+a^2}(-p\sin\infty - a\cos\infty) - \frac{e^0}{p^2+a^2}(-p\sin 0 - a\cos 0)\right]$$

$$= \left[0 - \frac{1}{p^2+a^2}(0-a)\right]$$

$\left[\because e^{-\infty}=0,\ e^0=1\right]$

$$= \frac{a}{p^2+a^2}$$

$$\Rightarrow \boxed{L\{\sin at\} = \frac{a}{p^2+a^2}} \quad p>0.$$

⑤ Prove $L\{\cos at\} = \frac{p}{p^2+a^2}$, $p>0$.

Proof:- we have $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$

$$\therefore L\{\cos at\} = \int_0^\infty e^{-pt} \cos at\,dt$$

$$= \left[\frac{e^{-pt}}{(-p)^2+a^2}\left\{-p\cos at + a\sin at\right\}\right]_0^\infty$$

$\left[\text{By } \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]\right]$

$$= \left[\frac{e^{-\infty}}{p^2+a^2}\{-p\cos\infty + a\sin\infty\} - \frac{e^0}{p^2+a^2}\{-p\cos 0 + a\sin 0\}\right]$$

$$= \left[0 - \frac{1}{p^2+a^2}(-p+0)\right]$$

$$= \frac{p}{p^2+a^2}$$

$$\Rightarrow \boxed{L\{\cos at\} = \frac{p}{p^2+a^2}}, \quad p>0$$

⑥ Prove $L\{\sinh at\} = \frac{a}{p^2-a^2}$, $p>|a|$

Proof:

$L\{\sinh at\} = L\{\frac{1}{2}(e^{at} - e^{-at})\}$ $\left[\because \sinh at = \frac{e^{at}-e^{-at}}{2}\right]$

$= \frac{1}{2} L\{e^{at}\} - \frac{1}{2} L\{e^{-at}\}$ (by Linear Property)

$= \frac{1}{2} \frac{1}{p-a} - \frac{1}{2} \frac{1}{p+a}$ By $L\{e^{at}\} = \frac{1}{p-a}, p>a$

$= \frac{1}{2}\left[\frac{(p+a)-(p-a)}{(p-a)(p+a)}\right]$

$= \frac{1}{2}\left[\frac{2a}{p^2-a^2}\right]$

$\Rightarrow \boxed{L\{\sinh at\} = \frac{a}{p^2-a^2}, \quad p>|a|}$

⑦ Prove $L\{\cosh at\} = \frac{p}{p^2-a^2}$, $p>|a|$.

Proof:

$L\{\cosh at\} = L\{\frac{1}{2}(e^{at} + e^{-at})\}$ $\because \cosh at = \frac{e^{at}+e^{-at}}{2}$

$= \frac{1}{2} L\{e^{at}\} + \frac{1}{2} L\{e^{-at}\}$ (by Linear property)

$= \frac{1}{2} \frac{1}{p-a} + \frac{1}{2} \frac{1}{p+a}$ By $L\{e^{at}\} = \frac{1}{p-a}, p>a$

$= \frac{1}{2}\left[\frac{(p+a)+(p-a)}{(p-a)(p+a)}\right]$

$= \frac{1}{2}\left[\frac{2p}{p^2-a^2}\right]$

$\boxed{L\{\cosh at\} = \frac{p}{p^2-a^2}, \quad p>|a|}$

Laplace Transform of Some Elementary functions

	$F(t)$	$L\{F(t)\}$
1	1	$\frac{1}{p}$, $p>0$
2	t	$\frac{1}{p^2}$, $p>0$
3	t^n (n is real number) $(n>-1)$	$\frac{\Gamma(n+1)}{p^{n+1}}$, $p>0$
	t^n (n is +ve integer)	$\frac{\lfloor n}{p^{n+1}}$, $p>0$
4	e^{at}	$\frac{1}{p-a}$, $p>a$
5	$\sin at$	$\frac{a}{p^2+a^2}$, $p>0$
6	$\cos at$	$\frac{p}{p^2+a^2}$, $p>0$
7	$\sinh at$	$\frac{a}{p^2-a^2}$, $p>\lvert a\rvert$
8	$\cosh at$	$\frac{p}{p^2-a^2}$, $p>\lvert a\rvert$
9	$C_1F_1(t)+C_2F_2(t)$	$C_1L\{F_1(t)\}+C_2L\{F_2(t)\}$ (Linear Properties)

Application of Laplace Transform Formulae →

① Find Laplace transform of

(i) $1+t+t^2+t^3$ (ii) $1+2\sqrt{t}+\frac{3}{\sqrt{t}}$.

Solⁿ:
(i) We have $L\{1\}=\frac{1}{p}$, $L\{t^n\}=\frac{\lfloor n}{p^{n+1}}$ For n is +ve integer

$=\frac{\Gamma(n+1)}{p^{p+1}}$, for n is real >-1

then
$$L\{1+t+t^2+t^3\}=L\{1\}+L\{t\}+L\{t^2\}+L\{t^3\}$$
$$=\frac{1}{p}+\frac{\lfloor 1}{p^{1+1}}+\frac{\lfloor 2}{p^{2+1}}+\frac{\lfloor 3}{p^{3+1}}$$
$$=\frac{1}{p}+\frac{1}{p^2}+\frac{2}{p^3}+\frac{6}{p^4}.$$

(ii)

$$L\{1+2\sqrt{t}+\frac{3}{\sqrt{t}}\} = L\{1\}+2L\{t^{1/2}\}+3L\{t^{-1/2}\}$$

$$= \frac{1}{p}+2\frac{\Gamma(\frac{1}{2}+1)}{p^{\frac{1}{2}+1}}+3\frac{\Gamma(-\frac{1}{2}+1)}{p^{-\frac{1}{2}+1}}$$

[By $L\{t^n\} = \frac{\Gamma(n+1)}{p^{n+1}}$, $n > -1$ (real)

$$= \frac{1}{p}+2\frac{\Gamma(3/2)}{p^{3/2}}+3\frac{\Gamma(\frac{1}{2})}{p^{1/2}}$$

$$= \frac{1}{p}+2\frac{\frac{1}{2}\sqrt{\pi}}{p^{3/2}}+3\frac{\sqrt{\pi}}{p^{1/2}}$$

[$\because$ $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$]

$$= \frac{1}{p}+\frac{\sqrt{\pi}}{p^{3/2}}+\frac{3\sqrt{\pi}}{p^{1/2}}.$$

Ex-2:- Find the Laplace transform of

$$7e^{2t}+9e^{-2t}+5\cos t+7t^3+5\sin 3t-2.$$

Solⁿ:-

$$L\{7e^{2t}+9e^{-2t}+5\cos t+7t^3+5\sin 3t-2\}$$

$$= 7L\{e^{2t}\}+9L\{e^{-2t}\}+5L\{\cos t\}+7L\{t^3\}+5L\{\sin 3t\}-2L\{1\}$$

$$= 7\frac{1}{p-2}+9\frac{1}{p+2}+5\frac{p}{p^2+1}+7\frac{\Gamma 3}{p^4}+5\frac{3}{p^2+3^2}-2\cdot\frac{1}{p}$$

$$= \frac{7}{p-2}+\frac{9}{p+2}+\frac{5p}{p^2+1}+\frac{42}{p^4}+\frac{15}{p^2+9}-\frac{2}{p}$$

[By $L\{e^{at}\} = \frac{1}{p-a}$, $L\{\cos at\} = \frac{p}{p^2+a^2}$, $L\{\sin at\} = \frac{a}{p^2+a^2}$

$L\{t^n\} = \frac{\Gamma n}{p^{n+1}}$, $L\{1\} = \frac{1}{p}$]

Ex-3:- Find the Laplace transform of

(i) $\sin 2t\cos 3t$ (ii) $\sin^3 2t$ (iii) $\cosh at - \sin at$

(iv) $\cosh^3 2t$

Solⁿ:- (i) Since $\sin 2t\cos 3t = \frac{1}{2}[2\cos 3t\sin 2t] = \frac{1}{2}(\sin 5t - \sin t)$

$\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$$\therefore L\{\sin 2t\cos 3t\} = \frac{1}{2}L\{\sin 5t - \sin t\}$$

$$= \frac{1}{2}L\sin 5t - \frac{1}{2}L\sin t = \frac{1}{2}\frac{5}{p^2+5^2} - \frac{1}{2}\frac{1}{p^2+1^2}$$

(ii) We know that $\sin 3t = 3\sin t - 4\sin^3 t$

then $\sin 6t = 3\sin 2t - 4\sin^3 2t$

$\Rightarrow 4\sin^3 2t = 3\sin 2t - \sin 6t$

$\Rightarrow \sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t$

$\Rightarrow L\{\sin^3 2t\} = L\{\frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t\}$

$= \frac{3}{4}L\{\sin 2t\} - \frac{1}{4}L\{\sin 6t\}$

$= \frac{3}{4}\frac{2}{p^2+2^2} - \frac{1}{4}\frac{6}{p^2+6^2}$

$= \frac{48}{(p^2+4)(p^2+36)}$.

Note → Hyperbolic formulae

(i) $\cosh^2 t - \sinh^2 t = 1$

(ii) $\cosh^2 t + \sinh^2 t = \cosh 2t$

(iii) $\cosh 2t = 1 + 2\sinh^2 t = 2\cosh^2 t - 1$

(iv) $\sinh 3t = 3\sinh t + 4\sinh^3 t$

(v) $\cosh 3t = 4\cosh^3 t - 3\cosh t$

(vi) $\cosh at = \frac{e^{at}+e^{-at}}{2}$, $\sinh at = \frac{e^{at}-e^{-at}}{2}$

(iv) we know that $\cosh 3t = 4\cosh^3 t - 3\cosh t$

$\Rightarrow \cosh 6t = 4\cosh^3 2t - 3\cosh 2t$

$\Rightarrow 4\cosh^3 2t = 3\cosh 2t + \cosh 6t$

$\Rightarrow \cosh^3 2t = \frac{3}{4}\cosh 2t + \frac{1}{4}\cosh 6t$

$\Rightarrow L\{\cosh^3 2t\} = \frac{3}{4}L\{\cosh 2t\} + \frac{1}{4}L\{\cosh 6t\}$

$= \frac{3}{4}\frac{p}{p^2-2^2} + \frac{1}{4}\frac{p}{p^2-6^2}$

By $L\{\cosh at\} = \frac{p}{p^2-a^2}$, $p > |a|$

Home Assignment

① Find (i) $L\{t^{-1/2}\}$ (ii) $L\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$

(iii) $L\{\sin t \cos t\}$ (iv) $L\{\sin 2t \sin 3t\}$

(v) $L\{4\cos^2 2t\}$ (vi) $L\{\cosh^2 2t\}$ (vii) $L\{\sinh^2 2t\}$

(vii) Prove $L\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3 = \sqrt{\pi}\left[\frac{3}{4p^{5/2}} - 2\sqrt{p} + \frac{3}{2p^{3/2}} + \frac{3}{\sqrt{p}}\right]$.

② Find the L.T of the funch

$$F(t) = (\sin t - \cos t)^2.$$

③ Find L.T of (i) $3\cosh 5t - 4\sinh 5t$

(ii) $\sin^2 3t$ (iv) $e^{-2t} - e^{-3t}$

(iii) $\frac{e^{at} - 1}{a}$ (v) $L\{1 + \cos 2t\}$.

Trigonometric formulae:

① $\cos^2 t + \sin^2 t = 1$

② $\cos 2t = 1 - 2\sin^2 t = 2\cos^2 t - 1 = \cos^2 t - \sin^2 t$

③ ~~2G~~ $2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \sin B = \sin(A+B) - \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

④ $\sin 3t = 3\sin t - 4\sin^3 t$

⑤ $\cos 3t = 4\cos^3 t - 3\cos t$

⑥ $\sin 2x = 2\sin x \cos x$

Ex: Show that $L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{p}}\, e^{-\frac{1}{4p}}$

Soln: We have $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots$

Now $\frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{\sqrt{t}}\left[1 - \frac{(\sqrt{t})^2}{\lfloor 2} + \frac{(\sqrt{t})^4}{\lfloor 4} - \frac{(\sqrt{t})^6}{\lfloor 6} + \cdots\right]$

$= t^{-1/2} - \frac{1}{\lfloor 2} t^{1/2} + \frac{1}{\lfloor 4} t^{3/2} - \frac{1}{\lfloor 6} t^{5/2} + \cdots$

$$\therefore L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = L\{t^{-1/2}\} - \frac{1}{\lfloor 2}L\{t^{1/2}\} + \frac{1}{\lfloor 4}L\{t^{3/2}\} - \frac{1}{\lfloor 6}L\{t^{5/2}\} + \cdots$$

$$= \frac{\Gamma(-\frac{1}{2}+1)}{p^{-\frac{1}{2}+1}} - \frac{1}{\lfloor 2}\frac{\Gamma(\frac{1}{2}+1)}{p^{1/2+1}} + \frac{1}{\lfloor 4}\frac{\Gamma(\frac{3}{2}+1)}{p^{3/2+1}} - \frac{1}{\lfloor 6}\frac{\Gamma(\frac{5}{2}+1)}{p^{5/2+1}} + \cdots$$

$$= \frac{\Gamma\frac{1}{2}}{p^{1/2}} - \frac{1}{\lfloor 2}\frac{\Gamma 3/2}{p^{3/2}} + \frac{1}{\lfloor 4}\frac{\Gamma 5/2}{p^{5/2}} - \frac{1}{\lfloor 6}\frac{\Gamma 7/2}{p^{7/2}} + \cdots$$

$$= \frac{\sqrt{\pi}}{\sqrt{p}} - \frac{1}{2}\frac{\frac{1}{2}\sqrt{\pi}}{p\cdot\sqrt{p}} + \frac{1}{4\cdot3\cdot2\cdot1}\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{p^2\cdot\sqrt{p}} - \frac{1}{6\cdot5\cdot4\cdot3\cdot2\cdot1}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{p^3\sqrt{p}} + \cdots$$

$$= \frac{\sqrt{\pi}}{\sqrt{p}}\left[1 - \frac{1}{\lfloor 1}\left(\frac{1}{4p}\right) + \frac{1}{\lfloor 2}\left(\frac{1}{4p}\right)^2 - \frac{1}{\lfloor 3}\left(\frac{1}{4p}\right)^3 + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{\sqrt{p}} e^{-\frac{1}{4p}} \qquad \left[\because e^{-x} = 1 - \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \cdots\right]$$

<u>Ex</u> Find $L\{3^t\}$.

<u>Solⁿ</u>

$$L\{3^t\} = L\{e^{\log 3^t}\}$$

$$= L\{e^{t\log 3}\}$$

$$= L\{e^{(\log 3)t}\}.$$

$$= \frac{1}{p - \log 3}$$

$$L\{e^{at}\} = \frac{1}{p-a}$$

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Existence of Laplace Transform →

Statement → If $F(t)$ is a function which is piece wise continuous on every finite interval in the range $t \geq 0$ and satisfies $|F(t)| \leq M e^{at}$ for all $t \geq 0$ & for some constants a and M, then the Laplace transform of $F(t)$ exists for all $p > a$. [AKTU - 2011, 2012, 2016].

Proof → We have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$$

$$= \int_0^{t_0} e^{-pt} F(t)\,dt + \int_{t_0}^\infty e^{-pt} F(t)\,dt = I_1 + I_2 \longrightarrow (1)$$

The integral I_1 exists since $F(t)$ is piece wise continuous on every finite interval $0 \leq t \leq t_0$.

$$\& \quad |I_2| = \left| \int_{t_0}^\infty e^{-pt} F(t)\,dt \right| \leq \int_{t_0}^\infty |e^{-pt} F(t)|\, dt$$

$$\leq \int_0^\infty |e^{-pt} F(t)|\,dt \qquad [|F(t)| \leq M e^{at}]$$

$$\leq \int_0^\infty e^{-pt} \cdot M e^{at}\, dt = M\int_0^\infty e^{-(p-a)t}\,dt$$

$$= \frac{M}{p-a}, \; p > a.$$

$$\Rightarrow \quad |I_2| \leq \frac{M}{p-a}, \quad p > a.$$

$\Rightarrow$ The integral I_2 is also exist for $p > a$.

Hence $L\{F(t)\}$ exists for $p > a$.

Ex-1 → Find $L\{F(t)\}$ if $F(t) = \begin{cases} t/K, & 0<t<K \\ 1, & t>K \end{cases}$. [AKTU - 2010]

Sol^n → We have $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$

$$= \int_0^K e^{-pt}\frac{t}{K}\,dt + \int_K^\infty e^{-pt}\cdot 1\,dt$$

$$= \frac{1}{K}\left[\int_0^K t\, e^{-pt}\,dt\right] + \int_K^\infty e^{-pt}\,dt$$

$$= \frac{1}{K}\left[\left(t\,\frac{e^{-pt}}{-p}\right)_0^K - \int_0^K 1\cdot\frac{e^{-pt}}{(-p)}\,dt\right] + \left[\frac{e^{-pt}}{-p}\right]_K^\infty$$

$$= \frac{1}{K}\left[\left(\frac{K e^{-pK}}{-p}\right) - \left(\frac{e^{-pt}}{(-p)^2}\right)_0^K\right] + \left[0 + \frac{e^{-pK}}{p}\right]$$

$$= \frac{1}{K}\left[\frac{K e^{-pK}}{-p} - \frac{e^{-pK}}{p^2} + \frac{1}{p^2}\right] + \frac{e^{-pK}}{p}$$

$$= -\frac{e^{-pK}}{p} - \frac{1}{Kp^2}\left(e^{-pK} - 1\right) + \frac{e^{-pK}}{p}.$$

Note → Sometime K is replaced by T in this question.

Ex-2 → Find the Laplace transform of

$$F(t) = \begin{cases} \cos t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$$

Sol^n → We have $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$

$$= \int_0^\pi e^{-pt}\cos t\,dt + \int_\pi^\infty e^{-pt}\cdot 0\,dt$$

$$= \int_0^\pi e^{-pt}\cos t\,dt$$

$$\left[\text{By } \int e^{at}\cos bt\,dt = \frac{e^{at}}{a^2+b^2}\left[a\cos bt + b\sin bt\right]\right]$$

$$= \left[\frac{e^{-pt}}{(-p)^2+1^2}\left(-p\cos t + \sin t\right)\right]_0^\pi$$

$$= \frac{e^{-p\pi}}{p^2+1}\left(-p\cos\pi + \sin\pi\right) - \frac{e^0}{p^2+1}\left(-p\cos 0 + \sin 0\right)$$

$$= \frac{p\,e^{-p\pi}}{p^2+1} + \frac{p}{p^2+1}$$

$$\left[\because \cos\pi = -1,\ \sin\pi = 0\right]$$

Ex-3: Find $L\{F(t)\}$, if $F(t) = \begin{cases} e^t, & 0<t<5 \\ 3, & t>5. \end{cases}$

Solⁿ: we have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^5 e^{-pt}\cdot e^t\,dt + \int_5^\infty e^{-pt}\cdot 3\,dt$$

$$= \int_0^5 e^{-(p-1)t}\,dt + 3\int_5^\infty e^{-pt}\,dt$$

$$= \left[\frac{e^{-(p-1)t}}{-(p-1)}\right]_0^5 + 3\left[\frac{e^{-pt}}{-p}\right]_5^\infty$$

$$= \left[-\frac{e^{-(p-1)5}}{p-1} + \frac{1}{p-1}\right] + 3\left[0 + \frac{e^{-5p}}{p}\right]$$

$$= \frac{1 - e^{-(p-1)5}}{p-1} + \frac{3e^{-5p}}{p}.$$

Ex-4: Find $L\{F(t)\}$, if $F(t) = \begin{cases} t^2, & 0<t<2 \\ t-1, & 2<t<3 \\ 7, & 3<t<\infty \end{cases}$ (AKTU-2011)

Solⁿ: we have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^2 e^{-pt}\cdot t^2\,dt + \int_2^3 e^{-pt}(t-1)\,dt + \int_3^\infty e^{-pt}\cdot 7\,dt = I_1 + I_2 + I_3 \rightarrow (1)$$

Now $I_1 = \int_0^2 t^2 e^{-pt}\,dt = \left(t^2\frac{e^{-pt}}{(-p)}\right)_0^2 - \int_0^2 2t\,\frac{e^{-pt}}{(-p)}\,dt$

$$= -\frac{4}{p}e^{-2p} + \frac{2}{p}\int_0^2 t\,e^{-pt}\,dt$$

$$= -\frac{4}{p}e^{-2p} + \frac{2}{p}\left[t\frac{e^{-pt}}{(-p)} - \frac{e^{-pt}}{p^2}\right]_0^2$$

$$= -\frac{4}{p}e^{-2p} + \frac{2}{p}\left[-\frac{2}{p}e^{-2p} + \frac{e^{-2p}}{p^2} + 0 + \frac{1}{p^2}\right].$$

$$= -\frac{4}{p}e^{-2p} - \frac{4}{p^2}e^{-2p} - \frac{2}{p^3}e^{-2p} + \frac{2}{p^3}$$

& $I_2 = \int_2^3 (t-1)\,e^{-pt}\,dt = \left[(t-1)\frac{e^{-pt}}{(-p)}\right]_2^3 - \int_2^3 1\cdot\frac{e^{-pt}}{(-p)}\,dt$

$$= \frac{2}{-p}e^{-3p} + \frac{1}{p}e^{-2p} - \left[\frac{e^{-pt}}{p^2}\right]_2^3$$

$$= \frac{2}{-p}e^{-3p} + \frac{1}{p}e^{-2p} - \frac{e^{-3p}}{p^2} + \frac{e^{-2p}}{p^2}.$$

& $I_3 = 7\int_3^{\infty} e^{-pt}dt = 7\left[\frac{e^{-pt}}{(-p)}\right]_3^{\infty} = 7\left[0+\frac{e^{-3p}}{p}\right] = \frac{7e^{-3p}}{p}.$

Using in (I), we get

$$L\{F(t)\} = -\frac{4}{p}e^{-2p} - \frac{4}{p^2}e^{-2p} - \frac{2}{p^3}e^{-2p} + \frac{2}{p^3} - \frac{2}{p}e^{-3p} + \frac{1}{p}e^{-2p} - \frac{e^{-3p}}{p^2} + \frac{e^{-2p}}{p^2} + 7\frac{e^{-3p}}{p}$$

$$= \frac{2}{p^3} + \left(5\frac{e^{-3p}}{p} - \frac{e^{-3p}}{p^2}\right) - \left(+\frac{2}{p^3}e^{-2p} + \frac{3}{p^2}e^{-2p} + \frac{3}{p}e^{-2p}\right)$$

$$= \frac{2}{p^3} + \frac{e^{-3p}}{p^2}(5p-1) - \frac{e^{-2p}}{p^3}(2+3p+3p^2).$$

Home Assignment

(1) Find $L\{F(t)\}$, where $F(t) = \begin{cases} 0 & , 0<t<1 \\ t & , 1<t<2 \\ 0 & , t>2. \end{cases}$

(2) Find $L\{F(t)\}$ where $F(t)$ defined as

i) $F(t) = \begin{cases} t & , 0<t<4 \\ 5 & , t>4 \end{cases}$ ii) $F(t) = \begin{cases} \sin t & 0<t<\pi \\ 0 & t>\pi \end{cases}$

iii) $F(t) = \begin{cases} e^t & , 0<t<1 \\ 0 & , t>1 \end{cases}$, (iv) $F(t) = \begin{cases} e^{t-a} & , t>a \\ 0 & , t<a \end{cases}$

(AKTU-2012)

(3) Find Laplace transform of $F(t)$, where

$$F(t) = \begin{cases} \frac{t}{\omega} & , 0<t<\omega \\ 1-t/\omega & , \omega<t<2\omega \\ 1 & , 2\omega<t<\infty \end{cases}$$ (AKTU-2010, 2013)

(4) $F(t) = \begin{cases} 1 & , 0\le t<1 \\ t & , 1\le t<2 \\ t^2 & , 2\le t<\infty \end{cases}$ (5) $F(t) = \begin{cases} \frac{t}{T} & , 0<t<T \\ 1 & , t>T. \end{cases}$

(AKTU-2015)

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First Translation Property or First Shifting Property → (AKTU-2010).

If $L\{F(t)\} = f(p)$ then

$$\boxed{L\{e^{at}F(t)\} = f(p-a)}$$

Note → ① $L\{e^{-at}F(t)\} = f(p+a)$

② $L\{e^{at}F(bt)\} = \frac{1}{b} f\left(\frac{p-a}{b}\right)$

Application of first Shifting Property →

① $L\{t^n\} = \frac{\lfloor n}{p^{n+1}}$ where n is a +ve integer.

then $L\{e^{at}t^n\} = \frac{\lfloor n}{(p-a)^{n+1}}$

② $L\{\sin bt\} = \frac{b}{p^2+b^2}$

then $L\{e^{at}\sin bt\} = \frac{b}{(p-a)^2+b^2}$

③ $L\{\cos bt\} = \frac{p}{p^2+b^2}$

then $L\{e^{at}\cos bt\} = \frac{p-a}{(p-a)^2+b^2}$

④ $L\{\sinh bt\} = \frac{b}{p^2-b^2}$

then $L\{e^{at}\sinh bt\} = \frac{b}{(p-a)^2-b^2}$

⑤ $L\{\cosh bt\} = \frac{p}{p^2-b^2}$

then $L\{e^{at}\cosh bt\} = \frac{p-a}{(p-a)^2-b^2}$

Ex-1: Find the Laplace transforms of $(1+te^{-t})^3$.

Sol^n: Let $F(t) = (1+te^{-t})^3$

$$= 1^3 + (te^{-t})^3 + 3\times1\times te^{-t}(1+te^{-t})$$

$$\Rightarrow F(t) = 1 + t^3e^{-3t} + 3te^{-t} + 3t^2e^{-2t}$$ [$\because (a+b)^3 = a^3+b^3+3ab(a+b)$]

$$\therefore L\{F(t)\} = L\{1\} + L\{e^{-3t}\cdot t^3\} + 3L\{e^{-t}\cdot t\} + 3L\{e^{-2t}\cdot t^2\}.$$

$$= \frac{1}{p} + \frac{\lfloor 3}{(p+3)^4} + 3\frac{\lfloor 1}{(p+1)^2} + 3\frac{\lfloor 2}{(p+2)^3}$$

$$= \frac{1}{p} + \frac{6}{(p+3)^4} + \frac{3}{(p+1)^2} + \frac{6}{(p+2)^3}.$$

$\left[\because L\{t^n\} = \frac{\lfloor n}{p^{n+1}}\right.$

$\left.\therefore L\{e^{at}t^n\} = \frac{\lfloor n}{(p-a)^{n+1}}\right]$

Ex-2: Find $L\{e^t t^{-1/2}\}$.

Sol^n: we have $L\{t^{-1/2}\} = \frac{\Gamma(-\frac{1}{2}+1)}{p^{-\frac{1}{2}+1}}$ $\left[\because L(t^n) = \frac{\Gamma(n+1)}{p^{n+1}}\right.$ for n is real$\left.\right]$

$$= \frac{\Gamma\frac{1}{2}}{p^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{p}}$$

$$\therefore L\{e^t t^{-1/2}\} = \frac{\sqrt{\pi}}{\sqrt{p-1}}$$ [by first shifting Theorem]

Ex-3: Find the Laplace transform of $e^{-3t}(\cos 4t + 3\sin 4t)$.

Sol^n: we have

$$L\{\cos 4t + 3\sin 4t\} = L\{\cos 4t\} + 3L\{\sin 4t\}$$

$$= \frac{p}{p^2+16} + 3\frac{4}{p^2+16}$$

Now by first shifting theorem, $= \frac{p+12}{p^2+16}$ $\left[\because L\{\sin at\} = \frac{a}{p^2+a^2}\right.$, $\left.L\{\cos at\} = \frac{p}{p^2+a^2}\right]$

$$L\{e^{-3t}(\cos 4t + 3\sin 4t)\} = \frac{(p+3)+12}{(p+3)^2+16}.$$

Ex-4: Find $L\{e^{-t}\cos t\cos 2t\}$.

Sol^n: we have $\cos t\cos 2t = \frac{1}{2}[2\cos 2t\cos t] = \frac{1}{2}(\cos 3t + \cos t)$.

$$\therefore L\{\cos t\cos 2t\} = \frac{1}{2}[L\{\cos 3t\} + L\{\cos t\}] = \frac{1}{2}\left[\frac{p}{p^2+9} + \frac{p}{p^2+1}\right]$$

Now by first shifting property,

$$L\{e^{-t}\cos t\cos 2t\} = \frac{1}{2}\left[\frac{(p+1)}{(p+1)^2+9} + \frac{(p+1)}{(p+1)^2+1}\right]$$

Ex-5 → Find the Laplace Transform of $\cosh at \cos bt$. [AKTU-2012, 2016]

Solⁿ → Let $F(t) = \cosh at \cos bt$

$$= \left[\frac{e^{at} + e^{-at}}{2}\right] \cos bt$$

$$\Rightarrow F(t) = \frac{1}{2}\left[e^{at}\cos bt\right] + \frac{1}{2}\left[e^{-at}\cos bt\right].$$

we have $L\{\cos bt\} = \frac{p}{p^2+b^2}$

then by first shifting Property,

$$L\{F(t)\} = \frac{1}{2}L\{e^{at}\cos bt\} + \frac{1}{2}L\{e^{-at}\cos bt\}$$

$$= \frac{1}{2}\frac{p-a}{(p-a)^2+b^2} + \frac{1}{2}\frac{p+a}{(p+a)^2+b^2}$$

Ex-6 → Find (i) $L\{t \sin at\}$ (ii) $L\{t^2 \cos at\}$

Solⁿ → we have $\sin at = \frac{e^{iat} - e^{-iat}}{2i}$ & $L\{t\} = \frac{1!}{p^2}$

Then

$$L\{t \sin at\} = L\left\{\left(\frac{e^{iat} - e^{-iat}}{2i}\right)t\right\}$$

$$= \frac{1}{2i}\left[L\{e^{iat}t\} - L\{e^{-iat}t\}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{(p-ia)^2} - \frac{1}{(p+ia)^2}\right]$$ [by first shifting property]

$$= \frac{1}{2i}\left[\frac{(p+ia)^2 - (p-ia)^2}{(p-ia)^2(p+ia)^2}\right]$$

$$= \frac{1}{2i}\left[\frac{p^2+(ia)^2+2pia-p^2-(ia)^2+2pia}{(p^2-i^2a^2)^2}\right]$$

$$= \frac{1}{2i}\left[\frac{4iap}{(p^2+a^2)^2}\right]$$

$$= \frac{2ap}{(p^2+a^2)^2}.$$

$$\boxed{\text{(ii)} \quad L\{t^2\cos at\} \rightarrow \text{Hint} \quad \cos at = \frac{e^{iat} + e^{-iat}}{2}}$$

Home Assignment

Find the Laplace Transform of

① $e^{-\alpha t}\cos\beta t$ (AKTU-2010) ② $t^3 e^{-3t}$ ③ $e^{-3t}(2\cos 5t - 3\sin 5t)$

④ $\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t$ (AKTU-2009, 2012) ⑤ $F(t) = \sinh kt \cos kt$ (AKTU-2016)

⑥ $F(t) = e^{4t}\cos t\sin 2t$ ⑦ $F(t) = e^{2t}\sin^4 t$

⑧ $F(t) = e^{-2t}\sin^3 t$ ⑨ $F(t) = \sinh at \sin at$

⑩ $F(t) = e^{-t}\sin^2 t$ ⑪ $F(t) = (t+2)^2 e^t$

⑫ $e^{-t}(3\sinh 2t - 5\cosh 2t)$ ⊛ $F(t) = \frac{\cos 2t \sin t}{e^t}$ (AKTU-2010, 2016)

⑬ Find $L\left\{e^{-at}\frac{t^{n-1}}{\lfloor n-1}\right\}$. (AKTU-2005)

Solⁿ we have

$$L\left\{\frac{t^{n-1}}{\lfloor n-1}\right\} = \frac{1}{\lfloor n-1} L\{t^{n-1}\} = \frac{1}{\lfloor n-1}\frac{\lfloor n-1}{p^{n-1+1}} = \frac{1}{p^n} = f(p)$$

$$\left[\because L\{t^n\} = \frac{\lfloor n}{p^{n+1}}\right]$$

∴ From first shifting theorem, we get

$$L\left\{e^{-at}\frac{t^{n-1}}{\lfloor n-1}\right\} = f(p+a) = \frac{1}{(p+a)^n}.$$

Change of Scale Property →

$$\boxed{\text{If } L\{F(t)\} = f(p) \quad \text{then} \quad L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)}$$

Ex-1→ If $L\{F(t)\} = \dfrac{p^2-p+1}{(2p+1)^2(p-1)}$, show that $L\{F(2t)\} = \dfrac{p^2-2p+4}{4(p+1)^2(p-2)}$

[AKTU-2010, 2014, 2018].

Solⁿ→ we have $L\{F(t)\} = \dfrac{p^2-p+1}{(2p+1)^2(p-1)} = f(p)$ (say)

∴ By the change of scale property, we have

$$L\{F(2t)\} = \frac{1}{2} f\left(\frac{p}{2}\right) = \frac{1}{2} \frac{\left(\frac{p}{2}\right)^2 - \frac{p}{2} + 1}{\left(2\frac{p}{2}+1\right)^2\left(\frac{p}{2}-1\right)}$$

$$= \frac{1}{2} \frac{\frac{p^2-2p+4}{4}}{\frac{(p+1)^2(p-2)}{2}}$$

$$= \frac{p^2-2p+4}{4(p+1)^2(p-2)}.$$

Ex-2→ If $L\{J_0(\sqrt{t})\} = \dfrac{e^{-1/4p}}{p}$, find $L\{J_0(2\sqrt{t})\}$.

Solⁿ→ Given $L\{J_0(\sqrt{t})\} = \dfrac{e^{-1/4p}}{p} = f(p)$ (say)

Then by change of scale property, we have

$$L\{J_0(2\sqrt{t})\} = L\{J_0(\sqrt{4t})\} = \frac{1}{4} f\left(\frac{p}{4}\right)$$

$$= \frac{1}{4} \frac{e^{-\frac{1}{4}/\frac{p}{4}}}{\frac{p}{4}} = \frac{e^{-1/p}}{p}.$$

Ex-3→ Given that $L\left(\dfrac{\sin t}{t}\right) = \tan^{-1}\left(\dfrac{1}{p}\right)$, find $L\left(\dfrac{\sin at}{t}\right)$. (AKTU –2013)

Solⁿ→ we have $L\left(\dfrac{\sin t}{t}\right) = \tan^{-1}\left(\dfrac{1}{p}\right) = f(p)$

then by change of scale property,

$$L\left(\frac{\sin at}{at}\right) = \frac{1}{a} f\left(\frac{p}{a}\right) = \frac{1}{a}\tan^{-1}\left(\frac{1}{p/a}\right)$$

$$\Rightarrow \frac{1}{\cancel{a}} L\left\{\frac{\sin at}{t}\right\} = \frac{1}{\cancel{a}}\tan^{-1}\frac{a}{p} \Rightarrow \boxed{L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\frac{a}{p}}$$

Ex-4: If $L\{F(t)\} = \frac{e^{-1/p}}{p}$, show that $L\{F(3t)\} = \frac{e^{-3/p}}{p}$. Also find $L\{e^{-t}F(3t)\}$. [AKTU-2005, 2014]

Solⁿ: Given $L\{F(t)\} = \frac{e^{-1/p}}{p} = f(p)$ (say)

then by change of scale property,

$$L\{F(3t)\} = \frac{1}{3} f\left(\frac{p}{3}\right) = \frac{1}{3} \frac{e^{-\frac{1}{p/3}}}{p/3} = \frac{e^{-3/p}}{p} = G(p) \text{ (say)}$$

Now by using first shifting property,

$$L\{e^{-t}F(3t)\} = G(p+1) = \frac{e^{-\frac{3}{p+1}}}{p+1}.$$

Ex-5: Applying change of scale property, find $L\{\sinh 3t\}$.

Solⁿ: We have $L\{\sinh t\} = \frac{1}{p^2-1} = f(p)$ (say) $\left[\because L\{\sinh at\} = \frac{a}{p^2-a^2}\right]$

then by change of scale property,

$$L\{\sinh 3t\} = \frac{1}{3} f\left(\frac{p}{3}\right) = \frac{1}{3} \frac{1}{\left(\frac{p}{3}\right)^3 - 1} = \frac{3}{p^2-9}.$$

Home Assignment

Ex-1: If $L(\cos^2 t) = \frac{p^2+2}{p(p^2+4)}$, find $L(\cos^2 at)$.

Ex-2: If $L\{J_0(t)\} = \frac{1}{\sqrt{1+p^2}}$, find $L\{J_0(at)\}$.

Ex-3: If $L\{F(t)\} = f(p)$, show that $L\{F\left(\frac{t}{a}\right)\} = a f(ap)$.

Ex-4: If $L\{F(t)\} = \frac{20-4p}{p^2-4p+20}$, applying the change of scale property, evaluate $L\{F(3t)\}$.

Ex-5: Find $L\{\cos 5t\}$ by change of scale property.

Ex-6: Find $L\{\cosh \frac{t}{\sqrt{2}}\}$ by change of scale. Also evaluat $L\{e^{-\sqrt{2}t}\cosh\frac{t}{\sqrt{2}}\}$. (AKTU-2010, 2011)

Solⁿ: We have $L\{\cosh t\} = \frac{p}{p^2-1} = f(p)$ (say)

Then by change of scale, $L\{\cosh \frac{1}{\sqrt{2}}t\} = \frac{1}{1/\sqrt{2}} f\left(\frac{p}{1/\sqrt{2}}\right) = \sqrt{2} f(\sqrt{2}p) = \sqrt{2} \frac{(\sqrt{2}p)}{(\sqrt{2}p)^2-1}$

$$= \frac{2p}{2p^2-1} = G(p)$$

Now by first shifting theorem,

$$L\{e^{-\sqrt{2}t}\cosh\frac{t}{\sqrt{2}}\} = G(p+\sqrt{2}) = \frac{2(p+\sqrt{2})}{2(p+\sqrt{2})^2-1}$$

Second shifting property / Second Translation property
Second Shifting Theorem / Heaviside's Shifting Theorem

If $L\{F(t)\} = f(p)$ and $G(t) = \begin{cases} F(t-a) & , t>a \\ 0 & , t<a \end{cases}$

then $L\{G(t)\} = e^{-ap} f(p)$ (AKTU - 2010, 2015)

Ex-1 → Find $L\{G(t)\}$, where $G(t) = \begin{cases} e^{t-a} & , t>a \\ 0 & , t<a \end{cases}$.

Solⁿ →

Let $F(t) = e^{t}$

then $L\{F(t)\} = L\{e^{t}\} = \dfrac{1}{p-1} = f(p)$ (say)

by $L\{e^{at}\} = \dfrac{1}{p-a}$

Now $G(t) = \begin{cases} F(t-a) = e^{t-a} & , t>a \\ 0 & , t<a \end{cases}$

∴ from second shifting theorem, we have

$$L\{G(t)\} = e^{-ap} f(p) = e^{-ap} \cdot \frac{1}{p-1} = \frac{e^{-ap}}{p-1}, \quad p>1.$$

Ex-2 → Find $L\{F(t)\}$, where $F(t) = \begin{cases} \cos\left(t - \frac{2}{3}\pi\right) & , t > \frac{2\pi}{3} \\ 0 & , t < \frac{2\pi}{3} \end{cases}$. [AKTU 2011, 2013]

Solⁿ →

Let $P(t) = \cos t$

then $L\{P(t)\} = L\{\cos t\} = \dfrac{p}{p^2+1} = f(p)$ (say)

By $L\{\cos at\} = \dfrac{p}{p^2+a^2}$

Now $F(t) = \begin{cases} P\left(t - \frac{2\pi}{3}\right) = \cos\left(t - \frac{2\pi}{3}\right) & , t > \frac{2\pi}{3} \\ 0 & , t < \frac{2\pi}{3} \end{cases}$.

∴ From second shifting theorem, we get

$$L\{F(t)\} = e^{-ap} f(p) = e^{-\frac{2\pi}{3}p} f(p) = e^{-\frac{2\pi}{3}p} \cdot \frac{p}{p^2+1}.$$

Ex-3: Find $L\{F(t)\}$ if $F(t) = \begin{cases} (t-1)^2 , & t>1 \\ 0 , & 0<t<1 . \end{cases}$

Sol^n: Let $P(t) = t^2$

then $L\{P(t)\} = L\{t^2\} = \frac{\lfloor 2}{p^{2+1}} = \frac{2}{p^3} = f(p)$ (say)

By $L(t^n) = \frac{\lfloor n}{p^{n+1}}$

Now $F(t) = \begin{cases} P(t-1) = (t-1)^2 , & t>1 \\ 0 , & t<1 \end{cases}$

then by second shifting property,

$L\{F(t)\} = e^{-ap} f(p) = e^{-p} \cdot \frac{2}{p^3}$.

Home Assignment

Ex-1: Find $L\{F(t)\}$, where $F(t) = \begin{cases} \sin\left(t - \frac{\pi}{3}\right) , & t > \frac{\pi}{3} \\ 0 , & t < \frac{\pi}{3} . \end{cases}$

Ex-2: Find the Laplace transform of

$$G(t) = (t-1)^3 , \quad t>1$$
$$= 0 , \quad t<1$$

Ex-3: Find the Laplace transform of

$$G(t) = \begin{cases} \cos(t-a) , & t>a \\ 0 , & t<a . \end{cases}$$

Ex-4: Find the Laplace transform of

$$\phi(t) = \begin{cases} \sinh(t-\pi) , & t>\pi \\ 0 , & t<\pi . \end{cases}$$

Laplace transform of the funcⁿ multiplied by t^n →

or Multiplication by t^n →

If $L\{F(t)\} = f(p)$ then

$$\boxed{L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p)}$$ where $n = 1, 2, 3, \ldots$

Note → ① If $n=1$, then

$$L\{t\,F(t)\} = (-1)^1 \frac{d^1}{dp^1} f(p)$$

$$\Rightarrow \boxed{L\{t\,F(t)\} = -f'(p)}$$ (multiplication by t)

② If $n=2$, then

$$L\{t^2 F(t)\} = (-1)^2 \frac{d^2}{dp^2} f(p)$$

$$\Rightarrow \boxed{L\{t^2 F(t)\} = f''(p)}$$ (multiplication by t^2).

Ex-1 → Find the Laplace Transform of $t \sin^2 3t$.

Solⁿ → we have $\cos 2t = 1 - 2\sin^2 t$

$$\Rightarrow \cos 6t = 1 - 2\sin^2 3t$$

$$\Rightarrow \sin^2 3t = \frac{1 - \cos 6t}{2}$$

$$\therefore L\{\sin^2 3t\} = L\left\{\frac{1-\cos 6t}{2}\right\} = \frac{1}{2}L\{1\} - \frac{1}{2}L\{\cos 6t\}$$

$$= \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+36}\right]$$

$\because L\{1\} = \frac{1}{p}$, $L\{\cos at\} = \frac{p}{p^2+a^2}$

$$= \frac{1}{2}\left[\frac{p^2+36-p^2}{p(p^2+36)}\right] = \frac{18}{(p^3+36p)} = f(p)$$

By $L\{t\,F(t)\} = -\frac{d}{dp} f(p)$

$$\Rightarrow L\{t \sin^2 3t\} = -\frac{d}{dp}\left[\frac{18}{p^3+36p}\right] = -\left[\frac{-18}{(p^3+36p)^2} \times (3p^2+36)\right] = \frac{54(p^2+12)}{p^3+36p}.$$

Ex-2→ find $L\{(t^2-3t+2)\sin 3t\}$.

Sol^n→ we have $L\{\sin 3t\} = \dfrac{3}{p^2+9} = f(p)$ by $L\{\sin at\} = \dfrac{a}{p^2+a^2}$

Now

$L\{(t^2-3t+2)\sin 3t\}$ By $L\{t^n F(t)\} = (-1)^n \dfrac{d^n}{dp^n} f(p)$

$$= L\{t^2 \sin 3t\} - 3L\{t\sin 3t\} + 2L\{\sin 3t\}$$

$$= (-1)^2 \frac{d^2}{dp^2}\left[\frac{3}{p^2+9}\right] - 3(-1)^1 \frac{d}{dp}\left[\frac{3}{p^2+9}\right] + 2\,\frac{3}{p^2+9}.$$

$$= \frac{d}{dp}\left[\frac{(-3)\times 2p}{(p^2+9)^2}\right] + 3\,\frac{(-3)\times 2p}{(p^2+9)^2} + 2\,\frac{3}{p^2+9}.$$

$$= \frac{(p^2+9)^2(-6) - (-6p)\,2(p^2+9)\times 2p}{(p^2+9)^4} - \frac{18p}{(p^2+9)^2} + \frac{6}{p^2+9}.$$

$$= \frac{-6(p^2+9) + 24p^2}{(p^2+9)^3} - \frac{18p}{(p^2+9)^2} + \frac{6}{(p^2+9)}$$

$$= \frac{18p^2-54}{(p^2+9)^3} - \frac{18}{(p^2+9)^2} + \frac{6}{(p^2+9)}.$$

Ex-3→ Find $L\{t\,e^{-t}\cosh t\}$ (AKTU-2014)

Sol^n→ $L\{e^{-t}\cosh t\} = L\left[e^{-t}\dfrac{(e^t+e^{-t})}{2}\right\} = \dfrac{1}{2}L\{1+e^{-2t}\}.$

$$= \frac{1}{2}\left[L\{1\} + L\{e^{-2t}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{p} + \frac{1}{p+2}\right] = f(p).$$

$\left[\because L\{e^{at}\} = \dfrac{1}{p-a}\right]$

By $L\{t\,F(t)\} = (-1)^1 \dfrac{d}{dp} f(p)$.

$$\Rightarrow L\{t\,e^{-t}\cosh t\} = (-1)^1 \frac{d}{dp}\left[\frac{1}{2}\left\{\frac{1}{p} + \frac{1}{p+2}\right\}\right]$$

$$= \frac{(-1)}{2}\left[\frac{(-1)}{p^2} + \frac{(-1)}{(p+2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{p^2} + \frac{1}{(p+2)^2}\right].$$

Ex-4: Find the Laplace Transform of $t\sqrt{1+\sin t}$. (AKTU-2010, 2011)

Solⁿ: We have $\sqrt{1+\sin t} = \sqrt{\cos^2\frac{t}{2} + \sin^2\frac{t}{2} + 2\sin\frac{t}{2}\cos\frac{t}{2}}$

$$= \sqrt{\left(\cos\frac{t}{2} + \sin\frac{t}{2}\right)^2} = \cos\frac{t}{2} + \sin\frac{t}{2}.$$

$$\therefore L\{\sqrt{1+\sin t}\} = L\{\cos\frac{t}{2} + \sin\frac{t}{2}\}$$

$$= L\{\cos\frac{t}{2}\} + L\{\sin\frac{t}{2}\}$$

$$= \frac{p}{p^2+(\frac{1}{2})^2} + \frac{\frac{1}{2}}{p^2+(\frac{1}{2})^2}$$

By
$L\{\sin at\} = \frac{a}{p^2+a^2}$
$L\{\cos at\} = \frac{p}{p^2+a^2}$

$$= \left[\frac{4p}{4p^2+1} + \frac{2}{4p^2+1}\right]$$

$$= \frac{4p+2}{4p^2+1} = f(p) \text{ (say)}$$

By $L\{t\,F(t)\} = (-1)^n \frac{d}{dp} f(p)$

$$\Rightarrow L\{t\sqrt{1+\sin t}\} = (-1)\frac{d}{dp}\left(\frac{4p+2}{4p^2+1}\right)$$

$$= (-1)\frac{(4p^2+1).4 - (4p+2)(8p)}{(4p^2+1)^2}$$

$$= (-1)\frac{(16p^2+4) - 32p^2 - 16p}{(4p^2+1)^2}$$

$$= 4\frac{(4p^2+4p-1)}{(4p^2+1)^2}$$

Ex-5: Find $L\{t\,e^{-2t}\sin 3t\}$ (AKTU-2015)

Solⁿ: We have $L\{\sin 3t\} = \frac{3}{p^2+9}$

By $L\{\sin at\} = \frac{a}{p^2+a^2}$

$$\therefore L\{t\sin 3t\} = (-1)^1\frac{d}{dp}\left[\frac{3}{p^2+9}\right]$$

By
$L\{t\,F(t)\} = (-1)^n \frac{d}{dp} f(p)$

$$= (-1)\frac{(-3)\times 2p}{(p^2+9)^2}$$

$$= \frac{6p}{(p^2+9)^2} = f(p)$$

Now by first shifting property,

$$L\{e^{-2t}(t\sin 3t)\} = f(p+2) = \frac{6(p+2)}{[(p+2)^2+9]^2}$$

Ex-6 → Find $L\{t^2 e^t \sin 4t\}$. (AKTU - 2010, 2011)

Solⁿ → we have $L\{\sin 4t\} = \frac{4}{p^2+16}$

By $L\{t^2 F(t)\} = (-1)^2 \frac{d^2}{dp^2} f(p)$

$\therefore L\{t^2 \sin 4t\} = (-1)^2 \frac{d^2}{dp^2}\left(\frac{4}{p^2+16}\right)$

$= \frac{d}{dp}\left[\frac{(-1)\times 4\times 2p}{(p^2+16)^2}\right]$

$= \frac{d}{dp}\left[\frac{-8p}{(p^2+16)^2}\right]$

$= -8 \frac{d}{dp} \frac{p}{(p^2+16)^2}$

$= -8\left[\frac{(p^2+16)^2 . 1 - p\, 2(p^2+16) . 2p}{(p^2+16)^4}\right]$

$= -8\left[\frac{(p^2+16) - 4p^2}{(p^2+16)^3}\right]$

$= 8\left[\frac{3p^2-16}{(p^2+16)^3}\right] = f(p)$

Now by first shifting theorem,

$L\{e^{at} F(t)\} = f(p-a)$

$\Rightarrow L\{e^t (t^2 \sin 4t)\} = f(p-1) = 8\left[\frac{3(p-1)^2 - 16}{[(p-1)^2+16]^3}\right]$

$= \frac{8[3p^2 - 6p - 13]}{[p^2 - 2p + 17]^3}$.

Ex-7 → Show that $\int_0^\infty t^2 e^{-4t} \sin 2t\, dt = \frac{11}{500}$. (AKTU - 2008, 2014)

Solⁿ → $L\{\sin 2t\} = \frac{2}{p^2+4}$

then $L\{t^2 \sin 2t\} = (-1)^2 \frac{d^2}{dp^2}\left[\frac{2}{p^2+4}\right] = \frac{d}{dp}\left[\frac{-4p}{(p^2+4)^2}\right]$

$= \frac{(p^2+4)^2(-4) - (-4p)\,2(p^2+4)2p}{(p^2+4)^4}$

$= \frac{-4(p^2+4) + 16p^2}{(p^2+4)^3}$

$= \frac{12p^2 - 16}{(p^2+4)^3}$

we have $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$ Put $F(t) = t^2 \sin 2t$

$\Rightarrow \int_0^\infty e^{-pt} t^2 \sin 2t\, dt = L\{t^2 \sin 2t\}$

$\Rightarrow \int_0^\infty e^{-pt} t^2 \sin 2t\, dt = \dfrac{12p^2 - 16}{(p^2+4)^3}$

Put $p = -4$ both side, we get

$$\int_0^\infty e^{-4t} t^2 \sin 2t\, dt = \frac{12 \times 16 - 16}{(16+4)^3} = \frac{16(12-1)}{(20)^3}$$

$$= \frac{\cancel{16}^{4} \times 11}{\cancel{20}_{5} \times \cancel{20}_{5} \times 20} = \frac{11}{500}$$

Hence $\boxed{\int_0^\infty e^{-4t} t^2 \sin 2t\, dt = \frac{11}{500}}$

Home Assignment

Find Laplace Transform of

1. (i) $t \sin 3t \cos 2t$ (ii) $t\sqrt{1-\sin t}$
2. $t^2 e^{-2t} \cos t$
3. $t e^{-t} \sin 2t$ (AKTU-2010)
4. $t^2 e^{t} \sin 4t$
5. $t e^{-2t} \sin 3t$
6. $t \sinh at$
7. $t \sin^2 t$
8. $t e^{at} \sin at$ (AKTU-2012)
9. (i) Prove $\int_0^\infty e^{-2t} \sin^3 t\, dt = \frac{6}{65}$ (ii) $\int_0^\infty e^{-2t} t^2 \sin 3t\, dt$ (AKTU-2016, 19, 15)
10. (i) Evaluate $\int_0^\infty t^3 e^{-t} \sin t\, dt$ (AKTU-2010)
11. Find $\int_0^\infty e^{-st} t^3 \cos t\, dt$ (ii) $\int_0^\infty e^{-3t} t \sin t\, dt$.

Laplace Transform of the funcⁿ divided by t →

or Division by t →

$$\boxed{\text{If } L\{F(t)\} = f(p), \text{ then } L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(p)\,dp}$$

provided the integral exists.

Ex-1 → Find the Laplace Transform $\frac{e^{-at} - e^{-bt}}{t}$.

Solⁿ → Since $L\{e^{-at} - e^{-bt}\}$

$$= L\{e^{-at}\} - L\{e^{-bt}\}$$

By $L\{e^{at}\} = \frac{1}{p-a}$

$$= \frac{1}{p+a} - \frac{1}{p+b} = f(p) \text{ (say)}$$

Then by $L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(p)\,dp$

$$\Rightarrow L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \int_p^\infty \left[\frac{1}{p+a} - \frac{1}{p+b}\right] dp$$

$$= \left[\log(p+a) - \log(p+b)\right]_p^\infty$$

$$= \left[\log\frac{(p+a)}{(p+b)}\right]_p^\infty$$

$$= \left[\log\left(\frac{1+\frac{a}{p}}{1+\frac{b}{p}}\right)\right]_p^\infty = \log 1 - \log\left(\frac{1+\frac{a}{p}}{1+\frac{b}{p}}\right)$$

$$= 0 - \log\frac{(p+a)}{(p+b)}$$

$$= \log\frac{p+b}{p+a}.$$

Ex-2 → Find $L\left\{\frac{\cos at - \cos bt}{t}\right\}$ (AKTU-2017)

Solⁿ → $L\{\cos at - \cos bt\} = L\{\cos at\} - L\{\cos bt\}$.

$$= \frac{p}{p^2+a^2} - \frac{p}{p^2+b^2} = f(p) \text{ (say)}.$$

Then by $L\left\{\frac{1}{t}F(t)\right\} = \int_p^\infty f(p)\,dp$

$$\Rightarrow L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_p^\infty \left[\frac{p}{p^2+a^2} - \frac{p}{p^2+b^2}\right]dp$$

$$= \frac{1}{2}\int_p^\infty \left[\frac{2p}{p^2+a^2} - \frac{2p}{p^2+b^2}\right]dp$$

$$= \frac{1}{2}\left[\log(p^2+a^2) - \log(p^2+b^2)\right]_p^\infty$$

$$= \frac{1}{2}\left[\log\frac{(p^2+a^2)}{(p^2+b^2)}\right]_p^\infty$$

$$= \frac{1}{2}\left[\log\frac{\left(1+\frac{a^2}{p^2}\right)}{\left(1+\frac{b^2}{p^2}\right)}\right]_p^\infty$$

$$= \frac{1}{2}\left[\log\frac{1+0}{1+0} - \log\frac{\left(1+\frac{a^2}{p^2}\right)}{\left(1+\frac{b^2}{p^2}\right)}\right]$$

$$= \frac{1}{2}\left[-\log\frac{(p^2+a^2)}{(p^2+b^2)}\right]$$

$$= \frac{1}{2}\log\frac{p^2+b^2}{p^2+a^2}.$$

Ex-34 find $L\left\{e^{-t}\frac{\sin t}{t}\right\}$.

Soln $L\{\sin t\} = \frac{1}{p^2+1}$

$\left[\text{by } L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(p)\,dp\right]$

Then $L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty \frac{1}{p^2+1}\,dp$

$$= \left(\tan^{-1}p\right)_p^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}p.$$

$$= \frac{\pi}{2} - \tan^{-1}p \qquad \left[\because \tan^{-1}p + \cot^{-1}p = \frac{\pi}{2}\right]$$

$$= \cot^{-1}p = f(p) \text{ say})$$

Then by first shifting theorem,

$L\left\{e^{-t}\frac{\sin t}{t}\right\} = f(p+1) = \cot^{-1}(p+1)$

By $L\{e^{at}F(t)\} = f(p-a)$

Ex-4 → Evaluate $\int_0^\infty e^{-t} \frac{\sin^2 t}{t} dt$. (AKTU-2010, 2011)

Solⁿ →

$$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\} = \frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4}\right] = f(p) \text{ say})$$

(By $L\{1\} = \frac{1}{p}$. $L\{\cos at\} = \frac{p}{p^2+a^2}$

$$\therefore L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(p)\,dp$$

$$\Rightarrow L\left\{\frac{\sin^2 t}{t}\right\} = \int_p^\infty \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4}\right] dp$$

$$= \cancel{\tfrac{1}{2}\{\log p} = \frac{1}{2} \times \frac{1}{2} \int_p^\infty \left[\frac{2}{p} - \frac{2p}{p^2+4}\right] dp.$$

$$= \frac{1}{4}\left[2\log p - \log(p^2+4)\right]_p^\infty$$

$$= \frac{1}{4}\left[\log p^2 - \log(p^2+4)\right]_p^\infty$$

$$= \frac{1}{4}\left[\log \frac{p^2}{p^2+4}\right]_p^\infty$$

$$= \frac{1}{4}\left[\log \frac{1}{1+\frac{4}{p^2}}\right]_p^\infty$$

$$= \frac{1}{4}\left[\log 1 - \log \frac{1}{1+\frac{4}{p^2}}\right]$$

$$= -\frac{1}{4} \log \frac{p^2}{p^2+4} = \frac{1}{4} \log \frac{p^2+4}{p^2}.$$

By definition,

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

$$\Rightarrow \int_0^\infty e^{-pt} F(t)\,dt = L\{F(t)\}$$

$$\Rightarrow \int_0^\infty e^{-pt} \frac{\sin^2 t}{t}\,dt = \frac{1}{4} \log \frac{(p^2+4)}{p^2}.$$

Put $p=1$,

$$\boxed{\int_0^\infty e^{-t} \frac{\sin^2 t}{t}\,dt = \frac{1}{4} \log 5}$$

Home Assignment

① Find $L\left\{\frac{e^{at}-\cos bt}{t}\right\}$

② find $L\left\{\frac{1-\cos t}{t}\right\}$

③ Find $L\left\{\frac{\sin 3t \cos t}{t}\right\}$.

④ find the Laplace transform of the functions

(i) $\frac{\sin 2t}{t}$ (ii) $\frac{1-e^{t}}{t}$ (iii) $\frac{\cos 2t-\cos 3t}{t}$

(iv) $\frac{\sinh t}{t}$

⑤ find Laplace transform

(i) $\int_0^\infty e^{-t}\frac{\sin\sqrt{3}t}{t}\,dt$ (AKTU-2010)

(ii) $\int_0^\infty \frac{e^{-2t}-e^{-4t}}{t}\,dt$

(iii) $\int_0^\infty \frac{\sin t}{t}\,dt$ (AKTU-2009)

(iv) $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt$ (AKTU-2010)

(v) $\int_0^\infty \frac{e^{-3t}-e^{-6t}}{t}\,dt$

(vi) $\int_0^\infty \frac{\cos 6t-\cos 4t}{t}$ (AKTU-2012)

(vii) i) $\int_0^\infty e^{-t}\frac{\sin t}{t}\,dt$ (AKTU-2013) ii) $\int_0^\infty e^{-3t}\frac{\sin t}{t}\,dt$ (AKTU-2015)

(viii) Prove

$$\int_0^\infty e^{-2t}\left(\frac{2\sin t-3\sinh t}{t}\right)dt = 2\cot^{-1}(2)+\frac{3}{2}\log\frac{1}{3}.$$

(iv) $\int_0^\infty \frac{\sin^2 t}{t^2}\,dt$ (AKTU-2015) (Apply two time).

Lec- No-18

Laplace Transform of Derivatives →

If $L\{F(t)\} = f(p)$, Then

$L\{F'(t)\} = p\,f(p) - F(0)$

$L\{F''(t)\} = p^2 f(p) - p\,F(0) - F'(0)$

$L\{F'''(t)\} = p^3 f(p) - p^2 F(0) - p\,F'(0) - F''(0)$

and so on.

Ex-14 Find the Laplace Transform of $\sin\sqrt{t}$. Hence find the $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$. (AKTU – 2010)

Sol^n

By $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots$

Let $F(t) = \sin\sqrt{t}$

$= \sqrt{t} - \frac{(\sqrt{t})^3}{\lfloor 3} + \frac{(\sqrt{t})^5}{\lfloor 5} - \cdots$

$\Rightarrow F(t) = t^{1/2} - \frac{t^{3/2}}{\lfloor 3} + \frac{t^{5/2}}{\lfloor 5} - \cdots$

$\therefore L\{F(t)\} = L\{t^{1/2}\} - \frac{L\{t^{3/2}\}}{\lfloor 3} + \frac{L\{t^{5/2}\}}{\lfloor 5} - \cdots$

$= \frac{\Gamma\frac{3}{2}}{p^{3/2}} - \frac{\Gamma\frac{5}{2}}{\lfloor 3\, p^{5/2}} + \frac{\Gamma\frac{7}{2}}{\lfloor 5\, p^{7/2}} - \cdots$

By $L\{t^n\} = \frac{\Gamma n+1}{p^{n+1}}$

$= \frac{\frac{1}{2}\sqrt{\pi}}{p^{3/2}} - \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{\lfloor 3\, p^{3/2}\cdot p^1} + \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{\lfloor 5\, p^{3/2}\cdot p^2} - \cdots$

$= \frac{\frac{1}{2}\sqrt{\pi}}{p^{3/2}}\left[1 - \left(\frac{1}{4p}\right) + \frac{1}{\lfloor 2}\left(\frac{1}{4p}\right)^2 - \cdots\right] = \frac{\frac{1}{2}\sqrt{\pi}}{p^{3/2}}\, e^{-1/4p} = f(p)$ (say)

Now $F'(t) = \frac{d}{dt}F(t) = \frac{d}{dt}\sin\sqrt{t} = \frac{\cos\sqrt{t}}{2\sqrt{t}}$ & $F(0) = \sin\sqrt{0} = 0$.

Then by $L\{F'(t)\} = p\,f(p) - F(0)$

$\Rightarrow L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = p\,\frac{\frac{1}{2}\sqrt{\pi}}{p^{3/2}}\, e^{-1/4p} - 0 = \frac{1}{2}\sqrt{\frac{\pi}{p}}\, e^{-1/4p}$

Hence $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{p}}\, e^{-1/4p}$

Ex-1 Given $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2p^{3/2}}\, e^{-1/4p}$, show that

$$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{p}}\cdot e^{-1/4p}.$$ (AKTU-2012, 2017)

(Try yourself).

Ex-2 Given $L\left\{2\frac{\sqrt{t}}{\sqrt{\pi}}\right\} = \frac{1}{p^{3/2}}$, show that

$$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{p}}.$$

Sol^n Let $F(t) = 2\sqrt{\frac{t}{\pi}}$ then $F(0) = 2\sqrt{\frac{0}{\pi}} = 0$.

$$\& \; F'(t) = \frac{2}{\sqrt{\pi}}\cdot\frac{1}{2}t^{-1/2} = \frac{1}{\sqrt{\pi t}}.$$

Given $L\{F(t)\} = \frac{1}{p^{3/2}} = f(p)$ (say)

$$\therefore\; L\{F'(t)\} = p f(p) - F(0)$$

$$\Rightarrow L\left\{\frac{1}{\sqrt{\pi t}}\right\} = p\cdot\frac{1}{p^{3/2}} - 0 = \frac{1}{\sqrt{p}}.$$

V.V.a

Ex-3 find $L\{F(t)\}$ and $L\{F'(t)\}$ of $F(t) = \frac{\sin t}{t}$.

(AKTU-2016)

Sol^n Given $F(t) = \frac{\sin t}{t}$

Let $G(t) = \sin t$ then $L\{G(t)\} = L\{\sin t\} = \frac{1}{p^2+1} = g(p)$ (say)

Then

$$L\{F(t)\} = L\left\{\frac{G(t)}{t}\right\} = L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty g(p)\,dp = \int_p^\infty \frac{1}{p^2+1}\,dp = \left(\tan^{-1}p\right)_p^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}p$$

$$= \frac{\pi}{2} - \tan^{-1}p.$$

$$= \cot^{-1}p = f(p) \text{ (say)}$$

Now $F'(t) = \frac{d}{dt}\left(\frac{\sin t}{t}\right)$,

$$\& \; F(0) = \lim_{t\to 0}\frac{\sin t}{t} \quad \left[\tfrac{0}{0} \text{ form}\right]$$

$$= \lim_{t\to 0}\frac{\cos t}{1} = \cos 0 = 1.$$

Then $\boxed{L\{F'(t)\} = p f(p) - F(0) = p\cot^{-1}p - 1}$

Home Assignment

Ex-1 Find $L\{F(t)\}$ & $L\{F'(t)\}$ of

$$F(t) = \begin{cases} t & , 0 \le t < 3 \\ 6 & , t > 3 . \end{cases}$$

Ex-2 Find $L\{F'(t)\}$ where

$$F(t) = \frac{1 - \cos 2t}{t} .$$

Laplace Transform of Integrals →

$$\boxed{\text{If } L\{F(t)\} = f(p) \text{ then } L\left\{\int_0^t F(t)\,dt\right\} = \frac{F(p)}{p}}$$

Ex-1: Find the Laplace Transform of $\int_0^t e^{t} \frac{\sin t}{t}\,dt$ (AKTU-2014)

Sol^n: Let $F(t) = e^{t} \frac{\sin t}{t} \longrightarrow (1)$

We have $L\{\sin t\} = \frac{1}{p^2+1}$ $\quad$ (By $L\{\sin at\} = \frac{a}{p^2+a^2}$)

Then by the division of t,

$$L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty L\{\sin t\}\,dp = \int_p^\infty \frac{1}{1+p^2}\,dp = \left(\tan^{-1} p\right)_p^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} p$$

$$= \frac{\pi}{2} - \tan^{-1} p$$

$$= \cot^{-1} p. \quad \left(\text{By } \tan^{-1} p + \cot^{-1} p = \frac{\pi}{2}\right)$$

$$\Rightarrow L\left\{\frac{\sin t}{t}\right\} = \cot^{-1} p$$

Then by first Shifting Theorem,

$$L\left\{e^{t} \frac{\sin t}{t}\right\} = \cot^{-1}(p-1) = f(p) \text{ (say)} \quad \left(\text{By } L\{e^{at} F(t)\} = f(p-a)\right)$$

$$\Rightarrow L\{F(t)\} = f(p).$$

Now by the Laplace Transform of integrals,

$$L\left\{\int_0^t F(t)\,dt\right\} = \frac{f(p)}{p}$$

$$\Rightarrow L\left\{\int_0^t e^{t} \frac{\sin t}{t}\,dt\right\} = \frac{\cot^{-1}(p-1)}{p}.$$

Ex-2: Find $L\left\{\int_0^t e^{-t} \cos t\,dt\right\}$. $\quad$ (By $L\{\cos at\} = \frac{p}{p^2+a^2}$)

Sol^n: We have $L\{\cos t\} = \frac{p}{p^2+1}$

$\therefore$ By first Shifting property, $\quad$ ($L\{e^{at}F(t)\} = f(p-a)$)

$$L\{F(t)\} = L\{e^{-t} \cos t\} = \frac{(p+1)}{(p+1)^2+1} = f(p)$$

Hence $$L\left\{\int_0^t F(t)\,dt\right\} = L\left\{\int_0^t e^{-t} \cos t\,dt\right\} = \frac{f(p)}{p} = \frac{1}{p} \frac{(p+1)}{(p+1)^2+1}.$$

Ex-3 → find $L\left\{\int_0^t \frac{\cos 2t - \cos 3t}{t} dt\right\}$ (AKTU-2014)

Sol^n → we have $L\{\cos 2t - \cos 3t\}$

$$= L\{\cos 2t\} - L\{\cos 3t\}$$

$$= \frac{p}{p^2+4} - \frac{p}{p^2+9}$$

By $L\{\cos at\} = \frac{p}{p^2+a^2}$

By the division of t,

$$L\{F(t)\} = L\left\{\frac{\cos 2t - \cos 3t}{t}\right\} = \int_p^\infty \left[\frac{p}{p^2+4} - \frac{p}{p^2+9}\right] dp.$$

$$= \frac{1}{2}\int_p^\infty \left[\frac{2p}{p^2+4} - \frac{2p}{p^2+9}\right] dp.$$

$$= \frac{1}{2}\left[\log(p^2+4) - \log(p^2+9)\right]_p^\infty$$

$$= \frac{1}{2}\left[\log\frac{(p^2+4)}{(p^2+9)}\right]_p^\infty$$

$$= \frac{1}{2}\left[\log\frac{(1+4/p^2)}{(1+9/p^2)}\right]_p^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log\frac{1+4/p^2}{1+9/p^2}\right]$$

$$= \frac{1}{2}\left[-\log\frac{(p^2+4)}{(p^2+9)}\right] = \frac{1}{2}\log\frac{(p^2+9)}{(p^2+4)}$$

$$= f(p) \text{ (say)}$$

Hence

$$L\left\{\int_0^t F(t)dt\right\} = L\left\{\int_0^t \frac{\cos 2t - \cos 3t}{t} dt\right\}$$ (Laplace of integral)

$$= \frac{f(p)}{p} = \frac{1}{2p}\log\frac{(p^2+9)}{(p^2+4)}.$$

Ex-4 → Find the Laplace transform of $e^{-4t}\int_0^t t\sin 3t\, dt$.

Sol^n → $L\{F(t)\}$

$$= L\{t\sin 3t\} = (-1)^1 \frac{d}{dp} L\{\sin 3t\} = -\frac{d}{dp}\frac{3}{p^2+9} = \frac{6p}{(p^2+9)^2} = f(p)$$

Hence $L\left\{\int_0^t F(t)dt\right\} = L\left\{\int_0^t t\sin 3t\, dt\right\} = \frac{f(p)}{p} = \frac{6}{(p^2+9)^2} = g(p)$ (say)

~~E~~ Now by first shifting property,

$$L\left\{e^{-4t}\int_0^t F(t)dt\right\} = g(p+4) = \frac{6}{[(p+4)^2+9]^2}.$$

Ex-5 : Find the Laplace Transform of

$$\int_0^t \int_0^t \int_0^t t\sin t \, dt\, dt\, dt$$

Sol^n :

Let $F(t) = t\sin t$

$$\therefore L\{F(t)\} = L\{t\sin t\} = (-1)^1 \frac{d}{dp} L\{\sin t\} = -\frac{d}{dp}\frac{1}{p^2+1} = \frac{2p}{(p^2+1)^2}$$

$$= f(p) \text{ (say)}$$

Then

$$L\left\{\int_0^t F(t)dt\right\} = L\left\{\int_0^t t\sin t\,dt\right\} = \frac{f(p)}{p} = \frac{2}{(p^2+1)^2} = g(p) \text{ (say)}$$

$$L\left\{\int_0^t \int_0^t F(t)\,dt\,dt\right\} = L\left\{\int_0^t \int_0^t t\sin t\,dt\,dt\right\} = \frac{g(p)}{p} = \frac{1\times 2}{p(p^2+1)^2} = h(p) \text{ (say)}$$

& $$L\left\{\int_0^t \int_0^t \int_0^t F(t)\,dt\,dt\,dt\right\} = L\left\{\int_0^t \int_0^t \int_0^t t\sin t\,dt\,dt\,dt\right\}$$

$$= \frac{h(p)}{p} = \frac{2}{p^2(p^2+1)^2}$$

Home Assignment

Ex-1 : Find Laplace Transform of

1. $\int_0^t \frac{\sin t}{t}\,dt$
2. $\int_0^t e^{-t} \frac{\sin^2 t}{t}\,dt$ (AKTU-2013)
3. $\int_0^t \frac{e^{-at} - e^{-bt}}{t}\,dt$
4. $\int_0^t \frac{1-\cos 2t}{t}\,dt$
5. $\int_0^t t\, e^{-t} \sin 4t\,dt$
6. $e^{-4t}\int_0^t \frac{\sin 3t}{t}\,dt$
7. $\int_0^t \int_0^t \int_0^t \cos at\,dt\,dt\,dt$
8. $\int_0^t \frac{1}{u} e^{-4u} \sin 3u\,du$ (AKTU-2009, 2013)
9. $\int_0^t t\cosh t\,dt$
10. $t\int_0^t e^{-4t}\sin 3t\,dt$
11. $\int_0^t t^2 \sin t\,dt$

Ex. Prove that $\int_{t=0}^{\infty}\int_{u=0}^{t} e^{-t}\frac{\sin u}{u}\,du\,dt = \frac{\pi}{4}$.

(AKTU-2011)

Solⁿ, we have $L\{\sin u\} = \frac{1}{p^2+1}$

By $L\{\sin au\} = \frac{a}{p^2+a^2}$

$\therefore L\left(\frac{\sin u}{u}\right) = \int_p^{\infty}\frac{1}{p^2+1}\,dp = \left(\tan^{-1}p\right)_p^{\infty}$

$= \tan^{-1}\infty - \tan^{-1}p$

$= \frac{\pi}{2} - \tan^{-1}p$

$= \cot^{-1}p = g(p)$ (say)

Let $F(t) = \int_{u=0}^{t}\frac{\sin u}{u}\,du$

Hence $L\{F(t)\} = L\left\{\int_{u=0}^{t}\frac{\sin u}{u}\,du\right\} = \frac{g(p)}{p} = \frac{\cot^{-1}p}{p} = f(p)$ (say)

Also, $L\{F(t)\} = \int_0^{\infty} e^{-pt}F(t)\,dt = f(p)$

$\Rightarrow \int_0^{\infty} e^{-pt}\left(\int_{u=0}^{t}\frac{\sin u}{u}\,du\right)dt = \frac{\cot^{-1}p}{p}$

put $p=1$,

$$\boxed{\int_{t=0}^{\infty}\int_{u=0}^{t} e^{-t}\frac{\sin u}{u}\,du\,dt = \frac{\cot^{-1}1}{1} = \frac{\pi}{4}.}$$

Laplace Transform of Periodic functions →

Let $F(t)$ be a periodic function with period $T > 0$ then
$F(t+T) = F(t)$, $F(t+2T) = F(t)$, etc.

Then

$$\boxed{L\{F(t)\} = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} F(t)\, dt}$$

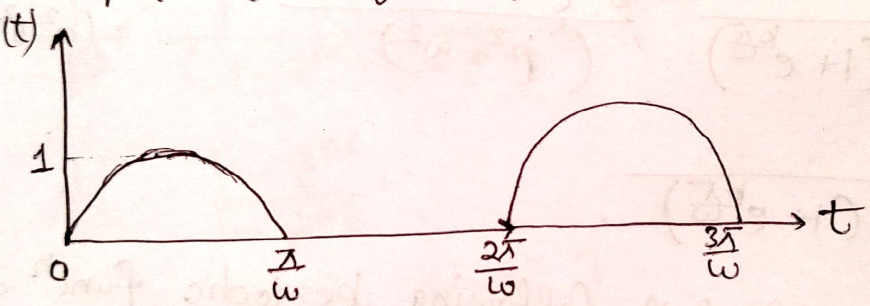
(AKTU-2010)

Ex-1 → Find the Laplace Transform of the rectified semi-wave funcⁿ defined by

$$F(t) = \begin{cases} \sin \omega t , & 0 < t \le \frac{\pi}{\omega} \\ 0 , & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

(AKTU-2010, 2011, 2013)

or Find the Laplace transform of following periodic function



Solⁿ Here $F(t)$ is a periodic function with period $\frac{2\pi}{\omega}$.

Then by $L\{F(t)\} = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} F(t) dt$ (Here $T = \frac{2\pi}{\omega}$)

$$= \frac{1}{1-e^{-p\frac{2\pi}{\omega}}} \int_0^{\frac{2\pi}{\omega}} e^{-pt} F(t)\, dt$$

$$= \frac{1}{1-e^{-\frac{2p\pi}{\omega}}} \left[\int_0^{\pi/\omega} e^{-pt} \sin \omega t\, dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-pt} \cdot 0\, dt \right]$$

$$= \frac{1}{1-e^{-\frac{2p\pi}{\omega}}} \int_0^{\pi/\omega} e^{-pt} \sin \omega t\, dt$$

$$= \frac{1}{1-e^{-\frac{2p\pi}{\omega}}}\left[\frac{e^{-pt}}{p^2+\omega^2}\left(-p\sin\omega t-\omega\cos\omega t\right)\right]_0^{\pi/\omega}$$

By $\int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2+b^2}\left[a\sin bt - b\cos bt\right]$

$$= \frac{1}{1-e^{-\frac{2p\pi}{\omega}}}\left[\frac{e^{-p\frac{\pi}{\omega}}}{p^2+\omega^2}\left(-p\sin\omega\tfrac{\pi}{\omega}-\omega\cos\omega\tfrac{\pi}{\omega}\right) - \frac{e^{0}}{p^2+\omega^2}\left(-p\sin 0-\omega\cos 0\right)\right]$$

where
$\sin\pi = 0$
$\cos\pi = -1$
$\sin 0 = 0$
$\cos 0 = 1$

$$= \frac{1}{1-e^{-\frac{2p\pi}{\omega}}}\left[\frac{\omega e^{-\frac{p\pi}{\omega}}}{p^2+\omega^2}+\frac{\omega}{p^2+\omega^2}\right]$$

$$= \frac{1}{1-e^{-\frac{2p\pi}{\omega}}}\;\frac{\omega\left(1+e^{-\frac{p\pi}{\omega}}\right)}{(p^2+\omega^2)}$$

$$= \frac{1}{\cancel{\left(1-e^{\frac{p\pi}{\omega}}\right)}\left(1+e^{\frac{p\pi}{\omega}}\right)}\;\frac{\omega\,\cancel{\left(1+e^{-\frac{p\pi}{\omega}}\right)}}{(p^2+\omega^2)}$$

By $(a^2-b^2)=(a-b)(a+b)$

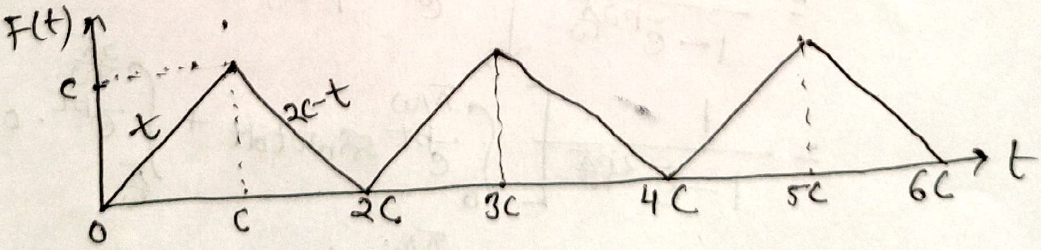
$$= \frac{\omega}{(p^2+\omega^2)\left(1+e^{\frac{p\pi}{\omega}}\right)}$$

~~Ex-27~~ Draw the graph of the following periodic funcⁿ and find its Laplace transform:

(AKTU-2009, 2014)

$$F(t) = \begin{cases} t & \text{for } 0 < t \le c \\ 2c - t & \text{for } c < t < 2c \end{cases}$$

Solⁿ: Here F(t) is a periodic funcⁿ with period $T = 2C$.



Now
$$L\{F(t)\} = \frac{1}{1-e^{-pT}}\int_0^T e^{-pt}F(t)\,dt$$
$$= \frac{1}{1-e^{-2cp}}\int_0^{2c} e^{-pt}F(t)\,dt.$$

$$= \frac{1}{1-e^{-2cp}}\left[\int_0^c e^{-pt}.\, t\, dt + \int_c^{2c} e^{-pt}(2c-t)\, dt\right]$$

$$= \frac{1}{1-e^{-2cp}}\left[\left\{t\,\frac{e^{-pt}}{(-p)}\right\}_0^c - \int_0^c 1.\frac{e^{-pt}}{(-p)}dt + \left\{(2c-t)\frac{e^{-pt}}{(-p)}\right\}_c^{2c} - \int_c^{2c}(-1)\frac{e^{-pt}}{(-p)}dt\right]$$

$$= \frac{1}{1-e^{-2cp}}\left[-\frac{c}{p}e^{-pc} - 0 - \left(\frac{e^{-pt}}{(-p)^2}\right)_0^c + \left\{0 - c\,\frac{e^{-pc}}{(-p)}\right\} + \left\{\frac{e^{-pt}}{(-p)^2}\right\}_c^{2c}\right]$$

$$= \frac{1}{1-e^{-2cp}}\left[-\cancel{\frac{c}{p}e^{-pc}} - \left(\frac{e^{-pc}}{p^2} - \frac{1}{p^2}\right) + \cancel{\frac{c}{p}e^{-pc}} + \left(\frac{e^{-p.2c}}{p^2} - \frac{e^{-pc}}{p^2}\right)\right]$$

$$= \frac{1}{1-e^{-2cp}}\left[\frac{-e^{-pc} + 1 + e^{-p.2c} - e^{-pc}}{p^2}\right]$$

$$= \frac{1}{1-e^{-2cp}}\left[\frac{1 + e^{-2pc} - 2e^{-pc}}{p^2}\right]$$

$$= \frac{1}{p^2}\,\frac{1}{(1-e^{-pc})(1+e^{-pc})} \times (1-e^{-pc})^2$$

$$= \frac{1}{p^2}\,\frac{1-e^{-pc}}{1+e^{-pc}}$$

$$= \frac{1}{p^2}\,\frac{e^{\frac{pc}{2}}\left[1-e^{-pc}\right]}{e^{\frac{pc}{2}}\left[1+e^{-pc}\right]}$$

$$= \frac{1}{p^2}\,\frac{e^{\frac{pc}{2}} - e^{-\frac{pc}{2}}}{e^{\frac{pc}{2}} + e^{-\frac{pc}{2}}}$$

By $\left[\tanh at = \frac{e^{at} - e^{-at}}{e^{at} + e^{-at}}\right]$

$$\boxed{L\{F(t)\} = \frac{1}{p^2}\tanh\frac{pc}{2}}$$

Ex-3: Find the Laplace Transform of "saw tooth wave" func^n $F(t)$ which is periodic with period 1 and defined as $F(t) = kt$, $0 < t < 1$. (AKTU - 2017)

Sol^n: Here $T = 1$.

$$L\{F(t)\} = \frac{1}{1-e^{-pT}}\int_0^T e^{-pt}F(t)\,dt = \frac{1}{1-e^{-p}}\int_0^1 e^{-pt}.\,F(t)\,dt$$

$$= \frac{1}{1-e^{-p}} \int_0^1 e^{-pt}\, Kt\, dt$$

$$= \frac{K}{1-e^{-p}} \left[\left(t\, \frac{e^{-pt}}{(-p)} \right)_0^1 - \int_0^1 1 \cdot \frac{e^{-pt}}{(-p)} dt \right]$$

$$= \frac{K}{1-e^{-p}} \left[\left(\frac{e^{-p}}{-p} - 0 \right) - \left\{ \frac{e^{-pt}}{(-p)^2} \right\}_0^1 \right]$$

$$= \frac{K}{1-e^{-p}} \left[\frac{e^{-p}}{-p} - \left\{ \frac{e^{-p}}{p^2} - \frac{e^0}{p^2} \right\} \right]$$

$$= \cancel{\frac{K}{1-e^{-p}} \left[\frac{-e^{-p}p - e^{-p} + 1}{p^2} \right]}$$

$$= \frac{K}{1-e^{-p}} \left[\frac{-e^{-p}}{p} + \frac{1-e^{-p}}{p^2} \right]$$

$$= \frac{-K e^{-p}}{p(1-e^{-p})} + \frac{K}{p^2}.$$

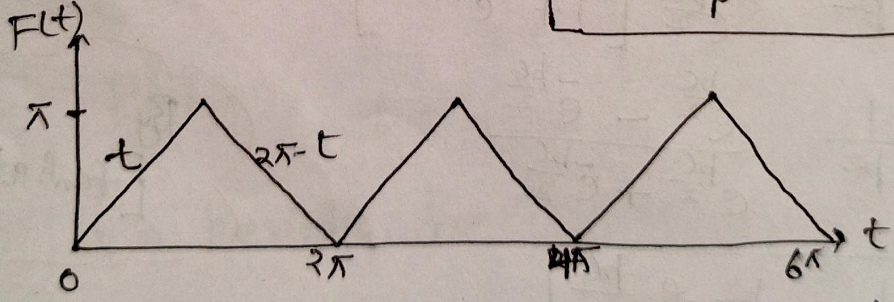
$\boxed{\text{Home Assignment}}$

Ex-1→ V.a. Draw the graph and find the Laplace Transform of the triangular wave func^n of period 2π given by

$$F(t) = \begin{cases} t & , 0<t\le\pi \\ 2\pi - t & , \pi<t<2\pi \end{cases}$$ (AKTU-2010, 2012, 2018)

$\boxed{\text{Ans } \frac{1}{p^2} \tanh \frac{\pi p}{2}}$

Graph→



Ex-2→ Draw the graph and Find L.T of the func^n with period $2a$'.

$$F(t) = \begin{cases} \frac{k}{a}t & , 0<t<a \\ \frac{k}{a}(2a-t) & , a<t<2a \end{cases}$$ (AKTU-2011) Ans $\boxed{\frac{k}{ap^2} \tanh \frac{ap}{2}}$

Ex-3→ Find L.T of

(i) $F(t) = \begin{cases} t & , 0<t\le 1 \\ 1 & , 1\le t\le 2 \end{cases}$ (AKTU 2013)

(ii) $F(t) = \begin{cases} \sin t & , 0<t<\pi \\ 0 & , \pi<t<2\pi \end{cases}$ (AKTU 2013) and $F(t+2\pi) = F(t)$

(iii) $F(t) = \begin{cases} 1 & , 0\le t\le \frac{a}{2} \\ -1 & , \frac{a}{2}<t<a \end{cases}$

(iv) $F(t) = \frac{t}{T}$, $0<t<T$ (AKTU-2011)

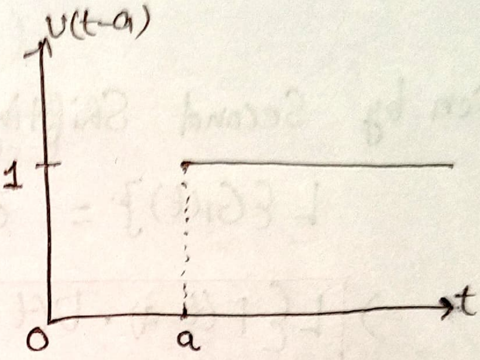
(v) $F(t) = \sin\left(\frac{\pi t}{a}\right)$, $0<t<a$

⊛ Unit Step function

or Heaviside's Unit Step func^n →

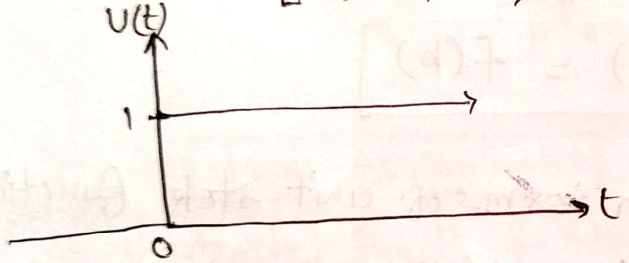
The unit step func^n $U(t-a)$ is defined as

$$\boxed{U(t-a) = \begin{cases} 0, & \text{for } t < a \\ 1, & \text{for } t \geq a \end{cases}}$$



Note → As a particular case, put $a=0$, then

$$U(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



Laplace Transform of Unit Step function →

we have $U(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$

we have $L\{G(t)\} = \int_0^\infty e^{-pt}\, G(t)\, dt$

$$\Rightarrow L\{U(t-a)\} = \int_0^\infty e^{-pt}\, U(t-a)\, dt$$

$$= \int_0^a e^{-pt} \cdot 0\, dt + \int_a^\infty e^{-pt} \cdot 1\, dt$$

$$= \int_a^\infty e^{-pt} dt = \left[\frac{e^{-pt}}{(-p)}\right]_a^\infty = \left[\frac{e^{-\infty}}{(-p)} - \frac{e^{-ap}}{(-p)}\right]$$

$$\Rightarrow \boxed{L\{U(t-a)\} = \frac{e^{-ap}}{p}} \quad p>0$$

Note → Put $a=0$ both side, we get $\boxed{L\{U(t)\} = \frac{1}{p}}$

Unit Step func^n with Second Shifting Theorem →

we have $$U(t-a) = \begin{cases} 0, & t<a \\ 1, & t \geq a \end{cases}$$

$$\therefore \; F(t-a) \cdot U(t-a) = \begin{cases} 0, & t<a \\ F(t-a), & t \geq a \end{cases} = G(t) \text{ (say)}$$

Then by Second Shifting property,

$$L\{G(t)\} = e^{-ap} \cdot L\{F(t)\} = e^{-ap} f(p)$$

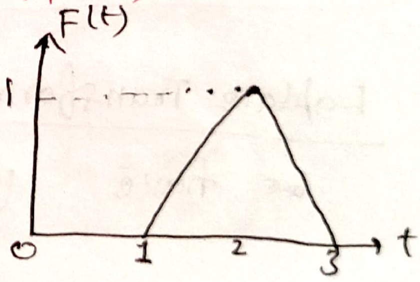
$$\Rightarrow \boxed{L\{F(t-a) \cdot U(t-a)\} = e^{-ap} f(p)}$$

Note → Particular Case, put $a=0$, we get

$$\boxed{L\{F(t)\, U(t)\} = e^{0} f(p) = f(p)}$$

Ex-1 → Express the following func^n in terms of unit step function and find its Laplace Transform: (AKTU-2010, 2015)

$$F(t) = \begin{cases} (t-1), & 1<t<2 \\ 3-t, & 2<t<3. \end{cases}$$



Sol^n we have $$F(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3. \end{cases}$$

$$\Rightarrow F(t) = (t-1)[U(t-1) - U(t-2)] + (3-t)[U(t-2) - U(t-3)]$$

$$= (t-1)U(t-1) - (t-1)U(t-2) \cancel{+(3-t)(U(t}$$
$$+ (3-t)U(t-2) + (t-3)U(t-3).$$

$$= (t-1)U(t-1) - 2(t-2)U(t-2) + (t-3)U(t-3)$$

$$\therefore L\{F(t)\} = L\{(t-1)U(t-1)\} - 2L\{(t-2)U(t-2)\} + L\{(t-3)U(t-3)\}.$$

$$= e^{-1p} \cdot \frac{1}{p^2} - 2\frac{e^{-2p}}{p^2} + \frac{e^{-3p}}{p^2}.$$

Here $L\{F(t)\} = L\{t\} = \frac{1}{p^2}$ [By $L\{F(t-a)U(t-a)\} = e^{-ap} f(p) = e^{-ap} L\{F(t)\}$]

Ex-44 Find the Laplace transform by Unit step $func^n$,

$$F(t) = \begin{cases} 0 & , 0<t<\pi/2 \\ \sin t & , t>\frac{\pi}{2}. \end{cases}$$

Sol^n

$$F(t) = 0\left[U(t-0) - U(t-\tfrac{\pi}{2})\right] + \sin t\left[U(t-\tfrac{\pi}{2})\right]$$

$$= \sin t\, U(t-\tfrac{\pi}{2})$$

$$= \sin\left[(t+\tfrac{\pi}{2}) - \tfrac{\pi}{2}\right] U(t-\tfrac{\pi}{2})$$

$$= \sin\left[\tfrac{\pi}{2} + (t-\tfrac{\pi}{2})\right] U(t-\tfrac{\pi}{2})$$

$$\Rightarrow F(t) = \cos(t-\tfrac{\pi}{2})\, U(t-\tfrac{\pi}{2})$$

$$\therefore L\{F(t)\} = e^{-\frac{\pi}{2}p} L\{\cos t\}$$

$$= e^{-\frac{\pi p}{2}} \frac{p}{p^2+1}$$

By
$L\{G(t-a)U(t-a)\} = e^{-ap} L\{G(t)\}$

$\because L\{\cos at\} = \frac{p}{p^2+a^2}$

$\boxed{\text{Home Assignment}}$

Express in Unit step $func^n$ & find Laplace Transform:

① $F(t) = (t-1)^2 U(t-1)$

② $\sin t\, U(t-\pi)$

③ $e^{-3t}\, U(t-2)$

④ $F(t) = \begin{cases} 2 & , 0<t<\pi \\ 0 & , \pi<t<2\pi \\ \sin t & , t>2\pi \end{cases}$

⑤ $F(t) = \begin{cases} 1 & , 0<t<2 \\ 2 & , 2<t<4 \\ 3 & , 4<t<6 \\ 0 & , t>0 \end{cases}$.

⑥ $e^{-t}\{1 - U(t-2)\}$

⑦ $\sin 2t\, U(t-\pi)$ (AKTU - 2014)

⑧ $F(t) = \begin{cases} \sin t & , 0<t<\pi \\ \sin 2t & , t<\pi \end{cases}$ (AKTU-2011)

⑨ $e^{t-2}\, U(t-2)$.

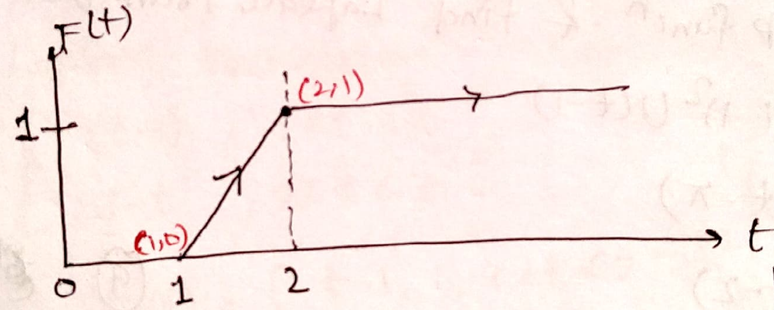
Ex-2→ Find the Laplace Transform of $t^2 U(t-3)$.

Sol^n→ We have

$$L\{t^2 U(t-3)\} = L\{(t+3-3)^2 U(t-3)\}.$$
$$= L\{F(t-3)\,U(t-3)\}.$$
$$= \bar{e}^{3p} L\{F(t)\} \qquad (\text{where } F(t) = (t+3)^2)$$
$$= \bar{e}^{3p} L\{(t+3)^2\}.$$
$$= \bar{e}^{3p} L\{t^2+6t+9\}.$$
$$= \bar{e}^{3p}\left[\frac{\lfloor 2}{p^3} + 6\frac{\lfloor 1}{p^2} + 9\cdot\frac{1}{p}\right].$$
$$= \bar{e}^{3p}\left[\frac{2}{p^3} + \frac{6}{p^2} + \frac{9}{p}\right]$$

$$\left(\text{By } L\{t^n\} = \frac{\lfloor n}{p^{n+1}}\right)$$

Ex-3→ Express the following func^n interms of unit step function and find its Laplace Transform: (AKTU-2002, 2012)



Sol^n→

Here
$$F(t) = \begin{cases} 0 & , \ 0 < t < 1 \\ t-1 & , \ 1 < t < 2 \\ 1 & , \ t > 2 \end{cases}$$

by
$$(y-y_1) = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$
$$[F(t)-0] = \frac{1-0}{2-1}(t-1)$$
$$F(t) = t-1$$

$$\Rightarrow F(t) = 0\,[U(t-0) - U(t-1)] + (t-1)[U(t-1) - U(t-2)] + 1\,[U(t-2)]$$
$$\Rightarrow F(t) = (t-1)\,U(t-1) - (t-1)\,U(t-2) + U(t-2)$$
$$\Rightarrow F(t) = (t-1)\,U(t-1) - (t-2)\,U(t-2)$$

$$\therefore L\{F(t)\} = L\{(t-1)U(t-1)\} - L\{(t-2)U(t-2)\}$$
$$= \bar{e}^{1p}\cdot\frac{1}{p^2} - e^{-2p}\cdot\frac{1}{p^2}.$$

By
$$[L\{G(t-a)\,U(t-a)\} = \bar{e}^{ap}\, g(p)]$$
Here
$$g(p) = L\{G(t)\}$$
$$= L\{t\} = \frac{1}{p^2}$$

Inverse Laplace Transform →

$$\boxed{\text{If } L\{F(t)\} = f(p) \text{ then } L^{-1}\{f(p)\} = F(t).}$$

i.e if $f(p)$ is the Laplace Transform of a func^n $F(t)$ then $F(t)$ is called inverse Laplace Transform of the func^n $f(p)$.

Here L^{-1} is called the inverse Laplace Transform operator.

Laplace Transform	Inverse Laplace Transform
① $L\{1\} = \frac{1}{p}$	① $L^{-1}\left\{\frac{1}{p}\right\} = 1$
② $L\{e^{at}\} = \frac{1}{p-a}$	② $L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$
③ (i) $L\{t^n\} = \frac{\lfloor n}{p^{n+1}}$ (n is +ve integer)	③ (i) $L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{\lfloor n}$ (n is +ve integer)
(ii) $L\{t^n\} = \frac{\Gamma n+1}{p^{n+1}}$ (n is real)	(ii) $L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{\Gamma n+1}$ (n is real)
④ $L\{\sin at\} = \frac{a}{p^2+a^2}$	④ $L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{\sin at}{a}$
⑤ $L\{\cos at\} = \frac{p}{p^2+a^2}$	⑤ $L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$
⑥ $L\{\sinh at\} = \frac{a}{p^2-a^2}$	⑥ $L^{-1}\left\{\frac{1}{p^2-a^2}\right\} = \frac{\sinh at}{a}$
⑦ $L\{\cosh at\} = \frac{p}{p^2-a^2}$	⑦ $L^{-1}\left\{\frac{p}{p^2-a^2}\right\} = \cosh at$
⑧ Linear Property → $L\{C_1F_1(t)+C_2F_2(t)\} = C_1L\{F_1(t)\}+C_2L\{F_2(t)\}$	⑧ Linear Property → $L^{-1}\{C_1 f_1(p)+C_2 f_2(p)\} = C_1L^{-1}\{f_1(p)\}+C_2L^{-1}\{f_2(p)\}$

Ex-1 → find $L^{-1}\left\{\frac{4}{p-2}\right\}$

Sol^n → $L^{-1}\left\{\frac{4}{p-2}\right\} = 4L^{-1}\left\{\frac{1}{p-2}\right\} = 4e^{2t}$

By $L\{e^{at}\} = \frac{1}{p-a}$

Ex-2: find $L^{-1}\left\{\frac{1}{\sqrt{p}}\right\}$.

By $L\{t^n\} = \frac{\Gamma(n+1)}{p^{n+1}}$

$L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}$

Sol^n: $L^{-1}\left\{\frac{1}{\sqrt{p}}\right\} = L^{-1}\left\{\frac{1}{p^{-\frac{1}{2}+1}}\right\}$

$$= \frac{t^{-1/2}}{\Gamma\left(-\frac{1}{2}+1\right)}$$

$$= \frac{t^{-1/2}}{\Gamma\frac{1}{2}} = \frac{t^{-1/2}}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi t}}.$$

Ex-3: Find $L^{-1}\left\{\frac{p}{p^2+2} + \frac{6p}{p^2-16} + \frac{3}{p-3}\right\}$.

Sol^n:

$$= L^{-1}\left\{\frac{p}{p^2+2}\right\} + 6\,L^{-1}\left\{\frac{p}{p^2-16}\right\} + 3\,L^{-1}\left\{\frac{1}{p-3}\right\}$$

(By Linear Property)

$$= L^{-1}\left\{\frac{p}{p^2+(\sqrt{2})^2}\right\} + 6\,L^{-1}\left\{\frac{p}{p^2-4^2}\right\} + 3\,L^{-1}\left\{\frac{1}{p-3}\right\}$$

$$= \cos\sqrt{2}t + 6\cosh 4t + 3e^{3t}$$

By $L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

$L^{-1}\left\{\frac{p}{p^2-a^2}\right\} = \cosh at$

$L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$

Ex-4: find $L^{-1}\left\{\frac{6}{2p-3} - \frac{3+4p}{9p^2-16} + \frac{8-6p}{16p^2+9}\right\}$. (AKTU-2001)

Sol^n:

$$L^{-1}\left\{\frac{6}{2p-3} - \frac{(3+4p)}{9p^2-16} + \frac{8-6p}{16p^2+9}\right\}$$

$$= L^{-1}\left\{\frac{6}{2p-3}\right\} - L^{-1}\left\{\frac{3}{9p^2-16}\right\} - L^{-1}\left\{\frac{4p}{9p^2-16}\right\} + L^{-1}\left\{\frac{8}{16p^2+9}\right\} - L^{-1}\left\{\frac{6p}{16p^2+9}\right\}.$$

$$= 3L^{-1}\left\{\frac{1}{p-\frac{3}{2}}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{p^2-\left(\frac{4}{3}\right)^2}\right\} - \frac{4}{9}L^{-1}\left\{\frac{p}{p^2-\left(\frac{4}{3}\right)^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{p^2+\left(\frac{3}{4}\right)^2}\right\} - \frac{3}{8}L^{-1}\left\{\frac{p}{p^2+\left(\frac{3}{4}\right)^2}\right\}$$

$$= 3e^{\frac{3}{2}t} - \frac{1}{3}\frac{\sinh\frac{4}{3}t}{\frac{4}{3}} - \frac{4}{9}\cosh\frac{4}{3}t + \frac{1}{2}\frac{\sin\frac{3}{4}t}{3/4} - \frac{3}{8}\cos\frac{3}{4}t$$

$$= 3e^{\frac{3}{2}t} - \frac{1}{4}\sinh\frac{4}{3}t - \frac{4}{9}\cosh\frac{4}{3}t + \frac{2}{3}\sin\frac{3}{4}t - \frac{3}{8}\cos\frac{3}{4}t.$$

Ex-5→ Prove that $L^{-1}\left\{\frac{1}{p^3+1}\right\} = \frac{t^2}{\lfloor 2} - \frac{t^5}{\lfloor 5} + \frac{t^8}{\lfloor 8} - \frac{t^{11}}{\lfloor 11} + \cdots$

Solⁿ→ we have $\frac{1}{1+p^3} = \frac{1}{p^3\left[1+\frac{1}{p^3}\right]}$

$$= \frac{1}{p^3}\left[1+\frac{1}{p^3}\right]^{-1}$$

$$= \frac{1}{p^3}\left[1-\frac{1}{p^3}+\frac{1}{p^6}-\frac{1}{p^9}+\cdots\right]$$

$$\Rightarrow \frac{1}{1+p^3} = \frac{1}{p^3}-\frac{1}{p^6}+\frac{1}{p^9}-\frac{1}{p^{12}}+\cdots$$

$$\therefore L^{-1}\left\{\frac{1}{1+p^3}\right\} = L^{-1}\left\{\frac{1}{p^3}-\frac{1}{p^6}+\frac{1}{p^9}-\frac{1}{p^{12}}+\cdots\right\}$$

$$= L^{-1}\left\{\frac{1}{p^3}\right\} - L^{-1}\left\{\frac{1}{p^6}\right\} + L^{-1}\left\{\frac{1}{p^9}\right\} - L^{-1}\left\{\frac{1}{p^{12}}\right\} + \cdots$$

$$= \frac{t^{3-1}}{\lfloor 3-1} - \frac{t^{6-1}}{\lfloor 6-1} + \frac{t^{9-1}}{\lfloor 9-1} - \frac{t^{12-1}}{\lfloor 12-1} + \cdots$$

$$= \frac{t^2}{\lfloor 2} - \frac{t^5}{\lfloor 5} + \frac{t^8}{\lfloor 8} - \frac{t^{11}}{\lfloor 11} + \cdots$$

By $L^{-1}\left\{\frac{1}{p^n}\right\} = \frac{t^{n-1}}{\lfloor n-1}$

Ex-6→ Find $L^{-1}\left\{\frac{1}{p}\cos\frac{1}{p}\right\}$.

Solⁿ→

By $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots$

we have

$$L^{-1}\left\{\frac{1}{p}\cos\frac{1}{p}\right\} = L^{-1}\left\{\frac{1}{p}\left(1-\frac{(\frac{1}{p})^2}{\lfloor 2}+\frac{(\frac{1}{p})^4}{\lfloor 4}-\frac{(1/p)^6}{\lfloor 6}+\cdots\right)\right\}$$

$$= L^{-1}\left\{\frac{1}{p} - \frac{1}{\lfloor 2}\frac{1}{p^3} + \frac{1}{\lfloor 4}\frac{1}{p^5} - \frac{1}{\lfloor 6}\frac{1}{p^7} + \cdots\right]$$

$$= L^{-1}\left\{\frac{1}{p}\right\} - \frac{1}{\lfloor 2}L^{-1}\left\{\frac{1}{p^3}\right\} + \frac{1}{\lfloor 4}L^{-1}\left\{\frac{1}{p^5}\right\} - \frac{1}{\lfloor 6}L^{-1}\left\{\frac{1}{p^7}\right\} + \cdots$$

$$= 1 - \frac{1}{\lfloor 2}\frac{t^{3-1}}{\lfloor 3-1} + \frac{1}{\lfloor 4}\frac{t^{5-1}}{\lfloor 5-1} - \frac{1}{\lfloor 6}\cdot\frac{t^{7-1}}{\lfloor 7-1} + \cdots$$

$$= 1 - \frac{t^2}{(\lfloor 2)^2} + \frac{t^4}{(\lfloor 4)^2} - \frac{t^6}{(\lfloor 6)^2} + \cdots$$

By $L^{-1}\left\{\frac{1}{p^n}\right\} = \frac{t^{n-1}}{\lfloor n-1}$

H.W

① Find the Laplace Transform of the functions:

i) $\frac{1}{p^4}$

(ii) $\frac{1}{p^2+4}$

(iii) $\frac{2p-5}{9p^2-25}$

(iv) $\frac{p}{p^2+2} + \frac{6p}{p^2-9} + \frac{5}{p-5} + \frac{7}{p^2-16}$

(v) $\frac{4}{p^2} + \frac{(\sqrt{p}-1)^2}{p^2} - \frac{5}{3p+4}$

(vi) Prove that

$$L^{-1}\left\{\frac{1}{p}\sin\frac{1}{p}\right\} = t - \frac{t^3}{(\lfloor 3)^2} + \frac{t^5}{(\lfloor 5)^2} - \frac{t^7}{(\lfloor 7)^2} + \cdots$$

(AKTU-2010, 2012)

$$\boxed{\left(\text{Use } \sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} \cdots\right)}$$

First Shifting Theorem / First Translation Theorem or First Shifting Property / First Shifting Theorem →

* If $L\{F(t)\} = f(p)$ then $L\{e^{at}F(t)\} = f(p-a)$

* If $L^{-1}\{f(p)\} = F(t)$ then $L^{-1}\{f(p-a)\} = e^{at}F(t)$

or $L^{-1}\{f(p-a)\} = e^{at} L^{-1}\{f(p)\}$.

Ex-1 → find $L^{-1}\left\{\frac{1}{(p+2)^5}\right\}$.

$L^{-1}\{f(p-a)\} = e^{at} L^{-1}\{f(p)\}$

By first Shifting Prop

Sol^n →

$$L^{-1}\left\{\frac{1}{(p+2)^5}\right\} = e^{-2t} L^{-1}\left\{\frac{1}{p^5}\right\}$$

$L^{-1}\left\{\frac{1}{p^n}\right\} = \frac{t^{n-1}}{\lfloor n-1}$

$$= e^{-2t}\frac{t^{5-1}}{\lfloor 5-1}$$

$$= e^{-2t}\frac{t^4}{\lfloor 4}.$$

Ex-2 → find $L^{-1}\left\{\frac{p}{(p+1)^{5/2}}\right\}$.

Sol^n →

$$L^{-1}\left\{\frac{p}{(p+1)^{5/2}}\right\} = L^{-1}\left\{\frac{(p+1)-1}{(p+1)^{5/2}}\right\}$$

By $L^{-1}\{f(p-a)\} = e^{at}L^{-1}f(p)$

$$= e^{-t} L^{-1}\left\{\frac{p-1}{p^{5/2}}\right\}$$

$$= e^{-t} L^{-1}\left\{\frac{1}{p^{3/2}} - \frac{1}{p^{5/2}}\right\}.$$

$$= e^{-t}\left[L^{-1}\left\{\frac{1}{p^{3/2}}\right\} - L^{-1}\left\{\frac{1}{p^{5/2}}\right\}\right]$$

$$= e^{-t}\left[\frac{t^{3/2-1}}{\Gamma 3/2} - \frac{t^{5/2-1}}{\Gamma 5/2}\right]$$

By $L^{-1}\left\{\frac{1}{p^n}\right\} = \frac{t^{n-1}}{\Gamma n}$

$$= e^{-t}\left[\frac{t^{1/2}}{\frac{1}{2}\cdot\sqrt{\pi}} - \frac{t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}\right]$$

$$= e^{-t}\left[2\frac{\sqrt{t}}{\sqrt{\pi}} - \frac{4}{3}\frac{t\sqrt{t}}{\sqrt{\pi}}\right]$$

Ex-3→ Find $L^{-1}\left\{\frac{1}{9p^2+6p+1}\right\}$.

Solⁿ→

$$L^{-1}\left\{\frac{1}{9p^2+6p+1}\right\} = L^{-1}\left\{\frac{1}{(3p+1)^2}\right\} = \frac{1}{9}L^{-1}\left\{\frac{1}{(p+\frac{1}{3})^2}\right\}$$

$$= \frac{1}{9}e^{-\frac{1}{3}t}\,L^{-1}\left\{\frac{1}{p^2}\right\}$$

By $L^{-1}\{f(p-a)\} = e^{at}L^{-1}\{f(p)\}$.

$$= \frac{1}{9}e^{-\frac{1}{3}t}\,\frac{t^{2-1}}{\lfloor 2-1}$$

$$= \frac{1}{9}t\,e^{-t/3}$$

By $L^{-1}\left\{\frac{1}{p^n}\right\} = \frac{t^{n-1}}{\lfloor n-1}$

Ex-4→ Find $L^{-1}\left\{\frac{1}{p^2-6p+10}\right\}$

Solⁿ→

$$L^{-1}\left\{\frac{1}{p^2-6p+10}\right\} = L^{-1}\left\{\frac{1}{(p-3)^2+1}\right\}$$

$$= e^{3t}L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

By $L^{-1}\{f(p-a)\} = e^{at}L^{-1}\{f(p)\}$

$$= e^{3t}\sin t$$

by $L^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$

Ex-5→

Evaluate $L^{-1}\left\{\frac{3p-2}{p^2-4p+20}\right\}$

Solⁿ→

$$L^{-1}\left\{\frac{3p-2}{p^2-4p+20}\right\} = L^{-1}\left\{\frac{3(p-\frac{2}{3})}{(p-2)^2+16}\right\}$$

$$= L^{-1}\left\{\frac{3(p-2+2-\frac{2}{3})}{(p-2)^2+16}\right\}$$

$$= L^{-1}\left\{\frac{3(p-2)+4}{(p-2)^2+16}\right\}$$

$$= e^{2t}L^{-1}\left\{\frac{3p+4}{p^2+16}\right\}$$

By $L^{-1}f(p-a) = e^{at}L^{-1}\{f(p)$

$$= e^{2t}\left[3L^{-1}\left\{\frac{p}{p^2+16}\right\} + L^{-1}\left\{\frac{4}{p^2+16}\right\}\right]$$

$$= e^{2t}\left[3\cos 4t + \sin 4t\right]$$

By $L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

$L^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$

Ex-6:- find $L^{-1}\left\{\frac{15}{p^2+4p+13}\right\}$ (AKTU-2015)

Sol^n:-

$$L^{-1}\left\{\frac{15}{p^2+4p+13}\right\} = L^{-1}\left\{\frac{15}{(p^2+4p+4)+9}\right\}$$

$$= L^{-1}\left\{\frac{15}{(p+2)^2+9}\right\}$$

$$= e^{-2t} L^{-1}\left\{\frac{15}{p^2+9}\right\}$$

By $L^{-1}\{f(p-a)\} = e^{at} L^{-1}\{f(p)\}$

$$= 15e^{-2t} L^{-1}\left\{\frac{1}{p^2+9}\right\}$$

by $L^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$

$$= 15\, e^{-2t}\, \frac{\sin 3t}{3}$$

$$= 5e^{-2t} \sin 3t.$$

Ex-7:- find $L^{-1}\left\{\frac{p+8}{p^2+4p+5}\right\}$. (AKTU-2018)

Sol:-

$$L^{-1}\left\{\frac{p+8}{p^2+4p+5}\right\} = L^{-1}\left\{\frac{(p+2)+6}{(p^2+4p+4)+1}\right\}$$

$$= L^{-1}\left\{\frac{(p+2)+6}{(p+2)^2+1}\right\}$$

$$= e^{-2t} L^{-1}\left\{\frac{p+6}{p^2+1}\right\}$$

By $L^{-1}\{f(p-a)\} = e^{at} L^{-1} f(p)$

$$= e^{-2t}\left[L^{-1}\left\{\frac{p}{p^2+1}\right\} + 6L^{-1}\left\{\frac{1}{p^2+1}\right\}\right]$$

$$= e^{-2t}\left[\cos t + 6\sin t\right]$$

By $L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$, $L^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$

Ex-8:- $L^{-1}\left\{\frac{3p+7}{p^2-2p-3}\right\}$.

Sol^n:-

$$L^{-1}\left\{\frac{3p+7}{p^2-2p-3}\right\} = L^{-1}\left\{\frac{3(p+\frac{7}{3})}{(p^2-2p+1)-4}\right\}$$

$$= L^{-1}\left\{\frac{3(p-1+1+\frac{7}{3})}{(p-1)^2-4}\right\} = L^{-1}\left\{\frac{3(p-1)+10}{(p-1)^2-4}\right\}$$

$$= e^{t} L^{-1}\left\{\frac{3p+10}{p^2-4}\right\} = e^{t}\left[3L^{-1}\left\{\frac{p}{p^2-4}\right\} + 10L^{-1}\left\{\frac{1}{p^2-4}\right\}\right]$$

$$= e^{t}\left[3\cosh 2t + 10\frac{\sinh 2t}{2}\right]$$

By $L^{-1}\left\{\frac{p}{p^2-a^2}\right\} = \cosh at$, $L^{-1}\left\{\frac{a}{p^2-a^2}\right\} = \sinh at$

Home Assignment

Ex-1 find Laplace inverse of the func^n:

(i) $L^{-1}\left\{\dfrac{1}{p^2+8p+16}\right\}$

(ii) $L^{-1}\left\{\dfrac{p}{(p+3)^2+4}\right\}$

(iii) $L^{-1}\left\{\dfrac{p}{p^2+6p+25}\right\}$

(iv) $L^{-1}\left\{\dfrac{p+b}{(p+b)^2+a^2}\right\}$.

(v) $L^{-1}\left\{\dfrac{1}{(p-4)^5}+\dfrac{5}{(p-2)^2+5^2}\right\}$

(vi) $L^{-1}\left\{\dfrac{1}{(p+a)^n}\right\}$

(vii) $L^{-1}\left\{\dfrac{p-1}{p^2-6p+25}\right\}$.

Second Shifting Property / Second Shifting Theorem →
or Second Translation Property →

(*) If $L\{F(t)\} = f(p)$ then $L\{G(t)\} = e^{-ap} f(p)$

where $G(t) = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$

(*) If $L^{-1}\{f(p)\} = F(t)$, then

$L^{-1}\{e^{-ap} f(p)\} = G(t)$ where $G(t) = \begin{cases} F(t-a), & t>a \\ 0, & t<a. \end{cases}$

Note→ In terms of Heaviside's Unit Step funcⁿ →

v.v.a.

$\boxed{\text{If } L^{-1}\{f(p)\} = F(t) \text{ then } L^{-1}\{e^{-ap} f(p)\} = F(t-a)\,U(t-a)}$

Ex-1→ Find $L^{-1}\left\{\dfrac{p\, e^{-\frac{2p\pi}{3}}}{p^2+9}\right\}$.

Solⁿ→ We have $L^{-1}\left\{\dfrac{p}{p^2+9}\right\} = \cos 3t = F(t)$ (say)

By $L^{-1}\left\{\dfrac{p}{p^2+a^2}\right\} = \cos at$

Now by 2ⁿᵈ Shifting Theorem,

$$L^{-1}\{e^{-ap} f(p)\} = F(t-a)\,U(t-a)$$

$$\Rightarrow L^{-1}\left\{e^{-\frac{2\pi}{3}p}\,\frac{p}{p^2+9}\right\} = \cos 3\left(t-\frac{2\pi}{3}\right) U\left(t-\frac{2\pi}{3}\right).$$

Ex-2→ Find $L^{-1}\left\{\dfrac{e^{4-3p}}{(p+4)^{5/2}}\right\}$.

Solⁿ→ We have $L^{-1}\left\{\dfrac{1}{(p+4)^{5/2}}\right\} = e^{-4t} L^{-1}\left\{\dfrac{1}{p^{5/2}}\right\}$

By $L^{-1}\{f(p-a)\} = e^{at}\cdot L^{-1}\{f(p)\}$

$$= e^{-4t}\,\frac{t^{5/2-1}}{\Gamma 5/2}$$

By $L^{-1}\left\{\dfrac{1}{p^n}\right\} = \dfrac{t^{n-1}}{\Gamma n}$

$$= e^{-4t}\,\frac{t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}$$

$$= \frac{4}{3\sqrt{\pi}}\, t^{3/2} e^{-4t} = F(t) \text{ (say)}$$

$$\therefore\ L^{-1}\left\{\frac{e^{4-3p}}{(p+4)^{5/2}}\right\} = e^{4} L^{-1}\left\{e^{-3p}\cdot\frac{1}{(p+4)^{5/2}}\right\}$$

$$= e^{4}\, F(t-3)\, U(t-3)$$

$$= e^{4}\left[\frac{4}{3\sqrt{\pi}}(t-3)^{3/2} e^{-4(t-3)}\right] U(t-3)$$

$$= \frac{4}{3\sqrt{\pi}}(t-3)^{3/2} e^{-4(t-4)}\, U(t-3).$$

By ~~$L^{-1}\{f(p-a)\}$~~ $L^{-1}\{e^{-ap} f(p)\} = F(t-a)\, U(t-a)$

Ex-3 → Find $L^{-1}\left\{\dfrac{p\, e^{-p/2} + \pi e^{-p}}{p^2+\pi^2}\right\}$ (AKTU-2007, 2010)

Sol^n →

$$L^{-1}\left\{\frac{p\, e^{-p/2}}{p^2+\pi^2} + \frac{\pi\, e^{-p}}{p^2+\pi^2}\right\}$$

$$= L^{-1}\left\{e^{-\frac{1}{2}p}\,\frac{p}{p^2+\pi^2}\right\} + L^{-1}\left\{e^{-p}\,\frac{\pi}{p^2+\pi^2}\right\} \longrightarrow (1)$$

Now we have $L^{-1}\left\{\dfrac{p}{p^2+\pi^2}\right\} = \cos\pi t = F_1(t)$ (say)

& $L^{-1}\left\{\dfrac{\pi}{p^2+\pi^2}\right\} = \sin\pi t = F_2(t)$ (say)

Then

$$L^{-1}\left\{\frac{p\, e^{-p/2}}{p^2+\pi^2} + \frac{\pi\, e^{-p}}{p^2+\pi^2}\right\} = F_1\left(t-\tfrac{1}{2}\right) U\left(t-\tfrac{1}{2}\right) + F_2(t-1)\, U(t-1)$$

$$= \cos\pi\left(t-\tfrac{1}{2}\right) U\left(t-\tfrac{1}{2}\right) + \sin\pi(t-1)\, U(t-1)$$

By $L^{-1}\{e^{-ap} f(p)\} = F(t-a)\, U(t-a)$

$$= \sin\pi t\ U\left(t-\tfrac{1}{2}\right) - \sin\pi t\ U(t-1)$$

$$= \sin\pi t\left[U\left(t-\tfrac{1}{2}\right) - U(t-1)\right].$$

By $\sin(\pi-\theta) = \sin\theta$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

Ex-4→ Find $L^{-1}\left\{\frac{(p+1)\,e^{-\pi p}}{p^2+p+1}\right\}$. (AKTU-2010)

Solⁿ→ we have

$$L^{-1}\left\{\frac{p+1}{p^2+p+1}\right\} = L^{-1}\left\{\frac{p+1}{(p^2+p+\frac{1}{4})+1-\frac{1}{4}}\right\} = L^{-1}\left\{\frac{(p+\frac{1}{2})+\frac{1}{2}}{(p+\frac{1}{2})^2+\frac{3}{4}}\right\}$$

$$= e^{-\frac{1}{2}t}\, L^{-1}\left\{\frac{p+\frac{1}{2}}{p^2+\frac{3}{4}}\right\}$$

$$= e^{-\frac{1}{2}t}\left[L^{-1}\left\{\frac{p}{p^2+(\frac{\sqrt{3}}{2})^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{p^2+(\frac{\sqrt{3}}{2})^2}\right\}\right]$$

By $L^{-1} f(p-a) = e^{at} L^{-1}\{f(p)\}$

$$= e^{-\frac{1}{2}t}\left[\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}\,\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}\right]$$

By $L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

$L^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$

$$= \frac{e^{-\frac{1}{2}t}}{\sqrt{3}}\left[\sqrt{3}\cos\frac{\sqrt{3}}{2}t + \sin\frac{\sqrt{3}}{2}t\right]$$

$$= F(t) \text{ (say)}$$

Now by $L^{-1}\{e^{-ap} f(p)\} = F(t-a)\,U(t-a)$

$$\Rightarrow L^{-1}\left\{e^{-\pi p}\,\frac{(p+1)}{p^2+p+1}\right\} = F(t-\pi)\,U(t-\pi)$$

$$= \frac{e^{-\frac{1}{2}(t-\pi)}}{\sqrt{3}}\left[\sqrt{3}\cos\frac{\sqrt{3}}{2}(t-\pi) + \sin\frac{\sqrt{3}}{2}(t-\pi)\right] \times U(t-\pi)$$

Ex-5→ Find $L^{-1}\left\{\frac{e^{-p}-3e^{-3p}}{p^2}\right\}$

Solⁿ→ we have $L^{-1}\left\{\frac{1}{p^2}\right\} = \frac{t^{n-1}}{\lfloor n-1} = \frac{t^{2-1}}{\lfloor 2-1} = t = F(t)$ (say)

Now $L^{-1}\left\{\frac{e^{-p}-3e^{-3p}}{p^2}\right\} = L^{-1}\left\{e^{-p}\cdot\frac{1}{p^2}\right\} - 3L^{-1}\left\{e^{-3p}\cdot\frac{1}{p^2}\right\}$

$$= F(t-1)U(t-1) - 3F(t-3)U(t-3)$$

$$= (t-1)U(t-1) - 3(t-3)U(t-3)$$

By $L^{-1}\{e^{-ap} f(p)\} = F(t-a)\,U(t-a)$

H.W

Ex-1 Find $L^{-1}\left\{\frac{e^{-3p}}{p^3}\right\}$. (AKTU (2013))

Ex-2 Find the inverse of L.T of $\left\{\frac{e^{-5p}}{(p-2)^4}\right\}$.

Ex-3 Find $L^{-1}\left\{\frac{3}{p} - \frac{4e^{-p}}{p^2} + \frac{4e^{-3p}}{p^2}\right\}$.

Ex-4 Find $L^{-1}\left\{\frac{e^{-\frac{\pi}{2}p} - 3e^{-3\frac{\pi}{2}p}}{p^2+1}\right\}$.

Ex-5 Prove $L^{-1}\left\{\frac{e^{-p\pi}}{p^2+1}\right\} = -\sin t\, U(t-\pi)$

Ex-6 find $L^{-1}\left\{\frac{e^{-p}}{(p+1)^3}\right\}$. (AKTU - 2011)

Ex-7 Prove $L^{-1}\left\{\frac{p\, e^{-ap}}{p^2-\omega^2}\right\} = \cosh \omega(t-a)\, U(t-a)$.

Ex-8 $L^{-1}\left\{\frac{e^{-p}}{\sqrt{p+1}}\right\}$ (AKTU-2015)

Change of Scale Property →

$$\boxed{\text{If } L^{-1}\{f(p)\} = F(t)\text{, then } L^{-1}\{f(ap)\} = \frac{1}{a}F\left(\frac{t}{a}\right)}$$

Ex→ If $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2}t\sin t$

, find $L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\}$

Solⁿ→ Give $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2}t\sin t = F(t)$ (say)

Now by change of scale property,

$L^{-1}\{f(p)\} = F(t)$ then $L^{-1}\{f(ap)\} = \frac{1}{a}F\left(\frac{t}{a}\right)$

$\therefore\ L^{-1}\left\{\frac{ap}{[(ap)^2+1]^2}\right\} = \frac{1}{a}F\left(\frac{t}{a}\right) = \frac{1}{a}\frac{1}{2}\frac{t}{a}\sin\frac{t}{a}$.

Put $a=4$,

$L^{-1}\left\{\frac{4p}{(16p^2+1)^2}\right\} = \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{t}{4}\sin\frac{t}{4}$.

$$\Rightarrow \boxed{L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\} = \frac{t}{4}\sin\frac{t}{4}}$$

Inverse Laplace Transform by Using Partial fraction→

For Partial Fraction of the Func^n $\frac{f(p)}{g(p)}$.

(i) deg $f(p)$ < deg $g(p)$

(ii) If deg $f(p) \geq$ deg $g(p)$ then first divide $f(p)$ by $g(p)$, then partial fraction.

$\boxed{\text{Type-1}}$

Ex-1→ Find $L^{-1}\left\{\frac{1}{(p+1)(p-2)}\right\}$.

Sol^n→

$$\frac{1}{(p+1)(p-2)} = \frac{-1/3}{p+1} + \frac{1/3}{p-2} \quad \text{[By Partial fraction]}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(p+1)(p-2)}\right\} = -\frac{1}{3}L^{-1}\left\{\frac{1}{p+1}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{p-2}\right\}$$

$$= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}$$

By $L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$

$\boxed{\text{H.W}}$

Ex-1→ Find $L^{-1}\left\{\frac{1}{p^2+3p+2}\right\}$

Ex-2→ find $L^{-1}\left\{\frac{1}{(p-\alpha)(p-\beta)}\right\}$

Ex-3→ (i) find $L^{-1}\left\{\frac{p^2+2p-3}{p(p-3)(p+2)}\right\}$ (ii) $L^{-1}\left\{\frac{2p^2-6p+5}{p^3-6p^2+11p-6}\right\}$ (AKTU -2010 2012)

Ex-4→ (i) Find $L^{-1}\left\{\frac{p^2+6}{(p^2+1)(p^2+4)}\right\}$ (ii) $L^{-1}\left\{\frac{2p^2-1}{(p^2+1)(p^2+4)}\right\}$.

Ex-5→ find $L^{-1}\left\{\frac{p^2}{(p^2+a^2)(p^2+b^2)}\right\}$.

Sol^n→

$$\frac{p^2}{(p^2+a^2)(p^2+b^2)} = \frac{\frac{a^2}{a^2-b^2}}{p^2+a^2} - \frac{\frac{b^2}{a^2-b^2}}{p^2+b^2} \quad \text{[By Partial fraction]}$$

$$\Rightarrow L^{-1}\left\{\frac{p^2}{(p^2+a^2)(p^2+b^2)}\right\} = \frac{a^2}{a^2-b^2}L^{-1}\left\{\frac{1}{p^2+a^2}\right\} - \frac{b^2}{a^2-b^2}L^{-1}\left\{\frac{1}{p^2+b^2}\right\}$$

$$= \frac{a^2}{a^2-b^2}\frac{\sin at}{a} - \frac{b^2}{a^2-b^2}\frac{\sin bt}{b}$$

$$= \frac{1}{a^2-b^2}\left[a\sin at - b\sin bt\right]$$

By $L^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$

Type-2

① Find $L^{-1}\left\{\dfrac{4p+5}{(p-1)^2(p+2)}\right\}$.

Solⁿ → we have

$$\frac{4p+5}{(p+2)(p-1)^2} = \frac{A}{p+2} + \frac{B}{(p-1)} + \frac{C}{(p-1)^2}$$

$$\therefore \quad \frac{4p+5}{(p+2)(p-1)^2} = \frac{-\frac{1}{3}}{p+2} + \frac{B}{p-1} + \frac{3}{(p-1)^2} \longrightarrow ①$$

Put $p=0$, in ①,

$$\frac{5}{2\times(-1)^2} = \frac{-1/3}{2} + \frac{B}{(-1)} + \frac{3}{(-1)^2}$$

$$\Rightarrow \frac{5}{2} = -\frac{1}{6} - B + 3$$

$$\Rightarrow B = -\frac{1}{6} - \frac{5}{2} + 3 = \frac{-1-15+18}{6} = \frac{2}{6} = \frac{1}{3}.$$

Hence

$$\frac{4p+5}{(p+2)(p-1)^2} = \frac{-\frac{1}{3}}{p+2} + \frac{1/3}{p-1} + \frac{3}{(p-1)^2}$$ [By Partial fraction]

$$\therefore L^{-1}\left\{\frac{4p+5}{(p+2)(p-1)^2}\right\} = -\frac{1}{3}L^{-1}\left\{\frac{1}{p+2}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{p-1}\right\} + 3L^{-1}\left\{\frac{1}{(p-1)^2}\right\}.$$

$$= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t} + 3e^{t}L^{-1}\left\{\frac{1}{p^2}\right\}.$$

$$= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t} + 3e^{t}\frac{t^{2-1}}{\lfloor 2-1}$$

$$= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t} + 3te^{t}$$

By $L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$

$L^{-1}f(p-a) = e^{at}L^{-1}\{f(p)\}$

H.W

Ex-1 → find $L^{-1}\left\{\dfrac{p+2}{p^2(p+3)}\right\}$

Ex-2 → find $L^{-1}\left\{\dfrac{p-1}{p^2(p-7)}\right\}$

Ex-3 → Find $L^{-1}\left\{\dfrac{4p+5}{(p-4)^2(p+3)}\right\}$

Ex-4 → V.Imp Evaluate $L^{-1}\left\{\dfrac{21p-33}{(p+1)(p-2)^3}\right\}$ (AKTU 2009).

Type-3:

Find $L^{-1}\left\{\dfrac{5p+3}{(p-1)(p^2+2p+5)}\right\}$.

Sol^n: We have

$$\frac{5p+3}{(p-1)(p^2+2p+5)} = \frac{A}{p-1} + \frac{Bp+C}{p^2+2p+5} \longrightarrow (*)$$

$$\Rightarrow \frac{5p+3}{(p-1)(p^2+2p+5)} = \frac{1}{p-1} + \frac{Bp+C}{p^2+2p+5} \longrightarrow (1)$$

Put $p=0$ in (1),

$$\frac{3}{(-1)(5)} = \frac{1}{0-1} + \frac{C}{5}$$

$$\Rightarrow -\frac{3}{5} = -1 + \frac{C}{5} \Rightarrow -3 = -5 + C \Rightarrow \boxed{C=2}$$

Using in (1)

$$\frac{5p+3}{(p-1)(p^2+2p+5)} = \frac{1}{p-1} + \frac{Bp+2}{p^2+2p+5} \longrightarrow (2)$$

Put $p=-1$ in (2),

$$\frac{-5+3}{(-1-1)(1-2+5)} = \frac{1}{-1-1} + \frac{(-B)+2}{1-2+5}$$

$$\Rightarrow \frac{-\cancel{2}}{-\cancel{2}\times 4} = -\frac{1}{2} + \frac{(-B)+2}{4} \Rightarrow \frac{1}{4}+\frac{1}{2}-\frac{1}{2} = -\frac{B}{4}$$

$$\Rightarrow \boxed{B=-1}$$

Hence (*) becomes,

$$\frac{5p+3}{(p-1)(p^2+2p+5)} = \frac{1}{p-1} + \frac{(-1)p+2}{p^2+2p+5}$$

$$= \frac{1}{p-1} - \frac{(p-2)}{(p^2+2p+1)+4}$$

$$= \frac{1}{p-1} - \left[\frac{(p+1)-3}{(p+1)^2+4}\right]$$

$$\therefore L^{-1}\left\{\frac{5p+3}{(p-1)(p^2+2p+5)}\right\} = L^{-1}\left\{\frac{1}{p-1}\right\} - L^{-1}\left\{\frac{(p+1)-3}{(p+1)^2+4}\right\}$$

$$= e^{t} - e^{-t} L^{-1}\left\{\frac{p-3}{p^2+4}\right\}$$

$$= e^{t} - e^{-t}\left[L^{-1}\left\{\frac{p}{p^2+4}\right\} - 3L^{-1}\left\{\frac{1}{p^2+4}\right\}\right]$$

$$= e^{t} - e^{-t}\left[\cos 2t - \frac{3}{2}\sin 2t\right]$$

By $L^{-1}\dfrac{1}{p-a} = e^{at}$ & $L^{-1}\{f(p-a)\} = e^{at}L^{-1}\{f(p)\}$

$\boxed{\text{H.W}}$

Ex-1→ Find $L^{-1}\left\{\frac{1}{p(p^2+4)}\right\}$

Ex-2→ Find $L^{-1}\left\{\frac{3p+1}{(p-1)(p^2+1)}\right\}$

Ex-3→ Find $L^{-1}\left\{\frac{1}{(p+1)(p^2+2p+2)}\right\}$

Ex-4→ Find $L^{-1}\left\{\frac{1}{p^3-a^3}\right\}$ or $L^{-1}\left\{\frac{1}{(p-a)(p^2+ap+a^2)}\right\}$.

Ex-5→ Find $L^{-1}\left\{\frac{p}{(p-1)(p^2+4)}\right\}$.

Ex-6→ Find $L^{-1}\left\{\frac{p+4}{p(p-1)(p^2+4)}\right\}$. (AKTU-2012)

$\boxed{\text{Type-4}}$

Evaluate $L^{-1}\left\{\frac{1}{\ldots}\right\}$ (AKTU-2010)

Type-4 → ① Evaluate $L^{-1}\left\{\frac{1}{(p^2+4)(p+1)^2}\right\}$.

Solⁿ → we have

$$\frac{1}{(p+1)^2(p^2+4)} = \frac{A}{(p+1)} + \frac{B}{(p+1)^2} + \frac{Cp+D}{p^2+4} \rightarrow ①$$

$$\frac{1}{(p+1)^2(p^2+4)} = \frac{A}{p+1} + \frac{1/5}{(p+1)^2} + \frac{Cp+D}{p^2+4} \rightarrow ②$$

Put $p=0$ in ②,

$$\frac{1}{4} = A + \frac{1}{5} + \frac{D}{4} \Rightarrow \boxed{4A + D = \frac{1}{5}} \rightarrow ③$$

Put $p=1$ in ②,

$$\cancel{\frac{1}{20}} = \frac{A}{2} + \cancel{\frac{1}{20}} + \frac{C+D}{5} \Rightarrow \boxed{5A + 2C + 2D = 0} \rightarrow ④$$

Put $p=-2$ in ②,

$$\frac{1}{8} = \frac{A}{-1} + \frac{1}{5} + \frac{-2C+D}{8} \Rightarrow \boxed{8A + 2C - D = \frac{3}{5}} \rightarrow ⑤$$

From ③, Put $D = \frac{1}{5} - 4A$ in ④ & ⑤, we get

$$5A + 2C + 2\left(\frac{1}{5} - 4A\right) = 0 \Rightarrow \boxed{3A - 2C = \frac{2}{5}} \rightarrow ⑥$$

$$\& \; 8A + 2C - \left(\frac{1}{5} - 4A\right) = \frac{3}{5} \Rightarrow \boxed{6A + C = \frac{2}{5}} \rightarrow ⑦$$

On solving eqⁿ ⑥ & ⑦

$$\boxed{A = \frac{2}{25}}, \quad \boxed{C = -\frac{2}{25}} \quad \& \quad \boxed{D = -\frac{3}{25}}$$

Using in ②,

$$\frac{1}{(p+1)^2(p^2+4)} = \frac{2}{25}\cdot\frac{1}{p+1} + \frac{1}{5}\frac{1}{(p+1)^2} - \frac{(2p+3)}{25(p^2+4)}.$$

$$\therefore L^{-1}\left\{\frac{1}{(p+1)^2(p^2+4)}\right\} = \frac{2}{25}L^{-1}\left\{\frac{1}{p+1}\right\} + \frac{1}{5}L^{-1}\left\{\frac{1}{(p+1)^2}\right\} - \frac{1}{25}\left[2L^{-1}\left\{\frac{p}{p^2+4}\right\} + 3L^{-1}\left\{\frac{1}{p^2+4}\right\}\right]$$

$$= \frac{2}{25}e^{-t} + \frac{1}{5}e^{-t}L^{-1}\left\{\frac{1}{p^2}\right\} - \frac{2}{25}\cos 2t - \frac{3}{25}\cdot\frac{\sin 2t}{2}.$$

$$= \frac{2}{25}e^{-t} + \frac{1}{5}te^{-t} - \frac{2}{25}\cos 2t - \frac{3}{50}\sin 2t$$

H.W   Find $L^{-1}\left\{\frac{p}{(p+1)^2(p^2+1)}\right\}$.

Type-5 → Prove that

Ex-1 → $$L^{-1}\left\{\frac{p}{(p^2-2p+2)(p^2+2p+2)}\right\} = \frac{1}{2}\sin t \sinh t.$$ (AKTU-2012)

Solⁿ → we have

$$\frac{p}{(p^2-2p+2)(p^2+2p+2)} = \frac{1}{4}\left[\frac{(p^2+2p+2)-(p^2-2p+2)}{(p^2-2p+2)(p^2+2p+2)}\right]$$

$$= \frac{1}{4}\frac{1}{p^2-2p+2} - \frac{1}{4}\frac{1}{p^2+2p+2}$$

$$= \frac{1}{4}\frac{1}{(p-1)^2+1} - \frac{1}{4}\frac{1}{(p+1)^2+1}$$

$$\Rightarrow L^{-1}\left\{\frac{p}{(p^2-2p+2)(p^2+2p+2)}\right\} = \frac{1}{4}L^{-1}\left\{\frac{1}{(p-1)^2+1}\right\} - \frac{1}{4}L^{-1}\left\{\frac{1}{(p+1)^2+1}\right\}.$$

$$= \frac{1}{4}e^{t}L^{-1}\left\{\frac{1}{p^2+1}\right\} - \frac{1}{4}e^{-t}L^{-1}\left\{\frac{1}{p^2+1}\right\}.$$

By $L^{-1}f(p-a) = e^{at}L^{-1}f(p)$

$$= \frac{1}{4}e^{t}\sin t - \frac{1}{4}e^{-t}\sin t$$

$$= \frac{\sin t}{2}\left(\frac{e^{t}-e^{-t}}{2}\right)$$

By $L^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$

$$= \frac{1}{2}\sin t \sinh t$$

H.W

1. Find $L^{-1}\left\{\frac{p^2+2p+3}{(p^2+2p+2)(p^2+2p+5)}\right\}$

2. Prove that $L^{-1}\left(\frac{p}{p^4+p^2+1}\right) = \frac{2}{\sqrt{3}}\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t.$

3. Show that $L^{-1}\left\{\frac{p^2}{p^4+4a^4}\right\} = \frac{1}{2a}\left[\cosh at \sin at + \sinh at \cos at\right]$

$\underline{\text{Sol}^n}$ (2) Prove that $L^{-1}\left\{\frac{p}{p^4+p^2+1}\right\} = \frac{2}{\sqrt{3}} \sinh\frac{t}{2} \sin\frac{\sqrt{3}}{2}t$.

$\underline{\text{Sol}^n}$ We have

$$\frac{1}{p^4+p^2+1} = \frac{1}{(p^4+2p^2+1)-p^2} = \frac{1}{(p^2+1)^2-p^2}$$

$$= \frac{1}{(p^2-p+1)(p^2+p+1)}$$

$$= \frac{1}{2p}\left[\frac{(p^2+p+1)-(p^2-p+1)}{(p^2-p+1)(p^2+p+1)}\right]$$

$$= \frac{1}{2p}\left[\frac{1}{p^2-p+1} - \frac{1}{p^2+p+1}\right]$$

$$\Rightarrow \frac{p}{p^4+p^2+1} = \frac{1}{2}\left[\frac{1}{p^2-p+1} - \frac{1}{p^2+p+1}\right]$$

$$\therefore L^{-1}\left\{\frac{p}{p^4+p^2+1}\right\} = \frac{1}{2} L^{-1}\left\{\frac{1}{p^2-p+1}\right\} - \frac{1}{2} L^{-1}\left\{\frac{1}{p^2+p+1}\right\}.$$

$$= \frac{1}{2} L^{-1}\left\{\frac{1}{(p^2-p+\frac{1}{4})+1-\frac{1}{4}}\right\} - \frac{1}{2} L^{-1}\left\{\frac{1}{(p^2+p+\frac{1}{4})+1-\frac{1}{4}}\right\}$$

$$= \frac{1}{2} L^{-1}\left\{\frac{1}{(p-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\} - \frac{1}{2} L^{-1}\left\{\frac{1}{(p+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\}$$

$$= \frac{1}{2} e^{\frac{1}{2}t} L^{-1}\left\{\frac{1}{p^2+(\frac{\sqrt{3}}{2})^2}\right\} - \frac{1}{2} e^{-\frac{1}{2}t} L^{-1}\left\{\frac{1}{p^2+(\frac{\sqrt{3}}{2})^2}\right\}$$

$$= \frac{1}{2} e^{\frac{1}{2}t} \frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}} - \frac{1}{2} e^{-\frac{1}{2}t} \frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}.$$

$$= \frac{e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}}{2} \cdot \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t$$

$$= \frac{2}{\sqrt{3}} \sinh\frac{1}{2}t \sin\frac{\sqrt{3}}{2}t.$$

Engineering Mathematics-II

By: Dr. Anuj Kumar

Lec-No-22

⊛ Inverse Laplace Transform of Derivatives →

If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{f^{n}(p)\} = L^{-1}\left[\frac{d^n}{dp^n} f(p)\right] = (-1)^n t^n L^{-1}\{f(p)\}$$

$$= (-1)^n t^n F(t)$$

$n = 1, 2, 3, \dots$

Ex-1 → Find the inverse Laplace Transform of

$\log\left(\frac{p+1}{p-1}\right)$. (AKTU-2012)

Sol^n → Let $f(p) = \log\left(\frac{p+1}{p-1}\right)$ and $L^{-1}[f(p)] = F(t)$ (say).

Now

$$L^{-1}\left\{\frac{d}{dp}\left(\log\frac{p+1}{p-1}\right)\right\} = (-1)^1 t^1 F(t)$$

By $L^{-1}\left\{\frac{d^n}{dp^n} f(p)\right\} = (-1)^n t^n F(t)$

$$\Rightarrow L^{-1}\left[\frac{d}{dp}\left(\log(p+1) - \log(p-1)\right)\right\} = -t F(t)$$

$$\Rightarrow L^{-1}\left[\frac{1}{p+1} - \frac{1}{p-1}\right\} = -t F(t)$$

$$\Rightarrow L^{-1}\left\{\frac{1}{p+1}\right\} - L^{-1}\left\{\frac{1}{p+1}\right\} = -t F(t)$$

By $L^{-1}\frac{1}{p-a} = e^{at}$

$$\Rightarrow e^{-t} - e^{t} = -t F(t)$$

$$\Rightarrow F(t) = \frac{e^{t} - e^{-t}}{t}$$

$$\Rightarrow F(t) = \frac{1}{t} \times 2 \times \frac{e^{t} - e^{-t}}{2}$$

By $\sinh at = \frac{e^{at} - e^{-at}}{2}$

$$\Rightarrow \boxed{F(t) = \frac{2\sinh t}{t}}$$

Ex-2 → Find $L^{-1}\left\{\log\left(\frac{p^2+4p+5}{p^2+2p+5}\right)\right\}$. (AKTU-2014)

Sol^n →

Let $f(p) = \log \frac{p^2+4p+5}{p^2+2p+5}$ then $L^{-1} f(p) = F(t)$ (say)

Now

$$L^{-1}\left[\frac{d}{dp} \log \frac{p^2+4p+5}{p^2+2p+5}\right] = (-1)^1 t^1 F(t)$$

By $L^{-1} \frac{d^n}{dp^n} f(p) = (-1)^n t^n F(t)$

$$\Rightarrow L^{-1}\left[\frac{d}{dp}\{\log(p^2+4p+5) - \log(p^2+2p+5)\}\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{(2p+4)}{p^2+4p+5} - \frac{(2p+2)}{p^2+2p+5}\right] = -t\,F(t)$$

$$\Rightarrow L^{-1}\left[\frac{2(p+2)}{(p+2)^2+1}\right] - L^{-1}\left[\frac{2(p+1)}{(p+1)^2+4}\right] = -t\,F(t)$$

$$\Rightarrow 2e^{-2t} L^{-1}\left\{\frac{p}{p^2+1}\right\} - 2e^{-t} L^{-1}\left\{\frac{p}{p^2+4}\right\} = -tF(t)$$

$$\Rightarrow 2e^{-2t}\cos t - 2e^{-t}\cos 2t = -tF(t)$$

$$\Rightarrow \boxed{F(t) = \frac{2}{t}\left(e^{-t}\cos 2t - e^{-2t}\cos t\right)}$$

Ex-3 → Find $L^{-1}\left\{\cot^{-1}\left(\frac{p+3}{2}\right)\right\}$. (AKTU-2013)

Sol^n →

Let $f(p) = \cot^{-1}\left(\frac{p+3}{2}\right)$ then $L^{-1} f(p) = F(t)$ (say)

$$\therefore L^{-1}\left[\frac{d}{dp}\cot^{-1}\left(\frac{p+3}{2}\right)\right] = (-1)^1 t^1 F(t)$$

By $L^{-1} \frac{d^n}{dp^n} f(p) = (-1)^n t^n F(t)$

$$\Rightarrow L^{-1}\left[\frac{-1}{1+\left(\frac{p+3}{2}\right)^2}\cdot\frac{1}{2}\right] = -tF(t)$$

By $L^{-1} f(p-a) = e^{at} L^{-1} f(p)$

$$\Rightarrow L^{-1}\left[\frac{-2}{(p+3)^2+4}\right] = -tF(t)$$

$L^{-1} \frac{a}{p^2+a^2} = \sin at$

$$\Rightarrow (-2)\, e^{-3t} L^{-1}\left\{\frac{1}{p^2+4}\right\} = -tF(t)$$

$$\Rightarrow (-2)\, e^{-3t}\frac{\sin 2t}{2} = -tF(t) \Rightarrow \boxed{F(t) = \frac{e^{-3t}\sin 2t}{t}}$$

Ex-4: Find $L^{-1}\left[\frac{p^2-a^2}{(p^2+a^2)^2}\right]$.

Sol^n: We have $L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$.

$\therefore L^{-1}\left\{\frac{d}{dp}\frac{p}{p^2+a^2}\right\} = (-1)^1 t^1 \cos at$

$\Rightarrow L^{-1}\left[\frac{(p^2+a^2).1 - p.2p}{(p^2+a^2)^2}\right] = -t\cos at$

$\Rightarrow L^{-1}\left[\frac{a^2-p^2}{(p^2+a^2)^2}\right] = -t\cos at$

$\Rightarrow L^{-1}\left\{\frac{p^2-a^2}{(p^2+a^2)^2}\right\} = t\cos at$.

Ex-5: Find $L^{-1}\left\{\frac{2ap}{(p^2+a^2)^2}\right\}$.

Sol^n: $L^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$

then $L^{-1}\left[\frac{d}{dp}\frac{a}{p^2+a^2}\right] = (-1)^1 t^1 \sin at$

$\Rightarrow L^{-1}\left[\frac{(p^2+a^2).0 - a.2p}{(p^2+a^2)^2}\right] = -t\sin at$

$\Rightarrow L^{-1}\left[\frac{-2ap}{(p^2+a^2)^2}\right] = -t\sin at$

$\Rightarrow L^{-1}\left[\frac{2ap}{(p^2+a^2)^2}\right] = t\sin at$.

$\boxed{\text{H.W}}$

1. Find $L^{-1}\left[\log\left(1+\frac{1}{p^2}\right)\right]$ (AKTU-2015)
2. Find $L^{-1}\tan^{-1}\frac{2}{p^2}$
3. Find $L^{-1}\left[\tan^{-1}\frac{1}{p}\right]$
4. Find $L^{-1}\left[\frac{p}{(p^2+a^2)^2}\right]$
5. Evaluate $L^{-1}\left[\cot^{-1}(1+p)\right]$
6. Evaluate $L^{-1}\left\{\log\left(1-\frac{1}{p^2}\right)\right\}$.
7. Find $L^{-1}\left\{\log\frac{p+a}{p+b}\right\}$ (AKTU-2003)
8. Find $L^{-1}\left\{\log\frac{1+p}{p}\right\}$.
9. $L^{-1}\left\{\frac{1}{2}\log\frac{p^2+b^2}{p^2+a^2}\right\}$.
10. $L^{-1}\left\{p\log\frac{p-1}{p+1}\right\}$. (AKTU-2014)

Inverse Laplace Transform multiplication by powers of $p \rightarrow$

$$\boxed{\begin{array}{l} \text{If } L^{-1}\{f(p)\} = F(t) \text{ and } F(0)=0 \\ \text{then } L^{-1}\{p\, f(p)\} = F'(t) \end{array}}$$

Note →

$$\boxed{\begin{array}{l} \text{If } L^{-1}\{f(p)\} = F(t) \text{ and } F(0)=0 \\ \text{then } L^{-1}\{p^n f(p)\} = F^n(t) = \frac{d^n}{dt^n} F(t) \end{array}}$$

Ex-1 → Find the Laplace Transform of $\frac{p}{(p-4)^5}$.

Sol^n → we have $L^{-1}\left\{\frac{1}{(p-4)^5}\right\} = e^{4t} L^{-1}\left\{\frac{1}{p^5}\right\}$

By $L^{-1} f(p-a) = e^{at} L^{-1}\{f(p)\}$

$$= e^{4t} \frac{t^{5-1}}{\lfloor 5-1}$$

$$= \frac{1}{24} e^{4t} \cdot t^4$$

By $L^{-1}\left\{\frac{1}{p^n}\right\} = \frac{t^{n-1}}{\lfloor n-1}$

$$= F(t) \text{ (say)}$$

Here $F(0)=0$

Then $L^{-1}\{p\, f(p)\} = F'(t)$

$$\Rightarrow L^{-1}\left\{p \cdot \frac{1}{(p-4)^5}\right\} = \frac{d}{dt}\left[\frac{e^{4t} \cdot t^4}{24}\right] = \frac{1}{24}\left[4e^{4t} \cdot t^4 + e^{4t} \cdot 4t^3\right]$$

$$= \frac{t^3 e^{4t}}{6}(t+1).$$

Ex-2 → find $L^{-1}\left\{\frac{p^2}{(p-1)^3}\right\}$.

Sol^n → we have $L^{-1}\left\{\frac{1}{(p-1)^3}\right\} = e^t L^{-1}\left\{\frac{1}{p^3}\right\} = e^t \cdot \frac{t^2}{\lfloor 2} = \frac{e^t \cdot t^2}{2}$.

Here $F(0)=0$

$$\therefore L^{-1}\{p^2 f(p)\} = F''(t)$$

$$\Rightarrow L^{-1}\left\{\frac{p^2}{(p-1)^3}\right\} = \frac{d^2}{dt^2}\left[\frac{e^t \cdot t^2}{2}\right]$$

$$= \frac{e^t}{2}(t^2+4t+2).$$

H.W

Ex-1 → Find $L^{-1}\left\{\frac{p}{(p+2)^2}\right\}$

Ex-2 → Find $L^{-1}\left\{\frac{p^2}{(p+4)^3}\right\}$.

Ex-3 → Find $L^{-1}\left\{p \log \frac{p-1}{p+1}\right\}$. (AKTU-2008)

Inverse Laplace Transform division by powers of p →

or Division by p →

If $L^{-1}\{f(p)\} = F(t)$ then

$$L^{-1}\left\{\frac{f(p)}{p}\right\} = \int_0^t F(t)\,dt$$

$$L^{-1}\left\{\frac{f(p)}{p^2}\right\} = \int_0^t \int_0^t F(t)\,dt\,dt$$

$$L^{-1}\left\{\frac{f(p)}{p^3}\right\} = \int_0^t \int_0^t \int_0^t F(t)\,dt\,dt\,dt \text{ etc..}$$

Ex-1 Evaluate $L^{-1}\left\{\frac{1}{p^3(p^2+1)}\right\}$.

Sol^n → We have $f(p) = \frac{1}{p^2+1}$ then $L^{-1}\left\{\frac{1}{p^2+1}\right\} = \sin t = F(t)$ (say)

Now $$L^{-1}\left[\frac{1}{p^3(p^2+1)}\right] = \int_0^t \int_0^t \int_0^t F(t)\,dt\,dt\,dt$$

$$= \int_0^t \int_0^t \left[\int_0^t \sin t\,dt\right] dt\,dt$$

$$= \int_0^t \int_0^t \left[-\cos t\right]_0^t dt\,dt$$

$$= \int_0^t \int_0^t (1-\cos t)\,dt\,dt$$

$$= \int_0^t \left[\int_0^t (1-\cos t)\,dt\right] dt$$

$$= \int_0^t [t - \sin t]_0^t \, dt$$

$$= \int_0^t [t - \sin t] \, dt$$

$$= \left[\frac{t^2}{2} + \cos t\right]_0^t$$

$$\Rightarrow L^{-1}\left\{\frac{1}{p^3(p^2+1)}\right\} = \frac{t^2}{2} + \cos t - 1$$

Ex-2 : Find the inverse Laplace Transform of $\frac{1}{p(p^2+2p+2)}$.

Solⁿ : Let $f(p) = \frac{1}{p^2+2p+2}$

$$\therefore L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{p^2+2p+2}\right\} = L^{-1}\left\{\frac{1}{(p+1)^2+1}\right\} = e^{-t} L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$= e^{-t}\sin t = F(t) \text{ (say)}$$

Then $L^{-1}\left\{\frac{f(p)}{p}\right\} = L^{-1}\left\{\frac{1}{p(p^2+2p+2)}\right\} = \int_0^t F(t)\, dt$

$$= \int_0^t e^{-t}\sin t \, dt$$

$$= \left[\frac{e^{-t}}{(-1)^2+1^2}(-\sin t - \cos t)\right]_0^t$$

By $\int e^{at}\sin bt\, dt = \left[\frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)\right]$

$$= \left[\frac{e^{-t}}{2}(-\sin t - \cos t) + \frac{1}{2}\right].$$

$$= \frac{1}{2}\left[1 - e^{-t}(\sin t + \cos t)\right].$$

H.W

1. Find $L^{-1}\left\{\frac{1}{s^2(1+s^2)}\right\}$
2. Find $L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\}$
3. Find $L^{-1}\left\{\frac{1}{p(p^2+4p+5)}\right\}$
4. Find $L^{-1}\left\{\frac{1}{p^2(p^2+a^2)}\right\}$
5. Find $L^{-1}\left\{\frac{1}{p^3(p+1)}\right\}$
6. Find $L^{-1}\left\{\frac{1}{p}\log\frac{p+2}{p+1}\right\}$.

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Lec- No- 23

Convolution of two func^n →

Let $F(t)$ and $G(t)$ be two functions, then the convolution of F and G is denoted by $F*G$ and is defined by

$$\boxed{F*G = \int_0^t F(x) \cdot G(t-x)\, dx}$$

Convolution Theorem → (For Inverse Laplace Transform)

(AKTU-2016, 2018)

Let $F(t)$ and $G(t)$ be any two func^n such that $L^{-1}\{f(p)\} = F(t)$ and $L^{-1}\{g(p)\} = G(t)$

Then $$\boxed{L^{-1}\{f(p)\cdot g(p)\} = \int_0^t F(x)\, G(t-x)\, dx = F*G.}$$

Ex-1 → Use the convolution theorem, to find $L^{-1}\left\{\frac{1}{p^2(p+1)^2}\right\}$.

(AKTU-2010, 2013)

Sol^n → Let $f(p) = \frac{1}{(p+1)^2}$ & $g(p) = \frac{1}{p^2}$.

Then $L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{(p+1)^2}\right\} = e^{-t} L^{-1}\left\{\frac{1}{p^2}\right\} = e^{-t}\cdot\frac{t^{2-1}}{\lfloor 2-1} = te^{-t} = F(t)$ (say)

& $L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{p^2}\right\} = \frac{t^{2-1}}{\lfloor 2-1} = t = G(t)$ (say)

By $L^{-1} f(p-a) = e^{at} L^{-1} f(p)$ & $L^{-1}\frac{1}{p^n} = \frac{t^{n-1}}{\lfloor n-1}$

Now by Convolution Theorem,

$$L^{-1}\{f(p)\cdot g(p)\} = \int_0^t F(x)\, G(t-x)\, dx$$

$$\Rightarrow L^{-1}\left\{\frac{1}{p^2(p+1)^2}\right\} = \int_0^t x e^{-x} (t-x)\, dx.$$

$$= \int_0^t (tx - x^2)\, e^{-x}\, dx$$

$$= \left[\left\{ (tx - x^2) \frac{e^{-x}}{(-1)} \right\}_0^t - \int_0^t (t - 2x) \frac{e^{-x}}{(-1)}\, dx \right]$$

$$= \left[(0 - 0) + \int_0^t (t - 2x)\, e^{-x}\, dx \right]$$

$$= \left[\left\{ (t - 2x) \frac{e^{-x}}{(-1)} \right\}_0^t - \int_0^t (-2) \frac{e^{-x}}{(-1)}\, dx \right]$$

$$= \left[\left\{ -(t) \frac{e^{-t}}{(-1)} + t \right\} - 2 \left\{ \frac{e^{-x}}{(-1)} \right\}_0^t \right]$$

$$= \left[t e^{-t} + t + 2 e^{-t} - 2 \right]$$

$$= (t+2) e^{-t} + t - 2 .$$

Ex-27 Evaluate $L^{-1}\left\{ \dfrac{p}{(p^2+1)(p^2+4)} \right\}$ by Convolution Theorem.

(AKTU - 2002, 2013, 2016)

Sol^n

Let $f(p) = \dfrac{1}{p^2+4}$, $g(p) = \dfrac{p}{p^2+1}$

Then $L^{-1}\{ f(p) \} = L^{-1}\left\{ \dfrac{1}{p^2+4} \right\} = \dfrac{\sin 2t}{2} = F(t)$ (say)

$L^{-1}\{ g(p) \} = L^{-1}\left\{ \dfrac{p}{p^2+1} \right\} = \cos t = G(t)$ (say)

By $L^{-1}\left\{ \dfrac{a}{p^2+a^2} \right\} = \sin at$

$L^{-1}\left\{ \dfrac{p}{p^2+a^2} \right\} = \cos at$

Now by convolution Theorem,

$$L^{-1}\{ f(p) \cdot g(p) \} = \int_0^t F(x)\, G(t-x)\, dx .$$

$$\Rightarrow L^{-1}\left\{ \frac{p}{(p^2+4)(p^2+1)} \right\} = \int_0^t \frac{\sin 2x}{2} \cos(t-x)\, dx .$$

$$= \frac{1}{2\times 2} \int_0^t 2 \sin 2x \cos(t-x)\, dx$$

$$= \frac{1}{4} \int_0^t \left[\sin(x+t) + \sin(3x - t) \right] dx .$$

$$= \cancel{\frac{1}{4} \int_0^t}$$

By $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$$= \frac{1}{4}\left[-\cos(x+t) - \frac{\cos(3x-t)}{3}\right]_0^t \qquad \text{By } \int \sin ax\,dx = -\frac{\cos ax}{a}$$

$$= \frac{1}{4}\left[-\cos 2t - \frac{\cos 2t}{3} + \cos t + \frac{\cos(t)}{3}\right]$$

$$= \frac{1}{4}\left[-\frac{4}{3}\cos 2t + \frac{4}{3}\cos t\right]$$

$$= \frac{1}{3}\left[\cos t - \cos 2t\right].$$

Ex-3: Use convolution Theorem to evaluate:

(AKTU-2012, 2018)

$$L^{-1}\left\{\frac{p^2}{(p^2+a^2)(p^2+b^2)}\right\}.$$

Solⁿ: Let $f(p) = \frac{p}{p^2+a^2}$, $g(p) = \frac{p}{p^2+b^2}$

$\therefore L^{-1}\{f(p)\} = L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at = F(t)$ (say)

$L^{-1}\{g(p)\} = L^{-1}\left\{\frac{p}{p^2+b^2}\right\} = \cos bt = G(t)$ (say)

Now by convolution theorem,

$$L^{-1}\{f(p)\cdot g(p)\} = \int_0^t F(x)\cdot G(t-x)\,dx$$

$$= \int_0^t \cos ax \cos b(t-x)\,dx$$

$$= \frac{1}{2}\int_0^t 2\cos ax \cos(bt-bx)\,dx \qquad \text{By } 2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$= \frac{1}{2}\int_0^t \left[\cos\{(a-b)x+bt\} + \cos\{(a+b)x-bt\}\right]dx$$

$$= \frac{1}{2}\left[\frac{\sin\{(a-b)x+bt\}}{(a-b)} + \frac{\sin\{(a+b)x-bt\}}{(a+b)}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at - \sin bt}{(a-b)} + \frac{\sin at + \sin bt}{(a+b)}\right] \qquad \text{By } \int \cos ax\,dx = \frac{\sin ax}{a}$$

$$= \frac{1}{2}\left[\frac{(a+b)[\sin at - \sin bt] + (a-b)[\sin at + \sin bt]}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at - 2b\sin bt}{a^2-b^2}\right] = \frac{a\sin at - b\sin bt}{(a^2-b^2)}.$$

Ex-4: Use convolution theorem to find

$$L^{-1}\left\{\frac{1}{(p^2+4)(p+2)}\right\}$$ (AKTU-2013, 2016)

Solⁿ: Let $f(p) = \frac{1}{p^2+4}$, $g(p) = \frac{1}{p+2}$

$\therefore L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{p^2+4}\right\} = \frac{\sin 2t}{2} = F(t)$ (say)

$L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{p+2}\right\} = e^{-2t} = G(t)$ (say)

By
$L^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$
$L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$

Now by convolution theorem.

$$L^{-1}\{f(p)\cdot g(p)\} = \int_0^t F(x)\, G(t-x)\, dx$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(p^2+4)(p+2)}\right\} = \int_0^t \frac{\sin 2x}{2}\cdot e^{-2(t-x)}\, dx$$

$$= \int_0^t \frac{\sin 2x}{2}\, e^{-2t}\cdot e^{2x}\, dx$$

$$= \frac{1}{2} e^{-2t} \int_0^t e^{2x} \sin 2x\, dx$$

$$= \frac{1}{2} e^{-2t}\left[\frac{e^{2x}}{2^2+2^2}(2\sin 2x - 2\cos 2x)\right]_0^t$$

By
$\int e^{ax} \sin bx\, dx$
$= \frac{e^{ax}}{a^2+b^2}[a \sin bx - b\cos bx]$

$$= \frac{1}{8} e^{-2t}\left[e^{2x}(\sin 2x - \cos 2x)\right]_0^t$$

$$= \frac{1}{8} e^{-2t}\left[e^{2t}(\sin 2t - \cos 2t) - e^0(\sin 0 - \cos 0)\right]$$

$$= \frac{1}{8}\left[\sin 2t - \cos 2t + e^{-2t}\right].$$

Ex-5: Use convolution theorem to find $L^{-1}\left\{\frac{16}{(p-2)(p+2)^2}\right\}$

(AKTU-2014)

Solⁿ: Let $f(p) = \frac{16}{(p+2)^2}$, $g(p) = \frac{1}{p-2}$

$\therefore L^{-1}\{f(p)\} = L^{-1}\left\{\frac{16}{(p+2)^2}\right\} = 16 e^{-2t} L^{-1}\left\{\frac{1}{p^2}\right\} = 16 e^{-2t}\cdot\frac{t^1}{\lfloor 1} = 16\, t\, e^{-2t}$

(F(t) say)

By
$L^{-1} f(p-a) = e^{at} L^{-1}\{f(p)\}$
$L^{-1}\left\{\frac{1}{p^n}\right\} = \frac{t^{n-1}}{\lfloor n-1}$
$L^{-1}\frac{1}{p-a} = e^{at}$

$L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{p-2}\right\} = e^{2t} = G(t)$ (say)

Now by convalution Theorem

$$L^{-1}\{f(p)\,g(p)\} = \int_0^t F(x)\,G(t-x)\,dx$$

$$= \int_0^t 16x\,e^{-2x}\,e^{2(t-x)}\,dx$$

$$= 16\int_0^t x\,e^{-2x}\cdot e^{2t}\cdot e^{-2x}\,dx$$

$$= 16\,e^{2t}\int_0^t x\,e^{-4x}\,dx$$

$$= 16\,e^{2t}\left[\left\{x\,\frac{e^{-4x}}{(-4)}\right\}_0^t - \int_0^t 1\cdot\frac{e^{-4x}}{(-4)}\,dx\right]$$

$$= 16\,e^{2t}\left[\left\{-\frac{t\,e^{-4t}}{4}+0\right\} - \left\{\frac{e^{-4x}}{(-4)^2}\right\}_0^t\right]$$

$$= 16\,e^{2t}\left[-\frac{t\,e^{-4t}}{4} - \frac{(e^{-4t}-1)}{16}\right]$$

$$= 16\,e^{2t}\left[\frac{-4t\,e^{-4t}+1-e^{-4t}}{16}\right]$$

$$= e^{2t}\left[1-e^{-4t}(4t+1)\right]$$

$$= e^{2t} - e^{-2t}(4t+1).$$

H.W

Home Assignment

Ex-1: Use convolution Theorem to evaluate:

$$L^{-1}\left\{\frac{p}{(p^2+4)^2}\right\}.$$ (AKTU-2010)

Ans $\frac{t}{4}\sin 2t$

Hint: Let $f(p) = \frac{1}{p^2+4}$, $g(p) = \frac{p}{p^2+4}$

Ex-2: Use the convalution theorem to find

$$L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\}$$

Ans $\frac{1}{2a}t\sin at$

Hint: Let $f(p) = \frac{1}{p^2+a^2}$, $g(p) = \frac{p}{p^2+a^2}$

Ex-3:- Find $L^{-1}\left\{\frac{1}{p(p^2+a^2)}\right\}$. (AKTU-2013)

Hint: Let $f(p) = \frac{1}{p^2+a^2}$, $g(p) = \frac{1}{p}$

Ans $\boxed{\frac{(1-\cos at)}{a^2}}$

Ex-4:- Apply convolution theorem to evaluate:

(i) $L^{-1}\left\{\frac{1}{p(p^2+4)}\right\}$ Ans $\boxed{\frac{1-\cos 2t}{4}}$

(ii) $L^{-1}\left\{\frac{1}{p^4(p^2+1)}\right\}$. Ans $\boxed{\frac{t^3}{6}+\sin t - t}$

(iii) $L^{-1}\left\{\frac{1}{p^3(p^2+1)}\right\}$ Ans $\boxed{\frac{t^2}{2}+\cos t - 1}$

(AKTU-2012)

(iv) Prove by C.T. (AKTU-2005)

$L^{-1}\left\{\frac{1}{(p^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)$ (AKTU-2015)

Hint: Let $f(p) = \frac{1}{p^2+a^2}$, $g(p) = \frac{1}{p^2+a^2}$

(v) $L^{-1}\left\{\frac{1}{(p+a)(p+b)}\right\}$ Ans $\boxed{\frac{e^{-bt}-e^{-at}}{a-b}}$

(vi) $L^{-1}\left\{\frac{1}{(p+1)(p+9)^2}\right\}$ Ans $\boxed{\frac{e^{-t}}{64}(1-e^{-8t}(1+8t))}$

New Ex-5:- Find the inverse Laplace Transform by Convalution Theorem of the func^n $\frac{8p}{(p^2+16)(p^2+1)^2}$. (AKTU-2009)(2015)

Ex-6:- find the $L^{-1}\left\{\frac{1}{(p+3)(p^2+2p+2)}\right\}$ by C.T.

(AKTU-2010, 2012)

Ans $\boxed{\frac{1}{2}\left[e^{-t}(2\sin t - \cos t) + e^{-3t}\right]}$.

Sol^n→ 5→ Let $f(p) = \frac{2p}{(p^2+1)^2}$, $g(p) = \frac{4}{p^2+16}$

we have $L^{-1}\left\{\frac{1}{p^2+1}\right\} = \sin t$

By $L^{-1}\left\{\frac{d^n}{dp^n}f(p)\right\} = (-1)^n t^n F(t)$

$\therefore L^{-1}\left\{\frac{d}{dp}\frac{1}{p^2+1}\right\} = (-1)^1 t^1 \sin t$

$\Rightarrow L^{-1}\left\{\frac{(-1)\times 2p}{(p^2+1)^2}\right\} = -t\sin t$

$\Rightarrow L^{-1}\{f(p)\} = L^{-1}\left\{\frac{2p}{(p^2+1)^2}\right\} = t\sin t = F(t)$ (say)

By $L^{-1}\frac{a}{p^2+a^2} = \sin at$

Also $L^{-1}\{g(p)\} = L^{-1}\left\{\frac{4}{p^2+16}\right\} = \sin 4t = G(t)$ (say)

Now by convolution theorem,

$$L^{-1}\{f(p)\cdot g(p)\} = \int_0^t F(x)G(t-x)\,dx$$

$$\Rightarrow L^{-1}\left\{\frac{8p}{(p^2+16)(p^2+1)^2}\right\} = \int_0^t x\sin x\sin(4t-4x)\,dx$$

$$= \frac{1}{2}\int_0^t x\,[2\sin x\sin(4t-4x)]\,dx$$

By $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$$= \frac{1}{2}\int_0^t x\,[\cos(5x-4t) - \cos(3x-4t)]\,dx$$

by $\cos(-\theta) = \cos\theta$

$$= \frac{1}{2}\left[\left\{x\left\{\frac{\sin(5x-4t)}{5} - \frac{\sin(3x-4t)}{3}\right\}\right\}_0^t - \int_0^t 1\left\{\frac{\sin(5x-4t)}{5} - \frac{\sin(3x-4t)}{3}\right\}dx\right]$$

$$= \frac{1}{2}\left[t\left\{\frac{\sin t}{5} + \frac{\sin t}{3}\right\} - \left\{-\frac{\cos(5x-4t)}{25} + \frac{\cos(3x-4t)}{9}\right\}_0^t\right].$$

$$= \frac{1}{2}\left[t\left\{\frac{\sin t}{5} + \frac{\sin t}{3}\right\} - \left\{-\frac{\cos t}{25} + \frac{\cos 4t}{25} + \frac{\cos t}{9} - \frac{\cos 4t}{9}\right\}\right]$$

$$= \frac{1}{2}\left[t\cdot\frac{8}{15}\sin t - \frac{16}{225}\cos t + \frac{16}{225}\cos 4t\right]$$

$$= \frac{4t\sin t}{15} - \frac{8}{225}\cos t + \frac{8}{225}\cos 4t$$

$$= \frac{60t\sin t - 8\cos t + 8\cos 4t}{225}$$

Ex-7→ Use convolution theorem, find $L^{-1}\left\{\frac{p}{(p^2+a^2)^3}\right\}$.

(AKTU-2010, 2012)

Hint Let $f(p) = \frac{p}{(p^2+a^2)^2}$, $g(p) = \frac{1}{(p^2+a^2)}$

Ans→ $\boxed{\frac{1}{8a^3}[\sin at - at\cos at]}$

Heaviside's Expansion Formula →

Let $f(p)$ and $g(p)$ be two polynomials in p such that $\deg f(p) < \deg g(p)$ and $g(p)$ has n distinct roots α_r, $r = 1, 2, \ldots n$. i.e

$$g(p) = (p-\alpha_1)(p-\alpha_2) \cdots (p-\alpha_n) \text{ then}$$

$$\boxed{L^{-1}\left[\frac{f(p)}{g(p)}\right] = \sum_{r=1}^{n} \frac{f(\alpha_r)}{g'(\alpha_r)} e^{\alpha_r t}}$$

Ex-1 → Using Heaviside's expansion formula find

$$L^{-1}\left\{\frac{2p^2-6p+5}{p^3-6p^2+11p-6}\right\}. \quad \text{(AKTU-2004)}$$

Sol^n → Let $f(p) = 2p^2-6p+5$

$$g(p) = p^3-6p^2+11p-6$$

$$\& \quad g'(p) = 3p^2-12p+11.$$

Put $g(p) = 0$

$$\Rightarrow p^3-6p^2+11p-6 = 0$$

$$\Rightarrow (p-1)(p-2)(p-3) = 0 \quad \Rightarrow \boxed{p = 1, 2, 3}$$

Let $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$

Now by the Heaviside expansion formula,

$$L^{-1}\left\{\frac{f(p)}{g(p)}\right\} = \sum_{r=1}^{3} \frac{f(\alpha_r)}{g'(\alpha_r)} e^{\alpha_r t}$$

$$\Rightarrow L^{-1}\left\{\frac{2p^2-6p+5}{p^3-6p^2+11p-6}\right\} = \frac{f(\alpha_1)}{g'(\alpha_1)} e^{\alpha_1 t} + \frac{f(\alpha_2)}{g'(\alpha_2)} e^{\alpha_2 t} + \frac{f(\alpha_3)}{g'(\alpha_3)} e^{\alpha_3 t}$$

$$= \frac{f(1)}{g'(1)} e^{t} + \frac{f(2)}{g'(2)} e^{2t} + \frac{f(3)}{g'(3)} e^{3t}$$

$$= \frac{1}{2} e^{t} - e^{2t} + \frac{5}{2} e^{3t}.$$

$\underline{Ex-1}$: find $L^{-1}\left\{\dfrac{p^2-6}{p^3+4p^2+3p}\right\}$

$\underline{Ex-2}$: Find $L^{-1}\left\{\dfrac{2p^2+5p-4}{p^3+p^2-2p}\right\}$

$\underline{Ex-3}$: find $L^{-1}\left\{\dfrac{3p+1}{(p-1)(p^2+1)}\right\}$.

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Lec No - 24

⊛ Application of Laplace Transform to solve ordinary diff. eqⁿ →

Step-1 → Take Laplace Transform of both sides of the given Differential eqⁿ and use the formulae of L·T.

& Use
$$L(y') = pL(y) - y(0) \qquad \text{(for derivatives)}$$
$$L(y'') = p^2 L(y) - p\,y(0) - y'(0)$$
$$L(y''') = p^3 L(y) - p^2 y(0) - p\,y'(0) - y''(0)$$
etc.

Step-2 → Solve the above eqⁿ and find $L(y)$ in terms of p.

Step-3 → Take inverse Laplace Transform of both sides. Find the solution y.

Ex-1 → Solve $\frac{d^2y}{dt^2} + y = 0$ under the conditions that $y = 1$, $\frac{dy}{dt} = 0$ when $t = 0$. (AKTU-2008)

Solⁿ → Given $y'' + y = 0 \rightarrow (1)$ & $y(0) = 1,\ y'(0) = 0 \rightarrow (2)$

Take Laplace Transform both side of (1),
$$L(y'') + L(y) = L(0)$$
$$\Rightarrow [p^2 L(y) - p\,y(0) - y'(0)] + L(y) = 0$$
$$\Rightarrow p^2 L(y) - p - 0 + L(y) = 0 \quad \text{(From (2))}$$
$$\Rightarrow (p^2+1) L(y) = p$$
$$\Rightarrow L(y) = \frac{p}{p^2+1}$$
$$\Rightarrow y = L^{-1}\left\{\frac{p}{p^2+1}\right\} = \cos t$$
$$\Rightarrow \boxed{y = \cos t}$$

By $L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

Ex-2: Using Laplace Transform, find the solution of the initial value problem $\frac{d^2y}{dt^2} + 9y = 6\cos 3t$; $y(0) = 2, y'(0) = 0$.

(AKTU-2007, 2015)

Sol^n: Given $y'' + 9y = 6\cos 3t \rightarrow (1)$ & $y(0) = 2, y'(0) = 0 \rightarrow (2)$

Taking Laplace Transform on both side of eq^n (1), we get

$$L(y'') + 9L(y) = 6L\{\cos 3t\}$$

By $L\cos at = \frac{p}{p^2+a^2}$

$$\Rightarrow [p^2L(y) - py(0) - y'(0)] + 9L(y) = 6\frac{p}{p^2+9}$$

$$\Rightarrow p^2L(y) - 2p + 9L(y) = \frac{6p}{p^2+9}$$

$$\Rightarrow (p^2+9)L(y) = \frac{6p}{p^2+9} + 2p$$

$$\Rightarrow L(y) = \frac{6p}{(p^2+9)^2} + \frac{2p}{p^2+9}$$

$$\Rightarrow y = L^{-1}\left\{\frac{6p}{(p^2+9)^2}\right\} + L^{-1}\left\{\frac{2p}{p^2+9}\right\} \longrightarrow (3)$$

Now we have

$$L^{-1}\left\{\frac{1}{p^2+9}\right\} = \frac{\sin 3t}{3}$$

By $L\left\{\frac{1}{p^2+a^2}\right\} = \frac{\sin at}{a}$

$$\therefore L^{-1}\left[\frac{d}{dp}\frac{1}{p^2+9}\right] = (-1)^1 t^1 \frac{\sin 3t}{3}$$

By $L^{-1}\{f'(p)\} = (-1)^n t^1 F(t)$

$$\Rightarrow L^{-1}\left\{\frac{(-1)\times 2p}{(p^2+9)^2}\right\} = -\frac{t\sin 3t}{3}$$

$$\Rightarrow L^{-1}\left\{\frac{6p}{(p^2+9)^2}\right\} = t\sin 3t$$

Then from (3),

$$y = L^{-1}\left\{\frac{6p}{(p^2+9)^2}\right\} + 2L^{-1}\left\{\frac{p}{p^2+9}\right\}$$

$$\boxed{y = t\sin 3t + 2\cos 3t}$$

Ex-3→ Solve the following diff. eqⁿ using Laplace Transform

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2e^t \text{ where } y(0)=1,\ \left(\frac{dy}{dt}\right)_{t=0}=0,\ \left(\frac{d^2y}{dt^2}\right)_{t=0}=-2.$$

(AKTU-2007, 2018)

Solⁿ→ Given

$$y''' - 3y'' + 3y' - y = t^2e^t \rightarrow (1) \quad \& \quad \left.\begin{aligned} y(0)&=1 \\ y'(0)&=0 \\ y''(0)&=-2 \end{aligned}\right\} \rightarrow (2)$$

Taking L.T on both side of eqⁿ (1),

$$L(y''') - 3L(y'') + 3L(y') - L(y) = L\{e^t.t^2\}$$

$$\Rightarrow \left[p^3L(y) - p^2y(0) - py'(0) - y''(0)\right] - 3\left[p^2L(y) - py(0) - y'(0)\right] + 3\left[pL(y) - y(0)\right] - L(y) = \frac{2}{(p-1)^3}.$$

By $L\{t^n\} = \frac{\lfloor n}{p^{n+1}}$

& $L\{e^{at}t^n\} = \frac{\lfloor n}{(p-a)^{n+1}}$

$$\Rightarrow \left[p^3L(y) - p^2 + 2\right] - 3\left[p^2L(y) - p\right] + 3\left[pL(y) - 1\right] - L(y) = \frac{2}{(p-1)^3}.$$

$$\Rightarrow (p^3 - 3p^2 + 3p - 1)L(y) - p^2 + 2 + 3p - 3 = \frac{2}{(p-1)^3}.$$

$$\Rightarrow (p-1)^3 L(y) = p^2 - 3p + 1 + \frac{2}{(p-1)^3}.$$

$$\Rightarrow L(y) = \frac{p^2 - 3p + 1}{(p-1)^3} + \frac{2}{(p-1)^6}$$

$$\Rightarrow L(y) = \frac{(p-1)^2 - p}{(p-1)^3} + \frac{2}{(p-1)^6}$$

$$= \frac{1}{(p-1)} - \frac{p}{(p-1)^3} + \frac{2}{(p-1)^6}.$$

$$= \frac{1}{(p-1)} - \frac{(p-1)+1}{(p-1)^3} + \frac{2}{(p-1)^6}$$

$$\Rightarrow L(y) = \frac{1}{p-1} - \frac{1}{(p-1)^2} - \frac{1}{(p-1)^3} + \frac{2}{(p-1)^6}$$

$$\Rightarrow y = L^{-1}\left\{\frac{1}{p-1}\right\} - L^{-1}\left\{\frac{1}{(p-1)^2}\right\} - L^{-1}\left\{\frac{1}{(p-1)^3}\right\} + 2L^{-1}\left\{\frac{1}{(p-1)^6}\right\}.$$

$$= e^t L^{-1}\left\{\frac{1}{p}\right\} - e^t L^{-1}\left\{\frac{1}{p^2}\right\} - e^t L^{-1}\left\{\frac{1}{p^3}\right\} + 2e^tL^{-1}\left\{\frac{1}{p^6}\right\}.$$

$$= e^t.1 - e^t\frac{t}{\lfloor 1} - e^t\frac{t^2}{\lfloor 2} + 2e^t\frac{t^5}{\lfloor 5}$$

By $L^{-1}\{f(p-a)\} = e^{at}L^{-1}f(p)$

$$\Rightarrow \boxed{y = e^t\left(1 - t - \frac{t^2}{2} + \frac{t^5}{60}\right)}$$

By $L^{-1}\frac{1}{p^n} = \frac{t^{n-1}}{\lfloor n-1}$

Ex-4: Using Laplace Transform, solve the following D.E :

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-t}\sin t, \text{ where } x(0)=0 \ \& \ x'(0)=1$$

(AKTU-2005, 2011)

Solⁿ: Give $x'' + 2x' + 5x = e^{-t}\sin t \rightarrow ①$ & $\left.\begin{array}{l} x(0)=0 \\ x'(0)=1 \end{array}\right\} \rightarrow ②$

Taking L.T of both side in eqⁿ ①,

$$L\{x''\} + 2L\{x'\} + 5L\{x\} = L\{e^{-t}\sin t\}$$

$$\Rightarrow [p^2L(x) - px(0) - x'(0)] + 2[pL(x) - x(0)] + 5L(x) = \frac{1}{(p+1)^2+1}$$

$$\Rightarrow (p^2+2p+5)L(x) - 1 = \frac{1}{p^2+2p+2}$$

By $L\{\sin t\} = \frac{1}{p^2+1}$ & $L\{e^{at}\sin t\} = \frac{1}{(p-a)^2+1}$

$$\Rightarrow (p^2+2p+5)L(x) = \frac{1}{(p^2+2p+2)} + 1$$

$$\Rightarrow L(x) = \frac{1}{(p^2+2p+2)(p^2+2p+5)} + \frac{1}{p^2+2p+5}$$

$$= \frac{1}{3}\left[\frac{(p^2+2p+5) - (p^2+2p+2)}{(p^2+2p+2)(p^2+2p+5)}\right] + \frac{1}{p^2+2p+5}$$

$$= \frac{1}{3}\frac{1}{p^2+2p+2} - \frac{1}{3}\frac{1}{p^2+2p+5} + \frac{1}{p^2+2p+5}$$

$$\Rightarrow L(x) = \frac{1}{3}\frac{1}{p^2+2p+2} + \frac{2}{3}\frac{1}{p^2+2p+5}$$

$$\Rightarrow x = \frac{1}{3}L^{-1}\left\{\frac{1}{p^2+2p+2}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{p^2+2p+5}\right\}.$$

$$\Rightarrow x = \frac{1}{3}L^{-1}\left\{\frac{1}{(p+1)^2+1^2}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{(p+1)^2+2^2}\right\}$$

$$\Rightarrow x = \frac{1}{3}e^{-t}L^{-1}\left\{\frac{1}{p^2+1^2}\right\} + \frac{2}{3}e^{-t}L^{-1}\left\{\frac{1}{p^2+2^2}\right\}.$$

$$\Rightarrow x = \frac{1}{3}e^{-t}\sin t + \frac{2}{3}e^{-t}\frac{\sin 2t}{2}$$

By $L^{-1}f(p-a) = e^{at}L^{-1}f(p)$

$$\Rightarrow \boxed{x = \frac{1}{3}e^{-t}(\sin t + \sin 2t)}$$

By $L^{-1}\frac{a}{p^2+a^2} = \frac{\sin at}{a}$

Ex-5: Using Laplace Transformation, solve the D.E

$$\frac{d^2x}{dt^2} + 9x = \cos 2t \quad \text{if } x(0) = 1,\ x\left(\frac{\pi}{2}\right) = -1. \qquad \text{(AKTU-2017)}.$$

Sol^n: Given $x'' + 9x = \cos 2t \rightarrow (1)$ & $\left.\begin{array}{l} x(0)=1 \\ x(\frac{\pi}{2})=-1 \end{array}\right\} \rightarrow (2)$.

Taking the Laplace Transform of both side, we get

$$L\{x''\} + 9L\{x\} = L\{\cos 2t\}.$$

$$\Rightarrow [p^2 L(x) - p\,x(0) - x'(0)] + 9L(x) = \frac{p}{p^2+4}$$

$$\Rightarrow (p^2+9)L(x) - p - A = \frac{p}{p^2+4}$$

$\boxed{\text{from (2) } \& \ x'(0) = A}$

$$\Rightarrow (p^2+9)L(x) = \frac{p}{p^2+4} + p + A$$

$$\Rightarrow L(x) = \frac{p}{(p^2+4)(p^2+9)} + \frac{p}{p^2+9} + \frac{A}{p^2+9}. \longrightarrow (3)$$

Now $\frac{1}{(p^2+4)(p^2+9)}$ take $p^2 = m$

$$\Rightarrow \frac{1}{(m+4)(m+9)} = \frac{1/5}{m+4} - \frac{1/5}{m+9} \qquad \text{By Partial fraction}$$

$$\Rightarrow \frac{1}{(p^2+4)(p^2+9)} = \frac{1}{5}\frac{1}{p^2+4} - \frac{1}{5}\frac{1}{p^2+9}$$

Then (3) becomes,

$$L(x) = \frac{1}{5}\frac{p}{p^2+4} - \frac{1}{5}\frac{p}{p^2+9} + \frac{p}{p^2+9} + \frac{A}{p^2+9}$$

$$\Rightarrow L(x) = \frac{1}{5}\frac{p}{p^2+4} + \frac{4}{5}\frac{p}{p^2+9} + \frac{A}{p^2+9}.$$

$$\Rightarrow x = \frac{1}{5}L^{-1}\left\{\frac{p}{p^2+4}\right\} + \frac{4}{5}L^{-1}\left\{\frac{p}{p^2+9}\right\} + A\,L^{-1}\left\{\frac{1}{p^2+9}\right\}$$

$$\Rightarrow \boxed{x(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + A\frac{\sin 3t}{3}} \rightarrow (4)$$

By
$L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$
$L^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$

Put $t = \frac{\pi}{2}$ in (4), $-1 = -\frac{1}{5} + 0 + \frac{A}{3}\cdot(-1) \Rightarrow \boxed{A = \frac{12}{5}}$

Hence

$$\boxed{x(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t}$$

Ex-6) Solve by Laplace Transform: (AKTU-2008, 2016)

$\frac{d^2y}{dt^2} + y = t\cos 2t$, $t>0$ given that $y = \frac{dy}{dt} = 0$ for $t=0$.

Sol^n Given $y'' + y = t\cos 2t \rightarrow ①$ & $y(0)=0,\ y'(0)=0] \rightarrow ②$

Then $L(y'') + L(y) = L(t\cos 2t)$

$\Rightarrow [p^2L(y) - p\,y(0) - y'(0)] + L(y) = (-1)^1 \frac{d}{dp} L(\cos 2t)$

$\Rightarrow p^2L(y) + L(y) = (-1)\frac{d}{dp}\left(\frac{p}{p^2+4}\right)$

by $L\{t\,F(t)\} = (-1)^1 \frac{d}{dp} L\{F(t)\}$.

$\Rightarrow (p^2+1)L(y) = (-1)\left[\frac{(p^2+4)\cdot 1 - p\cdot 2p}{(p^2+4)^2}\right]$

$\Rightarrow (p^2+1)L(y) = \frac{(-1)(4-p^2)}{(p^2+4)^2}$

$\Rightarrow (p^2+1)L(y) = \frac{p^2-4}{(p^2+4)^2}$

$\Rightarrow L(y) = \frac{(p^2-4)}{(p^2+1)(p^2+4)^2} \longrightarrow ①$

Now $\frac{p^2-4}{(p^2+1)(p^2+4)^2}$ (Put $p^2=m$)

$\Rightarrow \frac{m-4}{(m+1)(m+4)^2} = \frac{A}{(m+1)} + \frac{B}{(m+4)} + \frac{C}{(m+4)^2} \rightarrow ②$

$\therefore \frac{m-4}{(m+1)(m+4)^2} = \frac{-5/9}{m+1} + \frac{B}{m+4} + \frac{8/3}{(m+4)^2} \rightarrow ③$

Put $m=0$ in ③,

$\frac{-4}{16} = -\frac{5}{9} + \frac{B}{4} + \frac{8}{3\times 16}$

$\Rightarrow -\frac{1}{4} + \frac{5}{9} - \frac{1}{6} = \frac{B}{4} \Rightarrow \boxed{B = \frac{5}{9}}$

Hence ② gives,

$L(y) = \frac{p^2-4}{(p^2+1)(p^2+4)^2} = \frac{-5/9}{p^2+1} + \frac{5/9}{p^2+4} + \frac{8/3}{(p^2+4)^2}$.

$\Rightarrow y = -\frac{5}{9}L^{-1}\left\{\frac{1}{p^2+1}\right\} + \frac{5}{9}L^{-1}\left\{\frac{1}{p^2+4}\right\} + \frac{8}{3}L^{-1}\left\{\frac{1}{(p^2+4)^2}\right\}$.

$\Rightarrow y = -\frac{5}{9}\sin t + \frac{5}{9}\cdot\frac{\sin 2t}{2} + \frac{8}{3}\cdot\frac{1}{16}(\sin 2t - 2t\cos 2t)$

[By C.T. $L^{-1}\left\{\frac{1}{(p^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)$]

Q-7) Determine the response of damped mass-spring system under the ~~ea~~ square wave given by (AKTU - 2007, 2013, 2017)

$$y'' + 3y' + 2y = U(t-1) - U(t-2), \quad y(0)=0, \; y'(0)=0.$$

Solⁿ Given

$$y'' + 3y' + 2y = U(t-1) - U(t-2) \rightarrow ① \quad \& \quad \left.\begin{matrix} y(0)=0 \\ y'(0)=0 \end{matrix}\right\} \rightarrow ②$$

Taking Laplace both side of eqⁿ ①,

$$L(y'') + 3L(y') + 2L(y) = L\{U(t-1)\} - L\{U(t-2)\}$$

$$\Rightarrow [p^2 L(y) - p y(0) - y'(0)] + 3[pL(y) - y(0)] + 2L(y) = \frac{e^{-p}}{p} - \frac{e^{-2p}}{p}$$

$$\Rightarrow (p^2+3p+2)L(y) = \frac{e^{-p}}{p} - \frac{e^{-2p}}{p} \qquad \text{By } L\{U(t-a)\} = \frac{e^{-ap}}{p}$$

$$\Rightarrow L(y) = \frac{e^{-p}}{p(p^2+3p+2)} - \frac{e^{-2p}}{p(p^2+3p+2)} \longrightarrow \cancel{③}$$

$$\Rightarrow y = L^{-1}\left\{e^{-p}\frac{1}{p(p^2+3p+2)}\right\} - L^{-1}\left\{e^{-2p}\frac{1}{p(p^2+3p+2)}\right\} \longrightarrow ③$$

Let

$$f(p) = \frac{1}{p(p^2+3p+2)}$$

$$\Rightarrow f(p) = \frac{1}{p(p+1)(p+2)} = \frac{1/2}{p} - \frac{1}{p+1} + \frac{1/2}{p+2}$$

(by Partial fraction)

Then

$$L^{-1}\{f(p)\} = \frac{1}{2}L^{-1}\left\{\frac{1}{p}\right\} - L^{-1}\left\{\frac{1}{p+1}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{p+2}\right\}$$

$$= \frac{1}{2}\cdot 1 - e^{-t} + \frac{1}{2}e^{-2t} = F(t) \text{ (say)} \longrightarrow ④$$

Now from ③,

$$y = L^{-1}\{e^{-p}f(p)\} - L^{-1}\{e^{-2p}f(p)\}$$

$$= F(t-1)\,U(t-1) - F(t-2)\,U(t-2)$$

By $L^{-1}\{e^{-ap}f(p)\} = F(t-a)\,U(t-a)$

$$y = \left[\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right]U(t-1) - \left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right]U(t-2)$$

(From ④)

Ex-8: Solve by Laplace Transform

$$\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t \quad , \; y(0) = 1.$$

(AKTU-2015, 2017)

Solⁿ: Given $y' + 2y + \int_0^t y\,dt = \sin t \to (1)$ $\quad y(0) = 1 \to (2)$

Then $L(y') + 2L(y) + L\left\{\int_0^t y\,dt\right\} = L\{\sin t\}$

By $L\int_0^t F(t)\,dt = \frac{f(p)}{p} = \frac{L\{F(t)\}}{p}$

$$\Rightarrow [pL(y) - y(0)] + 2L(y) + \frac{L(y)}{p} = \frac{1}{p^2+1}$$

$$\Rightarrow \left[p + 2 + \frac{1}{p}\right] L(y) - 1 = \frac{1}{p^2+1}$$

$$\Rightarrow \frac{(p^2+2p+1)}{p} L(y) = \frac{1}{p^2+1} + 1 = \frac{p^2+2}{p^2+1}$$

$$\Rightarrow L(y) = \frac{p(p^2+2)}{(p^2+1)(p^2+2p+1)}$$

$$\Rightarrow L(y) = \frac{(p^3+2p)}{(p+1)^2(p^2+1)} = \frac{A}{p+1} + \frac{B}{(p+1)^2} + \frac{Cp+D}{p^2+1}$$

$$\Rightarrow L(y) = \frac{1}{p+1} - \frac{3}{2}\frac{1}{(p+1)^2} + \frac{1}{2}\frac{1}{p^2+1} \longrightarrow (3)$$

by Partial Fraction

$$\Rightarrow y = L^{-1}\left\{\frac{1}{p+1}\right\} - \frac{3}{2}L^{-1}\left\{\frac{1}{(p+1)^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{p^2+1}\right\}.$$

$$\Rightarrow y = e^{-t} - \frac{3}{2}e^{-t} L^{-1}\frac{1}{p^2} + \frac{1}{2}\sin t.$$

$$\Rightarrow \boxed{y = e^{-t} - \frac{3}{2}e^{-t}\, t + \frac{1}{2}\sin t}$$

Home Assignment

Ex-1 → Solve by using Laplace Transform

$y''(t) + 4y'(t) + 4y(t) = 6e^{-t}$; $y(0) = -2$, $y'(0) = 8$.

Ans $\boxed{y = 6e^{-t} - 8e^{-2t} - 2t\, e^{-2t}}$

Ex-2 → $y''' - 2y'' + 5y' = 0$, $y(0)=0$, $y'=1$ at $t=0$ and $y=1$ at $t=\frac{\pi}{8}$.

Ex-3 → $(D^2 + n^2)x = a \sin(nt + \alpha)$, $x = Dx = 0$ at $t=0$.

Ex-4 → $y''' + 2y'' - y' - 2y = 0$ where $y=1$, $\frac{dy}{dt} = 2$, $\frac{d^2y}{dt^2} = 2$ at $t=0$

Hint

Ans $L(y) = \frac{5}{3}\frac{1}{p-1} - \frac{1}{p+1} + \frac{1}{3(p+2)}$.

& $\boxed{y = \frac{5}{3}e^{t} - e^{-t} + \frac{1}{3}e^{-2t}}$

Ex-5 → $(D^3 - D^2 - D + 1)y = 8t\,e^{-t}$, $y(0)=0$, $y'(0)=1$, $y''(0)=0$.

Ans $\boxed{y = e^{-t}(1 + 2t + t^2) - e^{t}(1-t)}$ (AKTU-2014)

Ex-6 → $\frac{d^2x}{dt^2} + 16x = 2\sin 4t$; $x(0) = -\frac{1}{2}$, $x'(0) = 0$. (AKTU-2007, 2014)

Ans - $\boxed{x = -\left(\frac{1}{2} + \frac{t}{4}\right)\cos 4t + \frac{1}{16}\sin 4t}$

Ex-7 → $y'' + 2y' + y = t\,e^{-t}$; $y(0) = 1$, $y'(0) = -2$. (AKTU-2015)

Ans: $\boxed{y = \left(1 - t + \frac{t^3}{6}\right)e^{-t}}$

Ex-8 → $y'' + 9y = \sin 3t$, $y=0$, $\frac{dy}{dt} = 0$ at $t=0$ (AKTU-2012)

$\boxed{y = \frac{1}{18}(\sin 3t - 3t\cos 3t)}$

Ex-9 → $y'' + y = \sin t \cdot \sin 2t$; $y(0) = 1$, $y'(0) = 0$.

Ans $\boxed{y = \frac{15}{16}\cos t + \frac{t}{4}\sin t + \frac{1}{16}\cos 3t}$

Ex-10 → $y'' - 3y' + 2y = 4t + e^{3t}$, $y(0)=1$, $y'(0)=-1$.

Ans → $\boxed{y = 3 + 2t - \frac{1}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}}$

By: Dr. Anuj Kumar

Solution of Simultaneous Diff. eq^n by Laplace Transform →

① Take Laplace Transform both side of the give D. Eq^{n}s.
and use

$L(x') = pL(x) - x(0)$ | $L(y') = pL(y) - y(0)$
$L(x'') = p^2L(x) - px(0) - x'(0)$ | $L(y'') = p^2L(y) - py(0) - y'(0)$
etc. | etc.

② Put $L(x) = \bar{x}$ & $L(y) = \bar{y}$
and Solve both eq^n for $\bar{x}$ & $\bar{y}$.

③ Taking Laplace inverse on both side of $\bar{x}$ & $\bar{y}$.
Find the value of x and y.

Ex-1: The coordinates (x, y) of a particle moving along a plane curve at any time t are given by $\frac{dy}{dt} + 2x = \sin 2t$, $\frac{dx}{dt} - 2y = \cos 2t$ $(t > 0)$. It is given that at $t = 0$, $x = 1$ and $y = 0$. Show that using transforms that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.
(AKTU - 2017, 2012)

Sol^n: Given $y' + 2x = \sin 2t \rightarrow ①$ & $x' - 2y = \cos 2t \rightarrow ②$

Taking Laplace on both side of eq^n ① & ②, we get

$L(y') + 2L(x) = L\{\sin 2t\}$, $L(x') - 2L(y) = L\{\cos 2t\}$

$\Rightarrow pL(y) - y(0) + 2L(x) = \frac{2}{p^2+4}$, $pL(x) - x(0) - 2L(y) = \frac{p}{p^2+4}$

$\Rightarrow p\bar{y} - 0 + 2\bar{x} = \frac{2}{p^2+4}$, $p\bar{x} - 1 - 2\bar{y} = \frac{p}{p^2+4}$

where $\bar{y} = L(y)$
$\bar{x} = L(x)$

$$\Rightarrow \quad 2\bar{x} + p\bar{y} = \frac{2}{p^2+4} \rightarrow ③$$
$$p\bar{x} - 2\bar{y} = \frac{p}{p^2+4} + 1 \rightarrow ④$$

Multiplying eq^n ③ by p & ④ by 2, we get

$$\cancel{2p\bar{x}} + p^2\bar{y} = \frac{2p}{p^2+4}$$
$$\cancel{2p\bar{x}} - 4\bar{y} = \frac{2p}{p^2+4} + 2$$
$$- \qquad + \qquad - \qquad -$$

$$(p^2+4)\bar{y} = -2$$
$$\Rightarrow \bar{y} = \frac{-2}{p^2+4} \Rightarrow L(y) = -\frac{2}{p^2+4}$$
$$\Rightarrow y = -L^{-1}\left\{\frac{2}{p^2+4}\right\}$$
$$\Rightarrow \boxed{y = -\sin 2t} \longrightarrow (5)$$

Now Multiplying eq^n (3) by 2 & (4) by p, we get

$$4\bar{x} + \cancel{2p\bar{y}} = \frac{4}{p^2+4}$$
$$p^2\bar{x} - \cancel{2p\bar{y}} = \frac{p^2}{p^2+4} + p$$

$$(p^2+4)\bar{x} = \frac{4}{p^2+4} + \frac{p^2}{p^2+4} + p$$
$$\Rightarrow (p^2+4)\bar{x} = \frac{\cancel{4+p^2}}{\cancel{p^2+4}} + p$$
$$\Rightarrow (p^2+4)\bar{x} = 1 + p$$
$$\Rightarrow (p^2+4)L(x) = 1 + p$$
$$\Rightarrow L(x) = \frac{1+p}{p^2+4}$$
$$\Rightarrow L(x) = \frac{1}{p^2+4} + \frac{p}{p^2+4}$$
$$\Rightarrow x = L^{-1}\left\{\frac{1}{p^2+4}\right\} + L^{-1}\left\{\frac{p}{p^2+4}\right\}.$$
$$\Rightarrow \boxed{x = \frac{\sin 2t}{2} + \cos 2t} \longrightarrow (6)$$

Eq^n (5) & (6) are sol^n of (1) & (2).

Now $4x^2 + 5y^2 + 4xy = 4\left[\frac{\sin 2t}{2} + \cos 2t\right]^2 + 5[-\sin 2t]^2 + 4\left[\frac{\sin 2t}{2} + \cos 2t\right] \times (-\sin 2t)$

$$= 4\left[\frac{\sin^2 2t}{4} + \cos^2 2t + \sin 2t\cos 2t\right] + 5\sin^2 2t + 4\left[-\frac{\sin^2 2t}{2} - \sin 2t\cos 2t\right]$$

$$= 4(\sin^2 2t + \cos^2 2t) = 4.$$

$$\Rightarrow \boxed{4x^2 + 5y^2 + 4xy = 4}$$

Ex (2) Use Laplace Transform to solve:

$\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$ given that $x = 2$, $y = 0$ at $t = 0$. (AKTU - 2012, 2014)

Sol^n: Given $x' + y = \sin t \rightarrow (1)$ $y' + x = \cos t \rightarrow (2)$

Also $\left.\begin{matrix} x(0) = 2 \\ y(0) = 0 \end{matrix}\right\} \rightarrow (3)$

Taking Laplace of (1) & (2) both side,

$L(x') + L(y) = L\{\sin t\}$, $L(y') + L(x) = L(\cos t)$

$\Rightarrow p L(x) - x(0) + L(y) = \frac{1}{p^2+1}$, $p L(y) - y(0) + L(x) = \frac{p}{p^2+1}$

$\Rightarrow p\bar{x} - 2 + \bar{y} = \frac{1}{p^2+1}$, $p\bar{y} - 0 + \bar{x} = \frac{p}{p^2+1}$

$\Rightarrow p\bar{x} + \bar{y} = \frac{1}{p^2+1} + 2 \longrightarrow (4)$

$\bar{x} + p\bar{y} = \frac{p}{p^2+1} \longrightarrow (5)$

Multiplying eq^n (5) by p and subtracting from (4), we get

$$\cancel{p\bar{x}} + \bar{y} = \frac{1}{p^2+1} + 2$$
$$\underline{-\,\cancel{p\bar{x}} - p^2\bar{y} = -\frac{p^2}{p^2+1}}$$

$(1-p^2)\bar{y} = \frac{1}{p^2+1} - \frac{p^2}{p^2+1} + 2$

$\Rightarrow (1-p^2)\bar{y} = \frac{(1-p^2)}{p^2+1} + 2$

$\Rightarrow \bar{y} = \frac{1}{p^2+1} - \frac{2}{p^2-1}$

$\Rightarrow L(y) = \frac{1}{p^2+1} - 2\,\frac{1}{p^2-1}$

$\Rightarrow y = L^{-1}\left\{\frac{1}{p^2+1}\right\} - 2L^{-1}\left\{\frac{1}{p^2-1}\right\}$.

$\Rightarrow \boxed{y = \sin t - 2\sinh t} \longrightarrow (6)$

By $L^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$ & $L^{-1}\left\{\frac{a}{p^2-a^2}\right\} = \sinh at$

Now Multiplying eq^n (4) by p and take (4) - (5)

$$p^2\cancel{\bar{x}} + p\cancel{\bar{y}} = \frac{p}{p^2+1} + 2p$$
$$\underline{-\,\cancel{\bar{x}} - \cancel{p\bar{y}} = -\frac{p}{p^2+1}}$$

$(p^2-1)\bar{x} = 2p \Rightarrow L(x) = \bar{x} = \frac{2p}{p^2-1}$

$\Rightarrow x = 2L^{-1}\frac{p}{p^2-1} = 2\cosh t$

By $L^{-1}\left\{\frac{p}{p^2-a^2}\right\} = \cosh at$

$\Rightarrow \boxed{x = 2\cosh t} \rightarrow (7)$

Eq^n (6) & (7) are both sol^n of (1) & (2).

Ex-3: Solve the simultaneous equations: (AKTU-2010, 2011, 2015)

$\frac{dx}{dt} - y = e^t$, $\frac{dy}{dt} + x = \sin t$, given $x(0)=1$, $y(0)=0$.

Solⁿ: Given $x' - y = e^t \rightarrow (1)$ $y' + x = \sin t \rightarrow (2)$ & $\left.\begin{matrix} x(0)=1 \\ y(0)=0 \end{matrix}\right\} \rightarrow (3)$

Taking Laplace of eqⁿ (1) & (2) both,

$L(x') - L(y) = L(e^t)$, $L(y') + L(x) = L(\sin t)$

$\Rightarrow pL(x) - x(0) - L(y) = \frac{1}{p-1}$, $pL(y) - y(0) + L(x) = \frac{1}{p^2+1}$

$\Rightarrow p\bar{x} - \bar{y} = \frac{1}{p-1} + 1$, $p\bar{y} + \bar{x} = \frac{1}{p^2+1}$

$\Rightarrow p\bar{x} - \bar{y} = \frac{p}{p-1} \longrightarrow (4)$

$\bar{x} + p\bar{y} = \frac{1}{p^2+1} \longrightarrow (5)$

Multiplying eqⁿ (4) by p and adding with (5),

$p^2\bar{x} - \cancel{p\bar{y}} = \frac{p^2}{p-1}$

$\bar{x} + \cancel{p\bar{y}} = \frac{1}{p^2+1}$

$(p^2+1)\bar{x} = \frac{p^2}{(p-1)} + \frac{1}{(p^2+1)}$

$\Rightarrow \bar{x} = \frac{p^2}{(p-1)(p^2+1)} + \frac{1}{(p^2+1)^2}$

$\Rightarrow L(x) = \frac{1}{2}\left[\frac{1}{p-1} + \frac{1}{p^2+1} + \frac{p}{p^2+1}\right] + \frac{1}{(p^2+1)^2}$ (By Partial fraction)

$\Rightarrow x = \frac{1}{2}\left[L^{-1}\frac{1}{p-1} + L^{-1}\frac{1}{p^2+1} + L^{-1}\frac{p}{p^2+1}\right] + L^{-1}\frac{1}{(p^2+1)^2}$

$\Rightarrow x = \frac{1}{2}\left[e^t + \sin t + \cos t\right] + \frac{1}{2}(\sin t - t\cos t)$

$\Rightarrow \boxed{x = \frac{1}{2}(e^t + 2\sin t + \cos t - t\cos t)} \longrightarrow (6)$ By $\left[L^{-1}\left\{\frac{1}{(p^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)\right)$

Now multiplying eqⁿ (5) by p and subtracting from (4),

$\cancel{p\bar{x}} - \bar{y} = \frac{p}{p-1}$

$\cancel{p\bar{x}} + p^2\bar{y} = \frac{p}{p^2+1}$

$- \quad - \quad -$

$-(1+p^2)\bar{y} = \frac{p}{p-1} - \frac{p}{p^2+1}$

$\Rightarrow L(y) = \frac{p}{(p^2+1)^2} - \frac{p}{(p-1)(p^2+1)}$

$\Rightarrow L(y) = \frac{p}{(p^2+1)^2} - \frac{1}{2}\left[\frac{1}{p-1} + \frac{1}{p^2+1} - \frac{p}{p^2+1}\right]$ (by Partial fraction)

$\Rightarrow y = L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} - \frac{1}{2}\left[L^{-1}\frac{1}{p-1} + L^{-1}\frac{1}{p^2+1} - L^{-1}\frac{p}{p^2+1}\right]$

$\Rightarrow \boxed{y = \frac{1}{2}t\sin t - \frac{1}{2}\left[e^t - \cos t + \sin t\right]} \longrightarrow (7)$ $\left\{\text{By } L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = \frac{1}{2a}t\sin at\right\}$

Ex-41 Solve the simultaneous diff. eq^n

$$\frac{d^2x}{dt^2} + 5\frac{dy}{dt} - x = t \quad , \quad 2\frac{dx}{dt} - \frac{d^2y}{dt^2} + 4y = 2$$

given that when $t=0$, $x=0$, $y=0$, $\frac{dx}{dt}=0$, $\frac{dy}{dt}=0$. (AKTU-2011)

Sol^n Given $x'' + 5y' - x = t \rightarrow ①$ $\quad -y'' + 2x' + 4y = 2 \rightarrow ②$

& $\left.\begin{array}{l} x(0) = x'(0) = 0 \\ y(0) = y'(0) = 0 \end{array}\right\} \rightarrow ③$

Taking Laplace of eq^n ① & ② both side, we get

$$L(x'') + 5L(y') - L(x) = L(t) \quad , \quad -L(y'') + 2L(x') + 4L(y) = 2L(1)$$

$$\Rightarrow [p^2 L(x) - px(0) - x'(0)] + 5[pL(y) - y(0)] - L(x) = \frac{1}{p^2}$$

$$\& \; -[p^2 L(y) - py(0) - y'(0)] + 2[pL(x) - x(0)] + 4L(y) = \frac{2}{p}.$$

$$\Rightarrow p^2\bar{x} + 5p\bar{y} - \bar{x} = \frac{1}{p^2} \quad , \quad -p^2\bar{y} + 2p\bar{x} + 4\bar{y} = \frac{2}{p}.$$

$$\Rightarrow (p^2-1)\bar{x} + 5p\bar{y} = \frac{1}{p^2} \longrightarrow ④$$

$$2p\bar{x} + (4-p^2)\bar{y} = \frac{2}{p} \longrightarrow ⑤$$

Multiplying eq^n ④ by $2p$ & ⑤ by (p^2-1),

$$\cancel{2p(p^2-1)\bar{x}} + 10p^2\bar{y} = \frac{2}{p}$$

$$\cancel{2p(p^2-1)\bar{x}} + (p^2-1)(4-p^2)\bar{y} = \frac{2(p^2-1)}{p}$$

$$-\qquad\qquad -\qquad\qquad -$$

$$[10p^2 + (p^2-1)(p^2-4)]\bar{y} = \frac{2}{p} - 2p + \frac{2}{p}.$$

$$\Rightarrow [10p^2 + p^4 - 5p^2 + 4]\bar{y} = \frac{4-2p^2}{p}$$

$$\Rightarrow (p^4 + 5p^2 + 4)\bar{y} = \frac{4-2p^2}{p}$$

$$\Rightarrow L(y) = \frac{4-2p^2}{p(p^2+1)(p^2+4)}$$

$$\Rightarrow L(y) = \frac{1}{p} - \frac{2p}{p^2+1} + \frac{p}{p^2+4} \quad \text{(by Partial fraction)}$$

$$\Rightarrow y = L^{-1}\left(\frac{1}{p}\right) - 2L^{-1}\left(\frac{p}{p^2+1}\right) + L^{-1}\left(\frac{p}{p^2+4}\right)$$

$$\Rightarrow \boxed{y = 1 - 2\cos t + \cos 2t} \longrightarrow ⑥$$

Now eliminating $\bar{y}$ from ④ & ⑤, we get

$$\bar{x} = L(x) = \frac{11p^2-4}{p^2(p^2+1)(p^2+4)} = -\frac{1}{p^2} + \frac{5}{p^2+1} - \frac{4}{p^2+4}$$

$$\Rightarrow x = -L^{-1}\frac{1}{p^2} + 5L^{-1}\frac{1}{p^2+1} - 4L^{-1}\frac{1}{p^2+4} \quad \text{(By Partial fr}^n\text{)}$$

$$\Rightarrow \boxed{x = -t + 5\sin t - 2\sin 2t} \rightarrow ⑦$$

Eq^n ⑥ & ⑦ are both sol^n of ① & ②.

H.W

① Solve by Laplace Transform

$\frac{dx}{dt} + 4\frac{dy}{dt} - y = 0$, $\frac{dx}{dt} + 2y = e^{-t}$ with $x(0) = y(0) = 0$.

Ans $\boxed{x = \frac{1}{3} - \frac{5}{7}e^{-t} + \frac{8}{21}e^{3t/4}, \; y = \frac{1}{7}(e^{-t} - e^{\frac{3t}{4}})}$

② Solve by Laplace Transform,

$3\frac{dx}{dt} - y = 2t$, $\frac{dx}{dt} + \frac{dy}{dt} - y = 0$, with $x(0) = y(0) = 0$.

Ans $\boxed{x = \frac{t^2}{2} + \frac{t}{2} - \frac{3}{4}e^{2t/3} + \frac{3}{4}, \; y = t + \frac{3}{2} - \frac{3}{2}e^{2t/3}}$

③ Solve by L.T

$\frac{dx}{dt} + y = 0$, $\frac{dy}{dt} - x = 0$, $x(0) = 1$, $y(0) = 0$

Ans $\boxed{x = \cos t, \; y = \sin t}$

④ Solve by L.T

$\frac{dx}{dt} + \frac{dy}{dt} + x + y = 1$, $\frac{dy}{dt} = 2x + y$, $x(0) = 0, y(0) = 1$

(AKTU-2011)

Ans $\boxed{x = e^{-t} - 1, \; y = 2 - e^{-t}}$

⑤ Solve by L.T

$D^2x + y = -5\cos 2t$, $D^2y + x = 5\cos 2t$

$x(0) = x'(0) = y'(0) = 1$ & $y(0) = -1$.

Ans $\boxed{x = \sin t + \cos 2t, \; y = \sin t - \cos 2t}$.

⑥ Solve by L.T

$2x' + y' - x - y = e^{-t}$, $x' + y' + 2x + y = e^{t}$, $y(0) = 1$, $x(0) = 2$

Ans $\boxed{x = 2\cos t + 8\sin t, \; y = \cos t - 13\sin t + \sinh t}$