Engineering Mathematics - II

*** Lecture No-01 ***

By Dr. ANUJ KUMAR

Differential Equation →

The eq^n which express the relationship between dependent variables, independent variables and the derivatives of dependent variables w.r.t independent variables is called differential equation.

Mainly two types of differential eq^n are used :-

① Ordinary differential Eq^n (ODE) →

A differential eq^n contains differentials w.r.t only single independent variable is called ODE.

Ex→ (i) $\frac{dy}{dx} = \frac{1+x^2}{1+y^2}$ (ii) $\frac{dy}{dx} + \sin x \cdot y = e^x$.

② Partial differential eq^n (PDE) →

A differential eq^n which involves partial derivatives is called PDE.

Ex→ (i) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3x$ (ii) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

Order of differential Eq^n →

The order of the highest order derivative occurs in a differential eq^n is called the order of the differential eq^n.

Degree of differential Eq^n →

The degree of the differential eq^n is the highest power of the highest derivative which occurs in it, after the differential eq^n has been made free from radicals and fractions.

Note→ Radical & fractions ⇉
① The highest derivative should be free from sin, cos, tan, log etc
② The power of highest derivative should be an +ve integer.

Ex-1→ $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^x \rightarrow$ $\boxed{\text{order} = 2,\ \text{degree} = 1}$

Ex-2→ $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \rightarrow$ $\boxed{\text{order} = 2,\ \text{degree} = 2}$

Ex-3→ $\sin^{-1}\left(\frac{d^3y}{dx^3}\right) = x + y$

$\Rightarrow \frac{d^3y}{dx^3} = \sin(x+y) \rightarrow$ $\boxed{\text{order} = 3,\ \text{degree} = 1}$

Ex-4→ $\log\left(1 + \frac{dy}{dx}\right) = c \rightarrow$ $\boxed{\text{order} = 1,\ \text{degree does not excist}}$

Ex-5→ $e^{y''} + xy = 0 \rightarrow$ $\boxed{\text{order} = 2,\ \text{degree does not excist}}$

$\Rightarrow \left[1 + \frac{y''}{\lfloor 1} + \frac{(y'')^2}{\lfloor 2} + \frac{(y'')^3}{\lfloor 3} + \cdots\right] + xy = 0.$

Linear differential Eqⁿ→

A differential eeⁿ is called linear if

(i) dependent variables and its derivatives having one power. (i.e) (degree 1).

(ii) no product of dependent variable & derivatives occurs.

Otherwise it is called non. linear differential Eqⁿ.

Ex→ (i) $\frac{dy}{dx} + y = x$

(ii) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = e^x$ $\Bigg\}\rightarrow$ Linear

Ex→ i) $\left(\frac{dy}{dx}\right)^2 + y = x.$

(ii) $\frac{dy}{dx} = \sqrt{1 + \frac{d^2y}{dx^2}}$

(iii) $y\frac{dy}{dx} + x = 0$ $\Bigg\}\rightarrow$ Non-linear

Linear differential E_q^n of first order & first degree →

Type→1→ $\frac{dy}{dx} + Py = Q$ where P & Q are func^n of x.

(Linear in y)

Sol^n→ (i) $I.F = e^{\int P dx}$

↓

Integrating factor

(ii) Complete Sol^n,

$$y \times I.F = \int (Q \times I.F)\,dx + C$$

Ex→1→ Solve $\frac{dy}{dx} + \frac{y}{x} = x^2$.

Sol^n→ Given $\frac{dy}{dx} + \frac{1}{x}y = x^2$.

Here $P = \frac{1}{x}$, $Q = x^2$.

Now $I.F = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$.

Complete Sol^n,

$$y \times I.F = \int (I.F \times Q)\,dx + C$$

$$\Rightarrow y \cdot x = \int x \cdot x^2 dx + C$$

$$\Rightarrow xy = \int x^3 dx + C \Rightarrow \boxed{xy = \frac{x^4}{4} + C}$$

Type-II→ $\frac{dx}{dy} + P_1 x = Q_1$ where P_1 & Q_1 are func^n of y.

(Linear in x)

Sol^n→ (i) $I.F = e^{\int P_1 dy}$

(ii) Complete Sol^n

$$x \times I.F = \int (I.F \times Q_1)\,dy + C.$$

Ex→ Solve $(1+y^2)\,dx = (\tan^{-1} y - x)\,dy$.

Solution of D.E →

A solution or integral of a D.E is the relation b/w dependent variable & independent variables, which satisfies the given differential eqn.

Ex → $y = C_1x + C_2$ is the solution of the differential eqn

$$\frac{d^2y}{dx^2} = 0.$$

Note ÷ $\frac{d^2y}{dx^2} = 0$

on integrating, $\frac{dy}{dx} = C_1 \Rightarrow dy = C_1 dx$

Again integrating, $\boxed{y = C_1x + C_2}$

General Soln → The general soln of the D.E is the solution in which the number of arbitrary constants is equal to the order of given D.E.

Ex → $y = C_1x + C_2$ is the general soln of the D.E $\frac{d^2y}{dx^2} = 0$

Particular Soln → Particular soln of the differential eqn is the solution which is obtained from its general soln by giving particular values to arbitrary constants.

A particular soln of the differential eqn is the solution which is obtained from its general soln by giving particular values to arbitrary constants.

Ex → Taking $C_1 = 1$, $C_2 = 0$.

we get $y = x$ is the Particular soln of the D.E $\frac{d^2y}{dx^2} = 0$.

ENGINEERING MATHEMATICS-II

Module-I

ODE

* Lecture No- 02 *

By Dr. ANUJ KUMAR

Linear differential eq^n with constant Coefficients → (L.D.E)

A D. Eq^n of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots \cdots + a_n y = Q(x) \longrightarrow (1)$$

or $$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n) y = Q(x)$$

or $$f(D) y = Q(x)$$

is called Linear D. Eq^n with constant coefficients, in which $a_1, a_2, \cdots, a_n$ are constants & $Q(x)$ is a func^n of x.

General Sol^n of L.D.E →

General Sol^n

↓

$$y = C.F + P.I$$

Complementary func^n ↓ C.F involve n arabitrary constants & can be obtained from L.H.S of D. Eq^n.

Particular Integral or Sol^n ↓ P.I does not involve any arbitrary constants & can be obtained from R.H.S of D. Eq^n.

Rules to find C.F → (i) Convert the given D. Eq^n in the form

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n) y = Q(x) \quad \text{or} \quad f(D) y = Q(x)$$

(ii) Equating to zero the coefficient of y, make Auxilary Equation (A.E) i.e

$$D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n = 0 \quad \text{or} \quad f(D) = 0.$$

(iii) find the roots of Auxilary E_q^n or (A·E).

(iv) Use the given table for C.F (according to roots of A·E).

S.No	Nature of Roots	C.F
1	(i) One real root m_1	$C_1 e^{m_1 x}$
	(ii) Two real and different roots m_1, m_2	$C_1 e^{m_1 x} + C_2 e^{m_2 x}$
	(iii) Three real & different roots m_1, m_2, m_3	$C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$.
2-	(i) Two real & equal roots m_1, m_1	$(C_1 + C_2 x) e^{m_1 x}$
	(ii) Three real & equal roots m_1, m_1, m_1	$(C_1 + C_2 x + C_3 x^2) e^{m_1 x}$.
3-	(i) One pair of complex roots $\alpha \pm i\beta$	$e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$
	(ii) Two pairs of complex & equal roots $\alpha \pm i\beta$, $\alpha \pm i\beta$	$e^{\alpha x}[(C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x]$
4-	(i) One pair of surd roots $\alpha \pm \sqrt{\beta}$	$e^{\alpha x}[C_1 \cosh\sqrt{\beta}x + C_2 \sinh\sqrt{\beta}x]$ or $C_1 e^{(\alpha+\sqrt{\beta})x} + C_2 e^{(\alpha-\sqrt{\beta})x}$
	(ii) Two pairs of surd & equal roots $\alpha \pm \sqrt{\beta}$, $\alpha \pm \sqrt{\beta}$	$e^{\alpha x}[(C_1 + C_2 x)\cosh\sqrt{\beta}x + (C_3 + C_4 x)\sinh\sqrt{\beta}x]$

Ex-1- Solve $\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2} + 4y = 0$.

Sol^n- Given $\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2} + 4y = 0$

$\Rightarrow (D^4y - 5D^2y + 4y) = 0 \Rightarrow (D^4 - 5D^2 + 4)y = 0.$

Here A.E is

$D^4 - 5D^2 + 4 = 0 \Rightarrow D^4 - 4D^2 - D^2 + 4 = 0$

$\Rightarrow D^2(D^2-4) - 1(D^2-4) = 0$

$\Rightarrow (D^2-4)(D^2-1) = 0$

$\Rightarrow D^2-1=0 \quad , \quad D^2-4=0$

$\Rightarrow D = \pm 1 \quad , \quad D = \pm 2.$

$\therefore D = -1, 1, -2, 2$

$\therefore C.F = C_1 e^{-x} + C_2 e^{x} + C_3 e^{-2x} + C_4 e^{2x}$. & $P.I = 0$.

Hence solⁿ is

$$\boxed{y = C.F + P.I = C_1 e^{-x} + C_2 e^{x} + C_3 e^{-2x} + C_4 e^{2x}}$$

Ex-2: Solve the D. Eqⁿ

(i) $\frac{d^2y}{dx^2} - y = 0$ (ii) $(D^3 + 6D^2 + 11D + 6)y = 0$

(iii) $(D^2 - 3D + 2)y = 0$.

Ex-3: Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$.

Solⁿ: Given $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0 \Rightarrow (D^2 - 2D + 1)y = 0$.

Here A.E is $D^2 - 2D + 1 = 0 \Rightarrow (D-1)^2 = 0 \Rightarrow D = 1, 1$.

$\therefore C.F = (C_1 + C_2 x) e^{1.x}$. & $P.I = 0$.

Hence solⁿ is

$$\boxed{y = C.F + P.I = (C_1 + C_2 x) e^{x}}$$

Ex-4: Solve $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$.

Solⁿ: given $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$.

Here A.E is $D^4 - 6D^3 + 12D^2 - 8D = 0$

$\Rightarrow D(D^3 - 6D^2 + 12D - 8) = 0 \Rightarrow D(D-2)^3 = 0$

$\Rightarrow D = 0, \ D = 2, 2, 2.$

$\therefore\ y = C.F = C_1 e^{0x} + (C_2 + C_3 x + C_4 x^2)\, e^{2x}$

is the required solⁿ.

Ex-5 → Solve

(i) $(D^3 + 3D^2 + 3D + 1)y = 0$
or $(D+1)^3 y = 0$

(ii) $(D^4 + 2D^3 - 3D^2 - 4D + 4)y = 0$
Hint $(D = 1; 1, -2, -2)$

Ex-6 → Solve $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0$.

Solⁿ → Given $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0$

$\Rightarrow D^2 y + 2Dy + 5y = 0 \Rightarrow (D^2 + 2D + 5)y = 0.$

Here A.E is $D^2 + 2D + 5 = 0.$

$$\therefore\ D = \frac{-2 \pm \sqrt{(2)^2 - 4.1.5}}{2.1} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm i4}{2}$$

$\Rightarrow D = -1 \pm i2$

$\therefore\ y = C.F = e^{-1x}[C_1 \cos 2x + C_2 \sin 2x]$.

Ex-7 → Solve $\dfrac{d^4y}{dx^4} + m^4 y = 0$. (AKTU-2008)

Solⁿ → Given $\dfrac{d^4y}{dx^4} + m^4 y = 0 \Rightarrow (D^4 y + m^4 y) = 0$

$\Rightarrow (D^4 + m^4)y = 0$.

Here A.E is $D^4 + m^4 = 0.$

$\Rightarrow (D^4 + 2D^2m^2 + m^4) - 2D^2m^2 = 0$

$\Rightarrow (D^2 + m^2)^2 - (\sqrt{2}Dm)^2 = 0$

$\Rightarrow (D^2 + \sqrt{2}Dm + m^2)(D^2 - \sqrt{2}Dm + m^2) = 0$

$\Rightarrow D^2 + \sqrt{2}mD + m^2 = 0$, $D^2 - \sqrt{2}mD + m^2 = 0$.

$$\therefore\ D = \frac{-\sqrt{2}m \pm \sqrt{2m^2 - 4.1.m^2}}{2.1}, \quad D = \frac{\sqrt{2}m \pm \sqrt{2m^2 - 4.1.m^2}}{2.1}$$

$\Rightarrow D = -\dfrac{m}{\sqrt{2}} \pm i\dfrac{m}{\sqrt{2}}$, $D = \dfrac{m}{\sqrt{2}} \pm i\dfrac{m}{\sqrt{2}}$.

$\therefore y = C.F$

$$= e^{-\frac{m}{\sqrt{2}}x}\left[C_1 \cos\frac{m}{\sqrt{2}}x + C_2 \sin\frac{m}{\sqrt{2}}x\right] + e^{\frac{m}{\sqrt{2}}x}\left[C_3 \cos\frac{m}{\sqrt{2}}x + C_4 \sin\frac{m}{\sqrt{2}}x\right].$$

is the required sol^n.

Ex-8 → Solve the diff. Eq^n.

(i) $(D^2+u^2)y=0$ (ii) $(D^4+1)y=0$

(iii) $(D^4-m^4)y=0$ (iv) $(D^2+1)^2y=0$

(v) $(D^2-2D+5)^2y=0$ (vi) $(D^4+8D^2+16)y=0$

(vii) $(D^3-8)y=0$ (viii) $(D^2+1)^2(D^2+D+1)y=0$.

Sol^n → (viii) $(D^2+1)^2(D^2+D+1)y=0$

Here A.E is, $(D^2+1)^2(D^2+D+1)=0$

$\Rightarrow (D^2+1)^2=0$, $D^2+D+1=0$

$\Rightarrow D^2=-1,\ D^2=-1$, $D = \frac{-1 \pm \sqrt{1^2-4.1.1}}{2.1}$

$\Rightarrow D^2=i^2,\ D^2=i^2$, $D = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\Rightarrow D = 0 \pm i,\ 0 \pm i$

$\therefore y = C.F$

$$= e^{0x}\left[(C_1+C_2x)\cos x + (C_3+C_4x)\sin x\right] + e^{-\frac{1}{2}x}\left[C_5 \cos\frac{\sqrt{3}}{2}x + C_6 \sin\frac{\sqrt{3}}{2}x\right].$$

Ex-9 → Solve

$$\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$$

Hint → $(D^2-2D+2)^2=0$

Ex-10 → Solve the diff. eq^n

$\frac{d^2y}{dx^2} + y = 0$; given that $y(0)=2$ & $y(\frac{\pi}{2}) = -2$.

(AKTU – 2010, 2012).

Sol^n: Given $\frac{d^2y}{dx^2}+y=0$, $y(0)=2$, $y(\frac{\pi}{2})=-2$. $\longrightarrow$ ①

$\Rightarrow (D^2+1)y=0$

Here A.E is $D^2+1=0$

$\Rightarrow D^2=-1 \Rightarrow D^2=i^2 \Rightarrow D=0\pm i$.

$\therefore y = C.F = e^{0x}[C_1 \cos x + C_2 \sin x]$

$\left.\begin{array}{l}\Rightarrow y = C_1 \cos x + C_2 \sin x \\ \text{or } y(x) = C_1 \cos x + C_2 \sin x\end{array}\right\} \longrightarrow$ ②

Using eq^n ① in eq^n ②, we get

$y(0) = C_1 \cos 0 + C_2 \sin 0 \Rightarrow \boxed{2 = C_1}$

& $y(\frac{\pi}{2}) = C_1 \cos\frac{\pi}{2} + C_2 \sin\frac{\pi}{2} \Rightarrow \boxed{-2 = C_2}$

Hence ② gives $\boxed{y = 2\cos x - 2\sin x}$

Ex-11: A func^n $\eta(x)$ satisfies the differential eq^n $\frac{d^2\eta(x)}{dx^2} - \frac{\eta(x)}{L^2} = 0$, where L is a constant. The boundary cond^n are $\eta(0)=\alpha$ & $\eta(\infty)=0$. find the sol^n to this eq^n. (AKTU-2007, 2017).

Sol^n: Given $(D^2 - \frac{1}{L^2})\eta(x) = 0$

Here A.E is $D^2 - \frac{1}{L^2} = 0 \Rightarrow D = \pm\frac{1}{L}$

$\therefore \underbrace{\eta(x) = C_1 e^{\frac{1}{L}x} + C_2 e^{-\frac{1}{L}x}}_{= C.F} \longrightarrow$ ① & P.I = 0.

Given $\eta(0)=\alpha$ then ① given,

~~$\eta(0) = C_1 e$~~ $\eta(0) = C_1 + C_2 \Rightarrow \boxed{C_1 + C_2 = \alpha}$ $\rightarrow$ ②

Also, $\eta(\infty) = 0$,

from ①, $\eta(\infty) = C_1 e^{\infty} + C_2 e^{-\infty} \Rightarrow \boxed{0 = C_1} \rightarrow$ ③

from ② & ③, $\boxed{C_2 = \alpha.}$

Hence $\boxed{\eta(x) = \alpha e^{-\frac{1}{L}x}}$ is the required sol^n.

Ex-12→ Solve the diff. eqⁿ $\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$, where $R^2C = 4L$ and R, C, L are constants. (AKTU-2010)

Solⁿ→ The given eqⁿ is $\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$

$(D^2 + \frac{R}{L}D + \frac{1}{LC})\,i = 0$ where $D = \frac{d}{dt}$

Auxiliary eqⁿ is

$D^2 + \frac{R}{L}D + \frac{1}{LC} = 0$.

$\therefore D = \dfrac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - 4.1.\frac{1}{LC}}}{2.1}$ $\left[\because R^2C = 4L \Rightarrow R^2 = \frac{4L}{C}\right]$

$\Rightarrow D = \dfrac{-\frac{R}{L} \pm \sqrt{\frac{4L}{L^2C} - \frac{4}{LC}}}{2}$

$\Rightarrow D = \dfrac{-\frac{R}{L} \pm 2.\sqrt{\frac{1}{LC} - \frac{1}{LC}}}{2}$

$\Rightarrow D = -\frac{R}{2L} \pm 0$, $\Rightarrow D = -\frac{R}{2L}, -\frac{R}{2L}$

$\therefore C.F = (C_1 + C_2t)\,e^{-\frac{R}{2L}t}$ & $P.I = 0$.

Hence $i = C.F + P.I = (C_1 + C_2t)\,e^{-\frac{R}{2L}t}$.

Ex-13→ $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$, given that when $t=0$, $x=0$ & $\frac{dx}{dt} = 0$.

Ex-14→ $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 8y = 0$, under the condⁿ $y(0) = 0$, $y'(0) = 0$ & $y''(0) = 2$. (AKTU-2011).

Ex-15→ Solve $(D^2 - 8D + 9)\,y = 0$.

Here A.E is $D^2 - 8D + 9 = 0$

$\therefore D = \dfrac{8 \pm \sqrt{64 - 4.1.9}}{2.1}$

$\Rightarrow D = 4 \pm \sqrt{7}$ (Surd root)

Hence

$y = C.F = e^{4x}[C_1 \cosh\sqrt{7}x + C_2 \sinh\sqrt{7}x]$.

or

$y = C_1e^{(4+\sqrt{7})x} + C_2e^{(4-\sqrt{7})x}$

Method to find Particular Integral → (P.I).

Let the differential Eq^n $f(D)y = Q(x)$ or $y = \frac{1}{f(D)} Q(x)$.

Then $P.I = \frac{1}{f(D)} Q(x)$.

Method-1 → * If $Q(x) = e^{ax}$ then

$$\boxed{P.I = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \text{ provided } f(a) \neq 0}$$

* If $f(a) = 0$ then

$$\boxed{P.I = \frac{1}{f(D)} e^{ax} = x \cdot \frac{1}{f'(a)} e^{ax}, \text{ provided } f'(a) \neq 0.}$$

* If $f'(a) = 0$ then

$$\boxed{P.I = \frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}, \text{ provided } f''(a) \neq 0.}$$

and so on.

Ex-1 → Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$.

Sol^n → Given $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$

or $(D^2 + 6D + 9)y = 5e^{3x}$.

Here A.E is $D^2 + 6D + 9 = 0$

$\Rightarrow (D+3)^2 = 0$

$\Rightarrow D = -3, -3$.

$\therefore C.F = (C_1 + C_2 x) e^{-3x}$.

$$P.I = \frac{1}{D^2 + 6D + 9} 5e^{3x} = 5 \frac{1}{D^2 + 6D + 9} e^{3x}$$

$$= 5 \frac{1}{3^2 + 6\times3 + 9} e^{3x}$$

$$\Rightarrow P.I = \frac{5}{36} e^{3x}.$$

Hence general sol^n is $y = C.F + P.I = (C_1 + C_2 x) e^{-3x} + \frac{5}{36} e^{3x}$. Ans

Ex-2→ Solve $(D^3-2D^2+4D-8)y=8$

Solⁿ→ Here A.E is $D^3-2D^2+4D-8=0$

$\Rightarrow (D-2)(D^2+4)=0$

$\Rightarrow D-2=0,\quad D^2+4=0$ or $D^2=(2i)^2$

$\Rightarrow D=2,\quad D=0\pm 2i$

$\therefore C.F = C_1e^{2x} + e^{0x}[C_2\cos 2x + C_3\sin 2x]$

$\Rightarrow C.F = C_1e^{2x} + C_2\cos 2x + C_3\sin 2x$

$$P.I = \frac{1}{D^3-2D^2+4D-8}\,8 \quad = 8\frac{1}{D^3-2D^2+4D-8}e^{0x}$$

$$= 8\,\frac{1}{0-8}\cdot 1 \quad = (-1).$$

$\therefore$ General solⁿ is $y = C.F + P.I = C_1e^{2x} + C_2\cos 2x + C_3\sin 2x + (-1)$.

Ex-3→ Solve $(D-2)^3y = 17e^{2x}$ (AKTU-2011).

Solⁿ→ Here A.E is $(D-2)^3=0 \Rightarrow D=2,2,2$.

$\therefore C.F = (C_1+C_2x+C_3x^2)e^{2x}$.

$$\& \; P.I = \frac{1}{(D-2)^3}17e^{2x}$$

$$= 17\,\frac{1}{(D-2)^3}e^{2x} \qquad [\text{Case fail}]$$

$$= 17\,x\,\frac{1}{3(D-2)^2}e^{2x} \qquad [\text{Again Case fail}]$$

$$= 17\,x^2\,\frac{1}{6(D-2)}e^{2x} \qquad [\text{Again Case fail}.$$

$$= 17\,x^3\,\frac{1}{6\cdot 1}e^{2x}$$

$$= \frac{17}{6}x^3e^{2x}.$$

Hence general solⁿ is $y = C.F + P.I$

$\Rightarrow y = (C_1+C_2x+C_3x^2)e^{2x} + \frac{17}{6}x^3e^{2x}$.

Ex-4 Solve $y'' + 4y' + 13y = 18e^{-2x}$, $y(0)=0$, $y'(0)=9$.

Solⁿ Given $y'' + 4y' + 13y = 18e^{-2x}$

$\Rightarrow (D^2+4D+13)y = 18e^{-2x}$.

Here A.E is $D^2+4D+13=0$

$\therefore D = \dfrac{-4 \pm \sqrt{16-4\times1\times13}}{2\times1} = \dfrac{-4\pm\sqrt{-36}}{2}$

$\Rightarrow D = \dfrac{-4 \pm i6}{2} = -2 \pm 3i$

$\therefore C.F = e^{-2x}[C_1 \cos 3x + C_2 \sin 3x]$

& $P.I = \dfrac{1}{D^2+4D+13} 18e^{-2x}$

$= 18 \dfrac{1}{D^2+4D+13} e^{-2x}$

$= 18 \dfrac{1}{((-2)^2+4(-2)+13)} e^{-2x} = \dfrac{18}{9} e^{-2x} = 2e^{-2x}$.

$\therefore$ General solⁿ is,

$y(x) = C.F + P.I$

$\Rightarrow y(x) = e^{-2x}[C_1 \cos 3x + C_2 \sin 3x] + 2e^{-2x} \longrightarrow ①$

Put $x=0$ in ① & use $y(0)=0$, we get

$y(0) = C_1 + 2 \Rightarrow 0 = C_1 + 2 \Rightarrow \boxed{C_1 = -2} \longrightarrow ②$

from ①, On diff,

$y'(x) = e^{-2x}[-3C_1 \sin 3x + 3C_2 \cos 3x] + (-2e^{-2x})[C_1 \cos 3x + C_2 \sin 3x]$
$+ 2\times -2e^{-2x}$.

Put $x=0$ & use, $y'(0)=9$, we get

$y'(0) = 3C_2 - 2C_1 - 4 \Rightarrow 9 = 3C_2 + 4 - 4 \Rightarrow 3C_2 = 9 \Rightarrow \boxed{C_2 = 3} \longrightarrow ③$

Using ② & ③ in eqⁿ ①, we get

$$\boxed{y(x) = e^{-2x}[-2\cos 3x + 3\sin 3x] + 2e^{-2x}}.$$

which is required solⁿ.

Imp Note→ ① $\boxed{\cosh ax = \dfrac{e^{ax}+e^{-ax}}{2}}$ ② $\boxed{\sinh ax = \dfrac{e^{ax}-e^{-ax}}{2}}$

Ex-5: Solve $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Sol^n: Here A.E is, $(D+2)(D-1)^2 = 0$

$$\Rightarrow D = -2, 1, 1.$$

$$\therefore C.F = C_1 e^{-2x} + (C_2 + C_3 x)e^{x}.$$

& $P.I = \dfrac{1}{(D+2)(D-1)^2}(e^{-2x} + 2\sinh x)$

$$= \frac{1}{(D+2)(D-1)^2}\left[e^{-2x} + e^{x} - e^{-x}\right] \qquad \left[\because \sinh x = \frac{e^{x} - e^{-x}}{2}\right]$$

$$= \frac{1}{(D+2)(D-1)^2}e^{-2x} + \frac{1}{(D+2)(D-1)^2}e^{x} - \frac{1}{(D+2)(D-1)^2}e^{-x}.$$

$$= \frac{1}{(D+2)}\left[\frac{1}{(D-1)^2}e^{-2x}\right] + \frac{1}{(D-1)^2}\left[\frac{1}{D+2}e^{x}\right] - \frac{1}{(D+2)(D-1)^2}e^{-x}$$

$$= \frac{1}{D+2}\left[\frac{1}{(-2-1)^2}e^{-2x}\right] + \frac{1}{(D-1)^2}\left[\frac{1}{1+2}e^{x}\right] - \frac{1}{(-1+2)(-1-1)^2}e^{-x}$$

$$= \frac{1}{9}\,\frac{1}{D+2}e^{-2x} + \frac{1}{3}\,\frac{1}{(D-1)^2}e^{x} - \frac{1}{4}e^{-x}$$

$$= \frac{1}{9}\,x\,\frac{1}{1}e^{-2x} + \frac{1}{3}\,x\,\frac{1}{2(D-1)}e^{x} - \frac{1}{4}e^{-x}$$

$$= \frac{1}{9}x e^{-2x} + \frac{1}{3}x^2\frac{1}{2\cdot 1}e^{x} - \frac{1}{4}e^{-x}$$

$$\therefore P.I = \frac{1}{9}x e^{-2x} + \frac{1}{6}x^2 e^{x} - \frac{1}{4}e^{-x}.$$

Hence general sol^n is $\quad y = C.F + P.I$

$$\Rightarrow y = C_1 e^{-2x} + (C_2 + C_3 x)e^{x} + \frac{1}{9}x e^{-2x} + \frac{1}{6}x^2 e^{x} - \frac{1}{4}e^{-x}.$$

[Home Assignment]

Ex-1: Solve the differential eq^n,

(i) $(2D+1)^2 y = 4e^{-x/2}$ [Ans- $y = \left(C_1 + C_2 x + \frac{x^2}{2}\right)e^{-x/2}$.

(ii) $y_3 + y = 3 + 5e^{x}$ [Ans- $y = C_1 e^{-x} + e^{\frac{1}{2}x}\left(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right) + 3 + \frac{5}{2}e^{x}$.

(iii) $\frac{d^2y}{dx^2} - 4y = (1+e^x)^2$ [Ans. $y = C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{4} - \frac{2}{3}e^x + \frac{1}{4}xe^{2x}$

(iv) $y'' + 4y + 5y = -2\cosh x$ [Ans $y = e^{-2x}(C_1\cos x + C_2\sin x) - \frac{1}{10}e^{x} - \frac{1}{2}e^{-x}$

(v) $(D+2)(D-1)^3 y = e^x$

[Ans. $y = (C_1 + C_2 x + C_3 x^2)e^x + C_4 e^{-2x} + \frac{x^3 e^x}{18}$.

Ex-2: Solve the differential eq^n

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = e^x + 2$$

[Ans: $y = (C_1 + C_2 x + C_3 x^2)e^x + \frac{x^3}{6}e^x - 2$.

Ex-3: Solve $(D^2 + D + 1)y = (1+e^x)^2$.

[Ans: $y = e^{-x/2}\left[C_1\cos\frac{\sqrt{3}}{2}x + C_2\sin\frac{\sqrt{3}}{2}x\right] + 1 + \frac{1}{7}e^{2x} + \frac{2}{3}e^x$.

Ex-4: Solve $(D^2 - a^2)y = e^{ax} - e^{-ax}$ (AKTU-2009)

[Ans - $y = C_1 e^{ax} + C_2 e^{-ax} + \frac{x}{a}\cosh ax$].

Ex-5: Solve $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$

[Ans: $y = C_1 e^x + C_2 e^{2x} + C_3 e^{3x} - \frac{1}{120}(2e^{-2x} + e^{-3x})$.

Ex-6: Solve $(D^2+4)y = 5^x$.

Sol^n: Here A.E is $D^2+4=0 \Rightarrow D^2 = (2i)^2$

$\Rightarrow D = 0 \pm 2i$

$\therefore$ C.F $= C_1\cos 2x + C_2\sin 2x$.

& $P.I = \frac{1}{D^2+4}5^x = \frac{1}{D^2+4}e^{\log 5^x} = \frac{1}{D^2+4}e^{(\log 5).x}$.

$= \frac{1}{(\log 5)^2+4}e^{(\log 5)x}$

$= \frac{1}{(\log 5)^2+4}5^x$.

$\therefore$ General sol^n is $\boxed{y = C.F + P.I = C_1\cos 2x + C_2\sin 2x + \frac{1}{(\log 5)^2+4}5^x}$

Method-2 → If $f(D^2)y = Q(x)$

and [$Q(x) = \sin ax$ or $\sin(ax+b)$ / $\cos ax$ or $\cos(ax+b)$.]

(*) Then
$$\boxed{P.I = \frac{1}{f(D^2)}\sin ax \text{ or } \cos ax = \frac{1}{f(-a^2)}\sin ax \text{ or } \cos ax \quad \text{provided } f(-a^2) \neq 0.}$$

(*) If $f(-a^2) = 0$ then

$$\boxed{P.I = \frac{1}{f(D^2)}\sin ax / \cos ax = x\frac{1}{f'(-a^2)}\sin ax/\cos ax \quad \text{provided } f'(-a^2) \neq 0.}$$

(*) If $f'(-a^2) = 0$ then

$$\boxed{P.I = \frac{1}{f(D^2)}\sin ax/\cos ax = x^2\frac{1}{f''(-a^2)}\sin ax / \cos ax \quad \text{provided } f''(-a^2) \neq 0.}$$

and so on.

Ex-1 → Solve $(D^2+9)y = \cos 2x$

Solⁿ → Here A.E is $D^2+9 = 0 \Rightarrow D^2 = -9$

$\Rightarrow D^2 = i^2 9 \Rightarrow D^2 = (3i)^2$

$\Rightarrow D = 0 \pm 3i$

$\therefore$ C.F $= C_1\cos 3x + C_2 \sin 3x$.

& P.I $= \dfrac{1}{D^2+9}\cos 2x$

$= \dfrac{1}{-4+9}\cos 2x \quad$ [Put $D^2 = -a^2 = -2^2 = -4$

$= \dfrac{1}{5}\cos 2x$.

$\therefore$ General solⁿ is.

$y = C.F + P.I = C_1\cos 3x + C_2\sin 3x + \frac{1}{5}\cos 2x$.

Ex-2: Solve $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$ [AKTU-2010]

Solⁿ: Given $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$

$\Rightarrow (D^2+4)y = e^x + \sin 2x$

Here A.E is $D^2+4=0 \Rightarrow D = 0 \pm 2i$

$\therefore C.F = C_1 \cos 2x + C_2 \sin 2x.$

& $P.I = \frac{1}{D^2+4}[e^x + \sin 2x]$

$= \frac{1}{D^2+4} e^x + \frac{1}{D^2+4} \sin 2x$

$= \frac{1}{1^2+4} e^x + x \frac{1}{2D} \sin 2x$ [Case fail]
i.e $f(D^2) = f(-4) = 0$

$= \frac{1}{5} e^x + \frac{x}{2} \int \sin 2x \, dx$

$= \frac{e^x}{5} + \frac{x}{2}\left(-\frac{\cos 2x}{2}\right)$

$= \frac{e^x}{5} - \frac{x}{4} \cos 2x$

$\therefore$ General Solⁿ is $y = C.F + P.I = C_1 \cos 2x + C_2 \sin 2x + \frac{e^x}{5} - \frac{x}{4}\cos 2x.$

Ex-3: Solve $(D^2+5D-6)y = \sin 3x + \cos 2x$ [AKTU-2015, 2010]

Solⁿ: Here A.E is $D^2+5D-6=0$

$\Rightarrow D^2 + 6D - D - 6 = 0$

$\Rightarrow D(D+6) - 1(D+6) = 0$

$\Rightarrow (D-1)(D+6) = 0 \Rightarrow D = 1, -6.$

$\therefore C.F = C_1 e^x + C_2 e^{-6x}.$

$P.I = \frac{1}{D^2+5D-6}[\sin 3x + \cos 2x]$

$= \frac{1}{D^2+5D-6} \sin 3x + \frac{1}{D^2+5D-6} \cos 2x.$

$= \frac{1}{-9+5D-6} \sin 3x + \frac{1}{-4+5D-6} \cos 2x.$

$= \frac{1}{5D-15} \sin 3x + \frac{1}{5D-10} \cos 2x.$

$$= \frac{1}{5} \frac{(D+3)}{(D-3)(D+3)} \sin 3x + \frac{1}{5} \frac{(D+2)}{(D-2)(D+2)} \cos 2x$$

$$= \frac{1}{5} \frac{(D+3)}{D^2-9} \sin 3x + \frac{1}{5} \frac{(D+2)}{(D^2-4)} \cos 2x$$

$$= \frac{1}{5} \frac{(D+3)}{(-9-9)} \sin 3x + \frac{1}{5} \frac{(D+2)}{(-4-4)} \cos 2x$$

$$= -\frac{1}{90}(D+3)\sin 3x - \frac{1}{40}(D+2)\cos 2x$$

$$= -\frac{1}{90}[D\sin 3x + 3\sin 3x] - \frac{1}{40}[D\cos 2x + 2\cos 2x]$$

$$= -\frac{1}{90}[3\cos 3x + 3\sin 3x] - \frac{1}{40}[-2\sin 2x + 2\cos 2x]$$

$$= -\frac{1}{30}(\cos 3x + \sin 3x) + \frac{1}{20}(\sin 2x - \cos 2x).$$

Hence General solⁿ is $y = C.F + P.I$

$$\Rightarrow y = C_1 e^{x} + C_2 e^{-6x} - \frac{1}{30}(\cos 3x + \sin 3x) + \frac{1}{20}(\sin 2x - \cos 2x).$$

<u>Ex-44</u> Solve $\frac{d^2y}{dx^2} + 4y = \sin^2 2x$ with conditions $y(0)=0$, $y'(0)=0$.

[AKTU-2013]

<u>Solⁿ</u> → Given $\frac{d^2y}{dx^2} + 4y = \sin^2 2x \Rightarrow (D^2+4)y = \sin^2 2x.$

$$\Rightarrow (D^2+4)y = \frac{1-\cos 4x}{2} \qquad [\cos 4x = 1 - 2\sin^2 2x]$$

Here A.E is $D^2+4=0 \Rightarrow D^2 = i^2 2^2 \Rightarrow D = 0 \pm i2$.

$\therefore C.F = C_1 \cos 2x + C_2 \sin 2x.$

$$P.I = \frac{1}{D^2+4}\left[\frac{1-\cos 4x}{2}\right] = \frac{1}{2}\frac{1}{D^2+4} e^{0x} - \frac{1}{2}\frac{1}{D^2+4}\cos 4x$$

$$= \frac{1}{2}\frac{1}{0^2+4} e^{0x} - \frac{1}{2}\frac{1}{(-16+4)}\cos 4x$$

$$= \frac{1}{8} + \frac{1}{24}\cos 4x.$$

$\therefore$ General solⁿ is $y = C.F + P.I = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{8} + \frac{1}{24}\cos 4x \longrightarrow (1)$

Put $x=0$ in (1) & use $y(0)=0$, we get

$y(0) = C_1 + \frac{1}{8} + \frac{1}{24} \Rightarrow 0 = C_1 + \frac{1}{6} \Rightarrow \boxed{C_1 = -\frac{1}{6}} \longrightarrow (2)$

Diff. eqⁿ (1), we get

$y'(x) = -2C_1 \sin 2x + 2C_2 \cos 2x - \frac{4}{24}\sin 4x \longrightarrow (3)$ Put $x=0$ in (3) & use $y'(0)=0$.

$y(0) = 2C_2 \Rightarrow \quad 0 = 2C_2 \Rightarrow \boxed{C_2 = 0} \rightarrow (4)$

Using (2) & (4) in (1), we get

$$\boxed{y = -\frac{1}{6}\cos 2x + \frac{1}{8} + \frac{1}{24}\cos 4x}$$ which is required solⁿ.

Ex-5) Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y + 37\sin 3x = 0$ & find the value of y when $x = \frac{\pi}{2}$ being given that $y = 3$, $\frac{dy}{dx} = 0$ when $x = 0$. [AKTU-2011]

Solⁿ) Given $(D^2 + 2D + 10)y = -37\sin 3x$.

Here A.E is, $D^2 + 2D + 10 = 0$ then $D = \frac{-2 \pm \sqrt{4 - 4\times 1\times 10}}{2\times 1}$

$\Rightarrow \quad D = -1 \pm 3i$.

$\therefore$ C.F $= e^{-x}(C_1\cos 3x + C_2 \sin 3x)$.

$$P.I = \frac{1}{D^2 + 2D + 10}(-37\sin 3x)$$

$$= (-37)\frac{1}{D^2 + 2D + 10}\sin 3x$$

$$= (-37)\frac{1}{-9 + 2D + 10}\sin 3x$$

$$= (-37)\frac{1}{2D+1}\sin 3x$$

$$= (-37)\frac{2D-1}{(2D+1)(2D-1)}\sin 3x$$

$$= (-37)\frac{2D-1}{(4D^2-1)}\sin 3x$$

$$= (-37)\frac{(2D-1)\sin 3x}{(-36-1)}$$

$$= (2D-1)\sin 3x = 2[D\sin 3x] - (\sin 3x)$$

$$= 6\cos 3x - \sin 3x.$$

Hence the general solⁿ is

$$y = C.F + P.I = e^{-x}[C_1\cos 3x + C_2\sin 3x] + 6\cos 3x - \sin 3x \longrightarrow (1)$$

Put $x = 0$ in (1) & use $y(0) = 3$, we get

$y(0) = C_1 + 6 \Rightarrow 3 = C_1 + 6 \Rightarrow \boxed{C_1 = -3} \rightarrow (2)$.

Diff. eqⁿ (1),

$y'(x) = e^{-x}[-3C_1\sin 3x + 3C_2\cos 3x] - e^{-x}[C_1\cos 3x + C_2\sin 3x] - 18\sin 3x - 3\cos 3x$ (3)

Put $x=0$ in eqⁿ ③ , & use $y'(0)=0$, we get

$$y'(0) = 3C_2 - C_1 - 3 \Rightarrow 0 = 3C_2 - C_1 - 3 \Rightarrow 3C_2 = 0 \text{ (from 2)}$$

$$\Rightarrow \boxed{C_2 = 0} \rightarrow (4)$$

Using ② & ④ in ①, we get

$$y = e^{-x}[-3\cos 3x] + 6\cos 3x - \sin 3x \longrightarrow (5)$$

when $x = \frac{\pi}{2}$,

$$y = e^{-\pi/2}\left(-3\cos\frac{3\pi}{2}\right) + 6\cos\frac{3\pi}{2} - \sin\frac{3\pi}{2}$$

$$\Rightarrow y = 0 + 0 - (-1) = +1$$

$$\Rightarrow \boxed{y = 1}$$

Ex-6: Solve $(D^2+4)y = \cos x \cos 3x$. or $(D^2+4)y = \frac{1}{2}[2\cos 3x \cos x]$

(AKTU - 2007)

Solⁿ: $\Rightarrow (D^2+4)y = \frac{1}{2}[\cos 4x + \cos 2x]$.

Here A.E is $D^2+4=0 \Rightarrow D = 0 \pm 2i$

$\therefore$ C.F $= C_1\cos 2x + C_2 \sin 2x$

$$\text{P.I} = \frac{1}{D^2+4}\left[\frac{1}{2}\cos 4x + \frac{1}{2}\cos 2x\right]$$

$$= \frac{1}{2}\frac{1}{D^2+4}\cos 4x + \frac{1}{2}\frac{1}{D^2+4}\cos 2x.$$

$$= \frac{1}{2}\frac{1}{-16+4}\cos 4x + \frac{1}{2}x\frac{1}{2D}\cos 2x$$

$$= -\frac{1}{24}\cos 4x + \frac{1}{4}x\int \cos 2x\,dx$$

$$= -\frac{1}{24}\cos 4x + \frac{x}{4}\frac{\sin 2x}{2}.$$

$\therefore$ G. Solⁿ is $y = \text{C.F} + \text{P.I}$

$$\Rightarrow y = C_1\cos 2x + C_2\sin 2x - \frac{x}{8}\sin 2x - \frac{1}{24}\cos 4x.$$

Home Assignment

Ex-1: Solve $(D^2+4)y = \sin 3x + \cos 2x$.

Solⁿ: $y = C_1\cos 2x + C_2\sin 2x - \frac{1}{5}\sin 3x + \frac{x}{4}\sin 2x$.

Ex-2: Solve, $(D^3+1)y = \sin(2x+1)$

Ans: $\text{P.I} = \frac{1}{65}[\sin(2x+1) + 8\cos(2x+1)]$.

Ex-3: (i) Solve $\frac{d^2y}{dx^2} + a^2y = \sin ax$, Ans $y = C_1\cos ax + C_2 \sin ax - \frac{x}{2a}\cos ax$

(ii) $(D^2+4)y = \cos^2 x$, Ans $y = C_1\cos 2x + C_2\sin 2x + \frac{1}{8}(1 + x\sin 2x)$

(iii) Solve $\frac{d^4y}{dx^4} - m^4y = \cos mx$ [AKTU-2011]

Ans $y = C_1e^{mx} + C_2e^{-mx} + C_3\cos mx + C_4\sin mx - \frac{x}{4m^3}\sin mx$.

(iv) $(D^2 - 4D + 3)y = \sin 3x \cos 2x$

Ans. $y = C_1e^{x} + C_2e^{3x} + \frac{1}{884}(10\cos 5x - 11\sin 5x) + \frac{1}{20}(\sin x + 2\cos x)$.

(v) $\frac{d^3y}{dx^3} + a^2\frac{dy}{dx} = \sin ax$, (ST-1, 2019)

Ans $y = C_1 + C_2\cos ax + C_3\sin ax - \frac{x}{2a^2}\sin ax$.

Ex-4: Solve $(D^2 - 4D + 1)y = \cos x \cos 2x + \sin^2 x$.

Ans $y = e^{2x}[C_1\cosh\sqrt{3}x + C_2\sinh\sqrt{3}x] + \frac{1}{2} - \frac{1}{8}\sin x - \frac{1}{104}(3\sin 3x + 2\cos 3x) + \frac{1}{146}(8\sin 2x + 3\cos 2x)$.

Ex-5: Solve $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$. [AKTU-2012]

Hint:
$$P.I = \frac{1}{D^3 - 3D^2 + 4D - 2}[e^x + \cos x]$$
$$= \frac{1}{D^3 - 3D^2 + 4D - 2}e^x + \frac{1}{D^3 - 3D^2 + 4D - 2}\cos x$$
$$= x\frac{1}{3D^2 - 6D + 4}e^x + \frac{1}{D(-1) - 3(-1) + 4D - 2}\cos x$$
$$= x\frac{1}{3 - 6 + 4}e^x + \frac{1}{3D + 1}\cos x$$
$$= xe^x + \frac{(3D-1)}{(3D+1)(3D-1)}\cos x$$
$$= xe^x + \frac{(3D-1)}{9D^2 - 1}\cos x$$
$$= xe^x + \frac{(3D-1)\cos x}{-9-1}$$
$$= xe^x - \frac{1}{10}[3D\cos x - \cos x]$$
$$= xe^x + \frac{1}{10}(3\sin x + \cos x).$$

Ans: $y = C.F + P.I = C_1e^x + e^x(C_2\cos x + C_3\sin x) + xe^x + \frac{1}{10}(3\sin x + \cos x)$

Engineering Mathematic - II
KAS-203T
Lecture No- 04

* Module-I *
* (ODE) *
* By Dr. Anuj Kumar *

Method-III → If $Q(x) = x^n$, where m being +ve integer.

Then $P.I = \frac{1}{f(D)} x^n$, Convert it into

$$= [1+\phi(D)]^{-1} x^n \text{ or } [1-\phi(D)]^{-1} x^n$$

or

$$[1+\phi(D)]^{-2} x^n \text{ or } [1-\phi(D)]^{-2} x^n.$$

& Use the formulae

$$(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$$

$$(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots$$

$$(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \cdots$$

$$(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

Ex-1→ Solve $\frac{d^2y}{dx^2} - 4y = x^2$.

Solⁿ→ Give $(D^2-4)y = x^2$

Here A.E is $D^2-4=0 \Rightarrow D = \pm 2$.

$$\therefore C.F = c_1 e^{-2x} + c_2 e^{2x}$$

$$\& \; P.I = \frac{1}{D^2-4} x^2 = \frac{1}{-4\left[1-\frac{D^2}{4}\right]} x^2$$

$$= -\frac{1}{4}\left[1-\frac{D^2}{4}\right]^{-1} x^2$$

$$= -\frac{1}{4}\left[1+\frac{D^2}{4}+\frac{D^4}{16}+\cdots\right] x^2$$

$$= -\frac{1}{4}\left[x^2+\frac{D^2x^2}{4}+\frac{D^4x^2}{16}+\cdots\right]$$

$$= -\frac{1}{4}\left[x^2+\frac{2}{4}+0\right] = -\frac{1}{4}\left(x^2+\frac{1}{2}\right).$$

∴ General Solⁿ is

$$y = C.F + P.I$$

$$\Rightarrow \boxed{y = c_1 e^{-2x} + c_2 e^{2x} - \frac{1}{4}\left(x^2+\frac{1}{2}\right)}$$

Ex-2: Solve $(D^2-3D+2)y = x^2+2x+1$ [AKTU-2016]

Sol^n: Here A.E is $D^2-3D+2=0 \Rightarrow (D-2)(D-1)=0$

$\Rightarrow D=1,2.$

$\therefore CF = C_1e^{x}+C_2e^{2x}.$

& $P.I = \frac{1}{D^2-3D+2}(x^2+2x+1)$

$= \frac{1}{2\left[1+\frac{(D^2-3D)}{2}\right]}(x^2+2x+1)$

$= \frac{1}{2}\left[1+\frac{(D^2-3D)}{2}\right]^{-1}(x^2+2x+1)$

$= \frac{1}{2}\left[1-\frac{(D^2-3D)}{2}+\frac{(D^2-3D)^2}{4}-\cdots\right](x^2+2x+1)$

$= \frac{1}{2}\left[1-\frac{(D^2-3D)}{2}+\frac{(D^4+9D^2-6D^3)}{4}-\cdots\right](x^2+2x+1)$

$= \frac{1}{2}\left[(x^2+2x+1)-\frac{1}{2}\left\{D^2(x^2+2x+1)-3D(x^2+2x+1)\right\}+\frac{1}{4}\left\{9D^2(x^2+2x+1)\right\}\right]$

$= \frac{1}{2}\left[(x^2+2x+1)-\frac{1}{2}\left\{2-3(2x+2)\right\}+\frac{9}{4}\times 2\right]$

$= \frac{1}{2}\left[(x^2+2x+1)-1+\frac{3}{2}(2x+2)+\frac{9}{2}\right]$

$= \frac{1}{2}\left[x^2+5x+\frac{15}{2}\right]$

$\therefore$ General Sol^n is $y = C.F + P.I$

$\Rightarrow y = C_1e^{x}+C_2e^{2x}+\frac{1}{2}\left[x^2+5x+\frac{15}{2}\right].$

Ex-3: Solve $(D^3-1)y = 3x^4-2x^3$ [AKTU-2016].

Sol^n: Here A.E is $D^3-1=0$

$\Rightarrow (D-1)(D^2+D+1)=0$

$\Rightarrow D=1,\ D=\frac{-1\pm\sqrt{1-4\times1\times1}}{2} = \frac{-1}{2}\pm i\frac{\sqrt{3}}{2}$

$\therefore C.F = C_1e^{x}+e^{-\frac{1}{2}x}\left[C_2\cos\frac{\sqrt{3}}{2}x+C_3\sin\frac{\sqrt{3}}{2}x\right].$

& $P.I = \frac{1}{D^3-1} 3x^4 - 2x^3$

$= -[1-D^3]^{-1}(3x^4 - 2x^3)$

$= -[1+D^3+D^6+ \dots](3x^4-2x^3)$

$= -[(3x^4-2x^3)+(72x-12)]$

$= -[3x^4-2x^3+72x-12]$.

$\therefore y = C.F + P.I = C_1 e^{x} + e^{-x/2}\left[C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right] - [3x^4-2x^3+72x-12]$

Ex-4: Solve $\frac{d^2y}{dx^2} + y = e^{2x} + \cosh 2x + x^3$ [AKTU-2014].

Solⁿ: Given $(D^2+1)y = e^{2x} + \cosh 2x + x^3$

$\Rightarrow (D^2+1)y = e^{2x} + \frac{e^{2x} - e^{-2x}}{2} + x^3$ $\quad \left[\because \cosh ax = \frac{e^{ax}+e^{-ax}}{2}\right]$

$\Rightarrow (D^2+1)y = \frac{3}{2}e^{2x} + \frac{1}{2}e^{-2x} + x^3$.

Here A.E is $D^2+1=0 \Rightarrow D = 0 \pm i$

$\therefore C.F = C_1 \cos x + C_2 \sin x$.

& $P.I = \frac{1}{D^2+1}\left[\frac{3}{2}e^{2x} + \frac{1}{2}e^{-2x} + x^3\right]$

$= \frac{3}{2}\frac{1}{D^2+1}e^{2x} + \frac{1}{2}\frac{1}{D^2+1}e^{-2x} + \frac{1}{1+D^2}x^3$

$= \frac{3}{2}\frac{1}{2^2+1}e^{2x} + \frac{1}{2}\frac{1}{(-2)^2+1}e^{-2x} + [1+D^2]^{-1}x^3$

$= \frac{3}{10}e^{2x} + \frac{1}{10}e^{-2x} + [1-D^2+D^4 \dots]x^3$

$= \frac{3}{10}e^{2x} + \frac{1}{10}e^{-2x} + (x^3 - 6x)$

$\therefore \quad y = C.F + P.I = $ ____ + ____ is the G. Solⁿ.

Ex-5: Solve $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

Solⁿ: Here A.E is $(D-2)^2 = 0 \Rightarrow D = 2, 2$.

$\therefore C.F = (C_1 + C_2 x)e^{2x}$.

& $P.I = \frac{1}{(D-2)^2} 8[e^{2x} + \sin 2x + x^2]$

$= 8\frac{1}{(D-2)^2}e^{2x} + 8\frac{1}{D^2-4D+4}\sin 2x + 8\frac{1}{(D-2)^2}x^2$.

$$\Rightarrow P.I = 8x\frac{1}{2(D-2)}e^{2x} + 8\frac{1}{-4-4D+4}\sin 2x + 8\frac{1}{4}\left[1-\frac{D}{2}\right]^{-2}x^2$$

$$= 8x^2\frac{1}{2\times 1}e^{2x} + \frac{8}{-4}\int \sin 2x\,dx + 2\left[1 + 2\frac{D}{2} + 3\frac{D^2}{4} + \cdots\right]x^2$$

$$= 4x^2e^{2x} - 2\frac{\cos 2x}{-2} + 2\left[x^2 + 2x + \frac{3}{4}\times 2\right]$$

$$= 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

$\therefore$ General Sol^n is $y = C.F + P.I$

$$\Rightarrow y = (C_1 + C_2x)e^{2x} + 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3.$$

Home Assignment

Ex-1: Solve $(D^2 + 5D + 4)y = x^2 + 7x + 9$

Ex-2: Solve $(D^3 - D^2 - 6D)y = 1 + x^2$.

[Ans: $C_1 + C_2e^{-2x} + C_3e^{3x} - \frac{x}{18}\left(x^2 - \frac{x}{2} + \frac{25}{6}\right)$.

Ex-3: Solve $y'' - 2y' + 3y = \cos x + x^2$

Ans: $y = e^x[C_1\cos\sqrt{2}x + C_2\sin\sqrt{2}x) + \frac{1}{4}(\cos x - \sin x)$
$+ \frac{1}{27}(9x^2 + 12x + 2)$.

Ex-4: $(D^3 + 8)y = x^4 + 2x + 1$.

Ex-5: $(D^6 - D^4)y = x^2$

Ans $y = C_1 + C_2x + C_3x^2 + C_4x^3 + C_5e^x + C_6e^{-x} - \frac{x^4}{12} - \frac{x^6}{360}$.

Ex-6: $(D^2 - 4D + 4)y = x^2 + e^x + \cos 2x$

Ans: $y = (C_1 + C_2x)e^{2x} + \frac{1}{8}(2x^2 + 4x + 3) - \frac{1}{8}\sin 2x + e^x$.

Ex-7: $y_2 - y_1 + 4y = x^2 + e^x$

Ans $y = e^{x/2}\left[C_1\cos\frac{\sqrt{15}}{2}x + C_2\sin\frac{\sqrt{15}}{2}x\right) + \frac{1}{4}\left(e^x + x^2 + \frac{x}{2} - \frac{3}{8}\right)$.

Method-4→ If $Q(x) = e^{ax}.V$ where V is a func^n of x. Then

$$P.I = \frac{1}{f(D)} e^{ax}.V = e^{ax} \frac{1}{f(D+a)} V$$

[Put D = D+a & take e^{ax} back side]

& Solve $\frac{1}{f(D+a)} V$ by privious methods.

Ex-1→ Solve $(D^2-2D+1)y = e^x \sin x$ [AKTU-2016, 2017].

Sol^n→ Here A.E is $(D^2-2D+1) = 0$

$$\Rightarrow (D-1)^2 = 0$$

$$\Rightarrow D = 1, 1.$$

$$\therefore C.F = (C_1 + C_2 x)e^x.$$

$$\& \; P.I = \frac{1}{D^2-2D+1} e^x \sin x$$

$$= \frac{1}{(D-1)^2} e^x \sin x$$

$$= e^x \frac{1}{(D+1-1)^2} \sin x \quad [\text{Put } D = D+1]$$

$$= e^x \frac{1}{D^2} \sin x$$

$$= e^x \frac{1}{(-1)} \sin x \quad [\text{Put } D^2 = -a^2 = -1]$$

$$= -e^x \sin x.$$

Hence general Sol^n is $y = C.F + P.I = (C_1 + C_2 x)e^x - e^x \sin x$.

Ex-2→ Solve $(D-a)^2 y = e^{ax} f''(x)$ (AKTU-2013)

Hint $P.I = \frac{1}{(D-a)^2} e^{ax} f''(x) = e^{ax} \frac{1}{(D+a-a)^2} f''(x)$

$$= e^{ax} \frac{1}{D^2} f''(x) = e^{ax} f(x).$$

Ex-3: Solve $(D^2-2D+5)y = e^{2x}\cos x$ [AKTU-2013]

Solⁿ: Here A·E is

$D^2-2D+5=0$

$\Rightarrow D = \frac{2\pm\sqrt{4-4\times5}}{2\times1}$

$\Rightarrow D = \frac{2\pm i4}{2}$

$\Rightarrow D = 1\pm i2.$

$\therefore C.F = e^{x}(C_1\cos 2x + C_2\sin 2x)$

$P.I = \frac{1}{D^2-2D+5} e^{2x}\cos x$

$= e^{2x}\frac{1}{(D+2)^2-2(D+2)+5}\cos x$

$= e^{2x}\frac{1}{D^2+4D+4-2D-4+5}\cos x$

$= e^{2x}\frac{1}{D^2+2D+5}\cos x$

$= e^{2x}\frac{1}{-1+2D+5}\cos x$

$= e^{2x}\frac{1}{(2D+4)}\cos x$

$= \frac{e^{2x}}{2}\frac{(D-2)}{(D+2)(D-2)}\cos x$

$= \frac{e^{2x}}{2}\frac{(D-2)}{D^2-4}\cos x$

$= \frac{e^{2x}}{-10}(-\sin x - 2\cos x)$

Hence Genral Solⁿ is

$y = C.F + P.I$

$= e^{x}(C_1\cos 2x + C_2\sin 2x) + \frac{e^{2x}}{10}(\sin x + 2\cos x)$

Ex-4: Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2e^{3x}$

Solⁿ:

$C.F = (C_1+C_2x)e^{x}$

$P.I = \frac{1}{D^2-2D+1}e^{3x}.x^2 = \frac{1}{(D-1)^2}e^{3x}.x^2$

$= e^{3x}\frac{1}{(D+3-1)^2}x^2$

$= e^{3x}\frac{1}{(D+2)^2}x^2$

$= \frac{e^{3x}}{4}\left[1+\frac{D}{2}\right]^{-2}x^2$

$= \frac{e^{3x}}{4}\left[1-2\frac{D}{2}+3\frac{D^2}{4}+\cdots\right]x^2$

$= \frac{e^{3x}}{4}\left[x^2-2x+\frac{3}{2}\right].$

$\therefore$ General Solⁿ is

$y = C.F + P.I$

$\Rightarrow y = (C_1+C_2x)e^{x} + \frac{e^{3x}}{4}\left[x^2-2x+\frac{3}{2}\right].$

Ex-5 Solve $(D^2-2D+4)y = e^x \cos x + \sin x \cos 3x$. [AKTU-2010, 2018].

Solⁿ Given $(D^2-2D+4)y = e^x \cos x + \sin x \cos 3x$

$\Rightarrow (D^2-2D+4)y = e^x \cos x + \frac{1}{2}(2\cos 3x \sin x)$

$\Rightarrow (D^2-2D+4)y = e^x \cos x + \frac{1}{2}(\sin 4x - \sin 2x)$.

Here A·E is $(D^2-2D+4) = 0$

$\Rightarrow D = \dfrac{2 \pm \sqrt{4-4\times1\times4}}{2\cdot1} = \dfrac{2 \pm i2\sqrt{3}}{2} = 1 \pm i\sqrt{3}$.

$\therefore C.F = e^x[C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x]$

& $P.I = \dfrac{1}{D^2-2D+4}[e^x \cos x + \frac{1}{2}(\sin 4x - \sin 2x)]$

$= \dfrac{1}{D^2-2D+4} e^x \cos x + \dfrac{1}{2}\dfrac{1}{D^2-2D+4}\sin 4x - \dfrac{1}{2}\dfrac{1}{D^2-2D+4}\sin 2x$

$= e^x \dfrac{1}{(D+1)^2-2(D+1)+4}\cos x + \dfrac{1}{2}\dfrac{1}{-16-2D+4}\sin 4x - \dfrac{1}{2}\dfrac{1}{-4-2D+4}\sin 2x$

$= e^x \dfrac{1}{(D^2+2D+1)-2D-2+4}\cos x - \dfrac{1}{4}\dfrac{1}{(D+6)}\sin 4x + \dfrac{1}{4}\int \sin 2x\, dx$

$= e^x \dfrac{1}{D^2+3}\cos x - \dfrac{1}{4}\dfrac{(D-6)}{D^2-36}\sin 4x + \dfrac{1}{4}\dfrac{\cos 2x}{-2}$

$= e^x \dfrac{1}{-1+3}\cos x - \dfrac{1}{4}\dfrac{(D-6)\sin 4x}{-\cancel{1}6-36} + \dfrac{1}{-8}\cos 2x$

$= \dfrac{e^x}{2}\cos x + \dfrac{1}{4}\times\dfrac{(4\cos 4x - 6\sin 4x)}{+52} - \dfrac{1}{8}\cos 2x$.

$= \dfrac{e^x}{2}\cos x + \dfrac{1}{104}(2\cos 4x - 3\sin 4x) - \dfrac{1}{8}\cos 2x$.

$\therefore$ General Solⁿ is $\boxed{y = C.F + P.I}$

Home Assignment

Ex-1 find the complete solution of

$\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = xe^{3x} + \sin 2x$ [AKTU-2010].

Ans $\Rightarrow y = C_1e^x + C_2e^{2x} + \dfrac{e^{3x}}{2}(x-\frac{3}{2}) + \frac{1}{20}(3\cos 2x - \sin 2x)$.

Ex-2 → find P.I of $(D^2 - 3D + 2)y = 2e^x \cos\frac{x}{2}$.

Ans $P.I = -\frac{16}{5} e^x\left(\sin\frac{x}{2} + \frac{1}{2}\cos\frac{x}{2}\right)$.

Ex-3 → Solve $y'' - 2y' + 2y = x + e^x \cos x$.

Ans $y = e^x(C_1 \cos x + C_2 \sin x) + \frac{1}{2}(x+1) + \frac{xe^x}{2}\sin x$.

Ex-4 → Solve $\frac{d^2y}{dx^2} - 4y = x \sinh x$.

Ans ÷ $y = C_1 e^{2x} + C_2 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$.

Ex-5 → Solve $y'' + 2y' + y = \frac{e^{-x}}{x+2}$.

Ans $y = (C_1 + C_2 x)e^{-x} + e^{-x}[(x+2)\log(x+2) - x]$.

Ex-6 → Solve $y_2 + y = e^{-x} + \cos x + x^3 + e^x \sin x$. [AKTU-2004, 2007, 2011].

Ans $y = C_1 \cos x + C_2 \sin x + \frac{1}{2}e^{-x} + \frac{1}{2}x \sin x + x^3 - 6x - \frac{1}{5}e^x(2\cos x - \sin x)$.

Ex-7 → Solve $y'' + 2y = x^2 e^{3x} + e^x \cos 2x$.

Ans → $y = C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x + \frac{e^{3x}}{11}\left(x^2 - \frac{12}{11}x + \frac{50}{121}\right) + \frac{e^x}{17}(4\sin 2x - \cos 2x)$.

Ex-8 → Solve (i) $(D^2 - 2D + 4) = e^{2x}\cos x$

(ii) $(D^2 + 4)y = e^x \sin^2 x$.

Ex-9 → Solve $(D^2 + 6D + 9)y = \frac{e^{-3x}}{x^3}$. (AKTU - 2009).

Ex-10 → Solve $(D^4 - 1)y = e^x \cos x$.

Method-5:

If $Q(x) = x \cdot V(x)$

$$P.I = \frac{1}{f(D)} x.V(x)$$

$$= x \frac{1}{f(D)} V + \frac{d}{dD} \frac{1}{f(D)} V$$

or $$= x \frac{1}{f(D)} V - \frac{f'(D)}{[f(D)]^2} V$$

Ex-1: Solve $(D^2 - 2D + 1) y = x \cdot \sin x$ [AKTU - 2012]

Sol^n: Here A.E is $D^2 - 2D + 1 = 0 \Rightarrow (D-1)^2 = 0 \Rightarrow D = 1, 1.$

$\therefore C.F = (C_1 + C_2 x) e^x.$

& $$P.I = \frac{1}{D^2 - 2D + 1} x \cdot \sin x$$

$$= x \frac{1}{D^2 - 2D + 1} \sin x - \frac{2D - 2}{(D^2 - 2D + 1)^2} \sin x.$$

$$= x \frac{1}{-1 - 2D + 1} \sin x - \frac{(2D-2)}{(-1-2D+1)^2} \sin x.$$

$$= x \frac{1}{-2D} \sin x - \frac{(2D-2)}{4D^2} \sin x.$$

$$= \frac{x}{-2} \int \sin x \, dx - \frac{(2D-2)}{-4} \sin x.$$

$$= -\frac{x}{2} \times -\cos x + \frac{1}{2}(D-1) \sin x$$

$$= \frac{x}{2} \cos x + \frac{1}{2}[\cos x - \sin x].$$

$\therefore$ General Sol^n is $y = C.F + P.I$

$$\Rightarrow y = (C_1 + C_2 x) e^x + \frac{x}{2} \cos x + \frac{1}{2}[\cos x - \sin x].$$

[Home - Assignment]

Ex-1: Solve $(D^2 + 2D + 1) y = x \cos x$

Ans $y = (C_1 + C_2 x) e^{-x} + \frac{1}{2} \cos x + \frac{1}{2}(x-1) \sin x.$

Ex-2: Solve $(D^2 + 4) y = 3x \sin x$

Ex-3: Solve $(D^2 - 2D + 1) y = x e^x \cos x.$ [AKTU - 2011]

Ex-5 Solve $(D^2-2D+1)\,y = x\,e^x \sin x$ [AKTU-2012, 2009]

Sol^n Here A.E is $(D^2-2D+1)=0$

$\Rightarrow (D-1)^2=0 \Rightarrow D=1,1.$

$\therefore C.F = (C_1+C_2x)e^x.$

& $P.I = \frac{1}{(D^2-2D+1)} x e^x \sin x$

$= \frac{1}{(D-1)^2} e^x[x \sin x]$

$= e^x \frac{1}{(D+1-1)^2} x \sin x$ [Put $D = D+1$]

$= e^x \frac{1}{D^2} x \sin x.$

$= e^x\left[x \frac{1}{D^2}\sin x + \left(\frac{d}{dD}\frac{1}{D^2}\right)\sin x\right]$

$= e^x\left[x \frac{1}{(-1)}\sin x - \frac{2}{D^3}\sin x\right]$

$= e^x\left[-x \sin x + \frac{2}{D}\sin x\right]$

$= e^x\left[-x\sin x + 2\int \sin x\,dx\right]$

$= e^x\left[-x\sin x - 2\cos x\right]$

$\therefore$ G. Sol^n is $y = C.F + P.I = (C_1+C_2x)e^x - e^x[x\sin x + 2\cos x]$

Special Methods

Method-6 →

(i) $P.I = \frac{1}{f(D)} x^n \sin ax$

$=$ Imaginary part of $\frac{1}{f(D)} e^{iax}.x^n$

$=$ Img. ,, ,, ,, ,, $e^{iax}\frac{1}{f(D+ia)} x^n.$

(ii) $P.I = \frac{1}{f(D)} x^n \cos ax$

$=$ Real Part of $\frac{1}{f(D)} e^{iax}.x^n$

$=$ Real Part of $e^{iax}\frac{1}{f(D+ia)} x^n.$

v.v.imp Note → If in the method-5, dr. becomes zero then method-5 fail. In that case use method-6.

Ex-14 Solve $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x$. [AKTU-2008, 2010, 2012, 2016, 2015].

Solⁿ: Here A.E is $D^2-4D+4=0$

$\Rightarrow (D-2)^2=0 \Rightarrow D=2,2$.

$\therefore$ C.F $=(C_1+C_2x)e^{2x}$.

& $P.I = \frac{1}{D^2-4D+4} 8x^2e^{2x}\sin 2x$

$= 8\frac{1}{(D-2)^2} e^{2x}[x^2\sin 2x]$.

$= 8e^{2x}\frac{1}{(D+2-2)^2}x^2\sin 2x$ [Put $D=D+2$]

$= 8e^{2x}\frac{1}{D^2}x^2\sin 2x \longrightarrow (1)$

Now $\frac{1}{D^2}x^2\sin 2x$ = Img. Part of $\frac{1}{D^2}e^{i2x}.x^2$

= Img. Part of $e^{i2x}\frac{1}{(D+2i)^2}x^2$

= '' '' '' $\frac{e^{i2x}}{(2i)^2}\frac{1}{[1+\frac{D}{2i}]^2}x^2$

= '' '' '' $\frac{e^{i2x}}{(-4)}\left[1-\frac{Di}{2}\right]^{-2}x^2$

= '' '' '' $\frac{e^{i2x}}{(-4)}\left[1+2\frac{Di}{2}+3\frac{D^2i^2}{4}+\cdots\right]x^2$

= '' '' '' $\frac{e^{i2x}}{(-4)}\left[x^2+2\frac{i}{2}.2x-\frac{3}{4}.2\right]$

= '' '' '' $\frac{e^{i2x}}{(-4)}\left[(x^2-\frac{3}{2})+i2x\right]$

= Img. Part of $\left(\frac{\cos 2x+i\sin 2x}{(-4)}\right)\times\frac{[(2x^2-3)+i4x]}{2}$

$= -\frac{1}{8}\left[(2x^2-3)\sin 2x+4x\cos 2x\right]$

Using in (1), we get $P.I = 8e^{2x}\times -\frac{1}{8}\left[(2x^2-3)\sin 2x+4x\cos 2x\right]$

$= e^{2x}\left[(3-2x^2)\sin 2x-4x\cos 2x\right]$

Hence G. Solⁿ = C.F + P.I $= (C_1+C_2x)e^{2x}+e^{2x}\left[(3-2x^2)\sin 2x-4x\cos 2x\right]$.

Home Assignment

Ex-1 Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 e^{-x} \cos x$ [AKTU-2012, 2014.]

Ans $y = (C_1 + C_2 x)e^{-x} + e^{-x}(-x^2\cos x + 4x\sin x + 6\cos x)$.

Ex-2 Solve $(D^2-1)y = x^2 \cos x$

Ans → $y = C_1 e^{x} + C_2 e^{-x} + x\sin x + \left(\frac{1-x^2}{2}\right)\cos x$.

Ex-3 (i) Solve $(D^2-1)y = x^2 \cos x$ (ii) $(D^2+1)y = x^2 \sin 2x$.

Ex-4 Solve $(D^2+1)y = x\cos x$.

Sol^n Here A.E is $D^2+1=0$

$\therefore D = 0 \pm i$.

$\therefore C.F = C_1\cos x + C_2 \sin x$.

& $P.I = \frac{1}{D^2+1} x\cos x$ [$\because$ Note if $D^2=-1$ then Dr. becomes 0 so method-5 fail]

$= \text{Real Part of } \frac{1}{(D^2+1)} e^{ix} x$

$= R.P \text{ of } e^{ix} \frac{1}{(D+i)^2+1} x$

$= R.P \text{ of } e^{ix} \frac{1}{D^2+2iD+i^2+1} x$

$= R.P \text{ of } e^{ix} \frac{1}{D^2+2iD} x$.

$= R.P \text{ of } e^{ix} \frac{1}{2iD}\left[1+\frac{D}{2i}\right]^{-1} x$.

$= R.P \text{ of } e^{ix} \frac{i}{-2D}\left[1-\frac{Di}{2}\right]^{-1} x$

$= R.P \text{ of } e^{ix} \frac{i}{-2D}\left[1+\frac{Di}{2}+\frac{D^2i^2}{4}+\cdots\right] x$

$= R.P \text{ of } e^{ix} \frac{i}{(-2D)}\left(x+\frac{i}{2}\right)$

$= R.P \text{ of } e^{ix} \frac{i}{(-2)}\left(\frac{x^2}{2}+\frac{i}{2}x\right)$

$= R.P \text{ of } e^{ix}\left[i\frac{x^2}{(-4)}+\frac{1}{4}x\right]$

$= R.P \text{ of } [\cos x + i\sin x]\left[\frac{x}{4} - i\frac{x^2}{4}\right]$

$\Rightarrow P.I = \frac{x}{4}\cos x + \frac{x^2}{4}\sin x$. Hence $y = C.F + P.I$.

Method-7: (Special Method)

If $Q(x) = \text{Cosec}\,ax \,/\, \sec ax \,/\, \tan ax \,/\, \text{Cot}\,ax \,/$ other func^n of x.

Then this method is applicable.

Rule: ① Using Partial fraction convert $f(D)$ into linear factor as $(D+\alpha)$, $(D-\alpha)$.

② Use the formulas

(i) $\dfrac{1}{D-\alpha} Q(x) = e^{\alpha x} \int e^{-\alpha x} Q(x)\,dx$

(ii) $\dfrac{1}{D+\alpha} Q(x) = e^{-\alpha x} \int e^{\alpha x} Q(x)\,dx$.

Ex-1: Solve $(D^2+2D+2)y = e^{-x}\sec^3 x$ [AKTU-2017]

Sol^n: Here A.E is $D^2+2D+2=0$

$$\Rightarrow D = \frac{-2 \pm \sqrt{4-4.1.2}}{2} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$\Rightarrow D = -1 \pm i$$

$$\therefore \text{C.F} = e^{-1x}(C_1 \cos x + C_2 \sin x).$$

& $$\text{P.I} = \frac{1}{D^2+2D+2} e^{-x}\sec^3 x$$

$$= e^{-x} \frac{1}{(D-1)^2+2(D-1)+2} \sec^3 x \quad [\text{Put } D = D-1].$$

$$= e^{-x} \frac{1}{D^2-2D+1+2D-2+2} \sec^3 x$$

$$= e^{-x} \frac{1}{D^2+1} \sec^3 x$$

$$= e^{-x} \frac{1}{(D-i)(D+i)} \sec^3 x \quad [\text{By Partial fraction}]$$

$$= e^{-x} \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right]\sec^3 x$$

$$= \frac{e^{-x}}{2i}\left[\frac{1}{D-i}\sec^3 x - \frac{1}{D+i}\sec^3 x\right] \longrightarrow ①$$

Now $$\frac{1}{D-i}\sec^3 x = e^{ix}\int e^{-ix}\sec^3 x\,dx$$

$$= e^{ix}\int (\cos x - i\sin x)\sec^3 x\,dx$$

$$= e^{ix}\int [\sec^2 x - i\tan x \sec^2 x]\,dx.$$

$$= e^{ix}\left[\tan x - i\frac{\tan^2 x}{2}\right]. \longrightarrow ②$$

Similarly,

$$\frac{1}{D+i}\sec^3 x = e^{-ix}\left[\tan x + i\frac{\tan^2 x}{2}\right] \longrightarrow (3)$$ [Replace i by $-i$ in (2)]

Using (2) & (3) in (1), we get

$$P.I = \frac{e^{-x}}{2i}\left[e^{ix}\left\{\tan x - i\frac{\tan^2 x}{2}\right\} - e^{-ix}\left\{\tan x + i\frac{\tan^2 x}{2}\right\}\right]$$

$$= \frac{e^{-x}}{2i}\left[\tan x\,(e^{ix} - e^{-ix}) - i\frac{\tan^2 x}{2}(e^{ix} + e^{-ix})\right]$$

$$= e^{-x}\left[\tan x\,\frac{(e^{ix} - e^{-ix})}{2i} - \frac{\tan^2 x}{2}\frac{(e^{ix} + e^{-ix})}{2}\right]$$

$$= e^{-x}\left[\tan x \sin x - \frac{\tan^2 x}{2}\cos x\right]$$

$$= e^{-x}\left[\frac{\sin^2 x}{\cos x} - \frac{1}{2}\frac{\sin^2 x}{\cos x}\right]$$

$$= e^{-x}\,\frac{1}{2}\,\frac{\sin^2 x}{\cos x}$$

$$= \frac{e^{-x}}{2}\sin x \tan x.$$

$\therefore$ General Sol^n is $y = C.F + P.I$

$$\Rightarrow y = e^{-x}(C_1\cos x + C_2\sin x) + \frac{e^{-x}}{2}\sin x\tan x.$$

Ex-12: find the complete sol^n of $(D^2+a^2)\,y = \sec ax$

[AKTU-2011, 2017].

Sol^n: Here A.E is

$$D^2 + a^2 = 0 \Rightarrow D = 0 \pm ia$$

$$\therefore C.F = C_1\cos ax + C_2\sin ax.$$

$$\& \ P.I = \frac{1}{D^2+a^2}\sec ax$$

$$= \frac{1}{(D-ia)(D+ia)}\sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]\sec ax.$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia}\sec ax - \frac{1}{D+ia}\sec ax\right] \longrightarrow (1)$$

Now $\frac{1}{D-ia}\sec ax = e^{iax}\int e^{-iax}\sec ax\,dx.$

$$= e^{iax}\int [\cos ax - i\sin ax]\sec ax\,dx$$

$$= e^{iax}\int [1 - i\tan ax]\,dx$$

$$= e^{iax}\left[x + i\frac{\log\cos ax}{a}\right] \rightarrow ② \qquad \left[\because \int \tan ax\,dx = -\frac{\log\cos ax}{a}\right]$$

& $\frac{1}{D+ia}\sec ax = e^{-iax}\left[x - i\frac{\log\cos ax}{a}\right] \rightarrow ③$ [Replace i by $-i$ in eqⁿ ②]

Using ② & ③ in ① we get

$$P.I = \frac{1}{2ia}\left[e^{iax}\left\{x + i\frac{\log\cos ax}{a}\right\} - e^{-iax}\left\{x - i\frac{\log\cos ax}{a}\right\}\right]$$

$$= \frac{1}{2ia}\left[x\left(e^{iax} - e^{-iax}\right) + i\frac{\log\cos ax}{a}\left(e^{iax} + e^{-iax}\right)\right]$$

$$= \frac{x}{a}\left(\frac{e^{iax} - e^{-iax}}{2i}\right) + \frac{\log\cos ax}{a^2}\left(\frac{e^{iax} + e^{-iax}}{2}\right)$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}\log\cos ax \cdot \cos ax.$$

$\therefore$ Complete Solⁿ is

$y = C.F + P.I$

$$\Rightarrow \boxed{y = C_1\cos ax + C_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\log\cos ax \cdot \cos ax}$$

Home Assignment

Ex-1 → Solve (i) $(D^2+1)y = \sec x$

(ii) $(D^2+4)y = \sec 2x$.

Ex-2 → Solve (i) $y'' + a^2 y = \tan ax$

(ii) $y'' + 4y = \tan 2x$

Ans → $y = C_1\cos 2x + C_2\sin 2x - \frac{1}{4}\cos 2x\log(\sec 2x + \tan 2x)$

Ex-3 → Solve $(D^2+a^2)y = \operatorname{cosec} ax$

Ans $y = C_1\cos ax + C_2\sin ax + \frac{1}{a^2}\sin ax\log\sin ax - \frac{x}{a}\cos ax$.

(4) Solve $(D^2+a^2)y = \cot ax$.

(5) Find the General Solution of the differential equation

y.ar

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}. \quad \text{[AKTU- 2015]}$$

Solⁿ Given $(D^2+3D+2)y = e^{e^x}$.

Here A·E is $D^2+3D+2=0 \Rightarrow (D+1)(D+2)=0 \Rightarrow D=-1,-2$.

$\therefore$ C.F $= C_1e^{-x} + C_2e^{-2x}$.

& P.I $= \frac{1}{D^2+3D+2} e^{e^x}$

$= \frac{1}{(D+2)(D+1)} e^{e^x}$

$= \frac{1}{D+2}\left[\frac{1}{D+1} e^{e^x}\right]$

$= \frac{1}{D+2}\left[e^{-x}\int e^{x} e^{e^x} dx\right] \qquad \left[\because \frac{1}{D+\alpha} = e^{-\alpha x}\int e^{\alpha x}Q(x)\,dx\right]$

$= \frac{1}{D+2}\left[e^{-x} e^{e^x}\right] \qquad$ Let $e^x = t \Rightarrow e^x dx = dt$]

$= e^{-2x}\int e^{2x}\, e^{-x}\cdot e^{e^x} dx \qquad$ [Here $\alpha = 2$ & $Q(x) = e^{-x}e^{e^x}$]

$= e^{-2x}\int e^{x} e^{e^x} dx$

$= e^{-2x} e^{e^x}$.

$\therefore$ General Solⁿ is

$y =$ C.F + P.I

$\Rightarrow \boxed{y = C_1e^{-x} + C_2e^{-2x} + e^{-2x}e^{e^x}}$

Linear Differential Eq^n with variable Cofficient
OR
Homogeneous Linear Differential eq^n.
OR
Cauchy's Euler Equations.

The Diff. Eq^n of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n y = Q(x).$$

or $(x^n D^n + a_1 x^{n-1} D^{n-1} + a_2 x^{n-2} D^{n-2} + \cdots + a_n) y = Q(x)$

is called Homogeneous Linear Diff. Eq^n or Cauchy-Euler Eq^n, where $a_1, a_2, \ldots, a_n$ are constants & $Q(x)$ is a func^n of x.

Working Rule → (Solution by changing independent variable).

① Put $x = e^z$ or $z = \log x$.

② Replace
$$x\frac{d}{dx} = xD = D_1$$
$$x^2 \frac{d^2}{dx^2} = x^2 D^2 = D_1(D_1 - 1)$$
$$x^3 \frac{d^3}{dx^3} = x^3 D^3 = D_1(D_1 - 1)(D_1 - 2)$$
$$\vdots$$
and so on. where $D_1 = \frac{d}{dz}$

③ Now given D. Eq^n becomes Linear Diff. Eq^n with constant coefficients. Find $y = C.F + P.I$ in terms of z.

④ Put $e^z = x$ or $z = \log x$
find $y = C.F + P.I$ in terms of x.

Ex-1 Solve $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$.

Sol^n Given $(x^3 D^3 + 2x^2 D^2 + 2)y = 10\left(x + \frac{1}{x}\right) \longrightarrow ①$

Put $x = e^z$ or $z = \log x$ and

$xD = D_1$ where $D_1 = \frac{d}{dz}$.
$x^2 D^2 = D_1(D_1 - 1)$
$x^3 D^3 = D_1(D_1 - 1)(D_1 - 2)$ in eq^n ①. we get

$[D_1(D_1-1)(D_1-2) + 2D_1(D_1-1) + 2]y = 10(e^{z} + e^{-z})$

$\Rightarrow (D_1^3 - D_1^2 + 2)y = 10(e^{z} + e^{-z})$

Here A.E is $D_1^3 - D_1^2 + 2 = 0$

$\Rightarrow (D_1+1)(D_1^2 - 2D_1 + 2) = 0$

$\Rightarrow D_1 + 1 = 0, \quad D_1^2 - 2D_1 + 2 = 0$

$\Rightarrow D_1 = -1, \quad D_1 = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$\therefore$ C.F $= C_1 e^{-z} + e^{z}[C_2 \cos z + C_3 \sin z]$

& P.I $= \dfrac{1}{D_1^3 - D_1^2 + 2} 10(e^{z} + e^{-z})$

$= 10 \dfrac{1}{D_1^3 - D_1^2 + 2} e^{z} + 10 \dfrac{1}{D_1^3 - D_1^2 + 2} e^{-z}$

$= 10 \dfrac{1}{1^3 - 1^2 + 2} e^{z} + 10\, z \dfrac{1}{3D_1^2 - 2D_1} e^{-z}$

$= 5e^{z} + 10\, z \dfrac{1}{3(-1)^2 - 2(-1)} e^{-z}$

$= 5e^{z} + 10\, z \dfrac{e^{-z}}{5} = 5e^{z} + 2z e^{-z}$.

$\therefore$ G. Solⁿ is $y =$ C.F + P.I

$\Rightarrow y = C_1 e^{-z} + e^{z}[C_2 \cos z + C_3 \sin z] + 5e^{z} + 2z e^{-z}$

$\Rightarrow y = \dfrac{C_1}{x} + x[C_2 \cos(\log x) + C_3 \sin(\log x)] + 5x + 2\dfrac{\log x}{x}$.

Ex-2: Solve

$$\left(\frac{d}{dx} + \frac{1}{x}\right)^2 y = x^{-4} \qquad \text{[AKTU-2012]}$$

Sol^n: Give $\left(\frac{d}{dx} + \frac{1}{x}\right)^2 y = x^{-4}$ or $\left(\frac{d^2}{dx^2} + 2\frac{1}{x}\frac{d}{dx} + \frac{1}{x^2}\right)y = x^{-4}$

$\Rightarrow (x^2 D^2 + 2xD + 1)y = x^{-2} \longrightarrow (1)$

Put $x = e^{z}$ or $\log x = z$ & $xD = D_1$, $x^2D^2 = D_1(D_1-1)$ in eqⁿ (1), we get where $D_1 = \frac{d}{dz}$

$[D_1(D_1-1) + 2D_1 + 1]y = e^{-2z}$

$\Rightarrow [D_1^2 + D_1 + 1]y = e^{-2z}$.

Here A·E is $D_1^2+D_1+1=0$

$\therefore D_1 = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}$

$\therefore C.F = e^{-\frac{1}{2}z}\left[C_1 \cos\frac{\sqrt{3}}{2}z + C_2 \sin\frac{\sqrt{3}}{2}z\right]$

& $P.I = \dfrac{1}{D_1^2+D_1+1} e^{-2z}$

$= \dfrac{1}{(-2)^2+(-2)+1} e^{-2z} = \dfrac{1}{3} e^{-2z}$.

$\therefore$ General Sol^n is $y = C.F + P.I$

$\Rightarrow y = e^{-\frac{1}{2}z}\left[C_1 \cos\frac{\sqrt{3}}{2}z + C_2 \sin\frac{\sqrt{3}}{2}z\right] + \frac{1}{3}e^{-2z}$

$\Rightarrow y = x^{-\frac{1}{2}}\left[C_1 \cos\frac{\sqrt{3}}{2}(\log x) + C_2 \sin\frac{\sqrt{3}}{2}(\log x)\right] + \frac{1}{3}x^{-2}$.

Ex-3: Solve $(x^2D^2 - xD + 4)y = \cos(\log x) + x\sin(\log x) \longrightarrow (1)$

Sol^n: Put $x = e^z$ or $z = \log x$

& $xD = D_1$, $x^2D^2 = D_1(D_1-1)$ in eq^n (1), we get

$[D_1(D_1-1) - D_1 + 4]y = \cos z + e^z \sin z$

$\Rightarrow [D_1^2 - 2D_1 + 4]y = \cos z + e^z \sin z$

Here A·E is $D_1^2 - 2D_1 + 4 = 0$

$\Rightarrow D_1 = \dfrac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3}$.

$\therefore C.F = e^z[C_1 \cos\sqrt{3}z + C_2 \sin\sqrt{3}z]$.

& $P.I = \dfrac{1}{D_1^2-2D_1+4}[\cos z + e^z \sin z]$

$= \dfrac{1}{D_1^2-2D_1+4}\cos z + \dfrac{1}{D_1^2-2D_1+4} e^z \sin z$.

$= \dfrac{1}{-1-2D_1+4}\cos z + e^z \dfrac{1}{(D_1+1)^2-2(D_1+1)+4}\sin z$

$= \dfrac{1}{3-2D_1}\cos z + e^z \dfrac{1}{D_1^2+3}\sin z$

$= \dfrac{(3+2D_1)}{9-4D_1^2}\cos z + e^z \dfrac{1}{-1+3}\sin z$

$= \dfrac{1}{13}(3+2D_1)\cos z + \dfrac{e^z \sin z}{2} = \dfrac{1}{13}(3\cos z - 2\sin z) + \dfrac{e^z \sin z}{2}$

$\therefore$ General Sol^n is

$y = C.F + P.I$

$\Rightarrow y = e^{z}[C_1 \cos\sqrt{3}z + C_2 \sin\sqrt{3}z] + \frac{1}{13}(3\cos z - 2\sin z) + \frac{1}{2}e^{z}\sin z.$

$\Rightarrow y = x[C_1 \cos\sqrt{3}(\log x) + C_2 \sin\sqrt{3}(\log x)] + \frac{1}{13}[3\cos(\log x) - 2\sin(\log x)]$
$+ \frac{1}{2}x \sin(\log x).$

Ex-4: Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$ [AKTU - 2009]

Sol^n: Given $(x^2D^2 - xD - 3)y = x^2\log x \rightarrow ①$

Put $x = e^z$ or $\log x = z$ & $xD = D_1$, $x^2D^2 = D_1(D_1 - 1)$ in eq^n ①, we get

$[D_1(D_1-1) - D_1 - 3]y = e^{2z}.z$

$\Rightarrow [D_1^2 - 2D_1 - 3]y = e^{2z}.z.$

Here A.E is $D_1^2 - 2D_1 - 3 = 0 \Rightarrow D_1^2 - 3D_1 + D_1 - 3 = 0$

$\Rightarrow (D_1 - 3)(D_1 + 1) = 0 \Rightarrow D_1 = -1, 3.$

$\therefore C.F = C_1e^{-z} + C_2e^{3z}.$

& $P.I = \frac{1}{D_1^2 - 2D_1 - 3}e^{2z}.z$

$= e^{2z}\frac{1}{(D_1+2)^2 - 2(D_1+2) - 3}z$

$= e^{2z}\frac{1}{D_1^2 + 2D_1 - 3}z$

$= \frac{e^{2z}}{-3}\left[1 - \frac{(D_1^2 + 2D_1)}{3}\right]^{-1}z$

$= \frac{e^{2z}}{-3}\left[1 + \frac{(D_1^2 + 2D_1)}{3} + \cdots\right]z$

$= \frac{e^{2z}}{-3}\left[z + \frac{2}{3}\right]$

$\therefore$ G. Sol^n is $y = C.F + P.I = C_1e^{-z} + C_2e^{3z} - \frac{e^{2z}}{3}\left[z + \frac{2}{3}\right]$

$\Rightarrow y = \frac{C_1}{x} + C_2x^3 - \frac{x^2}{3}\left(\log x + \frac{2}{3}\right).$

Ex-5: Solve $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$ [AKTU-2013], 2012]

Sol: Given $(x^2D^2 + 4xD + 2)y = e^x \longrightarrow (1)$

Put $x = e^z$ or $\log x = z$ and $xD = D$ [Where $D_1 = \frac{d}{dz}$]

$x^2D^2 = D_1(D_1-1)$ in eqn (1), we get

$[D_1(D_1-1) + 4D_1 + 2]y = e^{e^z}$

$\Rightarrow [D_1^2 + 3D_1 + 2]y = e^{e^z}$

$\therefore$ Here A.E is $D_1^2 + 3D_1 + 2 = 0$

$\Rightarrow (D_1+1)(D_1+2) = 0$

$\Rightarrow D_1 = -1, -2.$

$\therefore$ C.F $= C_1e^{-z} + C_2e^{-2z}$

& P.I $= \frac{1}{D_1^2 + 3D_1 + 2} e^{e^z}$

$= \frac{1}{(D_1+2)(D_1+1)} e^{e^z}$

$= \frac{1}{(D_1+2)}\left[\frac{1}{D_1+1}\right]e^{e^z}$

$= \frac{1}{D_1+2}\left[e^{-z}\int e^z e^{e^z}dz\right]$ $\left[\because \frac{1}{D_1-\alpha}Q(z) = e^{\alpha z}\int e^{-\alpha z}Q(z)dz\right.$

$= \frac{1}{D_1+2}\left[e^{-z}e^{e^z}\right]$ [Put $e^z = t$ then $e^z dz = dt$]

$= e^{-2z}\int e^{2z}. e^{-z}e^{e^z}dz$

$= e^{-2z}\int e^z e^{e^z}dz = e^{-2z}e^{e^z}.$

$\therefore$ G. Soln is $y =$ C.F + P.I $= C_1e^{-z} + C_2e^{-2z} + e^{-2z}e^{e^z}.$

$\Rightarrow \boxed{y = \frac{C_1}{x} + C_2x^2 + x^{-2}e^x.}$

Ex-6: Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = (\log x)\sin(\log x)$

[AKTU-2015]

Sol: Given $(x^2D^2 + xD + 1)y = (\log x)\sin(\log x) \longrightarrow (1)$

Put $x = e^z$ or $\log x = z$ & $xD = D_1$ (where $D_1 = \frac{d}{dz}$)

$x^2D^2 = D_1(D_1-1)$ in eqn (1), we get

$[D_1(D_1-1) + D_1 + 1]y = z\sin z.$

$\Rightarrow (D_1^2+1)y = z\sin z.$

Here A.E is $D_1^2+1=0 \Rightarrow D_1 = 0 \pm i$ [$\therefore$ C.F $= C_1\cos z + C_2\sin z$]

Now P.I $= \frac{1}{D_1^2+1} z\sin z$.

$= \text{I. Part of } \frac{1}{D_1^2+1} e^{iz} \cdot z$.

$= \text{I. Part of } e^{iz} \frac{1}{(D_1+i)^2+1} z$.

$= \text{I.P of } e^{iz} \frac{1}{D_1^2+2iD_1} z$.

$= \text{I.P of } e^{iz} \frac{1}{2iD_1}\left[1+\frac{D_1}{2i}\right]^{-1} z$.

$= \text{I.P of } e^{iz} \frac{i}{-2D_1}\left[1-\frac{D_1 i}{2}\right]^{-1} z$.

$= \text{I.P of } e^{iz} \frac{i}{-2D_1}\left[1+\frac{D_1 i}{2}+\cdots\right] z$.

$= \text{I.P of } e^{iz} \frac{i}{-2D_1}\left[z+\frac{i}{2}\right]$.

$= \text{I.P of } \frac{e^{iz}}{-2} i\left[\frac{z^2}{2}+\frac{i}{2}z\right]$

$= \text{I.P of } (\cos z + i\sin z)\left[i\frac{z^2}{-4} - i^2\frac{z}{4}\right]$.

$= -\frac{z^2}{4}\cos z + \frac{z}{4}\sin z$

$\therefore$ G.Soln is $y = \text{C.F} + \text{P.I}$

$= C_1\cos z + C_2\sin z - \frac{z^2}{4}\cos z + \frac{z}{4}\sin z$.

$\Rightarrow y = C_1\cos z + C_2\sin z - \frac{z^2}{4}\cos z + \frac{z}{4}\sin z$

$\Rightarrow y = C_1\cos(\log x) + C_2\sin(\log x) - \frac{(\log x)^2}{4}\cos(\log x) + \frac{\log x}{4}\sin(\log x)$

Home Assignment

Ex-1 Solve $x^2y'' - 4xy' + 4y = 4x^2 - 6x^3$, $y(2)=4$, $y'(2)=1$ [AKTU-2009]

Ans $y = \frac{5}{3}x - \frac{23}{24}x^4 - 2x^2 + 3x^3$

Ex-2 Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \sin(\log x^2)$ [AKTU-2005]

Ans. $y = C_1\cos(\log x) + C_2\sin(\log x) - \frac{1}{3}\sin(\log x^2)$.

Ex-3 Solve $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x + \log x$.

Ans $y = C_1x^{-1} + \sqrt{x}\left\{C_2\cos\left(\frac{\sqrt{3}}{2}\log x\right) + C_3\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right\} + \frac{1}{2}x + \log x \ldots$

Ex-4→ Solve $x^2\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2\log x.$ [AKTU-2004]

Ans $y = C_1 + C_2\log x + C_3(\log x)^2 + \frac{x^3}{27}(\log x - 1).$

Ex-5→ Solve $(x^2D^2 - 3xD + 5)y = x\log x$

Ans $y = x^2[C_1\cos(\log x) + C_2\sin(\log x)] + \frac{x}{2}(1+\log x).$

Ex-6→

Solve $x^2y'' + xy' - y = x^3e^x$ [AKTU-2016]

Ans→ $y = C_1x + C_2\cdot\frac{1}{x} + \left(x - 3 + \frac{3}{x}\right)e^x.$

Ex-7→ Solve $x^2y'' + 2xy' - 12y = x^3\log x.$

Ans→ $y = C_1x^3 + C_2x^{-4} + \frac{x^3}{98}\log x(7\log x - 2).$

Ex→8→ Solve $x^2y'' - 3xy' + 5y = \sin(\log x).$

Ans $y = x^2[C_1\cos(\log x) + C_2\sin(\log x)]$
$+ \frac{1}{8}[\sin(\log x) + \cos(\log x)].$

Legendre's Homogeneous linear differential E_q^n →

A linear diff. E_q^n

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = Q(x) \longrightarrow ①$$

is called Legendre's Homogeneous linear differential E_q^n.

Rule → ① Put $(a+bx) = e^z$ or $z = \log(a+bx)$

& $(a+bx)\frac{d}{dx} = bD_1$

$(a+bx)^2 \frac{d^2}{dx^2} = b^2 D_1(D_1-1)$ where $D_1 = \frac{d}{dz}$.

$(a+bx)^3 \frac{d^3}{dx^3} = b^3 (D_1)(D_1-1)(D_1-2)$ & so on in eq^n ①.

② E_q^n ① reduces to linear diff. eq^n with constant coefficient, Then, find

$y = C.F + P.I$ in terms of z.

& then $y = C.F + P.I$ in terms of x.

Ex-1 → Solve $(3x+2)^2 \frac{d^2y}{dx^2} - (3x+2)\frac{dy}{dx} - 12y = 6x$.

[AKTU-2011]

Sol^n → Given $[(3x+2)^2 D^2 - (3x+2)D - 12]\, y = 6x \longrightarrow ①$

Put $3x+2 = e^z$ or $\log(3x+2) = z$ (where $D_1 = \frac{d}{dz}$).

& $(3x+2)D = 3D_1$, $(3x+2)^2 D^2 = 3^2 D_1(D_1-1)$ in ① we get

$$[9D_1(D_1-1) - 3D_1 - 12]\, y = 2e^z - 4.$$

$$\Rightarrow [9D_1^2 - 12D_1 - 12]\, y = 2e^z - 4.$$

Here A.E is $9D_1^2 - 12D_1 - 12 = 0$.

$\Rightarrow 3D_1^2 - 4D_1 - 4 = 0$

$\Rightarrow 3D_1^2 - 6D_1 + 2D_1 - 4 = 0$

$\Rightarrow 3D_1(D_1-2) - 2(D_1-2) = 0$

$\Rightarrow (D_1-2)(3D_1-2) = 0 \Rightarrow D_1 = 2, -\frac{2}{3}$.

$[\therefore C.F = C_1 e^{2z} + C_2 e^{-\frac{2}{3}z}]$.

$$P.I = \frac{1}{9D_1^2 - 12D_1 - 12}[2e^z - 4]$$

$$= 2\frac{1}{9D_1^2 - 12D_1 - 12}e^z - 4\frac{1}{9D_1^2 - 12D_1 - 12}e^{0z}.$$

$$= 2\frac{1}{9-12-12}e^z - 4\frac{1}{0-0-12}.1$$

$$= -\frac{2}{15}e^z + \frac{1}{13}.$$

G. Sol^n is

$\therefore\ y = C.F + P.I$

$$= C_1e^{2z} + C_2e^{-\frac{2}{3}z} - \frac{2}{15}e^z + \frac{1}{13}.$$

$$\boxed{y = C_1(3x+2)^2 + C_2(3x+2)^{-2/3} - \frac{2}{15}(3x+2) + \frac{1}{5}}$$

Home Assignment

Ex-1 → Solve $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1.$

Ans → $y = C_1(3x+2)^2 + \frac{C_2}{(3x+2)^2} + \frac{1}{108}\left[(3x+2)^2\log(3x+2) + 1\right].$

Ex-2 → Solve $(2x+1)^2\frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x.$ [AKTU-2009]

Ans → $\cancel{(1+4x^2)\frac{d^2y}{dx^2} + (1+4x)}$

$C_1(2x+1)^3 + C_2(2x+1)^{-1} - \frac{3}{16}(2x+1) + \frac{1}{4}.$

Ex-3 → Solve $(1+x)^2y'' + (1+x)y' + y = \sin 2\{\log(1+x)\}.$

Ans → $y = C_1\cos\{\log(1+x)\} + C_2\sin\{\log(1+x)\} - \frac{1}{3}\sin 2\{\log(1+x$

Ex-4 → Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos[\log(1+x)].$ [AKTU-2008]

Sol^n → Given $[(1+x)^2D^2 + (1+x)D + y] = 4\cos[\log(1+x)].$ $\longrightarrow$ ①

Put $1+x = e^z$ or $\log(1+x) = z.$

& $(1+x)D = 1.D_1,\ (1+x)^2D^2 = 1^2.D_1(D_1-1)$ where $D_1 = \frac{d}{dz}$ in ①.

$\cancel{[D_1(D_1-1)+D_1]y]=4\cos z}$. $[D_1(D_1-1)+D_1+1]y=4\cos z$

$\Rightarrow (D_1^2+1)\,y=4\cos z$.

Here A.E is $D_1^2+1=0$

$\Rightarrow D_1=0\pm i$

$\therefore$ C.F $=C_1\cos z+C_2\sin z$.

& P.I $=\dfrac{1}{D_1^2+1}4\cos z$.

$=4\dfrac{1}{D_1^2+1}\cos z$

$=4z\dfrac{1}{2D_1}\cos z$

$=2z\int\cos z\,dz=2z\sin z$.

$\therefore$ Complete Solution is

$y=$ C.F + P.I

$\Rightarrow y=C_1\cos z+C_2\sin z+2z\sin z$

$\Rightarrow \boxed{y=C_1\cos\log(1+x)+C_2\sin\log(1+x)+2\log(1+x)\sin\log(1+x)}$.

Ex 5. Solve $(x+1)^2\dfrac{d^2y}{dx^2}+(x+1)\dfrac{dy}{dx}=(2x+3)(2x+4)$

[AKTU-2011, 2006]

Ans. $y=C_1+C_2\log(x+1)+(x+1)^2+6(x+1)+[\log(x+1)]^2$.

Engineering Mathematics - II
KAS-203T
Lecture No - 7

Module - I
ODE
By. Dr. Anuj Kumar

Simultaneous Linear Differential Equation →

If two or more dependent variables are functions of a single independent variable, the equations involving their derivatives are called simultaneous linear differential Eq^n.

Ex→
$$\frac{dx}{dt} + 4y = t$$
$$\frac{dy}{dt} + 2x = e^t.$$

Note→ The method of solving these equations is based on the process of elimination, as we solve algebraic equations.

Ex-1→ The equations of motions of a particle are given by
$$\frac{dx}{dt} + \omega y = 0$$
$$\frac{dy}{dt} - \omega x = 0.$$ find the path of the particle and show that it is a circle. [AKTU-2009]

Sol^n→ Given
$$\frac{dx}{dt} + \omega y = 0 \longrightarrow (1)$$
$$\frac{dy}{dt} - \omega x = 0 \longrightarrow (2)$$

$$\Rightarrow \quad Dx + \omega y = 0 \longrightarrow (3)$$
$$Dy - \omega x = 0 \longrightarrow (4)$$

On multiplying (3) by ω & eq^n (4) by D, we get
$$\omega D\cancel{x} + \omega^2 y = 0$$
$$+ \quad \cancel{-\omega Dx} \;\underset{+}{+}\; D^2 y = 0$$
$$\overline{(D^2+\omega^2)y = 0.} \longrightarrow (5)$$

Here A·E is $D^2+\omega^2 = 0 \Rightarrow D^2 = -\omega^2$
$$\Rightarrow D^2 = i^2\omega^2 \Rightarrow D = 0 \pm i\omega \; \& \; P.I = 0$$
$$\therefore y = C.F + P.I$$
$$\Rightarrow y = C_1 \cos\omega t + C_2 \sin\omega t \longrightarrow (6)$$

From (6), $\frac{dy}{dt} = -C_1\omega \sin\omega t + C_2 \omega \cos\omega t \longrightarrow (7).$

Using eq^n (7) in eq^n (2), we get

$$x = \frac{1}{\omega}\frac{dy}{dt}.$$

$$\Rightarrow x = \frac{1}{\omega}\left[-C_1\omega\sin\omega t + C_2\omega\cos\omega t\right]$$

$$\Rightarrow x = -C_1\sin\omega t + C_2\cos\omega t \longrightarrow (8)$$

Hence eq^n (7) & (8) is the sol^n of given diff. Eq^n.

Squaring & adding eq^n (7) & (8), we get

$$x^2+y^2 = C_1^2(\cos^2\omega t + \sin^2\omega t) + C_2^2(\sin^2\omega t + \cos^2\omega t)$$

$$\Rightarrow x^2+y^2 = C_1^2+C_2^2$$

$$\Rightarrow \boxed{x^2+y^2=a^2} \quad \text{where } a^2 = C_1^2+C_2^2.$$

which is the eq^n of a circle.

Ex-2: Solve the following simultaneous diff. Eq^n

$$\frac{dx}{dt} = 3x+2y, \quad \frac{dy}{dt} = 5x+3y. \quad \text{[AKTU-2011]}.$$

Sol^n: Given $\quad \frac{dx}{dt} = 3x+2y \longrightarrow (1)$

$$\frac{dy}{dt} = 5x+3y \longrightarrow (2)$$

$$\Rightarrow (D-3)x - 2y = 0 \longrightarrow (3)$$

$$-5x + (D-3)y = 0 \longrightarrow (4)$$

Multiplying eq^n (3) by 5 & (4) by (D-3) & adding, we get

$$5(D-3)x - 10y = 0$$

$$-5(D-3)x + (D-3)^2y = 0$$

$$[(D-3)^2 - 10]\,y = 0$$

$$\Rightarrow (D^2-6D-1)\,y = 0.$$

Here A.E is $D^2-6D-1=0 \Rightarrow D = 3\pm\sqrt{10}$.

$$\therefore y = C.F = e^{3t}\left[C_1\cosh\sqrt{10}t + C_2\sinh\sqrt{10}t\right] \longrightarrow (5)$$

then (5), gives

$$\frac{dy}{dt} = e^{3t}\left[\sqrt{10}C_1\sinh\sqrt{10}t + \sqrt{10}C_2\cosh\sqrt{10}t\right] + e^{3t}\cdot 3\left[C_1\cosh\sqrt{10}t + C_2\sinh\sqrt{10}t\right] \longrightarrow (6)$$

Using eqn ⑥ in ②, we get

$$5x = \frac{dy}{dt} - 3y$$

$$= e^{3t}\sqrt{10}\,C_1 \sinh\sqrt{10}t + e^{3t}\sqrt{10}\,C_2 \cosh\sqrt{10}t$$
$$+ \cancel{3e^{3t}C_1\cosh\sqrt{10}t} + \cancel{3e^{3t}C_2\sinh\sqrt{10}t}$$
$$- \cancel{3e^{3t}C_1\cosh\sqrt{10}t} - \cancel{3e^{3t}C_2\sinh\sqrt{10}t}$$

$$\Rightarrow x = \frac{\sqrt{10}}{5} e^{3t}\left[C_1\sinh\sqrt{10}t + C_2\cosh\sqrt{10}t\right] \longrightarrow ⑦$$

Eqn ⑤ & ⑦ is the soln of given diff. eqn.

<u>Ex-3</u> Solve $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$ given that $x = y = 0$ when $t = 0$. [AKTU-2015].

<u>Soln</u> Given
$$\frac{dx}{dt} + 5x - 2y = t \longrightarrow ①$$
$$\frac{dy}{dt} + 2x + y = 0 \longrightarrow ②$$

$$\Rightarrow (D+5)x - 2y = t \longrightarrow ③$$
$$2x + (D+1)y = 0 \longrightarrow ④$$

Multiplying eqn ③ by D+1 & eqn ④ by ② & then adding, we get

$$(D+1)(D+5)x - 2\cancel{(D+1)y} = (D+1)t$$
$$+ 4x \mp \cancel{2(D+1)y} = 0$$

$$(D+1)(D+5)x + 4x = (D+1)t$$

$$\Rightarrow [D^2 + 5D + D + 5 + 4]x = (1+t)$$

$$\Rightarrow (D^2 + 6D + 9)x = (1+t) \longrightarrow ⑤$$

Here A-E is $(D^2+6D+9) = 0 \Rightarrow (D+3)^2 = 0 \Rightarrow D = -3, -3.$

$$\therefore C.F = (C_1 + C_2 t)e^{-3t}.$$

$$\& \; P.I = \frac{1}{(D+3)^2}(1+t) = \frac{1}{9}\left[1 + \frac{D}{3}\right]^{-2}(1+t)$$
$$= \frac{1}{9}\left[1 - 2\frac{D}{3} + \cdots\right](1+t) = \frac{1}{9}\left[(1+t) - \frac{2}{3}\right]$$
$$= \frac{1}{9}\left[t + \frac{1}{3}\right].$$

$\therefore\ x = C.F + P.I$

$$\Rightarrow \boxed{x = (C_1+C_2t)e^{-3t} + \frac{1}{9}\left(t+\frac{1}{3}\right)} \rightarrow ⑥$$

From ⑥, $\frac{dx}{dt} = (C_1+C_2t)e^{-3t}\times -3 + e^{-3t}C_2 + \frac{1}{9} \rightarrow ⑦$

Using ⑥ & ⑦ in eqⁿ ①, we get

$$2y = \frac{dx}{dt} + 5x - t$$

$$\Rightarrow 2y = -3(C_1+C_2t)e^{-3t} + C_2e^{-3t} + \frac{1}{9} + 5(C_1+C_2t)e^{-3t} + \frac{5}{9}\left(t+\frac{1}{3}\right) - t.$$

$$\Rightarrow 2y = +2(C_1+C_2t)e^{-3t} + C_2e^{-3t} - \frac{4}{9}t + \frac{8}{27}$$

$$\Rightarrow \boxed{y = (C_1+C_2t)e^{-3t} + \frac{C_2}{2}e^{-3t} - \frac{2}{9}t + \frac{4}{27}} \rightarrow ⑧$$

Put $t=0$ in eqⁿ ⑥ & use $x=0$, we get

$$0 = (C_1+C_2\cdot 0)e^{0} + \frac{1}{9}\left(0+\frac{1}{3}\right)$$

$$\Rightarrow \boxed{C_1 = -\frac{1}{27}}$$

Put $t=0$ in eqⁿ ⑧ & use $y=0$, we get

$$0 = (C_1+C_2\cdot 0)e^{0} + \frac{C_2}{2}e^{0} - \frac{2}{9}\cdot 0 + \frac{4}{27}$$

$$\Rightarrow C_1 + \frac{C_2}{2} + \frac{4}{27} = 0 \Rightarrow \frac{C_2}{2} = -C_1 - \frac{4}{27} = \frac{1}{27} - \frac{4}{27} = -\frac{3}{27}$$

$$\Rightarrow \boxed{C_2 = -\frac{2}{9}}.$$

Using the value of C_1 & C_2 in eqⁿ ⑥ & ⑧, we get

$$\boxed{\begin{aligned} x &= \left(-\frac{1}{27} + \frac{(-2)}{9}t\right)e^{-3t} + \frac{1}{9}\left(t+\frac{1}{3}\right) \\ y &= \left(-\frac{1}{27} - \frac{2}{9}t\right)e^{-3t} - \frac{1}{9}e^{-3t} - \frac{2}{9}t + \frac{4}{27} \end{aligned}}$$

Required Solⁿ.

Ex-4:- Solve $\frac{d^2x}{dt^2}+y=\sin t$, $\frac{d^2y}{dt^2}+x=\cos t$. [AKTU-2016]

Sol^n:- Given $\frac{d^2x}{dt^2}+y=\sin t \rightarrow (1)$

& $\frac{d^2y}{dt^2}+x=\cos t \rightarrow (2)$

$\Rightarrow \quad D^2x+y=\sin t \rightarrow (3)$

$x+D^2y=\cos t \rightarrow (4)$

Multiplying eq^n (3) by D^2, we get

$$D^4x+D^2y=D^2\sin t$$
$$\underline{-\;x+\underset{-}{D^2y}=\underset{-}{\cos t}}$$
$$(D^4-1)x=-\sin t-\cos t.$$

Here A.E is $D^4-1=0 \Rightarrow (D^2-1)(D^2+1)=0$.

$\Rightarrow D=\pm 1, 0\pm i$

$\therefore$ C.F $=C_1e^t+C_2e^{-t}+C_3\cos t+C_4\sin t$

& P.I $=\frac{1}{D^4-1}[-\sin t-\cos t]$

$=-\frac{1}{D^4-1}\sin t-\frac{1}{D^4-1}\cos t$

$=-t\frac{1}{4D^3}\sin t-t\frac{1}{4D^3}\cos t$

$=-t\frac{1}{-4D}\sin t-t\frac{1}{-4D}\cos t$ [Put $D^2=-1$]

$=\frac{t}{4}\int \sin t\,dt+\frac{t}{4}\int \cos t\,dt$

$=-\frac{t}{4}\cos t+\frac{t}{4}\sin t$.

$\therefore x=$ C.F + P.I

$\Rightarrow x=C_1e^t+C_2e^{-t}+C_3\cos t+C_4\sin t-\frac{t}{4}\cos t+\frac{t}{4}\sin t \rightarrow (5)$

from (5), $\frac{dx}{dt}=C_1e^t-C_2e^{-t}-C_3\sin t+C_4\cos t-\frac{1}{4}[-t\sin t+\cos t]$
$+\frac{1}{4}[t\cos t+\sin t]$

& $\frac{d^2x}{dt^2}=C_1e^t+C_2e^{-t}-C_3\cos t-C_4\sin t+\frac{1}{4}[t\cos t+\cancel{\sin t}]+\frac{1}{4}\sin t$
$+\frac{1}{4}[-t\sin t+\cos t]+\frac{1}{4}\cos t.$

$\Rightarrow \frac{d^2x}{dt^2}=C_1e^t+C_2e^{-t}-C_3\cos t-C_4\sin t+\frac{1}{4}t\cos t+\frac{1}{2}\sin t-\frac{t}{4}\sin t+\frac{1}{2}\cos t$
$\rightarrow (6)$

Using eqⁿ (6) in (1), we get

$$y = \sin t - \frac{d^2x}{dt^2}$$

$$= \sin t - c_1e^{t} - c_2e^{-t} + c_3\cos t + c_4\sin t - \frac{1}{4}t\cos t - \frac{1}{2}\sin t + \frac{t}{4}\sin t - \frac{1}{2}\cos t.$$

$$\Rightarrow y = -c_1e^{t} - c_2e^{-t} + c_3\cos t + c_4\sin t + \frac{t}{4}(\sin t - \cos t) + \frac{1}{2}(\sin t - \cos t) \quad \rightarrow (7)$$

Hence eqⁿ (5) & (7) are the required solⁿ.

Home Assignment

Ex-1→ Solve $\frac{dx}{dt} + 7x - y = 0$, $\frac{dy}{dt} + 2x + 5y = 0$. [AKTU-2010]

Ans→ $x = e^{-6t}(c_1\cos t + c_2\sin t)$, $y = e^{-6t}[(c_1+c_2)\cos t - (c_1-c_2)\sin t]$.

Ex-2→ Solve $\frac{dx}{dt} + x - 2y = 0$, $\frac{dy}{dt} + x + 4y = 0$; $x(0) = y(0) = 1$. [AKTU-2015].

Ans→ $x = 4e^{-2t} - 3e^{-3t}$, $y = -2e^{-2t} + 3e^{-3t}$.

Ex-3→ $\frac{dx}{dt} = 3x + 8y$, $\frac{dy}{dt} = -x - 3y$; $x(0) = 6$, $y(0) = -2$. [AKTU-2010]

Ans→ $x = 4e^{t} + 2e^{-t}$, $y = -e^{t} - e^{-t}$.

Ex-4→ Solve $\frac{dx}{dt} = y + 1$, $\frac{dy}{dt} = x + 1$.

Ans $x = c_1e^{t} + c_2e^{-t} - 1$, $y = c_1e^{t} - c_2e^{-t} - 1$.

Ex-5→ $\frac{dx}{dt} - y = e^{t}$, $\frac{dy}{dt} + x = \sin t$; $x(0) = 1$, $y(0) = 0$. [AKTU-2010, 2011]

Ans→ $x = 2\sin t + \frac{3}{2}\cos t + \frac{t}{2}\cos t - \frac{1}{2}e^{t}$, $y = \frac{1}{2}\cos t - \frac{3}{2}\sin t + \frac{t}{2}\sin t - \frac{1}{2}e^{t}$

Ex-6→ Solve

$$\frac{dx}{dt} + 2x - 3y = t, \quad \frac{dy}{dt} - 3x + 2y = e^{2t}.$$

[AKTU-2013].

Ans→

$$x = c_1e^{-5t} + c_2e^{t} + \frac{3}{7}e^{2t} - \frac{2}{5}t - \frac{13}{25}$$

$$y = c_1e^{-5t} + c_2e^{t} + \frac{4}{7}e^{2t} - \frac{3}{5}t - \frac{12}{25}.$$

Ex-7 Solve

$$\frac{d^2x}{dt^2} - 3x - 4y = 0 \quad , \quad \frac{d^2y}{dt^2} + x + y = 0.$$

Ans $x = (C_1 + C_2 t) e^{-t} + (C_3 + C_4 t) e^{t}$

$y = -\frac{1}{2}[C_1 + C_2(1+t)] e^{-t} + \frac{1}{2}[C_4(1-t) - C_3] e^{t}$.

Ex-8 Solve

$$\frac{dx}{dt} + 2x + 4y = 1 + 4t$$

$$\frac{dy}{dt} + x - y = \frac{3}{2}t^2 \quad \text{[AKTU-2013]}$$

Ans $x = C_1 e^{2t} + C_2 e^{-3t} + t + t^2$

$y = -C_1 e^{2t} + \frac{C_2}{4} e^{-3t} - \frac{t^2}{2}$.

Ex-9 Solve the simultaneous eq^n

$$\frac{dx}{dt} = -4(x+y)$$

$$\frac{dx}{dt} + 4\frac{dy}{dt} = -4y \quad \text{with cond}^n \; x(0) = 1, \; y(0) = 0$$

[AKTU-2014].

Ans $x = (1-2t) e^{-2t}$, $y = t e^{-2t}$.

Ex-10 Solve

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = y$$

$$\& \; \frac{d^2y}{dt^2} + 4\frac{dy}{dx} + 4y = 25x + 16e^{t} \quad \text{[AKTU-2010]}$$

Ex-11 Solve

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} + 3x = e^{-t}$$

$$\frac{d^2y}{dt^2} - 4\frac{dx}{dt} + 3y = \sin 2t \quad \text{[AKTU-2011]}$$

Ex-12 Solve the simultaneous equations:

$$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t$$

$$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t \quad \text{[AKTU-2011]}.$$

give the complete solution.

Example 7 *Solve the simultaneous differential equations*

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = y \quad and \quad \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 25x + 16e^t. \qquad \text{(A.K.T.U. 2018)}$$

Sol. Let $\frac{d}{dt} \equiv D$ then the given system of equations is

$$(D^2 - 4D + 4)\,x - y = 0 \qquad \text{...(1)}$$

$$-25x + (D^2 + 4D + 4)y = 16e^t \qquad \text{...(2)}$$

Operating (1) by $D^2 + 4D + 4$ and adding to (2), we get

$$(D^2 - 4D + 4)(D^2 + 4D + 4)x - 25x = 16\,e^t$$

$$\Rightarrow \quad (D^4 - 8D^2 - 9)x = 16\,e^t$$

Auxiliary equation is

$$m^4 - 8m^2 - 9 = 0$$

$$\Rightarrow \quad (m^2 - 9)(m^2 + 1) = 0 \quad \Rightarrow \quad m = \pm i, \pm 3$$

$$\therefore \quad \text{C.F.} = c_1e^{3t} + c_2e^{-3t} + c_3\cos t + c_4\sin t$$

$$\text{P.I.} = \frac{1}{D^4 - 8D^2 + 9}(16\,e^t) = 8\,e^t$$

$$\therefore \quad x = c_1e^{3t} + c_2e^{-3t} + c_3\cos t + c_4\sin t + 8e^t \qquad \text{...(3)}$$

$$\frac{dx}{dt} = 3c_1e^{3t} - 3\,c_2e^{-3t} + c_3(-\sin t) + c_4\cos t + 8e^t$$

$$\frac{d^2x}{dt^2} = 9c_1e^{3t} + 9c_2e^{-3t} - c_3\cos t - c_4\sin t + 8\,e^t$$

From (1),
$$y = \frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x$$

$$= 9c_1e^{3t} + 9c_2e^{-3t} - c_3\cos t - c_4\sin t + 8\,e^t$$
$$- 4(3c_1e^{3t} - 3c_2e^{-3t} - c_3\sin t + c_4\cos t + 8e^t)$$
$$+ 4(c_1e^{3t} + c_2e^{-3t} - c_3\cos t + c_4\sin t + 8\,e^t)$$

$$\Rightarrow \quad y = c_1e^{3t} + 25c_2e^{-3t} + (3c_3 - 4c_4)\cos t + (4c_3 + 3c_4)\sin t + 8\,e^t \quad \text{...(4)}$$

Eqns. (3) and (4) when taken together give the complete solution.

Example 9. *Solve:* $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} + 3x = e^{-t}$

$$\frac{d^2y}{dt^2} - 4\frac{dx}{dt} + 3y = \sin 2t.$$

[M.T.U. (SUM) 2011]

Sol. Let $D \equiv \dfrac{d}{dt}$ then we have

$$(D^2 + 3)\,x + Dy = e^{-t} \quad \text{...(1)}$$

$$-4Dx + (D^2 + 3)\,y = \sin 2t \quad \text{...(2)}$$

Operating (1) by $(D^2 + 3)$ and (2) by D then subtracting, we get

$$[(D^2 + 3)^2 + 4D^2]\,x = 4e^{-t} - 2\cos 2t$$

$$(D^4 + 10D^2 + 9)\,x = 4e^{-t} - 2\cos 2t$$

Auxiliary equation is

$$m^4 + 10m^2 + 9 = 0 \;\Rightarrow\; m = \pm i, \pm 3i$$

$$\text{C.F.} = c_1 \cos t + c_2 \sin t + c_3 \cos 3t + c_4 \sin 3t$$

$$\text{P.I.} = \frac{1}{D^4 + 10D^2 + 9}(4e^{-t}) - \frac{1}{D^4 + 10D^2 + 9}(2\cos 2t)$$

$$= \frac{1}{1 + 10 + 9}(4e^{-t}) - \frac{1}{16 - 40 + 9}(2\cos 2t) = \frac{1}{5}e^{-t} + \frac{2}{15}\cos 2t$$

$\therefore$
$$x = c_1 \cos t + c_2 \sin t + c_3 \cos 3t + c_4 \sin 3t + \frac{1}{5}e^{-t} + \frac{2}{15}\cos 2t \quad \text{...(3)}$$

Again operating (1) by 4D and (2) by $(D^2 + 3)$ then adding, we get

$$[(D^2 + 3)^2 + 4D^2]\,y = -4e^{-t} - \sin 2t$$

$$(D^4 + 10D^2 + 9)\,y = -4e^{-t} - \sin 2t$$

Auxiliary equation is

$$m^4 + 10m^2 + 9 = 0 \;\Rightarrow\; m = \pm i, \pm 3i$$

$$\text{C.F.} = c_5 \cos t + c_6 \sin t + c_7 \cos 3t + c_8 \sin 3t$$

$$\text{P.I.} = \frac{1}{D^4 + 10D^2 + 9}(-4e^{-t}) - \frac{1}{D^4 + 10D^2 + 9}(\sin 2t)$$

$$= -\frac{1}{5}e^{-t} + \frac{1}{15}\sin 2t$$

$\therefore$
$$y = c_5 \cos t + c_6 \sin t + c_7 \cos 3t + c_8 \sin 3t - \frac{1}{5}e^{-t} + \frac{1}{15}\sin 2t \quad \text{...(4)}$$

Equations (3) and (4), when taken together, give the complete solution.

Example 10. *Solve the simultaneous equations:*

$$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t$$

$$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t. \quad \text{(U.K.T.U. 20}$$

Sol. The given equations may be written as

$$Dx + (D-2)y = 2\cos t - 7\sin t$$

$$(D+2)x - Dy = 4\cos t - 3\sin t$$

Operating (1) by D and (2) by (D – 2), we get

$$D^2x + D(D-2)y = -2\sin t - 7\cos t$$

$$(D^2-4)x - D(D-2)y = -4\sin t - 8\cos t - 3\cos t + 6\sin t$$

Adding, we get

$$(2D^2-4)x = -18\cos t$$

$$\Rightarrow \quad (D^2-2)x = -9\cos t$$

Auxiliary equation is

$$m^2 - 2 = 0 \Rightarrow m = \pm\sqrt{2}$$

$$\text{C.F.} = c_1\cosh\sqrt{2}t + c_2\sinh\sqrt{2}t$$

$$\text{P.I.} = \frac{1}{D^2-2}(-9\cos t) = 3\cos t$$

$$\therefore \quad x = \text{C.F.} + \text{P.I.} = c_1\cosh\sqrt{2}t + c_2\sinh\sqrt{2}t + 3\cos t$$

Again, operating (1) by (D + 2) and (2) by D, we get

$$(D+2)Dx + (D^2-4)y = -2\sin t - 7\cos t + 4\cos t - 14\sin t$$

$$D(D+2)x - D^2y = -4\sin t - 3\cos t$$

Subtracting, we get

$$(2D^2-4)y = -12\sin t$$

$$(D^2-2)y = -6\sin t$$

$$\text{C.F.} = c_3\cosh\sqrt{2}t + c_4\sinh\sqrt{2}t$$

$$\text{P.I.} = \frac{1}{D^2-2}(-6\sin t) = 2\sin t$$

$$\therefore \quad y = \text{C.F.} + \text{P.I.} = c_3\cosh\sqrt{2}t + c_4\sinh\sqrt{2}t + 2\sin t$$

Eqns. (3) and (4), when taken together, give the complete solution.

Engineering Mathematics - II
KAS-203T
* Lecture No - 08 *

* Module - I *
ODE
By. DR. ANUJ KUMAR

Linear differential eqⁿ of second order →

A differential eqⁿ of the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$ is called a L.D.E of second order, where $P(x)$, $Q(x)$ & $R(x)$ are the funcⁿ of x.

Method-I (Solution by Method of Reduction of Order)

① Convert the given D.E in the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ and find P, Q, R.

② Applying the following condⁿ of P & Q. If any one of the condⁿ is satisfied, write down the corresponding part of C.F as 'u'.

		Condⁿ	Part of C.F
①	(i)	$1 + \frac{P}{a} + \frac{Q}{a^2} = 0$	$u = e^{ax}$
	(ii)	$1 + P + Q = 0$	$u = e^{x}$
	(iii)	$1 - P + Q = 0$	$u = e^{-x}$
②	(i)	$m(m-1) + Pmx + Qx^2 = 0$	$u = x^m$
	(ii)	$P + xQ = 0$	$u = x$
	(iii)	$2 + 2x.P + x^2 Q = 0$	$u = x^2$

③ Let $y = uv$ be the complete solⁿ of given D.E, where v can be determined by the formula

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

④ Put $\frac{dv}{dx} = p$ & $\frac{d^2v}{dx^2} = \frac{dp}{dx}$, then we get first order L.D.E & Salve it for p.

⑤ Put $p = \frac{dv}{dx}$ & integrate. We get v.

⑥ Complete Solⁿ is $y = u.v$.

Ex-14 Solve $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (1-\cot x)y = e^{x}\sin x$.

Soln Given $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (1-\cot x)y = e^x \sin x \rightarrow (1)$

Compairing eqn (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, we get

$P = -\cot x$, $Q = -1+\cot x$, $R = e^x \sin x$.

Here $1+P+Q = 1-\cot x - 1 + \cot x = 0$.

$\therefore$ $u = e^x$ is a part of C.F.

Let $y = u.v$ be the complete soln. Now v can be find by

$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$.

$\Rightarrow \frac{d^2v}{dx^2} + \left[-\cot x + \frac{2}{e^x}\cdot e^x\right]\frac{dv}{dx} = \frac{e^x \sin x}{e^x}$

$\Rightarrow \frac{d^2v}{dx^2} + (2-\cot x)\frac{dv}{dx} = \sin x$

$\Rightarrow \frac{dp}{dx} + (2-\cot x)p = \sin x \rightarrow (2)$ $\left[\text{Let } p = \frac{dv}{dx} \Rightarrow \frac{dp}{dx} = \frac{d^2v}{dx^2}\right]$

which is a L.D.E in p.

$\therefore$ I.F $= e^{\int(2-\cot x)dx} = e^{[2x - \log \sin x]}$

$= e^{2x}.e^{\log\frac{1}{\sin x}} = \frac{e^{2x}}{\sin x}$.

Now soln of (2) is,

$p.\text{I.F} = \int(\text{I.F} \times \sin x)dx + C_1$

$\Rightarrow p.\frac{e^{2x}}{\sin x} = \int \frac{e^{2x}}{\sin x}.\sin x \, dx + C_1$

$\Rightarrow p.\frac{e^{2x}}{\sin x} = \int e^{2x}dx + C_1$

$\Rightarrow p.\frac{e^{2x}}{\sin x} = \frac{e^{2x}}{2} + C_1 \Rightarrow p = \frac{1}{2}\sin x + C_1 e^{-2x}\sin x$.

$\Rightarrow \frac{dv}{dx} = \frac{1}{2}\sin x + C_1 e^{-2x}\sin x$

$\Rightarrow \int dv = \frac{1}{2}\int \sin x\, dx + C_1\int e^{-2x}\sin x\, dx$.

$\Rightarrow v = -\frac{\cos x}{2} + C_1 \cdot \frac{e^{-2x}}{(-2)^2+1^2}[-2\sin x - 1.\cos x] + C_2$

$\Rightarrow v = -\frac{\cos x}{2} - C_1\frac{e^{-2x}}{5}(\cos x + 2\sin x) + C_2$.

$\left[\because \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]\right]$.

Hence complete soln is $y = u.v$.

$\Rightarrow y = e^x\left[-\frac{\cos x}{2} - C_1\frac{e^{-2x}}{5}(\cos x + 2\sin x)\right] + C_2 e^x$

Ex-2→ Solve $xy'' - (2x-1)y' + (x-1)y = 0$.

Solⁿ→ Give $xy'' - (2x-1)y' + (x-1)y = 0 \Rightarrow y'' - \frac{(2x-1)}{x}y' + \frac{(x-1)}{x}y = 0 \longrightarrow (1)$

On comparing eqⁿ (1) with $y'' + Py' + Qy = R$, we get

$$P = -\frac{(2x-1)}{x}, \quad Q = \frac{x-1}{x}, \quad R = 0.$$

Here $1 + P + Q = 1 - 2 + \frac{1}{x} + 1 - \frac{1}{x} = 0$.

$\Rightarrow u = e^x$ is a part of C.F.

Now $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

$\Rightarrow \frac{d^2v}{dx^2} + \left[-2 + \frac{1}{x} + \frac{2}{e^x}\cdot e^x\right]\frac{dv}{dx} = 0$

$\Rightarrow \frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx} = 0 \qquad \left[\text{Let } p = \frac{dv}{dx} \Rightarrow \frac{dp}{dx} = \frac{d^2v}{dx^2}\right]$

$\Rightarrow \frac{dp}{dx} + \frac{1}{x}p = 0$

$\Rightarrow \frac{dp}{dx} = -\frac{1}{x}p \Rightarrow \frac{dp}{p} = -\frac{dx}{x}$

On integrating,

$\log p = -\log x + \log c_1$

$\Rightarrow \log p = \log\frac{c_1}{x} \Rightarrow p = \frac{c_1}{x}$.

$\Rightarrow \frac{dv}{dx} = \frac{c_1}{x} \Rightarrow dv = c_1\frac{1}{x}dx$.

On integrating,

$v = c_1\log x + c_2$.

Hence complete solⁿ is

$y = u \cdot v$

$\Rightarrow y = e^x[c_1\log x + c_2]$.

Ex-3→ Solve $\sin^2 x \frac{d^2y}{dx^2} = 2y$, given that $y = \cot x$ is a solution of it.

Solⁿ→ Given $\sin^2 x\frac{d^2y}{dx^2} = 2y$

$\Rightarrow \frac{d^2y}{dx^2} = 2\operatorname{cosec}^2 x \cdot y$.

$\Rightarrow \frac{d^2y}{dx^2} + 0\frac{dy}{dx} - 2\operatorname{cosec}^2 x \cdot y = 0 \longrightarrow (1)$

Here $P = 0$, $Q = -2\operatorname{cosec}^2 x$, $R = 0$.

Given $y = \cot x$ is a solⁿ of it. So let $\boxed{u = \cot x}$.

Now $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$.

$\Rightarrow \frac{d^2v}{dx^2} + \left[0 + \frac{2}{\cot x} \times -\operatorname{cosec}^2 x\right]\frac{dv}{dx} = 0$.

$\Rightarrow \frac{d^2v}{dx^2} - \frac{2\operatorname{cosec}^2 x}{\cot x}\frac{dv}{dx} = 0$. $\left[\text{Let } p = \frac{dv}{dx} \Rightarrow \frac{dp}{dx} = \frac{d^2v}{dx^2}\right]$

$\Rightarrow \frac{dp}{dx} = \frac{2\operatorname{cosec}^2 x}{\cot x} p$

$\Rightarrow \frac{dp}{p} = 2\frac{\operatorname{cosec}^2 x}{\cot x} dx$

On integrating,

$\log p = 2x - \log \cot x + \log C_1$ [Let $\cot x = t$
$-\operatorname{cosec}^2 x\, dx = dt$.

$\Rightarrow \log p = \log \tan^2 x \cdot C_1$

$\Rightarrow p = C_1 \tan^2 x$

$\Rightarrow \frac{dv}{dx} = C_1 \tan^2 x$

$\Rightarrow dv = C_1(\sec^2 x - 1)\,dx$

On integrating, $v = C_1(\tan x - x) + C_2$.

Hence complete Solⁿ is $y = C.F + P.I$ $\left[C_1 \tan x - C_1 x + C_2\right]$.

$\Rightarrow y = \cot x\left[C_1 \tan x - C_1 x + C_2\right]$.

[Home Assignment]

Ex-1 → Solve $xy'' - (3+x)y' + 3y = 0$ [Ans $y = -C_1(x^3 + 3x^2 + 6x + 6) + C_2 e^x$

Ex-2 → Solve $(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)e^x$

[Ans $y = y = -e^x - \frac{1}{4}C_1(2x+5) + C_2 e^{2x}$].

Ex-3 → Solve $y'' - 4xy' + (4x^2 - 2)y = 0$ given $y = e^{x^2}$ is a solⁿ.

Ans: $y = e^{x^2}(C_1 x + C_2)$.

Ex-4 → $x^2 y'' - (x^2 + 2x)y' + (x+2)y = x^3 e^x$ of which $y = x$ is a solⁿ.

Ans - $y = x(x-1)e^x + C_1 x e^x + C_2 x$.

Ex-5 → Solve $(x\sin x + \cos x)y'' - x\cos x\, y' + y\cos x = 0$ of which $y = x$ is a solⁿ. [Ans $y = -C_1\cos x + C_2 x$

Ex-6 → Solve $(1-x^2)y'' + xy' - y = x(1-x^2)^{3/2}$

Ans: $y = -x\frac{(1-x^2)^{3/2}}{9} - C_1\left[\sqrt{1-x^2} + x\sin^{-1}x\right] + C_2 x$.

Engineering Mathematics - II
KAS - 203T
* Lecture No - 09 *

Module - I
* ODE *
By Dr. ANUJ KUMAR

Linear Differential Equation of second order →

Method - II → By changing into Normal Form
OR
Removal of first derivative

① Compare the given D. Eqⁿ with $y'' + Py' + Qy = R$ & find P, Q & R.

② Find $u = e^{-\frac{1}{2}\int P\,dx}$.

③ Make the D. Eqⁿ $\frac{d^2v}{dx^2} + Q_1 v = R_1$ ..(Normal form)

where $Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$

& $R_1 = \frac{R}{u}$.

and solve this eqⁿ for v.

④ Putting the values of v in $\boxed{y = u \cdot v}$ to get complete Solⁿ.

Note:- When the part of C.F cannot obtain by previous method then use this.

Ex-1 → Solve the D. Eqⁿ
$y'' + 2x\,y' + (x^2 - 8)\,y = x^2 e^{-x^2/2}$ by Normal form.
[AKTU-2012, 2013]

Solⁿ →
Given $y'' + 2x\,y' + (x^2-8)\,y = x^2 e^{-x^2/2} \longrightarrow ①$

Comparing eqⁿ ① with $y'' + Py' + Qy = R$ we get
$P = 2x$, $Q = x^2 - 8$, $R = x^2 e^{-x^2/2}$.

Now $u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int 2x\,dx} = e^{-x^2/2}$.

& $Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = (x^2-8) - \frac{1}{2}\cdot 2 - \frac{1}{4}\cdot 4x^2$

$\Rightarrow Q_1 = x^2 - 9 - x^2 = -9$.

& $R_1 = \frac{R}{u} = \frac{x^2 e^{-x^2/2}}{e^{-x^2/2}} = x^2$.

Thus $\frac{d^2v}{dx^2} + Q_1 v = R_1 \Rightarrow \frac{d^2v}{dx^2} - 9v = x^2$.

$\Rightarrow (D^2-9)v = x^2$.

Here A.E is $D^2-9=0 \Rightarrow D=\pm 3$.

$\therefore$ C.F $= C_1 e^{-3x} + C_2 e^{3x}$.

& P.I $= \dfrac{1}{D^2-9} x^2 = -\dfrac{1}{9}\left[1-\dfrac{D^2}{9}\right]^{-1} x^2$

$\Rightarrow$ P.I $= -\dfrac{1}{9}\left[1+\dfrac{D^2}{9}+\dfrac{D^4}{81}+\cdots\right] x^2$

$= -\dfrac{1}{9}\left[x^2+\dfrac{2}{9}\right]$

$\therefore$ $v =$ C.F + P.I

$\Rightarrow v = C_1 e^{-3x} + C_2 e^{3x} - \dfrac{1}{9}\left[x^2+\dfrac{2}{9}\right]$.

Now the complete sol^n of given D.E is

$y = u.v$

$\Rightarrow y = e^{-x^2/2}\left[C_1 e^{-3x} + C_2 e^{3x} - \dfrac{1}{9}\left(x^2+\dfrac{2}{9}\right)\right]$.

Ex-2: Solve $\dfrac{d^2y}{dx^2} - 2\tan x \dfrac{dy}{dx} + 5y = e^x \sec x$ [AKTU-2015].

Sol^n: Given $y'' - 2\tan x\, y' + 5y = e^x \sec x \rightarrow (1)$

Here $P = -2\tan x$, $Q = 5$, $R = e^x \sec x$.

$\therefore u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -2\tan x\, dx} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$

& $Q_1 = Q - \dfrac{1}{2}\dfrac{dP}{dx} - \dfrac{1}{4}P^2$

$= 5 - \dfrac{1}{2}\cdot(-2\sec^2 x) - \dfrac{1}{4}\cdot 4\tan^2 x$.

$= 5 + \sec^2 x - \tan^2 x$

$= 5 + 1 + \tan^2 x - \tan^2 x$

$= 6$.

& $R_1 = \dfrac{R}{u} = \dfrac{e^x \cdot \sec x}{\sec x} = e^x$.

Thus $\dfrac{d^2v}{dx^2} + Q_1 v = R_1 \Rightarrow \dfrac{d^2v}{dx^2} + 6v = e^x \Rightarrow (D^2+6)v = e^x$.

Here A.E is $D^2+6=0 \Rightarrow D^2=-6 \Rightarrow D^2 = i^2 6$

$\Rightarrow D = 0 \pm i\sqrt{6}$.

$\therefore$ C.F $= C_1 \cos\sqrt{6}x + C_2 \sin\sqrt{6}x$.

& $P.I = \frac{1}{D^2+6} e^x = \frac{1}{1^2+6} e^x = \frac{e^x}{7}$.

$\therefore$ $v = C.F + P.I$

$\Rightarrow v = C_1 \cos\sqrt{6}x + C_2 \sin\sqrt{6}x + \frac{e^x}{7}$.

Hence complete solⁿ is

$y = u.v = \sec x [C_1 \cos\sqrt{6}x + C_2 \sin\sqrt{6}x + \frac{e^x}{7}]$.

Ex-3 → Solve $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2+2)y = e^{\frac{1}{2}(x^2+2x)}$ by Normal form. [AKTU-2011]

Solⁿ → Given $y'' - 2xy' + (x^2+2)y = e^{\frac{1}{2}(x^2+2x)}$ → ①

On comparing eqⁿ ① with $y'' + Py' + Qy = R$, we get

$P = -2x$, $Q = x^2+2$, $R = e^{\frac{1}{2}(x^2+2x)}$

Now $u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -2x dx} = e^{x^2/2}$.

& $Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = (x^2+2) - \frac{1}{2}(-2) - \frac{1}{4}4x^2$

$\Rightarrow Q_1 = x^2 + 2 + 1 - x^2 = 3$.

& $R_1 = \frac{R}{u} = \frac{e^{\frac{1}{2}(x^2+2x)}}{e^{x^2/2}} = \frac{e^{x^2/2}.e^x}{e^{x^2/2}} = e^x$.

Thus $\frac{d^2v}{dx^2} + Q_1 v = R_1 \Rightarrow \frac{d^2v}{dx^2} + 3v = e^x \Rightarrow (D^2+3)v = e^x$.

Here A.E is $D^2+3=0 \Rightarrow D = 0 \pm i\sqrt{3}$

$\therefore C.F = C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x$.

& $P.I = \frac{1}{D^2+3} e^x = \frac{1}{1^2+3} e^x = \frac{1}{4}e^x$.

$\therefore$ $v = C.F + P.I$

$\Rightarrow v = C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x + \frac{1}{4}e^x$.

Hence complete solⁿ is

$y = u.v$

$\Rightarrow y = e^{x^2/2}[C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x + \frac{e^x}{4}]$.

Ex-4→ Solve $\frac{d}{dx}\left(\cos^2 x \frac{dy}{dx}\right) + y\cos^2 x = 0$ by Normal form.

[(Hint)

$$\frac{d}{dx}\left(\cos^2 x \frac{dy}{dx}\right) + y\cos^2 x = 0$$

$$\Rightarrow \cos^2 x \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 2\cos x \times -\sin x + \cos^2 x\, y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + [redacted]x \cdot y = 0 \;]$$

Ans→ $y = \sec x\,[C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x]$.

$\boxed{\text{Home Assignment}}$

Ex-1→ Solve $x^2y'' - 2(x^2+x)y' + (x^2+2x+2)y = 0$ by Normal form.

Ans÷ $y = (C_1x + C_2)\,xe^{x}$. [AKTU-2019]

Ex-2→ Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-1)y = -3e^{x^2}\sin 2x$ by removal of first order derivative.

Ans÷ $y = e^{x^2}[C_1\cos x + C_2\sin x + \sin 2x]$.

Ex-3→ Solve $y'' - 4xy' + (4x^2-3)y = e^{x^2}$ [AKTU-2006]

Ans→ $y = e^{x^2}[C_1e^{x} + C_2e^{-x} - 1]$.

Ex-4→ Solve $y'' + \frac{2}{x}y' = n^2y$. [Ans÷ $y = \frac{1}{x}[C_1e^{nx} + C_2e^{-nx}]$.

Ex-5→ Solve $x\frac{d}{dx}\left(x\frac{dy}{dx} - y\right) - 2x\frac{dy}{dx} + 2y + x^2y = 0$.

Ans→ $y = x(C_1\cos x + C_2\sin x)$.

Ex-6→ Solve $\left(\frac{d^2y}{dx^2} + y\right)\cot x + 2\left(\frac{dy}{dx} + y\tan x\right) = \sec x$ [AKTU-2009]

Ans÷ $y = \frac{1}{2}(\sin x - x\cos x) + C_1x\cos x + C_2\cos x$.

Ex-7→ Solve $y'' + \frac{1}{x^{1/3}}y' + \left(\frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2}\right)y = 0$. [AKTU-2010].

Ex-8→ Solve $y'' - \frac{2}{x}y' + \left(n^2 + \frac{2}{x^2}\right)y = 0$

Ans $y = (C_1\cos nx + C_2\sin nx)\,x$.

Engineering Mathematics - II
* Lecture - 09 *
* KAS-203T *

* Module - I *
ODE
By. Dr. ANUJ KUMAR

Linear Differential eq^n of second order →

Method-III → (Solution by changing the independent variable).

Rule → ① find P, Q, R by the given D.Eq^n.

② Taking Q_1 = any constant & use the formula

$$\frac{Q(x)}{\left(\frac{dz}{dx}\right)^2} = Q_1, \text{ then solve it for } \frac{dz}{dx}, z, \frac{d^2z}{dx^2}.$$

③ Find $P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$, $R_1 = \dfrac{R}{\left(\frac{dz}{dx}\right)^2}$.

④ form the D.Eq^n $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$, by changing the independent variable x into z & solve it to find

$$y = C.F + P.I \text{ in } z.$$

⑤ Put the value of z & find the complete sol^n.

V. Imp

Note → If the coefficient of y in the given D.Eq^n is perfect square then use the method of changing the independent variable.

Ex-1 → By changing the independent variable, solve the diff. Eq^n

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4 \quad \text{[AKTU-2015]}.$$

Sol^n → Given $y'' - \frac{1}{x}y' + 4x^2y = x^4 \longrightarrow ①$

On comparing eq^n ① with $y'' + Py' + Qy = R$, we get

$$P = -\frac{1}{x}, \quad Q = 4x^2, \quad R = x^4.$$

Let $Q_1 = 4$.

Then $\frac{Q(x)}{\left(\frac{dz}{dx}\right)^2} = Q_1 \Rightarrow \frac{4x^2}{\left(\frac{dz}{dx}\right)^2} = 4 \Rightarrow \left(\frac{dz}{dx}\right)^2 = x^2.$

$$\Rightarrow \frac{dz}{dx} = x \quad \text{Then } \left.\begin{array}{l}\frac{d^2z}{dx^2} = 1 \\ \& \ z = \frac{x^2}{2}.\end{array}\right\} \rightarrow ②$$

Here,
$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{1 - \frac{1}{x}\cdot x}{\left(\frac{dz}{dx}\right)^2} = 0$$

& $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{x^2} = x^2 = 2z$ (from ②)

Now $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$

$\Rightarrow \frac{d^2y}{dz^2} + 4y = 2z \Rightarrow (D_1^2+4)y = 2z \longrightarrow$ ③

Here A.E. is $D_1^2+4=0 \Rightarrow D_1 = 0 \pm 2i$

$\therefore C.F = C_1\cos 2z + C_2\sin 2z$.

& $P.I = \frac{1}{D_1^2+4}2z = \frac{1}{4}\left[1+\frac{D_1^2}{4}\right]^{-1}2z$

$= \frac{1}{4}\left[1 - \frac{D_1^2}{4} + \cdots\right]2z$

$= \frac{2z}{4} = \frac{z}{2}$.

$\therefore y = C.F + P.I$

$\Rightarrow y = C_1\cos 2z + C_2\sin 2z + \frac{z}{2}$.

$\Rightarrow \boxed{y = C_1\cos x^2 + C_2\sin x^2 + \frac{x^2}{4}}$ (from ②) $\left[\because z = \frac{x^2}{2}\right]$

Ex-2→ Solve $\frac{d^2y}{dx^2} - \cot x\frac{dy}{dx} - y\sin^2 x = \cos x - \cos^3 x$.

[AKTU-2011, 2013]

Solⁿ Given $y'' - \cot x\, y' - \sin^2 x\, y = \cos x - \cos^3 x \longrightarrow$ ①

Here $P = -\cot x$, $Q = -\sin^2 x$, $R = \cos x - \cos^3 x = \cos x\cdot\sin^2 x$.

Let $Q_1 = -1$.

Then $\frac{Q}{\left(\frac{dz}{dx}\right)^2} = Q_1 \Rightarrow \frac{-\sin^2 x}{\left(\frac{dz}{dx}\right)^2} = -1 \Rightarrow \left(\frac{dz}{dx}\right)^2 = \sin^2 x$.

$$\left.\begin{aligned} \Rightarrow \frac{dz}{dx} = \sin x \quad \therefore\ z &= -\cos x \\ \& \frac{d^2z}{dx^2} &= \cos x \end{aligned}\right\} \longrightarrow ②$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x - \cot x \cdot \sin x}{\sin^2 x} = 0.$$

$$\& \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x \cdot \sin^2 x}{\sin^2 x} = \cos x = -z \quad (\text{from } ②)$$

$$\therefore \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

$$\Rightarrow \frac{d^2y}{dz^2} - y = -z \quad \Rightarrow (D_1^2 - 1)y = -z \longrightarrow ③$$

Here A.E is $D_1^2 - 1 = 0 \Rightarrow D_1 = \pm 1$.

$$\therefore C.F = C_1 e^{-z} + C_2 e^{z}.$$

$$\& \quad P.I = \frac{1}{D_1^2 - 1}(-z) = [1 - D_1^2]^{-1} z.$$

$$= [1 + D_1^2 + D_1^4 + \cdots] z = z$$

Complete Sol^n is $y = C.F + P.I$

$$= C_1 e^{-z} + C_2 e^{z} + z.$$

$$\Rightarrow \boxed{y = C_1 e^{\cos x} + C_2 e^{-\cos x} - \cos x} \quad [\text{from } ② \; z = -\cos x].$$

Ex-3: Solve by changing the independent variable

$$x y'' + (4x^2 - 1)y' + 4x^3 y = 2x^3 \quad [\text{AKTU} - 2013]$$

Sol^n: Given $y'' + (4x - \frac{1}{x})y' + 4x^2 y = 2x^2 \longrightarrow ①$

Here $P = 4x - \frac{1}{x}$, $Q = 4x^2$, $R = 2x^2$.

Let $Q_1 = 4$. Then $\frac{Q}{\left(\frac{dz}{dx}\right)^2} = Q_1 \Rightarrow \frac{4x^2}{\left(\frac{dz}{dx}\right)^2} = 4 \Rightarrow \left(\frac{dz}{dx}\right)^2 = x^2$.

$$\Rightarrow \frac{dz}{dx} = x \quad \therefore \quad z = \frac{x^2}{2} \;\&\; \frac{d^2z}{dx^2} = 1.$$

Also, $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{1 + (4x - \frac{1}{x})x}{x^2} = \frac{4x^2}{x^2} = 4.$$

$$\& \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{2x^2}{x^2} = 2.$$

$$\therefore \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \Rightarrow \frac{d^2y}{dz^2} + 4\frac{dy}{dz} + 4y = 2.$$

$\Rightarrow (D_1^2+4D_1+4)y = 2$

Here A.E is $D_1^2+4D_1+4=0 \Rightarrow (D_1+2)^2=0 \Rightarrow D=-2,-2.$

$\therefore$ C.F $= (C_1+C_2 z)e^{-2z}$.

& P.I $= \frac{1}{(D_1+2)^2} 2 = 2\frac{1}{(D_1+2)^2} e^{0z} = \frac{1}{2}$.

$\therefore$ Complete Sol^n is $y =$ C.F + P.I

$\Rightarrow y = (C_1+C_2 z)e^{-2z} + \frac{1}{2}$

$\Rightarrow y = (C_1 + C_2 \frac{x^2}{2}) e^{-x^2} + \frac{1}{2}$.

Home Assignment

Solve by change of independent variable.

Ex-1) $\frac{d^2y}{dx^2} + (3\sin x - \cot x)\frac{dy}{dx} + 2y\sin^2 x = e^{-\cos x}.\sin^2 x$ [AKTU-2015]

Ans $y = C_1 e^{\cos x} + C_2 e^{-\cos x} + \frac{e^{-\cos x}}{6}$.

Ex-2) Solve $(1+x)^2 y'' + (1+x)y' + y = 4\cos[\log(1+x)]$.

[Ans: $y = C_1 \cos\log(1+x) + C_2 \sin\log(1+x) + 2\log(1+x)\sin\log(1+x)$].

Ex-3) $(1+x^2)^2 y'' + 2x(1+x^2)y' + 4y = 0$.

Ans $y = C_1\cos(2\tan^{-1}x) + C_2\sin(2\tan^{-1}x)$

Ex-4) $y'' + \cot x\, y' + 4y\,\mathrm{cosec}^2 x = 0$. [AKTU-2012]

$y = C_1\cos\left(2\log\tan\frac{x}{2}\right) + C_2\sin\left(2\log\tan\frac{x}{2}\right)$

Ex-5) $x^6 y'' + 3x^5 y' + a^2 y = \frac{1}{x^2}$ [AKTU-2014].

Ans $y = C_1\cos\left(\frac{a}{2x^2}\right) + C_2\sin\left(\frac{a}{2x^2}\right) + \frac{1}{a^2x^2}$

Ex-6) $xy'' - y' - 4x^3 y = 8x^3 \sin x^2$

[Ans $y = C_1 e^{x^2} + C_2 e^{-x^2} - \sin x^2$]

Ex-7) $\cos x\, y'' + \sin x\, y' - 2y\cos^3 x = 2\cos^5 x$.

$y = C_1\cosh\sqrt{2}(\sin x) + C_2\sinh\sqrt{2}(\sin x) + \sin^2 x$

Ex-8) $y'' + \tan x\, y' + y\cos^2 x = 0$.

Engineering Mathematics - II   Module-1
* KAS - 203T *   ODE
Lecture - No - 10   By - Dr. Anuj Kumar

Linear Diff. Eq^n of Second order →

Method - IV → Method of Variation of Parameter

Rule → Given $y'' + Py' + Qy = R$

Problem based on L.D.E with constant coefficient

① Find P, Q, R.

② Find C.F of the D. Eq^n $y'' + Py' + Qy = 0$.

③ Let $C.F = C_1 u + C_2 v \longrightarrow ①$

where $W = W(u,v) = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} \neq 0$

④ Find $f(x) = -\int \frac{Rv}{W} dx$ & $g(x) = \int \frac{Ru}{W} dx$.

then $P.I = u f(x) + v g(x) \longrightarrow ②$

⑤ Complete Sol^n is $y = C.F + P.I$

Ex-1 → Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + a^2 y = \sec ax. \quad \text{[AKTU - 2013, 2014, 2015]}.$$

Sol^n → Given $y'' + a^2 y = \sec ax \longrightarrow ①$

On comparing eq^n ① with $y'' + Py' + Qy = R$, we get

$P = 0$, $Q = a^2$, $R = \sec ax$.

Take $y'' + a^2 y = 0$ or $(D^2 + a^2) y = 0$. The A.E is $D^2 + a^2 = 0$ or $D = 0 \pm ia$.

$\therefore \boxed{C.F = C_1 \cos ax + C_2 \sin ax} \longrightarrow ②$

Let $u = \cos ax$ & $v = \sin ax$.

Then $W = W(u,v) = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$

$= a\cos^2 ax + a\sin^2 ax$

$\Rightarrow \boxed{W = a \neq 0} \longrightarrow ③$.

Now

$$f(x) = -\int \frac{R\,v}{W}\,dx$$

$$= -\int \frac{\sec ax}{a}\cdot \sin ax\,dx$$

$$= -\frac{1}{a}\int \tan ax\,dx = -\frac{1}{a}\left[-\frac{1}{a}\log\cos ax\right]$$

$$= \frac{1}{a^2}\log\cos ax$$

$$\& \quad g(x) = \int \frac{R\,u}{W}\,dx = \int \frac{\sec ax}{a}\cos ax\,dx$$

$$= \frac{1}{a}\int dx = \frac{x}{a}.$$

$\therefore$ $P.I = u\,f(x) + v\,g(x)$

$$= \cos ax\left[\frac{1}{a^2}\log\cos ax\right] + \sin ax\cdot\frac{x}{a}. \longrightarrow (4)$$

Thus Complete solⁿ is

$y = C.F + P.I$

$$\Rightarrow \boxed{y = C_1\cos ax + C_2\sin ax + \frac{1}{a^2}\cos ax\log\cos ax + \frac{x}{a}\sin ax.}$$

Ex-2→ Solve by the method of variation of parameters

i) $y'' + y = \sec x$ (ii) $y'' + 4y = 2\sec 2x$.

Ex-3→ Solve the D.Eqⁿ by the method of variation of parameters

i) $y'' + a^2 y = \tan ax$

(ii) $y_2 + 4y = 4\tan 2x$

(iii) $y'' + 4y = \tan 2x$.

(iv) $y'' + y = \tan x$ [AKTU-2011, 2015, 2019].

Solⁿ→ 3(iv) → Given $y'' + y = \tan x \longrightarrow (1)$

Here $P = 0$, $Q = 1$, $R = \tan x$.

Take $y'' + y = 0 \Rightarrow (D^2+1)y = 0$.

Here A-E is $D^2+1 = 0 \Rightarrow D = 0 \pm i$ $\quad\therefore \boxed{C.F = C_1\cos x + C_2\sin x}$

Let $u = \cos x$, $v = \sin x$.

Then $W = W(u, v) = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$$= \cos^2 x + \sin^2 x = 1 \neq 0.$$

$$\Rightarrow \boxed{W = 1} \longrightarrow (3)$$

ow $f(x) = -\int \frac{Rv}{W} dx = -\int \frac{\tan x \cdot \sin x}{1} dx = -\int \frac{\sin^2 x}{\cos x} dx.$

$$\cancel{= -\int \cos x \, dx = -\sin x.}$$

$$\cancel{\& \; g(x) = \int \frac{Ru}{W} dx = \int \frac{\tan x \cdot \cos x}{1} dx}$$

$$= -\int \frac{(1 - \cos^2 x)}{\cos x} dx$$

$$= -\int (\sec x - \cos x) \, dx$$

$$\Rightarrow f(x) = -\log(\sec x + \tan x) + \sin x.$$

& $g(x) = \int \frac{Ru}{W} dx = \int \frac{\tan x \cdot \cos x}{1} dx$

$$= \int \sin x \, dx = -\cos x.$$

$\therefore$ P.I $= u f(x) + v g(x)$

$$= \cos x \left[-\log(\sec x + \tan x) + \sin x \right] + \sin x (-\cos x).$$

$$= -\cos x \log(\sec x + \tan x) + \cancel{\cos x \sin x} - \cancel{\sin x \cos x}$$

$$\boxed{\text{P.I} = -\cos x \log(\sec x + \tan x)} \longrightarrow (4).$$

Hence complete solⁿ is

$$y = \text{C.F} + \text{P.I}$$

$$\Rightarrow \boxed{y = c_1 \cos x + c_2 \sin x - \cos x \log(\sec x + \tan x)}$$

Home Assignment

Ex→1 Solve by the method of variation of parameters

i) $y'' + a^2 y = \text{Cosec}\, ax$

ii) $y'' + y = \text{Cosec}\, x$ [AKTU-2005].

[Ans $y = C_1 \cos x + C_2 \sin x - x\cos x + \sin x \log \sin x$.

Ex-2→ Solve

i) $y_2 + a^2 y = \cot ax$ ii) $y_2 + 4y = \cot 2x$

Ans:- $y = C_1 \cos ax + C_2 \sin ax + \frac{1}{a^2} \log \tan \frac{ax}{2}$.

Ex-3→ Solve by the method of variation of parameters:

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x} \cdot$$ [AKTU-2010]

Soln→ Given $y'' - y = \frac{2}{1+e^x} \rightarrow (1)$

Here $P = 0,\ Q = -1,\ R = \frac{2}{1+e^x}$.

Take $y'' - y = 0 \Rightarrow (D^2 - 1)y = 0$.

Here A·E is $D^2 - 1 = 0$

$\Rightarrow D = \pm 1$.

$$\therefore \boxed{C.F = C_1 e^x + C_2 e^{-x}} \rightarrow (2)$$

Let $u = e^x,\ v = e^{-x}$.

Then $W = W(u,v) = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2 \neq 0$.

$\Rightarrow \boxed{W = -2} \rightarrow (3)$

Now $f(x) = -\int \frac{Rv}{W} dx = -\int \frac{\frac{2}{1+e^x} \cdot e^{-x}}{-2} dx$

$= \int \frac{e^{-x}}{1+e^x} dx = \int \frac{e^{-x}\, e^{-x}\, dx}{e^{-x} + 1}$

$= -\int \frac{(t-1)\,dt}{t} = -\int \left[1 - \frac{1}{t}\right] dt$ [Let $e^{-x} + 1 = t$

$-e^{-x} dx = dt$].

$= -[t - \log t]$

$= -(e^{-x} + 1) + \log(e^{-x} + 1) = \log\frac{(1+e^x)}{e^x} - e^{-x} - 1$.

$\therefore\ y(x) = \int \frac{Ry}{W}\,dx = \int \frac{\frac{2}{1+e^x}\cdot e^x}{-2}\,dx$

$= -\int \frac{e^x}{1+e^x}\,dx$

$= -\log(1+e^x).$

$\therefore$ P.I of eqⁿ ①

$= u\,f(x) + v\,g(x)$

$= e^x\left[\log\frac{(1+e^x)}{e^x} - e^{-x} - 1\right] + e^{-x}\left[-\log(1+e^x)\right]$

Hence complete solⁿ is

$y = C.F + P.I$

$\Rightarrow y = c_1e^x + c_2e^{-x} + e^x\left[\log\frac{(1+e^x)}{e^x} - e^{-x} - 1\right] - e^{-x}\log(1+e^x).$

Home Assignment

Ex-17 Solve by method of variation of parameters

(i) $y_2 - 4y_1 + 3y = \frac{e^x}{1+e^x}$. [AKTU - 2006, 2012].

Ans: $y = c_1e^x + c_2e^{2x} - e^x - xe^{2x} + e^x\log(e^{-x}+1) + e^{2x}\log(1+e^x).$

(ii) $y'' + 2y' + y = e^{-x}\log x$

(iii) $(D^2-1)y = 2(1-e^{-2x})^{-1/2}$ [AKTU-2011]

Ans → $y = c_1e^x + c_2e^{-x} - e^x\sin^{-1}(e^{-x}) - e^{-x}(e^{2x}-1)^{1/2}.$

(iv) $\frac{dy}{dx} - 3\frac{dy}{dx} + 2y = e^{2x} + x^2$ [AKTU-2014]

Ans $y = c_1e^x + c_2e^{2x} + xe^{2x} + \frac{3}{2}x + \frac{7}{4} + \frac{1}{2}x^2 - e^{2x}.$

(v) Solve $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$ [AKTU-2017]

Ans: $y = (c_1x + c_2)e^{3x} - e^{3x}\log x.$

(vi) $y'' - 3y' + 2y = \sin e^{-x}$ [AKTU-2012]

Ans: $y = c_1e^x + c_2e^{2x} - e^{2x}\sin e^{-x}.$

(vi)

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 3x = \frac{e^t}{1+e^t}$$ [AKTU-2013]

Ans:- $y = C_1e^t + C_2e^{3t} + \frac{e^t}{2}\log(e^{-t}+1) - \frac{e^{3t}}{4}(e^{-t}+1)^2$

$- \frac{e^{3t}}{2}\log(e^{-t}+1)$

$+ e^{3t}(e^{-t}+1)$.

(vii) $y'' - 2y' + 2y = e^x \tan x$.

Ans:- $y = e^x(C_1\cos x + C_2\sin x) - e^x\cos x \log(\sec x + \tan x)$

(viii) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x$ [AKTU-2005].

Sol^n Given $y'' - 2y' + 0y = e^x\sin x \rightarrow (1)$ Here $R = e^x \sin x$.

Take $y'' - 2y' = 0 \Rightarrow (D^2 - 2D)y = 0$. Here A.E is $D^2 - 2D = 0$

$\Rightarrow D(D-2) = 0 \Rightarrow D = 0, 2$.

$\therefore \boxed{C.F = C_1e^{0x} + C_2e^{2x} = C_1 + C_2e^{2x}.} \rightarrow (2)$

Let $u = 1$, $v = e^{2x}$.

Then $W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x} \neq 0$.

Now $f(x) = -\int \frac{Rv}{W}dx = -\int \frac{e^x \sin x \cdot e^{2x}}{2e^{2x}}dx$

$= -\frac{1}{2}\int e^x \sin x \, dx$.

$= -\frac{1}{2} \frac{e^x}{1^2+1^2}[1\sin x - 1\cos x]$.

$= -\frac{1}{4}e^x(\sin x - \cos x)$ $\left[\because \int e^{ax}\sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]\right]$

& $g(x) = \int \frac{Ru}{W}dx$

$= \int \frac{e^x \sin x}{2e^{2x}} \cdot 1 \, dx$

$= \frac{1}{2}\int e^{-x}\sin x \, dx = -\frac{e^{-x}}{4}(\sin x + \cos x)$.

$\therefore$ P.I $= u f(x) + v g(x)$.

$= \frac{e^x}{4}(\sin x - \cos x) + e^{2x}\cdot\frac{(-e^{-x})}{4}(\sin x + \cos x)$

$= -\frac{1}{2}e^x \sin x$. Hence C. Sol^n is $y = C.F + P.I$

Problem based on Cauchy-Euler's Equation

Ex-4 Use the variation of parameter method to solve the D. Eqⁿ

$$x^2y'' + xy' - y = x^2e^x \quad \text{(AKTU-2018)}$$

Solⁿ: Given $x^2y'' + xy' - y = x^2e^x$

$$\Rightarrow y'' + \frac{1}{x}y' - \frac{1}{x^2}y = e^x \longrightarrow (1)$$

Here $P = \frac{1}{x}$, $Q = -\frac{1}{x^2}$, $R = e^x$.

Take $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0 \Rightarrow x^2y'' + xy' - y = 0$

$$\Rightarrow (x^2D^2 + xD - 1)y = 0 \longrightarrow (2)$$

Put $x = e^z$ or $\log x = z$ & $xD = D_1$, $x^2D^2 = D_1(D_1-1)$ in eqⁿ (2), we get

$$[D_1(D_1-1) + D_1 - 1]y = 0$$

$$\Rightarrow (D_1^2 - 1)y = 0.$$

Here A.E i.e $D_1^2 - 1 = 0 \Rightarrow D_1 = \pm 1$.

$$\therefore C.F = C_1e^z + C_2e^{-z}$$

$$\Rightarrow \boxed{C.F = C_1x + C_2 \cdot \frac{1}{x}} \longrightarrow (3)$$

Let $u = x$ & $v = \frac{1}{x}$.

Then $W = W(u,v) = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{1}{x} - \frac{1}{x} = -\frac{2}{x} \neq 0$

$$\Rightarrow \boxed{W = -\frac{2}{x}} \longrightarrow (4)$$

Now $f(x) = -\int \frac{Rv}{W}dx = -\int \frac{e^x \cdot \frac{1}{x}}{-\frac{2}{x}}dx = \frac{1}{2}\int e^x dx = \frac{1}{2}e^x$.

& $g(x) = \int \frac{Ru}{W}dx = \int \frac{e^x \cdot x}{-\frac{2}{x}}dx$

$$= -\frac{1}{2}\int x^2e^x dx$$

$$= -\frac{1}{2}\left[x^2e^x - \int 2x \cdot e^x dx\right]$$

$$= -\frac{1}{2}\left[x^2e^x - 2xe^x + 2e^x\right].$$

$$= -\frac{1}{2}x^2e^x + xe^x - e^x = -\frac{1}{2}x^2e^x + (x-1)e^x.$$

$\therefore$ P.I $= u\,f(x) + v\,g(x)$

$= x\,\frac{1}{2}e^x + \frac{1}{x}\left[-\frac{1}{2}x^2e^x + (x-1)e^x\right]$

$= \frac{x}{2}e^x - \frac{x}{2}e^x + \left(1-\frac{1}{x}\right)e^x$

$= \left(1-\frac{1}{x}\right)e^x.$

Hence Complete sol^n is $y = C.F + P.I$

$\Rightarrow \boxed{y = C_1x + C_2\frac{1}{x} + \left(1-\frac{1}{x}\right)e^x}$

Home Assignment

Solve by the method of variation of parameters

Ex-1 → $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = 0$ (AKTU-2016)

Ans → $\boxed{y = C_1x^3 + C_2x^{-4}}$

Ex-2 → $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$ (AKTU-2012).

Ans → $y = C_1\frac{1}{x} + C_2\frac{1}{x^2} + \frac{e^x}{x} + (1-x)e^x.\frac{1}{x^2}$.

Ex-3 → $x^2y'' + xy' - 9y = 48x^5$ (AKTU-2010)

Ans → $y = C_1x^3 + C_2x^{-3} + 4x^2 \times x^3 - x^8 \times x^{-3}$.

Ex-4 → $x^2y'' + 2xy' - 12y = x^3\log x$ [AKTU-2007, 2011].