5) Properties of a norm $\|x\|$

(i) $\|x\| \geq 0$; $\|x\| = 0 \iff x = \vec{0}$

(ii) $\|\alpha x\| = |\alpha| \times \|x\|$ where α is a scalar.

(iii) $\|x+y\| \leq \|x\| + \|y\|$

it is ~~easy~~ clear that our norm $\|x\| = \sqrt{\langle x, x\rangle}$ satisfies (i) and (ii). To show it satisfied (iii) we shall use the following Lemma.

Lemma: $Re(\langle u, v\rangle)^2 \leq \langle u,u\rangle \langle v,v\rangle \rightarrow (1)$

Proof: Consider $\langle u-kv, u-kv\rangle \geq 0$ for some real no. k.

by using Linearity and conjugate symmetry we have:

$$k^2 \langle v,v\rangle - k\left(\langle v,u\rangle + \overline{\langle v,u\rangle}\right) + \langle u,u\rangle \geq 0$$

Since k is real; $\Rightarrow$ ~~the discriminant~~ the equation has no real roots.

$\therefore$ discriminant ≤ 0; $\langle v,u\rangle + \overline{\langle v,u\rangle} = 2Re(\langle v,u\rangle)$

$$\Rightarrow \boxed{Re(\langle v,u\rangle)^2 \leq \langle u,u\rangle \langle v,v\rangle}$$

Now $\|x+y\| \leq \|x\| + \|y\|$ for $\|u\| = \sqrt{\langle u,u\rangle}$

$\Leftrightarrow \|x+y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$

$\Leftrightarrow \langle x+y, x+y\rangle \leq \langle x,x\rangle + \langle y,y\rangle + 2\sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}$

$\Leftrightarrow \langle x,y\rangle + \overline{\langle x,y\rangle} \leq 2\sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}$

$\Leftrightarrow Re(\langle x,y\rangle)^2 \leq \langle x,x\rangle\langle y,y\rangle$

Which is our Lemma.

Rules for a metric $d(x,y)$

(i) $d(x,x)=0$ (ii) $d(x,y)>0$; (iii) $d(x,y)=d(y,x)$.

$$d(x,z)\le d(x,y)+d(y,z)$$

for $d(x,y)= |x-y| = \cancel{\sqrt{\quad}}$ $\|x-y\|$.

$\therefore$ to show: $\|x-z\| \le \|x-y\| + \|y-z\|$

this is simply what we showed just now

$$f_n(t) = \begin{cases} 1 & t\in[-1,0] \\ 1-nt & t\in[0,\frac{1}{n}] \\ 0 & \text{otherwise} \end{cases}$$

$$\langle f_n, f_m\rangle = \int_{-1}^{1} f_n f_m \, dt \quad ; \text{WLOG let } n>m.$$

$$= \int_{-1}^{0} \cancel{\phi}\, 1\times 1\, dt + \int_{0}^{\frac{1}{n}} (1-nt)(1-mt)\, dt + 0.$$

$$= 1 + \int_0^{\frac{1}{n}} 1 - (n+m)t + nmt^2 \, dt$$

$$= 1 + \frac{1}{n} \cancel{\phi} - \frac{(n+m)}{2n^2} + \frac{nm}{3n^3}$$

$$= 1+\frac{1}{n} - \left(\frac{n+m}{2n^2}\right) + \frac{m}{3n^2}.$$

$$= \boxed{1+\frac{1}{2n} - \frac{m}{6n^2}.}$$

~~Sol~~

To show that $\{f_n\}$ is ~~converge~~ cauchy we simply consider $d(f_n, f_m)$; $n>m$

$$= \cancel{\Rightarrow} \sqrt{\langle f_n - f_m, f_n - f_m \rangle}$$

$$= \sqrt{\langle f_n | f_n \rangle + \langle f_m | f_m \rangle - 2\langle f_n | f_m \rangle}$$

$$= \sqrt{\frac{1}{3m} + \frac{m}{3n^2} - \frac{2}{3n}}$$

$$< \sqrt{\frac{1}{3m} + \frac{m}{3n^2}}$$

$$< \sqrt{\frac{1}{3m} + \frac{1}{3m}} \quad [n>m].$$

$$= \frac{1}{\sqrt{3m}}$$

$\frac{1}{\sqrt{3m}}$ ~~is a dec~~ tends to zero. $\therefore$ there exists a M such that $\frac{1}{\sqrt{3m}} < \epsilon$ ~~for all ϵ~~ for all $m > M$

$\Rightarrow$ for all real $\epsilon > 0$ there exists a M such that $d(f_n, f_m) < \epsilon$ for all $\boxed{n, m > M}$

These functions converge to $f(x) = \begin{cases} 1 & x \in [-1, 0] \\ 0 & \text{otherwise} \end{cases}$

$f(x)$ is clearly not continuous. $\therefore$ This inner product space is not complete as this cauchy sequence converges to a function outside the space.

$\Rightarrow$ Consider a cauchy sequence of vectors.

$\{v_i\}$.

Since V is finite dimensional, it has an orthogonal basis $= \{e_1, e_2 \cdots e_n\}$.

$\underline{\{v_i = \sum \alpha_k e_k}$

~~consid.~~

For this sequence to be cauchy; for all $\epsilon > 0$; there exists a M such that

$$d(v_i, v_j) < \epsilon \text{ for all } \underline{i, j > M}$$

$$\Rightarrow \langle v_i - v_j | v_i - v_j \rangle < \epsilon^2.$$

let $v_i = \sum \alpha_k e_k$, $v_j = \sum \beta_k e_k$.

$$\Rightarrow \sum (\alpha_k - \beta_k)^2 < \epsilon^2.$$

let $\gamma_k = \alpha_k - \beta_k$.

$\Rightarrow \sum_{k=1}^{n} \gamma_k^2 < \epsilon^2$; Now WLOG

let $\boxed{\gamma_i > \gamma_{i-1}}$

i.e numbering is taken by order of value.

$\therefore$ for any γ_i

$$i \gamma_i^2 < \sum_{k=1}^{n} \gamma_k^2 < \epsilon^2.$$

$$\Rightarrow \gamma_i < \frac{\epsilon}{\sqrt{i}} = \epsilon'$$

$\therefore \gamma_i < \epsilon$

$\Rightarrow |\alpha_i - \beta_i| < \epsilon$

$\therefore$ ~~$\alpha_i - \beta_i$~~ $\Rightarrow$ the coefficients of e_i form a cauchy sequence. but coefficients of e_i are real numbers. $\therefore$ this cauchy sequence converges to a real number.

$\therefore$ Each coefficient converges to a real no.

$\therefore$ ~~e~~ Vectors converge

$\Rightarrow$ Space is complete.