THE EVOLUTION OF A RANDOM VORTEX FILAMENT

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Abstract. We study an evolution problem in the space of continuous loops in three-dimensional Euclidean space modelled upon the dynamics of vortex lines in 3d incompressible and inviscid fluids. We establish existence of a local solution starting from Hölder regular loops with index greater than 1/3. When the Hölder regularity of the initial condition X is smaller or equal $1/2$ we require X to be a *rough path* in the sense of Lyons [Lyo98, LQ02]. The solution will then live in an appropriate space of rough paths. In particular we can construct (local) solution starting from almost every Brownian loop.

1. Introduction

The aim of this work is to study the well-posedness of the evolution problem for a model of a random *vortex filament* in three dimensional incompressible fluid. If u is the velocity field of the fluid, the vorticity $\omega : \mathbb{R}^3 \to \mathbb{R}^3$ is a solenoidal field defined as $\omega = \operatorname{curl} u$. A vortex filament is a field of vorticity ω which is strongly concentrated around a three-dimensional closed curve γ described parametrically as a continuous function $\gamma : [0,1] \to \mathbb{R}^3$ such that $\gamma_0 = \gamma_1$. Ideally, neglecting the transverse size of the filament, we can describe the vorticity field ω^γ generated by γ formally as the distribution

$$\omega^\gamma(x) = \Gamma \int_0^1 \delta(x - \gamma_\xi) d_\xi \gamma_\xi, \qquad x \in \mathbb{R}^3 \tag{1}$$

where $\Gamma > 0$ is the intensity of the filament. In $\mathbb{R}^3$, the velocity field associated to ω can be reconstructed with the aid of the Biot-Savart formula:

$$u^\gamma(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \wedge \omega^\gamma(y)\, dy. \tag{2}$$

which is the solution of $\operatorname{curl} u^\gamma = \omega^\gamma$ with enough decay at infinity.

Then

$$u^\gamma(x) = -\frac{\Gamma}{4\pi} \int_0^1 \frac{(x-\gamma_\xi)}{|x-\gamma_\xi|^3} \wedge\, d_\xi \gamma_\xi. \tag{3}$$

where $a \wedge b$ is the vector product of the vectors $a, b \in \mathbb{R}^3$. The evolution in time of the infinitely thin vortex filament is obtained by imposing that the curve γ is transported by the velocity field u^γ:

$$\frac{d}{dt}\gamma(t)_\xi = u^{\gamma(t)}(\gamma(t)_\xi), \qquad \xi \in [0,1] \tag{4}$$

and this gives the initial value problem

$$\frac{d}{dt}\gamma(t)_\xi = -\frac{\Gamma}{4\pi} \int_0^1 \frac{(\gamma(t)_\xi - \gamma(t)_\eta)}{|\gamma(t)_\xi - \gamma(t)_\eta|^3} \wedge\, d_\eta \gamma(t)_\eta, \qquad \xi \in [0,1] \tag{5}$$

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Even if the curve γ is smooth this expression is not well defined since the integral is divergent if γ has non-zero curvature.

To overcome this divergence a natural approach is that of Rosenhead [Ros30], who suggested the following approximate equation of motion based on a regularized kernel

$$\frac{d}{dt}\gamma(t)_\xi = -\frac{\Gamma}{4\pi}\int_0^1 \frac{(\gamma(t)_\xi - \gamma(t)_\eta)}{[|\gamma(t)_\xi - \gamma(t)_\eta|^2 + \mu^2]^{3/2}} \wedge d_\eta \gamma(t)_\eta, \qquad \xi \in [0,1] \tag{6}$$

for some $\mu > 0$. This model has clear advantages and been used in some numerical calculation of aircraft trailing vortexes by Moore [Moo72].

We will consider a generalization of the Rosenhead model where the function γ is not necessarily smooth. This is natural when we want to study models of *random* vortex filaments. Indeed for simple models of random vortex lines the curve γ is rarely smooth or even of bounded variation. Here we are thinking at taking as initial condition a typical trajectory of a Browian loop (since the path must be closed) or other simple models as Fractional Brownian Loops (to be described precisely in Sec. 5.1). As we will see later, a major problem is then the interpretation of equation (6).

The study of the dynamics of random vortex lines is suggested by some work of A. Chorin [Cho94] and G. Gallavotti [Gal02]. The main justification for the adoption of a probabilistic point of view comes from two different directions. Chorin builds discrete models of random vortex filaments to explain the phenomenology of turbulence by the statistical mechanics of these coherent structures. Gallavotti instead suggested the use of very irregular random functions to provide a natural regularization of eq. (5). Both approaches are on a physical level of rigor.

On the mathematical side in the recent year there have been some interest in the study of the statistical mechanics of continuous models for vortex filaments. P.L. Lions and A.J. Majda [LM00] proposed a statistical model of quasi-3d random vortex lines which are constrained to remain parallel to a given direction and thus cannot fold. Flandoli [Fla02] rigorously studied the problem of the definition of the energy for a random vortex filament modeled over a Brownian motion and Flandoli and Gubinelli [FG02] introduced a probability measure over Brownian paths to study the statistical mechanics of vortex filaments. The study of the energy of filament configurations has been extended to models based on fractional Brownian motion by Nualart, Rovira and Tindel [NRT03] and Flandoli and Minnelli [FM01]. Moreover a model of Brownian vortex filaments capable of reproducing the multi-fractal character [Fri95] of turbulent velocity fields has been introduced in [FG04].

The problem (6) define a natural flow on three dimensional closed curves. The study of this kind of flows has been recently emphasized by Lyons and Li in [LL04] where they study a class of flows of the form

$$\frac{dY}{dt} = F(I_f(Y)), \qquad Y_0 = X \tag{7}$$

where Y takes values in a (Banach) space of functions, F is a smooth vector field and $I_f(Y)$ is an Itô map, i.e. the map $Y \mapsto Z$ where Z is the solution of the differential equation

$$dZ_\sigma = f(Z_\sigma) dY_\sigma$$

driven by the path Y. They prove that, under suitable conditions, I_f is a smooth map and then that eq. (7) has (local) solutions and thus as a by-product that $F \circ I_f$ can be effectively considered a vector-field on a space of paths.

Our evolution equation does not match the structure of the flows considered by Lyons and Li. A very important difference is that eq. (7), under suitable assumptions on the initial condition X (e.g. X a semi-martingale path) , can be solved with standard tools of stochastic analysis (essentially Itô stochastic calculus) while eq. (6) has a structure which is not well adapted to a filtered probability space and prevents event to (easily) set-up the problem in a space of semi-martingale paths. To our opinion this peculiarity makes the problem interesting from the point of view of stochastic analysis and was one of our main motivation to start its study. Using Lyons' rough paths we will show that it is possible to give a meaning to the evolution problem (6) starting form a (Fractional or Standard) Browian Loop and that this problem has always a local solution (recall that the existence of a global solution, to our knowledge, has not been proven even in the case of a smooth curve).

The paper is organized as follows: in Sec. 2 we describe precisely the model we are going to analyze and we make some preliminary observations on the structure of covariation (in the sense of stochastic analysis) of the solution (assuming the initial condition has finite covariations), moreover we introduce the functional spaces in which we will set-up the problem of existence of solutions. In Sec. 3 we build a local solution for initial conditions which are Hölder continuous with exponent greater that 1/2 and for which the line integrals can be understood *à la* Young [You36]. In Sec. 4 we build a solution in a class of rough paths (introduced in [Gub04]) for initial conditions which are rough paths of Hölder regularity greater that 1/3 (which essentially are p-rough paths for $p < 3$, in the terminology of [LQ02]). Finally, in Sec.5 we apply these deterministic results to obtain the evolution of random initial conditions of Brownian Loop type or its Fractional variant. Appendix A collect some estimates used in the proofs.

2. The Model

2.1. **The evolution equation.** Our aim is to start a study of the true tridimensional evolution of random vortex filaments by the analysis of the well-posedness of the regularized dynamical equations. Inspired by the Rosenhead model we will be interested in studying the evolution described by

$$\frac{\partial Y(t)_\xi}{\partial t} = V^{Y(t)}(Y(t)_\xi), \qquad Y(0) = X \tag{8}$$

with initial condition X belonging to the set $\mathcal{C}$ of closed and continuous curve in $\mathbb{R}^3$ parametrized by $\xi \in [0,1]$. For any $Z \in \mathcal{C}$, V^Z is the vector-field given by the line integral

$$V^Z(x) = \int_Z A(x-y)dy = \int_0^1 A(x - Z_\xi)dZ_\xi, \qquad x \in \mathbb{R}^3 \tag{9}$$

where $A : \mathbb{R}^3 \to \mathbb{R}^3 \otimes \mathbb{R}^3$ is a matrix-valued field. In this setting the Rosenhead model is obtained by taking A of the form

$$A(x)^{ij} = -\frac{\Gamma}{4\pi}\epsilon_{ijk}\frac{x^j}{[|x|^2 + \mu^2]^{3/2}}, \qquad i, j = 1, 2, 3$$

where ϵ_{ijk} is the completely antisymmetric tensor in $\mathbb{R}^3$ normalized such that $\epsilon_{123} = 1$ and $\mu > 0$ is a fixed constant.

2.2. **A first approach using covariations for random initial conditions.** Even if the kernel of the paper is fully pathwise, before studying existence and uniqueness (in some sense to be precised) of the equation, we would like to insert a preamble concerning the stability of the quadratic variation in space, of the solution.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. In order to simplify a bit the proofs we have chosen to use the notion of covariation introduced, for instance, by [RV95].

Given two processes $X = (X_x)_{x\in[0,1]}$ and $Y = (Y_x)_{x\in[0,1]}$, the covariation $[X, Y]$ is defined (if it exists) as the limit of

$$\int_0^x (X_{x+\varepsilon} - X_x)(Y_{x+\varepsilon} - Y_x)\frac{dx}{\varepsilon}$$

in the uniform convergence in probability sense (ucp). If X and Y are classical continuous semi-martingales, it is well-known that previous $[X, Y]$ coincides with the classical covariation.

A vector $(X^1, \dots, X^n)$ of stochastic processes is said to have *all its mutual covariations* if $[X^i, X^j]$ exist for every $i, j = 1, \dots, n$. Generally here we will consider $n = 3$. Moreover, given a matrix or vector v we denote v^* its transpose.

It is sometimes practical to have a matrix notations. If $M_x = \{m^{ij}_x\}_{i,j}$, $N_x = \{n^{ij}_x\}_{i,j}$, are matrices of stochastic processes such they are compatible for the matrix product, we denote

$$[M, N]_x = \left\{\sum_{k=1}^n [m^{ik}, n^{kj}]_x\right\}_{i,j}$$

Remark 1. *The following result can be easily deduced from* [RV95]. *Let Φ_1, Φ_2 be of class $C^1(\mathbb{R}^3; \mathbb{R}^3)$, $X = (X^1, X^2, X^3)^*, Z = (Z^1, Z^2, Z^3)^*$ (intendend as row vectors in the matrix calculus) such that (X, Z) has all its mutual covariations. Then $(\Phi_1(X), \Phi_2(Z))$ has all its mutual covariations and*

$$[\Phi_1(X), \Phi_2(Z)]_x = \int_0^x d[X, Z]_y.(\nabla\Phi_1)(Z_y)^*(\nabla\Phi_2)(X_y).$$

Remark 2. *In reality, we could have chosen the modified Föllmer* [Föl81] *approach appearing in* [ERV02]*, based on discretization procedures for which the common reader would be more accustomed.*

In that case the same results stated in Remark 1 and Proposition 1 will be valid also in this discretization framework. We recall briefly that context.

Consider a family of subdivisions $0 = x^n_0 < \ldots < x^n_N = 1$ of the interval $[0, 1]$ or simply $0 = x_0 < \ldots < x_N = 1$ of the interval $[0, 1]$, when the step n is implicit. We will say that the mesh of the subdivision converges to zero if $\lim_{n\to\infty} x^n_{i+1} - x^n_i$ go to zero.

In this framework, the covariation $[X, Y]$ is defined (if it exists) as the limit of

$$\sum_{i=1}^N (X_{x_{i+1}\wedge x} - X_{x_i\wedge x})(Y_{x_{i+1}\wedge x} - Y_{x_i\wedge x})$$

in the ucp (uniform convergence in probability) sense with respect to x and the limit does not depend on the chosen family of subdivisions.

We will consider the following framework.

Suppose there exists a sub Banach space B of $C([0,1];\mathbb{R}^3)$ and let $V:(\gamma,y)\to V^\gamma(y)$ be a Borel map from $B\times\mathbb{R}^3$ to $\mathbb{R}^3$ such that

V1) for fixed $\gamma\in B$, $y\to V^\gamma(y)$ is $C^1_{\mathrm{b}}(\mathbb{R}^3;\mathbb{R}^3)$;
V2) the application $\gamma\to\|\nabla V^\gamma\|_\infty$ is locally bounded from B to $\mathbb{R}$.

The main motivation for this abstract framework comes from the setting described in the following section. Indeed, as we will see, there exists natural Banach sub-spaces B of $C([0,1],\mathbb{R}^3)$ such that the map V defined as

$$V^\gamma(x)=\int_{\mathbb{R}} A(x-\gamma_\xi)d^*\gamma_\xi,$$

where d^* denotes some kind of path integration defined for every $\gamma\in B$, satisfy the above hypoteses V1) and V2).

Proposition 1. *Suppose that a random field $(Y(t)_x)$ is a continuous solution of*

$$Y(t)_x=X_x+\int_0^t V^{Y(s)}(Y(s)_x)ds; \tag{10}$$

and $t\to Y(t)$ from $[0,T]\to B$ with initial condition X having all its mutual covariations. Then at each time t the path $Y(t)$ has all its mutual covariations. Moreover

$$[Y(t),Y^*(t)]_y=\int_0^y M(t)_\xi d[X,X^*]_\xi(M(t)_\xi)^* \tag{11}$$

where

$$M(t)_\xi:=\exp\left[\int_0^t(\nabla V^{Y(s)})(Y(s)_\xi)ds\right]. \tag{12}$$

Remark 3. *Note the following:*

(1) *Since we are in the multidimensional (3-dimensional) case, we recall that $E_\xi(t)=\exp\left[\int_0^t Q_\xi(s)ds\right]$ is defined as the matrix-valued function satisfying the differential equation*
$$\frac{dE_\xi(t)}{dt}=Q_\xi(t)E_\xi(t),\qquad E_\xi(0)=Id.$$
(2) *A typical case of initial condition of process having all its mutual covariation is a 3-dimensional Brownian loop (bridge). In this case, as we will see in Sec.4, the function $(t,\xi)\mapsto M(t)_\xi$ will be relevant in the construction of the solution in a space of rough paths.*
(3) *It is possible to adapt this proof to the situation where the solution exists up to a random time.*

Proof. For simplicity, we prolongate the processes X parametrized by $[0,1]$ setting $X_\xi=X_1,\xi\geq 1$.

Let $\varepsilon>0$. For $t\in[0,T],\xi\in[0,1]$, we write

$$Z^\varepsilon(t)_\xi=Y(t)_{\xi+\varepsilon}-Y(t)_\xi,\quad X^\varepsilon_\xi=X_{\xi+\varepsilon}-X_\xi,$$

so that

$$\begin{aligned} Z^\varepsilon(t)_\xi &= X^\varepsilon_\xi + \int_0^t [V^{Y(s)}(Y(s)_{\xi+\varepsilon}) - V^{Y(s)}(Y(s)_\xi)]ds \\ &= X^\varepsilon_\xi + \int_0^t (\nabla V^{Y(s)})(Y(s)_\xi) Z^\varepsilon(s)_\xi \, ds \\ &\quad + \int_0^t R^\varepsilon(s)_\xi Z(s)_\xi \, ds \end{aligned}$$

where

$$R^\varepsilon(s)_\xi = \int_0^t ds \int_0^1 da \left[(\nabla V^{Y(s)})(Y(s)_\xi + aZ^\varepsilon(s)_\xi) - (\nabla V^{Y(s)})(Y(s)_\xi)\right]$$

so that

$$\sup_{s\le T} |R^\varepsilon(s)_\xi| \to 0, \quad a.s.$$

Therefore,

$$Z^\varepsilon(t)_\xi = \exp\left[\int_0^t ds (\nabla V^{Y(s)})(Y(s)_\xi) + \int_0^t ds R^\varepsilon(s)_\xi\right] X^\varepsilon_\xi = M(t)^\varepsilon_\xi X^\varepsilon_\xi.$$

Multipling both sides by their transposed, dividing by ε and integrating from 0 to y we get

$$\int_0^y \frac{Z^\varepsilon(t)_\xi (Z^\varepsilon)^*(t)_\xi}{\varepsilon} d\xi = \int_0^y M(t)^\varepsilon_\xi \frac{X^\varepsilon_\xi (X^\varepsilon)^*_\xi}{\varepsilon} (M(t)^\varepsilon_\xi)^* d\xi.$$

Then since, as $\varepsilon \to 0$,

$$M(t)^\varepsilon_\xi \to \exp\left[\int_0^t (\nabla V^{Y(s)}(Y(s)_\xi) ds\right] = M(t)_\xi$$

uniformly in t and ξ almost surely, using Lebesgue dominated convergence theorem, and similar arguments to Proposition 2.1 of [RV95], it is enough to show that

$$\int_0^y \exp\left[\int_0^t ds (\nabla V^{Y(s)})(Y(s)_\xi)\right] \frac{X^\varepsilon_\xi (X^\varepsilon_\xi)^*}{\varepsilon} \exp\left[\int_0^t ds (\nabla V^{Y(s)})(Y(s)_\xi)\right]^* d\xi$$

converges ucp to the right member of (11).

This is obvious since

$$\int_0^y \frac{X^\varepsilon_\xi (X^\varepsilon_\xi)^*}{\varepsilon} d\xi \to [X, X^*](y)$$

ucp with respect to y and so, by means of subsequence extraction, we can make use of the weak $\star$-topology. □

Remark 4. *We make the same assumptions as the assumptions before (10) Suppose that the initial condition has all its $n-$ mutual covariations $n \geq 3$, see for this* [ER03]. *Proceeding in a similar way as before, it is possible to show that $Y(t, \cdot)$ has all its finite mutual n - covariations.*

The scalar analogous of Remark 3 2) would be the following. If the initial condition X has strong n-finite variation then at each time t.

A typical example of process having a strong finite n-variation is fractional Brownian motion with Hurst index $H = 1/n$.

2.3. **The functional space framework.** The actual differential evolution problem when X has finite-length has been previously studied in [BB02] where it is proved that under some regularity conditions on A there exists a unique local solution living in the space H^1_c of closed curves with L^1 derivative.

We would like to be able to solve the Cauchy problem (8) starting from a random curve X like a 3d Brownian Loop (since it must be closed) or a Fractional Brownian Loop. In these cases X is a.s. not in H^1_c and, as a consequence, we need a sensible definitions to the path-integral appearing in eq. (9).

Even if X is a Brownian Loop do not exists a simple strategy to give a well defined meaning to the evolution problem (8) through the techniques of stochastic calculus. Indeed we can think to define the integral in V^Y as an Itô or Stratonovich integral which requires Y to be a semi-martingale with respect to some filtration $\mathcal{F}$ (e.g. the filtration generated by X). However we readily note that the problem has no relationship with any natural filtration $\mathcal{F}$ since for example to compute the velocity field $V^Y(x)$ in some point x we need information about the whole trajectory of Y.

A viable (and relatively straightforward) strategy to give a well defined meaning to the problem is then that of using a path-wise approach . We then require that the initial data has γ-Hölder regularity.

When $\gamma > 1/2$ the line integral appearing in the definition (9) of the instantaneous velocity field $V^{Y(t)}$ can be understood *à la* Young [You36]. The corresponding results will be presented in Sec. 3.

When $1/2 \geq \gamma > 1/3$ an appropriate notion of line integral has been formulated by Lyons in [Lyo98, LQ02]. In Sec. 4 we will show that given an initial γ-Hölder path X (and an associated *area process* $\mathbb{X}^2$) there exists a unique local solution of the problem (13) in the class $\mathcal{D}_X$ of paths *weakly-controlled* by X. The class $\mathcal{D}_X$ has been introduced in [Gub04] to provide an alternative formulation of Lyons' theory of integration and corresponds to paths $Z \in \mathcal{C}$ which locally behaves as X in the sense that

$$Z_\xi - Z_\eta = F_\eta(X_\xi - X_\eta) + \mathrm{o}(|\xi-\eta|^\gamma)$$

where $F \in C([0,1], \mathbb{R}^3 \otimes \mathbb{R}^3)$ is a path taking values in the bounded endomorphisms of $\mathbb{R}^3$.

In particular these results provide solutions of the problem when X is a Fractional Brownian Loop of Hurst-index $H > 1/3$ (see Sec.5).

3. Evolution for γ-Hölder curves with $\gamma > 1/2$

3.1. **Setting and notations.** For any $X \in \mathcal{C}$ let

$$\|X\|_\gamma := \sup_{\xi,\eta\in[0,1]} \frac{|X_\xi - X_\eta|}{|\xi-\eta|^\gamma}$$

and

$$\|X\|_\infty := \sup_{\xi\in[0,1]} |X_\xi|$$

moreover

$$\|X\|^*_\gamma := \|X\|_\infty + \|X\|_\gamma$$

Denote $\mathcal{C}^\gamma$ the set of paths $X \in \mathcal{C}$ such that $\|X\|^*_\gamma < \infty$.

All along this section we will assume that γ is a fixed number greater than 1/2. In this case the following result states that there exists a unique extension to the Riemann-Stieltjes integral $\int f dg$ defined for smooth functions f, g to all $f, g \in \mathcal{C}^\gamma$.

Proposition 2 (Young's integral)**.** *Let $X, Y \in \mathcal{C}^\gamma$, then $\int_\eta^\xi X_\rho dY_\rho$ is well defined, coincide with the Riemann-Stieltjes integral when the latter exists and satisfy the following bound*

$$\left| \int_\eta^\xi (X_\rho - X_\eta) dY_\rho \right| \leq C_\gamma \|X\|_\gamma \|Y\|_\gamma |\xi - \eta|^{2\gamma}$$

for all $\xi, \eta \in [0,1]$ where C_γ is a constant depending only on γ.

Proof. See e.g.[You36, Lyo98]. □

It will be convenient to introduce the integrated form of (8) as

$$Y(t)_\xi = X_\xi + \int_0^t V^{Y(s)}(Y(s)_\xi) ds \tag{13}$$

Consider the Banach space $\mathbf{X}_T := C([0,T], \mathcal{C}^\gamma)$ with norm

$$\|Y\|_{\mathbf{X}_T} =: \sup_{t \in [0,T]} \|Y(t)\|_\gamma^*, \qquad Y \in \mathbf{X}_T.$$

Solution of (13) will then be found as fixed points of the non-linear map $F : \mathbf{X}_T \to \mathbf{X}_T$ defined as

$$F(Y)(t)_\xi = X_\xi + \int_0^t V^{Y(s)}(Y(s)_\xi)\, ds, \qquad t \in [0,T], \xi \in [0,1]. \tag{14}$$

where the application $Z \mapsto V^Z$ is defined for any $Z \in \mathcal{C}^\gamma$ as in eq. (9) with the line integral understood according to proposition 2 and with the matrix field A satisfying regularity conditions which will be shortly specified.

On m-tensor field $\varphi : \mathbb{R}^3 \to \mathbb{R}^m$ and for any integer $n \geq 0$ we define the following norm:

$$\|\varphi\|_n := \sum_{k=0}^n \|\nabla^n \varphi\|$$

where

$$\|\varphi\| := \sup_{x \in \mathbb{R}^3} |\varphi(x)|.$$

Then we can state:

Theorem 1. *Assume $\|\nabla A\|_2 < \infty$ and $X \in \mathcal{C}^\gamma$. Then there exists a time $\overline{T}$ depending only on $\|\nabla A\|_2, X, \gamma$ such that the equation (13) has a unique solution bounded in $\mathcal{C}^\gamma$.*

Proof. Lemma 2 below together with lemma 3 prove that on a small enough time interval F is a strict contraction on $\mathbf{X}_T$ having a unique fixed point. □

Before giving the lemmas used in the proof we will state useful results which will allow to control the velocity field V^Y in terms of the regularity of Y and of A.

Lemma 1. *Let $Y, \widetilde{Y} \in \mathcal{C}^\gamma$. For any integer n the following estimates holds:*

$$\|\nabla^n V^Y\|_\infty \leq C_\gamma \|\nabla^{n+1} A\|_\infty \|Y\|_\gamma^2 \tag{15}$$

$$\|\nabla^n V^Y - \nabla^n V^{\widetilde{Y}}\|_\infty \leq C_\gamma \|Y - \widetilde{Y}\|_\gamma^* \|\nabla^{n+1} A\|_1 (\|Y\|_\gamma + \|\widetilde{Y}\|_\gamma + 3\|\widetilde{Y}\|_\gamma^2) \tag{16}$$

Proof. See Appendix, section A.1. □

3.2. **Local existence and uniqueness.** In the proof we will need often to use Taylor expansions with integral remainders, so for convenience we introduce a special notation: given $X \in \mathcal{C}$ let $X_{\eta\xi} := X_\eta - X_\xi$ and $X^r_{\eta\xi} := J_r(X_\eta, X_\xi)$ where $J_r(x,y)$ is the linear interpolation

$$J_r(x,y) = (x-y)r + y$$

for $r \in [0,1]$.

Lemma 2. *Assume $\|\nabla A\|_1 < \infty$. For any initial datum $X \in \mathcal{C}^\gamma$ there exists a time T_0 such that for any time $T < T_0$ the set*

$$Q_T := \{Y \in \mathbf{X}_T : \|Y\|_{\mathbf{X}_T} \le B_T\}$$

where B_T is a suitable constant, is invariant under F.

Proof. Let us compute

$$|F(Y)(t)_\xi| \le |X_\xi| + \int_0^t |V^{Y(s)}(Y(s)_\xi)| ds \le |X_\xi| + \int_0^t \|V^{Y(s)}\|_\infty ds$$

so

$$\|F(Y)(t)\|_\infty \le \|X\|_\infty + \int_0^t \|V^{Y(s)}\|_\infty \|Y(s)\|_\gamma \, ds.$$

The γ-Hölder norm of the path $F(Y)(t)$ can be estimated starting from the Taylor expansion

$$V^{Y(s)}(Y(s)_\xi) - V^{Y(s)}(Y(s)_\rho) = \int_0^1 \nabla V^{Y(s)}(Y(s)^r_{\xi\rho}) dr \, Y(s)_{\xi\rho}$$

so that

$$\|V^{Y(s)}(Y(s).)\|_\gamma \le \|\nabla V^{Y(s)}\|_\infty \|Y(s)\|_\gamma$$

then

$$\|F(Y)\|_{\mathbf{X}_T} \le \|X\|^*_\gamma + \int_0^T \|V^{Y(s)}\|_1 \|Y(s)\|^*_\gamma ds$$

and, using lemma 1,

$$\|F(Y)\|_{\mathbf{X}_T} \le \|X\|^*_\gamma + C_\gamma T \|\nabla A\|_1 (\|Y\|^2_{\mathbf{X}_T} + \|Y\|^3_{\mathbf{X}_T})$$

Let B_T be a solution of

$$B_T \le \|X\|^*_\gamma + C_\gamma T \|\nabla A\|_1 (B_T^2 + B_T^3)$$

which exists for any $T < T_0$ where T_0 is a constant depending only on $\|X\|^*_\gamma$, $\|\nabla A\|_1$ and γ. Then if $\|Y\|_{\mathbf{X}_T} \le B_T$ we have $\|F(Y)\|_{\mathbf{X}_T} \le B_T$ and Q_T is invariant under F.

□

Lemma 3. *Assume $\|\nabla A\|_2 < \infty$. There exists a time $\overline{T} < T_0$ such that the map F is a strict contraction on $Q_{\overline{T}}$.*

Proof. To prove that the map F is a strict contraction proceed as follows: for any $T < T_0$ take $Y, \widetilde{Y} \in Q_T$ Then

$$\begin{aligned}
&|V^{Y(s)}(Y(s)_\xi) - V^{\widetilde{Y}(s)}(\widetilde{Y}(s)_\xi)| \\
&\qquad \le |V^{Y(s)}(Y(s)_\xi) - V^{\widetilde{Y}(s)}(Y(s)_\xi)| + |V^{Y(s)}(Y(s)_\xi) - V^{Y(s)}(\widetilde{Y}(s)_\xi)| \\
&\qquad \le \|V^{Y(s)} - V^{\widetilde{Y}(s)}\|_\infty + \|\nabla V^{Y(s)}\| \|Y(s) - \widetilde{Y}(s)\|_\infty
\end{aligned}$$

where we used Lemma 1 to estimate the norms of V.

Next,

$$\begin{aligned}
&[V^{Y(s)}(Y(s)_\xi) - V^{\widetilde{Y}(s)}(\widetilde{Y}(s)_\xi)] - [V^{Y(s)}(Y(s)_\rho) - V^{\widetilde{Y}(s)}(\widetilde{Y}(s)_\rho)] \\
&\qquad = V^{Y(s)}(Y(s)_\xi) - V^{\widetilde{Y}(s)}(\widetilde{Y}(s)_\xi) - V_s(Y(s)_\rho) + V^{\widetilde{Y}(s)}(\widetilde{Y}(s)_\rho) \\
&\qquad = \int_0^1 \nabla V^{Y(s)}(Y^r(s)_{\xi\rho}) dr Y(s)_{\xi\rho} - \int_0^1 \nabla V^{\widetilde{Y}(s)}(\widetilde{Y}^r(s)_{\xi\rho}) dr \widetilde{Y}(s)_{\xi\rho} \\
&\qquad = \int_0^1 \nabla V^{Y(s)}(Y^r(s)_{\xi\rho}) dr (Y(s)_{\xi\rho} - \widetilde{Y}(s)_{\xi\rho}) \\
&\qquad\qquad + \int_0^1 \left[\nabla V^{Y(s)}(Y^r(s)_{\xi\rho}) - \nabla V^{\widetilde{Y}(s)}(\widetilde{Y}^r(s)_{\xi\rho})\right] dr \widetilde{Y}(s)_{\xi\rho} \qquad (17) \\
&\qquad = \int_0^1 \nabla V^{Y(s)}(Y^r(s)_{\xi\rho}) dr (Y(s)_{\xi\rho} - \widetilde{Y}(s)_{\xi\rho}) \\
&\qquad\qquad + \int_0^1 \left[\nabla V^{Y(s)}(Y^r(s)_{\xi\rho}) - \nabla V^{Y(s)}(\widetilde{Y}^r(s)_{\xi\rho})\right] dr \widetilde{Y}(s)_{\xi\rho} \\
&\qquad\qquad + \int_0^1 \left[\nabla V^{Y(s)}(\widetilde{Y}^r(s)_{\xi\rho}) - \nabla V^{\widetilde{Y}(s)}(\widetilde{Y}^r(s)_{\xi\rho})\right] dr \widetilde{Y}(s)_{\xi\rho} \\
&\qquad = I_1 + I_2 + I_3
\end{aligned}$$

Let us first estimate I_1:

$$|I_1| \le \|\nabla V^{Y(s)}\|_\infty \|Y(s) - \widetilde{Y}(s)\|_\gamma |\xi - \eta|^\gamma$$

Next,

$$\begin{aligned}
|I_2| &= \left| \int_0^1 dr \int_0^1 dw \nabla^2 V^{Y(s)}(J_w(Y(s)^r_{\xi\rho}, \widetilde{Y}(s)^r_{\xi\rho}))(Y(s)^r_{\xi\rho} - \widetilde{Y}(s)^r_{\xi\rho}) \widetilde{Y}(s)_{\xi\rho} \right| \\
&\le 3 \|\nabla^2 V^{Y(s)}\|_\infty \|Y(s) - \widetilde{Y}(s)\|_\infty \|\widetilde{Y}(s)\|_\gamma |\xi - \eta|^\gamma
\end{aligned}$$

Finally,

$$|I_3| \le \|\nabla V^{Y(s)} - \nabla V^{\widetilde{Y}(s)}\|_\infty \|\widetilde{Y}(s)\|_\gamma |\xi - \eta|^\gamma$$

Putting all together we end up with

$$\begin{aligned}
\|V^{Y(s)}(Y(s).) - V^{\widetilde{Y}(s)}(\widetilde{Y}(s).)\|_\gamma \le{}& \|\nabla V^{Y(s)}\|_\infty \|Y(s) - \widetilde{Y}(s)\|_\gamma \\
&+ 3\|\nabla^2 V^{Y(s)}\|_\infty \|Y(s) - \widetilde{Y}(s)\|_\infty \|\widetilde{Y}(s)\|_\gamma \\
&+ \|\nabla V^{Y(s)} - \nabla V^{\widetilde{Y}(s)}\|_\infty \|\widetilde{Y}(s)\|_\gamma
\end{aligned}$$

Then

$$\begin{aligned}\|V^{Y(s)}(Y(s).) - V^{\widetilde{Y}(s)}(\widetilde{Y}(s).)\|_\gamma^* &\leq \|V^{Y(s)} - V^{\widetilde{Y}(s)}\|_\infty + \|\nabla V^{Y(s)}\|\|Y(s) - \widetilde{Y}(s)\|_\infty \\ &\quad + \|\nabla V^{Y(s)}\|_\infty \|Y(s) - \widetilde{Y}(s)\|_\gamma \\ &\quad + 3\|\nabla^2 V^{Y(s)}\|_\infty \|Y(s) - \widetilde{Y}(s)\|_\infty \|\widetilde{Y}(s)\|_\gamma \\ &\quad + \|\nabla V^{Y(s)} - \nabla V^{\widetilde{Y}(s)}\|_\infty \|\widetilde{Y}(s)\|_\gamma \\ &\leq C_\gamma \|\nabla A\|_2 B_T(4 + 8B_T + 3B_T^2)\|Y(s) - \widetilde{Y}(s)\|_\gamma^*\end{aligned}$$

and

$$\|F(Y)(t) - F(\widetilde{Y})(t)\|_\gamma^* \leq \int_0^t \|V^{Y(s)}(Y(s).) - V^{\widetilde{Y}(s)}(\widetilde{Y}(s).)\|_\gamma^* ds$$

This implies that

$$\|F(Y) - F(\widetilde{Y})\|_{\mathbf{X}_T} \leq C_\gamma T \|\nabla A\|_2 B_T (4 + 8B_T + 3B_T^2)\|Y - \widetilde{Y}\|_{\mathbf{X}_T}$$

There exists $\overline{T} \leq T_0$ such that

$$C_\gamma \overline{T} \|\nabla A\|_2 B_{\overline{T}}(4 + 8B_{\overline{T}} + 3B_{\overline{T}}^2) < 1$$

and F is a strict contraction on $Q_{\overline{T}}$ with a unique fixed-point.

□

Remark 5. *Here and in the proofs for the case $\gamma > 1/3$ some conditions on A can be slightly relaxed using better estimates. For example, in the proof of lemma 2 the condition $\|\nabla A\|_1 < \infty$ can be relaxed to require ∇A to be a Hölder continuous function of index $(1-\gamma+\epsilon)/\gamma$ for some $\epsilon > 0$, etc... However these refinements are not able to improve qualitatively the results.*

3.3. **Blow-up estimate.** From the previous results it is clear that if the norm $\|Y(t)\|_\gamma$ of a solution Y is bounded by some number M in some interval $[0, \overline{T}]$, then the solution can be extended on a strictly larger interval $[0, \overline{T} + \delta_M]$ with δ_M depending only on M (and on the data of the problem). This implies that the only case in which we cannot find a global solution (for any positive time) is when there is some time $\hat{t}_\gamma$ such that $\lim_{t \to \hat{t}_\gamma -} \|Y(t)\|_\gamma = +\infty$. This time is an *epoch of irregularity* for the evolution in the class $\mathcal{C}^\gamma$. Near this epoch we can establish a lower bound for the norm $\|Y(t)\|_\gamma$.

Proposition 3. *Assume $\hat{t}_\gamma > 0$ is the smallest epoch of irregularity for a solution Y in the class $\mathcal{C}^\gamma$. Then we have*

$$\|Y(t)\|_\gamma \geq \frac{C_\gamma}{(\hat{t}_\gamma - t)^{1/2}} \tag{18}$$

for any $t \in [0, \hat{t}_\gamma)$.

Proof.

$$\begin{aligned}\|Y(t)\|_\gamma - \|Y(s)\|_\gamma &\leq \int_0^t \|V^{Y(u)}(Y(u).)\|_\gamma du \\ &\leq \int_s^t \|\nabla V^{Y(u)}\|_\infty \|Y(u)\|_\gamma du\end{aligned} \tag{19}$$

and using lemma 1 we have

$$\|Y(t)\|_\gamma - \|Y(s)\|_\gamma \leq C \int_s^t \|Y(u)\|_\gamma^3 du$$

for some constant C depending only on A and γ, so that

$$\frac{d}{dt}\|Y(t)\|_\gamma \leq C\|Y(t)\|_\gamma^3$$

letting $y(t) = \|Y(t)\|_{B,\gamma}$ and integrating the differential inequality between times $t > s$ we obtain

$$\frac{1}{y(s)^2} - \frac{1}{y(t)^2} \leq 2C(t-s)$$

now, assume that there exists a time $\hat{t}_\gamma$ such that $\lim_{t\to\hat{t}_\gamma -} y(t) = +\infty$, then for any $s < \hat{t}_\gamma$ we have the following lower bound for the explosion of the $\mathcal{C}^\gamma$ norm of Y:

$$\|Y(s)\|_\gamma = y(s)^{1/2} \geq \frac{1}{(2C)^{1/2}(\hat{t}_\gamma - s)^{1/2}}.$$

□

The estimate (19) used in the previous proof implies also that we have

$$z(t) \leq z(0) + \int_0^t \|\nabla V^{Y(s)}\|_\infty z(s) ds \tag{20}$$

where $z(t) = \sup_{s\in[0,t]} \|Y(s)\|_\gamma$ and by Gronwall lemma

$$z(t) \leq z(0) \exp\left(\int_0^t \|\nabla V^{Y(s)}\|_\infty ds\right)$$

This bound allows the continuation of any solution on the interval $[0,t]$ if the integral $\int_0^t \|\nabla V^{Y(s)}\|_\infty ds$ is finite. Then if $\hat{t}_\gamma$ is the first irregularity epoch for the class $\mathcal{C}^\gamma$ we must have that $\hat{t}_\gamma = \hat{t} = \sup_{1/2<\gamma\leq 1} \hat{t}_\gamma$ for any $1/2 < \gamma \leq 1$. Indeed is easy to see that for any $t < \hat{t}$ there exists a finite constant M_t such that $\sup_{s\in[0,t]} \|V^{Y(s)}\|_\infty \leq M_t$.

Corollary 1. *Let $X \in \mathcal{C}^{\gamma_*}$ with $\gamma_* > 1/2$, then for any $1/2 < \gamma \leq \gamma_*$ there exists a unique solution $Y^\gamma \in C([0,\hat{t}_\gamma), \mathcal{C}^\gamma)$ with initial condition X. Moreover the first irregularity epoch $\hat{t}_\gamma$ for the solution in $\mathcal{C}^\gamma$ does not depend on γ.*

4. Evolution for $\gamma > 1/3$

4.1. **Rough path-integrals.** When $\gamma \leq 1/2$ there are difficulties in defining the path-integral appearing in the expression (9) for the velocity field V. A successful approach to such irregular integrals has been found by Lyons to consist in enriching the notion of *path* (see e.g. [LQ02, Lyo98] and for some recent contributions [Fri04, Gub04, FDLP04]).

Here a γ-*rough path* (of degree two) is a couple $(X, \mathbb{X}^2)$ where $X \in \mathcal{C}^\gamma$ with $\gamma > 1/3$ and $\mathbb{X}^2 \in C([0,1]^2, \mathbb{R}^3 \otimes \mathbb{R}^3)$ is a matrix-valued function (called the *area process*) on the square $[0,1]^2$ verifying the following compatibility condition with X:

$$\mathbb{X}^{2,ij}_{\xi\rho} - \mathbb{X}^{2,ij}_{\xi\eta} - \mathbb{X}^{2,ij}_{\eta\rho} = (X^i_\xi - X^i_\eta)(X^j_\eta - X^j_\rho), \qquad \xi, \eta, \rho \in [0,1]^2 \tag{21}$$

($i, j = 1, 2, 3$ are vector indexes) and such that

$$\|\mathbb{X}^2\|_{2\gamma} := \sup_{\xi,\eta\in[0,1]} \frac{|\mathbb{X}^2_{\xi\eta}|}{|\xi-\eta|^{2\gamma}} < \infty. \tag{22}$$

Remark 6. *If $\gamma > 1/2$ then a natural choice for the area process $\mathbb{X}^2$ is the* geometric *one given by*

$$(\mathbb{X}^2_{geom})^{ij}_{\xi\eta} = \int_\xi^\eta (X_\rho - X_\xi)^i dX^j_\rho$$

which naturally satisfy eq. (21) (as can be directly checked) and eq. (22) (using lemma 2).

As shown by Lyons [Lyo98], when $\gamma > 1/3$ any integral of the form

$$\int \varphi(X) dX$$

can be defined to depend in a continuous way on the rough path $(X, \mathbb{X}^2)$ for sufficiently regular φ.

In [Gub04] it is pointed out that any rough path X define a natural class of paths for which path-integrals are meaningful. Define the Banach space $\mathcal{D}_X$ of paths *weakly-controlled by* X as the set of paths Y that can be decomposed as

$$Y_\xi - Y_\eta = Y'_\eta(X_\xi - X_\eta) + R^Y_{\eta\xi} \tag{23}$$

with $Y' \in C^\gamma([0,1], \mathbb{R}^3 \otimes \mathbb{R}^3)$ and $\|R^Y\|_{2\gamma} < \infty$. Define the norm for $Y \in \mathcal{D}_X$ as

$$\|Y\|_D := \|Y'\|_\gamma + \|R^Y\|_{2\gamma} + \|Y'\|_\infty$$

Moreover let

$$\|Y\|^*_D := \|Y\|_D + \|Y\|_\infty$$

Since we will need to consider only closed paths we will require for $Y \in \mathcal{D}_X$ that $Y_0 = Y_1$. Then it is easy to show that

$$\|Y\|_\gamma \le \|Y\|_D(1 + \|X\|_\gamma)$$

and that $\mathcal{D}_X \subseteq \mathcal{C}^\gamma$.

The next lemma states that $\mathcal{D}_X$ behaves nicely under maps by regular functions:

Lemma 4. *If φ is a C^2 function and $Y \in \mathcal{D}_X$ then*

$$\|\varphi(Y)\|_D \le \|\nabla\varphi\|_1 \|Y\|_D. \tag{24}$$

Proof. See [Gub04]. □

The main result about weakly-controlled paths is that they can be integrated one against the other with a good control of the resulting object:

Lemma 5 (Integration of weakly-controlled paths)**.** *If $Y, Z \in \mathcal{D}_X$ then the integral*

$$\int_\xi^\eta Y dZ := Y_\xi(Z_\eta - Z_\xi) + Y'_\xi Z'_\xi \mathbb{X}^2_{\eta\xi} + Q_{\xi\eta}, \qquad \eta, \xi \in [0,1]$$

is well defined with

$$\|Q\|_{3\gamma} \le C'_\gamma C_X \|Y\|_D \|Z\|_D$$

where $C'_\gamma > 1$ and

$$C_X = (1 + \|X\|_\gamma + \|\mathbb{X}^2\|_{2\gamma}).$$

Moreover if $\widetilde{Y}, \widetilde{Z} \in \mathcal{D}_X$*, then*

$$\int_\xi^\eta Y dZ - \int_\xi^\eta \widetilde{Y} d\widetilde{Z} = Y_\xi(Z_\eta - Z_\xi) - \widetilde{Y}_\xi(\widetilde{Z}_\eta - \widetilde{Z}_\xi) + (Y'_\xi Z'_\xi - \widetilde{Y}'_\xi \widetilde{Z}'_\xi)\mathbb{X}^2_{\eta\xi} + Q_{\xi\eta} - \widetilde{Q}_{\xi\eta}$$

and

$$\|Q - \widetilde{Q}\|_{3\gamma} \leq C_X(\|Y\|_D \epsilon_Y + \|Z\|_D \epsilon_Z)$$

with

$$\epsilon_Y = \|Y' - \widetilde{Y}'\|_\infty + \|Y' - \widetilde{Y}'\|_\gamma + \|R^Y - R^{\widetilde{Y}}\|_{2\gamma}$$
$$\epsilon_Z = \|Z' - \widetilde{Z}'\|_\infty + \|Z' - \widetilde{Z}'\|_\gamma + \|R^Z - R^{\widetilde{Z}}\|_{2\gamma}$$

Proof. See [Gub04]. □

Remark 7. *The* rough integral *so defined coincides with the corresponding Riemann-Stieltjes integral when both exists. Moreover it can be shown that* $\int_\xi^\eta Y dZ$ *is the limit of the following "renormalized" finite sums*

$$\sum_{i=0}^{n-1} \left[Y_{\xi_i}(Z_{\xi_{i+1}} - Z_{\xi_i}) + Y'_{\xi_i} Z'_{\xi_i} \mathbb{X}^2_{\xi_{i+1}\xi_i} \right]$$

(where $\xi_0 = \xi < \xi_1 < \cdots \xi_n = \eta$ *is a finite partition of* $[\xi, \eta]$*) as the size of the partition goes to zero.*

Provided V is defined through rough integrals we can obtain the following bounds on its regularity:

Lemma 6. *Let* $Y, \widetilde{Y} \in \mathcal{D}_X$ *, for any integer* $n \geq 0$*:*

$$\|\nabla^n V^Y\|_\infty \leq 4C'_\gamma \|\nabla^{n+1} A\|_1 C_X^3 \|Y\|_D^2 (1 + \|Y\|_D) \tag{25}$$

and

$$\|\nabla^n V^Y - \nabla^n V^{\widetilde{Y}}\|_\infty \leq 16 C'_\gamma C_X^3 \|\nabla^{n+1} A\|_2 \|Y - \widetilde{Y}\|_D^* (1 + \|Y\|_D)^2 \|Y\|_D \tag{26}$$

Proof. See Appendix, section A.2. □

4.2. **Local existence and uniqueness.** Given $T > 0$, consider the Banach space $\mathbf{D}_{X,T} = C([0,T], \mathcal{D}_X)$ endowed with the norm

$$\|Y\|_{\mathbf{D}_{X,T}} := \sup_{t \in [0,T]} \|Y(t)\|_D^*$$

and, as above, the application $F : \mathbf{D}_{X,T} \to \mathbf{D}_{X,T}$ defined as

$$F(Y)(t)_\xi := X_\xi + \int_0^t V^{Y(s)}(Y(s)_\xi) ds$$

with

$$V^Y(x) := \int_0^1 A(x - Y_\eta) dY_\eta$$

understood as a rough integral. Actually, for $F(Y)$ to be a well-defined path in $\mathcal{D}_X$ we must specify its "derivative" $F(Y)'$. Then we set (with explicit vector notation)

$$[F(Y)(t)']^{ij}_\xi := \delta_{ij} + \sum_{k=1}^3 \int_0^t \nabla_k [V^{Y(s)}(Y(s)_\xi)]^i [Y(s)'_\xi]^{kj} \, ds \tag{27}$$

where δ_{ij} is the Kronecker symbol.

We will state now the main result of this section, namely the existence and uniqueness of solutions to the vortex line equation in the space $\mathcal{D}_X$.

Theorem 2. *Assume $\|\nabla A\|_4 < \infty$ and $(X, \mathbb{X}^2)$ is a rough path in $\mathcal{C}^\gamma$. Then there exists a time $\overline{T}$ depending only on $\|\nabla A\|_4, X, \mathbb{X}^2, \gamma$ such that the equation (13) has a unique solution bounded in $\mathcal{D}_X$ fot any $T < \overline{T}$.*

Proof. Lemma 7 and lemma 8 prove that on a small enough time interval $[0, T]$ the map F is a strict contraction on some ball of $\mathbf{D}_{X,T}$ having a unique fixed point. ☐

Lemma 7. *Assume $\|\nabla A\|_3 < \infty$. For any initial rough path $(X, \mathbb{X}^2)$ with $\gamma > 1/3$ there exists a time T_0 such that for any time $T < T_0$ the set*

$$Q_T := \{Y \in \mathbf{D}_{X,T} : \|Y\|_{\mathbf{D}_{X,T}} \leq B_T\}$$

where B_T is a suitable constant, is invariant under F.

Proof. Fix a time $T > 0$. First of all we have, for any $t \in [0, T]$

$$\begin{aligned}
|F(Y)(t)_\xi| &\leq |X_\xi| + \int_0^t \|V^{Y(s)}\|_\infty ds \\
&\leq \|X\|_\infty + \int_0^t ds 4C'_\gamma \|\nabla A\|_1 C_X^3 \|Y(s)\|_D^2 (1 + \|Y(s)\|_D) \\
&\leq \|X\|_\infty + 4C'_\gamma T C_X^3 \|\nabla A\|_1 \|Y\|^2_{\mathbf{D}_{X,T}} (1 + \|Y\|_{\mathbf{D}_{X,T}})
\end{aligned}$$

so that

$$\sup_{t\in[0,T]} \|F(T)(t)\| \leq \|X\|_\infty + 4TC'_\gamma C_X^3 \|\nabla A\|_1 \|Y\|^2_{\mathbf{D}_{X,T}} (1 + \|Y\|_{\mathbf{D}_{X,T}})$$

Next,

$$\begin{aligned}
\|F(Y)(t)\|_D &\leq \|X\|_D + \int_0^t \|V^{Y(s)}(Y(s).)\|_D ds \\
&\leq \|X\|_D + \int_0^t \|\nabla V^{Y(s)}\|_1 \|Y(s)\|_D ds
\end{aligned}$$

Given this we obtain

$$\begin{aligned}
\|F(Y)\|_{\mathbf{D}_{X,T}} &\leq \|X\|_D^* + T \left(\sup_{0\leq s\leq T} \|\nabla V^{Y(s)}\|_1 \|Y\|_{\mathbf{D}_{X,T}} + \sup_{0\leq s\leq T} \|V^{Y(s)}\|_\infty \right) \\
&\leq \|X\|_D^* + T \sup_{0\leq s\leq T} \|V^{Y(s)}\|_2 (1 + \|Y\|_{\mathbf{D}_{X,T}}) \\
&\leq \|X\|_\infty + 1 + 12C'_\gamma C_X^5 T \|\nabla A\|_3 \|Y\|^3_{\mathbf{D}_{X,T}} (1 + \|Y\|_{\mathbf{D}_{X,T}})^2
\end{aligned}$$

and for T small enough ($T \leq T_0$ with T_0 depending only on $\|X\|_D^*, C_X$ and $\|\nabla A\|_3$) we have that there exists a constant B_T such that if $\|Y\|_{\mathbf{D}_{X,T}} \leq B_T$ then $\|F(Y)\|_{\mathbf{D}_{X,T}} \leq B_T$. ☐

Lemma 8. *Assume $\|\nabla A\|_4 < \infty$. There exists a time $\overline{T} < T_0$ such that the map F is a strict contraction on $Q_{\overline{T}}$.*

Proof. Let $Z(t) := F(Y)(t)$ and $\widetilde{Z}(t) := F(\widetilde{Y})(t)$, with $Y, \widetilde{Y} \in Q_T$. Let $H := Z - \widetilde{Z}$.We start with the estimation of the sup norm of $H(t)$:

$$\begin{aligned} H(t)_\xi = Z(t)_\xi - \widetilde{Z}(t)_\xi &= \int_0^t ds\left[V^{Y(s)}(Y(s)_\xi) - V^{\widetilde{Y}(s)}(\widetilde{Y}(s)_\xi)\right] \\ &= \int_0^t ds\left[V^{Y(s)}(Y(s)_\xi) - V^{Y(s)}(\widetilde{Y}(s)_\xi) + V^{Y(s)}(\widetilde{Y}(s)_\xi) - V^{\widetilde{Y}(s)}(\widetilde{Y}(s)_\xi)\right] \end{aligned}$$

which gives

$$\begin{aligned} \|H(t)\|_\infty &\le \int_0^t ds\left[\|\nabla V^{Y(s)}\|_\infty \|Y(s) - \widetilde{Y}(s)\|_\infty + \|V^{Y(s)} - V^{\widetilde{Y}(s)}\|_\infty\right] \\ &\le C'_\gamma T\|Y - \widetilde{Y}\|_{\mathbf{D}_{X,T}}\left[C_X^3\|\nabla^2 A\|_1 B_T^2(1+B_T) + 16C_X^3\|\nabla A\|_2 B_T(1+B_T)^2\right] \\ &\le 20C'_\gamma T C_X^3\|\nabla A\|_2 B_T(1+B_T)^2\|Y - \widetilde{Y}\|_{\mathbf{D}_{X,T}}. \end{aligned} \tag{28}$$

Next we need to estimate the $\mathcal{D}_X$ norm of $H(t)$. To be able to do that we need an expression for the decomposition 23 of $H(t)$:

$$\begin{aligned} H(t)_\xi - H(t)_\eta &= [Z(t)_\xi - \widetilde{Z}(t)_\xi] - [Z(t)_\eta - \widetilde{Z}(t)_\eta] \\ &= \int_0^t ds\left[V^{Y(s)}(Y(s)_\xi) - V^{Y(s)}(Y(s)_\eta) - V^{\widetilde{Y}(s)}(\widetilde{Y}(s)_\xi) + V^{\widetilde{Y}(s)}(\widetilde{Y}(s)_\eta)\right] \\ &= \int_0^t ds \int_0^1 dr\left[\nabla V^{Y(s)}(Y(s)^r_{\xi\eta})Y(s)_{\xi\eta} - \nabla V^{\widetilde{Y}(s)}(\widetilde{Y}(s)^r_{\xi\eta})\widetilde{Y}(s)_{\xi\eta}\right]. \end{aligned}$$

from which we obtain

$$\begin{aligned} H(t)_\xi - H(t)_\eta &= (Z'(t)_\eta - \widetilde{Z}'(t)_\eta)X_{\xi\eta} + R^Z_{\xi\eta} - R^{\widetilde{Z}}_{\xi\eta} \\ &= H(t)'_\eta X_{\xi\eta} + R^H(t)_{\xi\eta} \end{aligned}$$

with

$$\begin{aligned} H(t)'_\eta &= Z(t)'_\eta - \widetilde{Z}(t)'_\eta \\ &= \int_0^t ds\left[\nabla V^{Y(s)}(Y(s)_\eta)Y'(s)_\eta - \nabla V^{\widetilde{Y}(s)}(\widetilde{Y}(s)_\eta)\widetilde{Y}'(s)_\eta\right] \\ &= \int_0^t ds\Big[\nabla V^{Y(s)}(Y(s)_\eta)(Y'(s)_\eta - \widetilde{Y}'(s)_\eta) \\ &\qquad + (\nabla V^{Y(s)}(Y(s)_\eta) - \nabla V^{\widetilde{Y}(s)}(\widetilde{Y}(s)_\eta))\widetilde{Y}'(s)_\eta\Big] \\ &= \int_0^t ds\Big\{\nabla V^{Y(s)}(Y(s)_\eta)(Y'(s)_\eta - \widetilde{Y}'(s)_\eta) \\ &\qquad\quad + [\nabla V^{Y(s)}(Y(s)_\eta) - \nabla V^{\widetilde{Y}(s)}(Y(s)_\eta)]\widetilde{Y}'(s)_\eta \\ &\qquad\quad + [\nabla V^{\widetilde{Y}(s)}(Y(s)_\eta) - \nabla V^{\widetilde{Y}(s)}(\widetilde{Y}(s)_\eta)]\widetilde{Y}'(s)_\eta\Big\} \end{aligned}$$

and

$$\begin{aligned}R^H(t)_{\xi\eta} &= R^Z(t)_{\xi\eta} - R^{\widetilde{Z}}(t)_{\xi\eta}\\&= \int_0^t ds \Big[\nabla V^{Y(s)}(Y(s)_\eta)R^Y(s)_{\xi\eta} + \int_0^1 dr\int_0^r dw \nabla^2 V^{Y(s)}(Y(s)^w_{\xi\eta})Y(s)_{\xi\eta}Y(s)_{\xi\eta}\\&\quad -\nabla V^{\widetilde{Y}(s)}(\widetilde{Y}(s)_\eta)R^{\widetilde{Y}}(s)_{\xi\eta} + \int_0^1 dr\int_0^r dw\nabla^2 V^{\widetilde{Y}(s)}(\widetilde{Y}(s)^w_{\xi\eta})\widetilde{Y}(s)_{\xi\eta}\widetilde{Y}(s)_{\xi\eta}\Big].\end{aligned}$$

Using Lemma 6 repeatedly we can bound

$$\begin{aligned}\|H(t)'\|_\infty &\le \int_0^t ds\Big[\|\nabla V^{Y(s)}\|_\infty\|Y'(s)-\widetilde{Y}'(s)\|_\infty + \|\nabla V^{Y(s)} - \nabla V^{\widetilde{Y}(s)}\|_\infty\|\widetilde{Y}'(s)\|_\infty\\&\quad +\|\nabla^2 V^{\widetilde{Y}(s)}\|_\infty\|Y(s)-\widetilde{Y}(s)\|_\infty\|\widetilde{Y}'(s)\|_\infty\Big]\\&\le C'_\gamma T\|Y-\widetilde{Y}\|_{\mathbf{D}_{X,T}}\left[4\|\nabla^2 A\|_1 C_X^3 B_T^2(1+B_T) + 16 C_X^3\|\nabla^2 A\|_2 B_T^2(1+B_T)\right.\\&\quad\left.+4\|\nabla^3 A\|_1 C_X^3 B_T^3(1+B_T)\right]\\&\le 20 C'_\gamma T\|\nabla^2 A\|_2 C_X^3 B_T^2(1+B_T)^2\|Y-\widetilde{Y}\|_{\mathbf{D}_{X,T}}\end{aligned}\tag{29}$$

and in the same way

$$\begin{aligned}\|H(t)'\|_\gamma &\le \int_0^t ds\Big[\|\nabla V^{Y(s)}\|_\infty\|Y'(s)-\widetilde{Y}'(s)\|_\gamma + \|V^{Y(s)}(Y(s).) - V^{\widetilde{Y}(s)}(Y(s).)\|_\gamma\|\widetilde{Y}(s)'\|_\infty\\&\quad +\|V^{Y(s)}(Y(s).) - V^{\widetilde{Y}(s)}(Y(s).)\|_\infty\|\widetilde{Y}(s)'\|_\gamma\\&\quad +\|V^{\widetilde{Y}(s)}(Y(s).) - V^{\widetilde{Y}(s)}(\widetilde{Y}(s).)\|_\gamma\|\widetilde{Y}(s)'\|_\infty\\&\quad +\|V^{\widetilde{Y}(s)}(Y(s).) - V^{\widetilde{Y}(s)}(\widetilde{Y}(s).)\|_\infty\|\widetilde{Y}(s)'\|_\gamma\Big]\\&\le TC'_\gamma\|Y-\widetilde{Y}\|_{\mathbf{D}_{X,T}}\left[4\|\nabla^2 A\|_1 C_X^4 B_T^2(1+B_T) + 16 C_X^4\|\nabla^2 A\|_2(1+B_T)^2 B_T^3\right.\\&\quad +16 C_X^3\|\nabla A\|_2 B_T^2(1+B_T)^2 + 12 C_X^4 B_T^3(1+B_T)^2\\&\quad\left.+4\|\nabla^2 A\|_1 C_X^3 B_T^3(1+B_T)\right]\\&\le 48 TC'_\gamma\|\nabla A\|_3 C_X^4 B_T^3(1+B_T)^3\|Y-\widetilde{Y}\|_{\mathbf{D}_{X,T}}\end{aligned}\tag{30}$$

where we used the following four inequalities:

$$\begin{aligned}\|V^{Y(s)}(Y(s).) - V^{\widetilde{Y}(s)}(Y(s).)\|_\gamma &\le \|\nabla V^{Y(s)} - \nabla V^{\widetilde{Y}(s)}\|_\infty\|Y(s)\|_\gamma\\&\le 16 C'_\gamma C_X^4\|\nabla^2 A\|_2(1+B_T)^2 B_T^2\|Y-\widetilde{Y}\|_{\mathbf{D}_{X,T}};\end{aligned}$$

$$\begin{aligned}\|V^{Y(s)}(Y(s).) - V^{\widetilde{Y}(s)}(Y(s).)\|_\infty &\le \|V^{Y(s)} - V^{\widetilde{Y}(s)}\|_\infty\\&\le 16 C'_\gamma C_X^3\|\nabla A\|_2 B_T(1+B_T)^2\|Y-\widetilde{Y}\|_{\mathbf{D}_{X,T}};\end{aligned}$$

$$\begin{aligned}\|V^{\widetilde{Y}(s)}(Y(s).) - V^{\widetilde{Y}(s)}(\widetilde{Y}(s).)\|_\infty &\le \|\nabla V^{\widetilde{Y}(s)}\|_\infty\|Y(s)-\widetilde{Y}(s)\|_\infty\\&\le 4C'_\gamma\|\nabla^2 A\|_1 C_X^3\|Y(s)\|_D^2(1+\|Y(s)\|_D)\|Y(s)-\widetilde{Y}(s)\|_D^*\\&\le 4C'_\gamma\|\nabla^2 A\|_1 C_X^3 B_T^2(1+B_T)\|Y-\widetilde{Y}\|_{\mathbf{D}_{X,T}};\end{aligned}$$

and

$$
\begin{aligned}
&\|V^{\widetilde{Y}(s)}(Y(s).) - V^{\widetilde{Y}(s)}(\widetilde{Y}(s).)\|_\gamma \\
&\quad \leq \|\nabla V^{\widetilde{Y}(s)}\|_\infty \|Y(s) - \widetilde{Y}(s)\|_\gamma + 3\|\nabla^2 V^{\widetilde{Y}(s)}\|_\infty \|\widetilde{Y}(s)\|_\gamma \|Y(s) - \widetilde{Y}(s)\|_\infty \\
&\quad \leq \|Y(s) - \widetilde{Y}(s)\|_D^* [4C'_\gamma \|\nabla^2 A\|_1 C_X^4 \|\widetilde{Y}(s)\|_D^2 (1 + \|\widetilde{Y}(s)\|_D) \\
&\qquad + 12C'_\gamma \|\nabla^3 A\|_1 C_X^4 \|\widetilde{Y}(s)\|_D^3 (1 + \|\widetilde{Y}(s)\|_D)] \\
&\quad \leq 12C'_\gamma C_X^4 \|\widetilde{Y}(s)\|_D^2 (1 + \|\widetilde{Y}(s)\|_D)^2 \|Y(s) - \widetilde{Y}(s)\|_D^* \\
&\quad \leq 12C'_\gamma C_X^4 B_T^2 (1 + B_T)^2 \|Y - \widetilde{Y}\|_{\mathbf{D}_{X,T}}
\end{aligned}
$$

This last bound is the result of considering the expansion

$$
\begin{aligned}
\phi(Y_\eta) - \phi(Y_\xi) - (\phi(\widetilde{Y}_\eta) - \phi(\widetilde{Y}_\xi)) &= \int_0^1 dr \left[\nabla\phi(Y^r_{\eta\xi}) Y_{\eta\xi} - \nabla\phi(\widetilde{Y}^r_{\eta\xi}) \widetilde{Y}_{\eta\xi}\right] \\
&= \int_0^1 dr \left[\nabla\phi(Y^r_{\eta\xi})(Y_{\eta\xi} - \widetilde{Y}_{\eta\xi}) + (\nabla\phi(Y^r_{\eta\xi}) - \nabla\phi(\widetilde{Y}^r_{\eta\xi}))\widetilde{Y}_{\eta\xi}\right] \\
&= \int_0^1 dr \left[\nabla\phi(Y^r_{\eta\xi})(Y_{\eta\xi} - \widetilde{Y}_{\eta\xi}) + \int_0^1 dw \nabla^2\phi(J_w(Y^r_{\eta\xi}, \widetilde{Y}^r_{\eta\xi}))(Y^r_{\eta\xi} - \widetilde{Y}^r_{\eta\xi})\widetilde{Y}_{\eta\xi}\right]
\end{aligned}
$$

where $\phi(y) = A(x - y)$.

At last,

$$
\begin{aligned}
\|R^H(t)\|_{2\gamma} \leq \int_0^t ds \Big[&\|\nabla^2 V^{Y(s)} - \nabla^2 V^{\widetilde{Y}(s)}\|_\infty \|Y(s) - \widetilde{Y}(s)\|_\infty \|R^Y(s)\|_{2\gamma} \\
&+ \|\nabla V^{\widetilde{Y}(s)}\|_\infty \|R^Y(s) - R^{\widetilde{Y}}(s)\|_{2\gamma} \\
&+ \|Y(s)\|_\gamma^2 (\|\nabla^3 V^{Y(s)}\|_\infty \|Y(s) - \widetilde{Y}(s)\|_\infty + \|\nabla^2 V^{Y(s)} - \nabla^2 V^{\widetilde{Y}(s)}\|_\infty) \\
&+ 2\|\nabla^2 V^{Y(s)}\|_\infty \|Y(s)\|_\gamma \|Y(s) - \widetilde{Y}(s)\|_\gamma \Big]
\end{aligned}
$$

where, as above,

$$
\begin{aligned}
\|R^H(t)\|_{2\gamma} &\leq C'_\gamma T \|Y - \widetilde{Y}\|_{\mathbf{D}_{X,T}} \big[16 C_X^3 \|\nabla^3 A\|_2 B_T^2 (1 + B_T)^2 \\
&\qquad + 4C_X^3 \|\nabla^2 A\|_1 B_T^2 (1 + B_T) + 4C_X^5 \|\nabla^4 A\|_1 B_T^4 (1 + B_T) \\
&\qquad + 16 C_X^5 \|\nabla^3 A\|_2 B_T^3 (1 + B_T)^2 + 8C_X^5 \|\nabla^3 A\|_1 B_T^3 (1 + B_T)\big] \\
&\leq C'_\gamma T \|Y - \widetilde{Y}\|_{\mathbf{D}_{X,T}} C_X^5 \|\nabla^2 A\|_3 [32 B_T^2 (1 + B_T)^3 + 4B_T^4 (1 + B_T)] \\
&\leq 36 C'_\gamma T C_X^5 \|\nabla^2 A\|_3 B_T^2 (1 + B_T)^3 \|Y - \widetilde{Y}\|_{\mathbf{D}_{X,T}}
\end{aligned} \tag{31}
$$

Finally, collecting together the bounds (28), (29) (30) and (31) we obtain

$$
\|Z - \widetilde{Z}\|_{\mathbf{D}_{X,T}} = \|H\|_{\mathbf{D}_{X,T}} \leq 48 C'_\gamma T C_X^5 \|\nabla A\|_4 B_T (1 + B_T)^5 \|Y - \widetilde{Y}\|_{\mathbf{D}_{X,T}}
$$

Proving that for $\overline{T} < T_0$ small enough so that

$$
48 C'_\gamma \overline{T} C_X^5 \|\nabla A\|_4 B_{\overline{T}} (1 + B_{\overline{T}})^5 < 1
$$

F is a contraction in the ball $Q_{\overline{T}} \subset \mathbf{D}_{X,T}$. $\square$

Remark 8. *By imposing enough regularity on A and requiring that X can be completed to a geometric rough path with bounded p-variation (in the sense of Lyons) it is likely that the above proof of existence and uniqueness can be extended to cover the case of rougher initial conditions (e.g. paths living in some $\mathcal{C}^\gamma$ with $\gamma < 1/3$).*

Remark 9. *The solution (Y, Y') in $\mathbf{D}_{X,\overline{T}}$, satisfy (compare with eq. (27))*

$$Y(t)'_\eta = \mathrm{Id} + \int_0^t \nabla V^{Y(s)}(Y(s)_\eta) Y'(s)_\eta \, ds \tag{32}$$

and, as can be easily verified,

$$\begin{aligned}[R^Y(t)]^i_{\xi\eta} = \int_0^t ds \Big[& \nabla^k V^{Y(s)\,i}(Y(s)_\eta) R^{Y\,k}(s)_{\xi\eta} + \\ & + \int_0^1 dr \int_0^r dw \nabla^k \nabla^l V^{Y(s)\,i}(Y(s)^w_{\xi\eta}) Y(s)^k_{\xi\eta} Y(s)^l_{\xi\eta} \Big].\end{aligned} \tag{33}$$

4.3. **Dynamics of the covariations.** Recall the framework described in Sec. 2.2 on the covariation structure of the solution. If we assume that the initial condition $(X, \mathbb{X}^2)$ is a random variable a.s. with values in the space of γ-rough paths (with $\gamma > 1/3$) and that X is a process with all its mutual covariations, then the solution $Y(t)$ at any instant of time t less that a random time $\overline{T}$ (depending on the initial condition) it is still a process with all its mutual covariations (due to Prop.1).

The covatiations of Y satisfy the equation

$$[Y(t)^i, Y(t)^j]_\eta = \int_0^\eta (Y(t)')^{ik}_\rho (Y(t)')^{jl}_\rho d_\rho [X^k, X^l]_\rho \tag{34}$$

Indeed, comparing eq. (32) with eq. (12) we can identify the function $M(t)_\xi$ in Prop. 1 with $Y(t)'_\xi$.

Remark 10. *The same result can be obtained noting that, for our solution,*

$$\sum_i |Y(t)_{\xi_{i+1}\xi_i}|^2 = \sum_i Y(t)'_{\xi_i} X_{\xi_{i+1}\xi_i} Y'(t)_{\xi_i} X_{\xi_{i+1}\xi_i} + O(|\xi_{i+1} - \xi_i|^{3\gamma}).$$

Eq. (34) has a differential counterpart in the following equation

$$\frac{d}{dt} W(t)_\xi = \int_0^\xi (H(t)_\rho d_\rho W(t)_\rho + d_\rho W(t)_\rho H(t)^*_\rho) \tag{35}$$

where we let $W(t)_\xi := [Y(t), Y(t)^*]_\xi$ as a matrix valued process, and $H(t)_\xi := \nabla V^{Y(t)}(Y(t)_\xi)$. To understand better this evolution equation let us split the matrix $H(t)_\xi$ into its symmetric S and anti-symmetric T components:

$$H(t)_\xi = S(t)_\xi + T(t)_\xi, \qquad S(t)_\xi = S(t)^*_\xi, T(t)_\xi = -T(t)^*_\xi.$$

Moreover define $Q(t)_\xi$ as the solution of the Cauchy problem

$$\frac{d}{dt} Q(t)_\xi = -Q(t)_\xi T(t)_\xi, \qquad Q(0)_\xi = \mathrm{Id}$$

i.e.

$$Q(t)_\xi = \exp\left[-\int_0^t T(s)_\xi ds \right].$$

Since T is antisymmetric, the matrix Q is orthogonal, i.e. $Q(t)_\xi^{-1} = Q(t)_\xi^*$. This matrix describe the rotation of the local frame of reference at the point $Y(t)_\xi$ induced by the motion of the curve.

Then define

$$\widetilde{W}(t)_\xi := \int_0^\xi Q(t)_\rho d_\rho W(t)_\rho Q(t)_\rho^{-1}$$

and analogously $\widetilde{T}(t)_\xi := Q(t)_\xi T(t)_\xi Q(t)_\xi^{-1}$ and $\widetilde{S}(t)_\xi = Q(t)_\xi S(t)_\xi Q(t)_\xi^{-1}$, and compute the following time-derivative:

$$\begin{aligned}
\frac{d}{dt} d_\xi \widetilde{W}(t)_\xi &= \frac{dQ(t)_\xi}{dt} Q(t)_\xi^{-1} d_\xi \widetilde{W}(t)_\xi + d_\xi \widetilde{W}(t)_\xi Q(t)_\xi \frac{dQ(t)_\xi^{-1}}{dt} + Q(t)_\xi \left(\frac{d}{dt} d_\xi W(t)_\xi \right) Q(t)_\xi^{-1} \\
&= -\widetilde{T}(t)_\xi d_\xi \widetilde{W}(t)_\xi + d_\xi \widetilde{W}(t)_\xi \widetilde{T}(t)_\xi + Q(t)_\xi [H(t)_\xi d_\xi W(t)_\xi + d_\xi W(t)_\xi H(t)_\xi^*] Q(t)_\xi^{-1} \\
&= \widetilde{S}(t)_\xi d_\xi \widetilde{W}(t)_\xi + d_\xi \widetilde{W}(t)_\xi \widetilde{S}(t)_\xi
\end{aligned}$$

This result implies that $d_\xi W(t)_\xi$ can be decomposed as

$$d_\xi W(t)_\xi = Q(t)_\xi^{-1} \exp\left[\int_0^t \widetilde{S}(s)_\xi ds \right] d_\xi W(0)_\xi \exp\left[\int_0^t \widetilde{S}(s)_\xi ds \right]^* Q(t)_\xi \tag{36}$$

The relevance of this decomposition is the following. Modulo rotations, the matrix $\widetilde{S}(t)_\xi$, corresponds to the symmetric part of the tensor field $\nabla V^{Y(t)}(x)$ in the point $x = Y(t)_\xi$. This symmetric component descride the stretching of the volume element around x due to the flow generated by the (time-dependent) vector field $V^{Y(t)}$. The magnitude of the covariation then varies with time, due to this stretching contribution, according to eq. (36).

5. Random vortex filaments

5.1. **Fractional Brownian Loops with $H > 1/2$.** Consider the following probabilistic model of Gaussian vortex filament. Let $(\tilde{X}_\xi)_{\xi \in [0,1]}$ a 3d Fractional Brownian Motion (FBM) of Hurst index H, i.e. a centered Gaussian process on $\mathbb{R}^3$ defined on the probability space $(\Omega, \mathbb{P}, \mathcal{F})$ such that

$$\mathbb{E}\, \tilde{X}_\xi^i \tilde{X}_\eta^j = \frac{\delta_{ij}}{2} (|\xi|^{2H} + |\eta|^{2H} - |\xi - \eta|^{2H})$$

with $H > 1/2$ and $X_0 = 0$. Consider the Gaussian process $(X_\xi)_{\xi \in [0,1]}$ defined as

$$X_\xi = \tilde{X}_\xi - \frac{C(\xi, 1)}{C(1,1)} \tilde{X}_1$$

where $C(\xi, \eta) = (|\xi|^{2H} + |\eta|^{2H} - |\xi - \eta|^{2H})$. Then $X_0 = X_1 = 0$ a.s. moreover the process $(X_\xi)_\xi$ is independent of the r.v. $\tilde{X}_1$. We call X a Fractional Brownian Loop (FBL). Using the standard Kolmogorov criterion it is easy to show that X in a.s. Hölder continuous for any index $\gamma < H$. Since $H > 1/2$ then we can choose $\gamma > 1/2$ and apply the results of Sec. 3 to obtain the evolution of a random vortex filament modeled on a FBL.

5.2. **Evolution of Brownian loops.** As an example of application of Theorem 2 we can consider the evolution of an initial random curve whose law is that of a Brownian Bridge on $[0,1]$ starting at an arbitrary point x_0. A standard three-dimensional Brownian Bridge $\{B_\xi\}_{\xi\in[0,1]}$ such that $B_0 = B_1 = x_0 \in \mathbb{R}^3$ is a stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose law is the law of a Brownian motion starting at x_0 and conditioned to reach x_0 at "time" 1. As in the previous section, it can be obtained starting from a standard Brownian motion $\{\tilde{B}\}_{\xi\in[0,1]}$ as

$$B_\xi = \tilde{B}_\xi - \xi \tilde{B}_1, \qquad \xi \in [0,1].$$

The Brownian Bridge is a semi-martingale with respect to its own filtration $\{\mathcal{F}^B_\xi : 0 \le \xi \le 1\}$ with decomposition

$$dB_\xi = \frac{B_\xi - x_0}{1-\xi} d\xi + d\beta_\xi$$

where $\{\beta_\xi\}_{\xi\in[0,1]}$ is a standard 3d Brownian motion. Using the results in [Gub04], it is easy to see that B is a γ-Hölder rough path if we consider it together with the area process defined as

$$\mathbb{B}^{2,ij}_{\xi\eta} = \int_\xi^\eta (B^i_\rho - B^i_\eta) \circ dB^j_\rho \tag{37}$$

where the integral is understood in Stratonovich sense. Indeed there exists a version of the process $(\xi,\eta) \mapsto \mathbb{B}^2_{\xi\eta}$ which is continuous in both parameters and such that $\|\mathbb{B}\|_{2\gamma}$ is almost surely finite (also all moments are finite). Then outside an event of $\mathbb{P}$-measure zero the couple $(B, \mathbb{B})$ is a γ-Hölder rough path and by theorem 2 there exists a solution of the problem (8) starting at B. Of course, in this case, the solution depends a priori on the choice (37) we made for the area process. Indeed if in eq. (37) we consider, for example, the Itô integral (for which the regularity result on $\mathbb{B}$ still holds) we would have obtained a different solution, even if the path B is unchanged.

Consider the discussion in Sec. 4.3 and noting that, for our Brownian loop B the covariation is $[B,B]_\xi = \mathrm{Id}\cdot\xi$ we can say that the covariation of the solution Y starting at B will be

$$d_\xi[Y(t), Y(t)^*]_\xi = Q(t)^{-1}_\xi \exp\left[\int_0^t \widetilde{S}(s)_\xi ds\right] \exp\left[\int_0^t \widetilde{S}(s)_\xi ds\right]^* Q(t)_\xi$$

(the notations are the same as in Sec. 4.3).

Remark 11. *In* [CQ02] *the authors give, in particular, a construction of the* area process $\mathbb{X}^2$ *in the case where X is a FBM of Hurst index $H > 1/4$. By suitably adapting this construction would be possible to build a closed γ-rough path $(X, \mathbb{X}^2)$ where X is a FBL with $H > 1/3$ for any $\gamma < H$.*

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Appendix A. Proofs of some lemmas

A.1. **Proof of lemma 1.**

Proof. It is enough to consider $n = 0$, the proof for general n being similar. The Taylor expansion

$$A(Y_\xi) - A(Y_\eta) = Y_{\xi\eta} \int_0^1 dr \nabla A(J_r(Y_\xi, Y_\eta))$$

and the related inequality

$$|A(Y_\xi) - A(Y_\eta)| \le \|\nabla A\|_\infty |Y_{\xi\eta}|.$$

gives

$$\|\nabla^n A(Y_\cdot)\|_\gamma \le \|Y\|_\gamma \|\nabla^{n+1} A\|_\infty. \tag{38}$$

Next, if we consider the decomposition

$$\begin{aligned}
[A(Y_\xi) - A(\widetilde{Y}_\xi)] - [A(Y_\eta) - A(\widetilde{Y}_\eta)] &= Y_{\xi\eta} \int_0^1 \nabla A(Y^r_{\xi\eta}) dr - \widetilde{Y}_{\xi\eta} \int_0^1 \nabla A(\widetilde{Y}^r_{\xi\eta}) dr \\
&= (Y_{\xi\eta} - \widetilde{Y}_{\xi\eta}) \int_0^1 \nabla A(Y^r_{\xi\eta}) dr + \widetilde{Y}_{\xi\eta} \left[\int_0^1 \nabla A(Y^r_{\xi\eta}) dr - \int_0^1 \nabla A(\widetilde{Y}^r_{\xi\eta}) dr \right] \\
&= (Y_{\xi\eta} - \widetilde{Y}_{\xi\eta}) \int_0^1 \nabla A(Y^r_{\xi\eta}) dr - \widetilde{Y}_{\xi\eta} \int_0^1 dr (\widetilde{Y}^r_{\xi\eta} - Y^r_{\xi\eta}) \int_0^1 dw \nabla^2 A(J_w(\widetilde{Y}^r_{\xi\eta}, Y^r_{\xi\eta}))
\end{aligned}$$

where $Y^r_{\xi\eta} := J_r(Y_\xi, Y_\eta)$, we obtain

$$\|A(Y_\cdot) - A(\widetilde{Y}_\cdot)\|_\gamma \le \|Y - \widetilde{Y}\|_\gamma \|\nabla A\|_\infty + 3\|\widetilde{Y}\|_\gamma \|Y - \widetilde{Y}\|_\infty \|\nabla^2 A\|_\infty$$

which implies

$$\|\nabla^n A(Y_\cdot) - \nabla^n A(\widetilde{Y}_\cdot)\|_\gamma \le \|Y - \widetilde{Y}\|_\gamma \|\nabla^{n+1} A\|_1 (1 + 3\|\widetilde{Y}\|_\gamma). \tag{39}$$

Using the lemma 2 and the bounds (38) and (39) the estimates (15) and (16) on the velocity vector-field follow as:

$$\begin{aligned}
|V^{Y(s)}(x)| = \left| \int_0^1 A(x - Y(s)_\eta) dY(s)_\eta \right| &\le C_\gamma \|A(x - Y(s)_\cdot)\|_\gamma \|Y(s)\|_\gamma \\
&\le C_\gamma \|\nabla A\|_\infty \|Y(s)\|^2_\gamma.
\end{aligned}$$

Moreover for the difference $V^Y - V^{\widetilde{Y}}$ we have the decomposition

$$\begin{aligned}
V^Y(x) - V^{\widetilde{Y}}(x) &= \int_0^1 \left[A(x - Y_\eta) dY_\eta - A(x - \widetilde{Y}_\eta) d\widetilde{Y}_\eta \right] \\
&= \int_0^1 A(x - Y_\eta) d(Y - \widetilde{Y})_\eta + \int_0^1 \left[A(x - Y_\eta) - A(x - \widetilde{Y}_\eta) \right] d\widetilde{Y}_\eta
\end{aligned}$$

which in turn can be estimated as

$$\begin{aligned}
|V^Y(x) - V^{\widetilde{Y}}(x)| &\le C_\gamma \|A(x - Y_\cdot)\|_\gamma \|Y - \widetilde{Y}\|_\gamma \\
&\quad + C_\gamma \|A(x - .) - A(x - \widetilde{Y}_\cdot)\|_\gamma \|\widetilde{Y}\|_\gamma \\
&\le C_\gamma \|\nabla A\|_\infty \|Y\|_\gamma \|Y - \widetilde{Y}\|_\gamma \\
&\quad + C_\gamma \|\widetilde{Y}\|_\gamma (\|\nabla A\|_\infty \|Y - \widetilde{Y}\|_\gamma + 3\|\widetilde{Y}\|_\gamma \|Y - \widetilde{Y}\|_\infty \|\nabla^2 A\|_\infty) \\
&\le C_\gamma \|Y - \widetilde{Y}\|^*_\gamma \|\nabla A\|_1 (\|Y\|_\gamma + \|\widetilde{Y}\|_\gamma + 3\|\widetilde{Y}\|^2_\gamma)
\end{aligned}$$

□

A.2. **Proof of lemma 6.**

Proof. Consider the case $n=0$, the general case being similar. The path $Z_\xi = A(x-Y_\xi)$ belongs to $\mathcal{D}_X$ and has the following decomposition

$$\begin{aligned}
Z_{\xi\eta} &= \nabla A(x-Y_\eta)Y_{\xi\eta} + Y_{\xi\eta}Y_{\xi\eta}\int_0^1 dr\int_0^r dw \nabla^2 A(x-Y^w_{\xi\eta}) \\
&= \nabla A(x-Y_\eta)Y'_\eta X_{\xi\eta} + \nabla A(x-Y_\eta)R^Y_{\xi\eta} + Y_{\xi\eta}Y_{\xi\eta}\int_0^1 dr\int_0^r dw\nabla^2 A(x-Y^w_{\xi\eta}) \\
&= Z'_\eta X_{\xi\eta} + R^Z_{\xi\eta}
\end{aligned}$$

with

$$Z'_\eta = \nabla A(x-Y_\eta)Y'_\eta$$

and

$$R^Z_{\xi\eta} = \nabla A(x-Y_\eta)R^Y_{\xi\eta} + Y_{\xi\eta}Y_{\xi\eta}\int_0^1 dr\int_0^r dw\nabla^2 A(x-Y^w_{\xi\eta})$$

Then

$$\begin{aligned}
\|Z\|_D &= \|Z'\|_\infty + \|Z'\|_\gamma + \|R^Z\|_{2\gamma} \\
&\le \|\nabla A\|_\infty \|Y'\|_\infty + \|\nabla^2 A\|_\infty \|Y\|_\gamma + \|\nabla A\|_\infty\|Y'\|_\gamma \\
&\qquad + \|\nabla A\|_\infty \|R^Y\|_{2\gamma} + \|Y\|_\gamma^2 \|\nabla^2 A\|_\infty \\
&\le \|\nabla A\|_1 (\|Y\|_D + \|Y\|_\gamma + \|Y\|^2_\gamma) \\
&\le \|\nabla A\|_1 [(1+C_X)\|Y\|_D + C_X^2\|Y\|_D^2] \\
&\le C_X^2 \|\nabla A\|_1 [2\|Y\|_D + \|Y\|^2_D]
\end{aligned} \tag{40}$$

where we used the fact that

$$\begin{aligned}
\|Y\|_\gamma &\le \|Y'\|_\infty \|X\|_\gamma + \|R^Y\|_\gamma \\
&\le \|Y'\|_\infty \|X\|_\gamma + \|R^Y\|_{2\gamma} \\
&\le (1+\|X\|_\gamma)\|Y\|_D \le C_X \|Y\|_D
\end{aligned} \tag{41}$$

$$V^Y(x) = \int_0^1 A(x-Y_\eta)dY_\eta = \int_0^1 Z_\eta dY_\eta Z_0(Y_1-Y_0) + Z'_0 Y'_0 \mathbb{X}^2_{01} + Q_{01}$$

with

$$\|Q\|_{3\gamma} \le C'_\gamma C_X \|Z\|_D \|Y\|_D$$

Then

$$\begin{aligned}
|V^Y(x)| &\le \|Z'\|_\infty \|Y'\|_\infty \|\mathbb{X}^2\|_{2\gamma} + \|Q\|_{3\gamma} \\
&\le 2C'_\gamma C_X \|Z\|_D\|Y\|_D \\
&\le 4C'_\gamma \|\nabla A\|_1 C_X^3 \|Y\|^2_D (1+\|Y\|_D)
\end{aligned}$$

where we used the fact that $C'_\gamma \ge 1$.

To bound $V^Y(x) - V^{\widetilde{Y}}(x)$ we need the $\mathcal{D}_X$ norm of the difference $A(x-Y_\cdot) - A(x-\widetilde{Y}_\cdot)$. Let $\phi(y) = A(x-y)$ and consider the expansion

$$\begin{aligned}
&\phi(Y_\eta) - \phi(Y_\xi) - (\phi(\widetilde{Y}_\eta) - \phi(\widetilde{Y}_\xi)) \\
&= \left[\nabla\phi(Y_\xi)Y_{\eta\xi} - \nabla\phi(\widetilde{Y}_\xi)\widetilde{Y}_{\eta\xi}\right] + \int_0^1 dr\int_0^r dw\left[\nabla^2\phi(Y^w_{\eta\xi})Y_{\eta\xi}Y_{\eta\xi} - \nabla^2\phi(\widetilde{Y}^w_{\eta\xi})\widetilde{Y}_{\eta\xi}\widetilde{Y}_{\eta\xi}\right]
\end{aligned}$$

which by arguments similar to those leading to eq. (40) gives a related estimate:

$$
\begin{aligned}
\|\phi(Y_\cdot)-\phi(\widetilde{Y}_\cdot)\|_D \leq & \|\nabla\phi\|_\infty\|Y-\widetilde{Y}\|_D+\|\nabla^2\phi\|_\infty\|Y-\widetilde{Y}\|_\infty\|Y\|_D \\
& +3\|\nabla^3\phi\|\|Y-\widetilde{Y}\|_\infty\|Y\|_\gamma^2+2\|\nabla^2\phi\|_\infty\|Y-\widetilde{Y}\|_\gamma\|Y\|_\gamma \\
& \leq 6\|\nabla\phi\|_2 C_X^2(1+\|Y\|_D)^2\|Y-\widetilde{Y}\|_D^*
\end{aligned}
$$

so that

$$
\|A(x-Y_\cdot)-A(x-\widetilde{Y}_\cdot)\|_D \leq 6\|\nabla A\|_2 C_X^2(1+\|Y\|_D)^2\|Y-\widetilde{Y}\|_D^*
$$

Now,

$$
\begin{aligned}
V^Y(x)-V^{\widetilde{Y}}(x) &= \int_0^1 A(x-Y_\eta)dY_\eta-\int_0^1 A(x-\widetilde{Y}_\eta)d\widetilde{Y}_\eta \\
&= \int_0^1 [A(x-Y_\eta)-A(x-\widetilde{Y}_\eta)]dY_\eta+\int_0^1 A(x-\widetilde{Y}_\eta)d(Y-\widetilde{Y})_\eta
\end{aligned}
$$

So

$$
\begin{aligned}
|V^Y(x)-V^{\widetilde{Y}}(x)| \leq & \, 2C'_\gamma C_X(\|A(x-Y_\cdot)-A(x-\widetilde{Y}_\cdot)\|_D\|Y\|_D \\
& +\|A(x-Y_\cdot)\|_D\|Y-\widetilde{Y}\|_D) \\
& \leq 16C'_\gamma C_X^3\|\nabla A\|_2\|Y-\widetilde{Y}\|_D^*(1+\|Y\|_D)^2\|Y\|_D
\end{aligned}
$$

□

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