Applications of controlled paths

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Nancy, May 9th 2012

Outline

I will exhibith various applications of the idea of a "controlled path".

- Pre-history
- Rough path theory
- Averaging by oscillations
- Stochastic Burgers equation with derivative white noise perturbation
- NSE with random dispersion
- Korteweg–de Vries equation with distributional initial condition

Pre-historic controlled paths

If $f : \mathbb{R} \to \mathbb{R}$ is γ-Hölder we know that a suitable generalization when $\gamma \in (1, 2)$ is to require

$$f_t - f_s = g_s(t - s) + O(|t - s|^\gamma)$$

for some given function $g : \mathbb{R} \to \mathbb{R}$. Then we know also that

$$f_t - f_s = \int_s^t g_r \mathrm{d}r = \lim_{|\Pi_{s,t}| \to 0} \sum_{t_i \in \Pi_{s,t}} g_{t_i}(t_{i+1} - t_i)$$

Young integral

Let f, g two smooth function and consider the bilinear form

$$I(f,g)_t = \int_0^t f_t \mathrm{d}g_r = \int_0^t f_r \partial_r g_r \mathrm{d}r = f_t g_t - f_0 g_0 - \int_0^t g_r \partial_r f_r \mathrm{d}r.$$

Then

$$I : C \times H^1 \to H^1 \text{ and } I : H^1 \times C \to C$$

The interpolation space $X_2 = [C, H^1]_{1/2}$ allows $I : X_2 \times X_2 \to X_2$. In practice it is enough to take C^γ for $\gamma > 1/2$ and more generally, if $\gamma + \rho > 1$

$$I : C^\rho \times C^\gamma \to C^\gamma$$

Moreover $h = I(f,g)$ is the unique function which satisfy

$$h_t - h_s = f_s(g_t - g_s) + O(|t-s|^{\gamma+\rho}) \quad \text{or} \quad h_t - h_s = \lim_{|\Pi_{s,t}| \to 0} \sum_{t_i \in \Pi_{s,t}} f_{t_i}(g_{t_{i+1}} - g_{t_i})$$

Remark: This result say that $\partial_t g_t$ is a distribution for which the product $f_r \partial_r g_r$ is still a well-defined distribution.

Beyond Young: Controlled paths

Let $f \in C^\rho$ and $g \in C^\gamma$ and assume that the following equation

$$\Phi_{s,t} - \Phi_{s,u} - \Phi_{u,t} = (f_s - f_u)(g_u - g_t), \qquad i,j \in \{1,\dots,d\}, 0 \leqslant s \leqslant u \leqslant t$$

has a solution $\Phi(f,g) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^d \otimes \mathbb{R}^d$ such that $|\Phi(f,g)_{st}| \lesssim |t-s|^{\rho+\gamma}$, then if $\gamma + \rho + \theta > 1$, for any function h such that

$$h_t - h_s = h'_s(f_t - f_s) + O(|t-s|^{\rho+\theta})$$

with $h' \in C^\theta$ there exists a unique solution to the requirement

$$z_t - z_s = h_s(g_t - g_s) + h'_s \Phi(f,g)_{s,t} + O(|t-s|^{\gamma+\rho+\theta})$$

and moreover it holds that

$$z_t - z_s = \lim_{|\Pi_{s,t}| \to 0} \sum_{t_i \in \Pi_{s,t}} h_{t_i}(g_{t_{i+1}} - g_{t_i}) + h'_{t_i} \Phi(f,g)_{t_i,t_{i+1}} = \int_s^t h_r \mathrm{d} g_r$$

Remark: The integration of controlled paths can be interpreted as a definition for the product of distributions.

Averaging along a Brownian motion

A. Davie has showed that if $b : \mathbb{R}^d \to \mathbb{R}$ is a bounded function and B a d-dimensional Brownian motion. The average of b along the Brownian trajectory given by

$$\sigma_t(y) = \int_0^t b(B_s + x)\mathrm{d}s$$

satisfy

$$\mathbb{E}|\sigma_t(y) - \sigma_t(x)|^{2p} \leqslant C_p|x-y|^{2p}t^p$$

From this it is possible to deduce that the ODE

$$X_t = x + \int_0^t b(X_s)\mathrm{d}s + B_t$$

has a unique continuous solution for almost every sample path of B.

Averaging along an fBm

Let $\mathcal{F}L^{\alpha}$ the set of distribution $b : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$N_{\alpha}(b) = \int_{\mathbb{R}^d} (1 + |\xi|)^{\alpha} |\hat{b}(\xi)| \mathrm{d}\xi < +\infty.$$

Then it is possible to show that if $(w_t)_{t \geqslant 0}$ is the sample path of a d-dim. fractional Brownian motion and $x \in Q^w_{\gamma} \subset C(\mathbb{R}; \mathbb{R}^d)$ is *controlled* by w in the sense that

$$x_t - x_s = w_t - w_s + O(|t - s|^{\rho})$$

for some $\rho > 1/2$, for all $b \in \mathcal{F}L^{\alpha}$ with $\alpha > 1 - 1/2H$ the integral

$$\lim_{n \to \infty} \int_0^t b_n(x_s) \mathrm{d}s =: \int_0^t b(x_s) \mathrm{d}s$$

is well defined for any sequence of smooth function $(b_n)_{n \geqslant 1}$ such that $N_{\alpha}(b - b_n) \to 0$ and independent of the sequence. Moreover the map $t \mapsto \int_0^t b(x_s) \mathrm{d}s$ is C^{γ} for some $\gamma > 1/2$.

[joint work with R. Catellier]

Regularization by oscillations

If $\alpha > 2 - 1/2H$ the map

$$y \mapsto \int_0^t b(x_s + y)\mathrm{d}s$$

is Lipshitz:

$$\left| \int_s^t b(x_r + y)\mathrm{d}r - \int_s^t b(x_r + z)\mathrm{d}r \right| \lesssim_{x,w} N_\alpha(b)|y - z||t - s|^\gamma.$$

The previous results allows to study the the ODE in $\mathbb{R}^d$

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}s + w_t$$

where $b \in \mathcal{F}L^\alpha$.

- Existence in Q_γ^w for $\alpha > 1 - 1/2H$
- Uniqueness in Q_γ^w for $\alpha > 2 - 1/2H$ + Lipshitz flow.
- If b is not random we can have uniqueness for $\alpha > 1 - 1/2H$.

Stochastic Burgers equation

[joint work with M. Jara]

Here the stochastic Burgers equation on $\mathbb{T} = [-\pi, \pi]$

$$\mathrm{d}u_t = \frac{1}{2}\partial_\xi^2 u_t(\xi)\mathrm{d}t + \frac{1}{2}\partial_\xi (u_t(\xi))^2 \mathrm{d}t + \partial_\xi \mathrm{d}W_t$$

where $\mathrm{d}W_t$ is space-time white noise.

The solution u would like to be the derivative of the solution of the Kardar–Parisi–Zhang equation

$$\mathrm{d}h_t = \frac{1}{2}\partial_\xi^2 h_t(\xi)\mathrm{d}t + \frac{1}{2}(\partial_\xi h_t(\xi))^2 \mathrm{d}t + \mathrm{d}W_t.$$

which captures the macroscopic behavior of a large class of surface growth phenomena.

Problems with the weak formulation

For sufficiently smooth test functions $\varphi : \mathbb{T} \to \mathbb{R}$ look for solutions of

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u_s(\partial_\xi^2 \varphi) \mathrm{d}s + \int_0^t \langle \partial_\xi \varphi, B(u_s) \rangle \mathrm{d}s + W_t(\partial_\xi \varphi)$$

where $B(u_s)(\xi) = (u_s(\xi))^2$.

- We would like to start the equation from initial condition u_0 which is space white noise, this is expected to be an invariant measure.
- The linearized equation

$$X_t(\varphi) = u_0(\varphi) + \int_0^t X_s(\partial_\xi^2 \varphi) \mathrm{d}s + W_t(\partial_\xi \varphi)$$

  has trajectories which looks like white noise in space.

$\Rightarrow$ The nonlinear term $B(u_s)$ is not defined.

"Lazy" smoothing estimation

Here a controlled process y is such that

$$y_t(\varphi) = y_0(\varphi) + \int_0^t v_s(\partial_\xi^2 \varphi)\mathrm{d}s + \mathcal{A}_t(\varphi) + W_t(\partial_\xi \varphi)$$

where

- $\mathcal{A}_t(\varphi)$ is a zero-quadratic variation process
- y_t is space-time white noise at all times
- The reversed process $\hat{y}_t = y_{T-t}$ has the same properties with drift $\widehat{\mathcal{A}} = -\mathcal{A}$.

Formulation of the equation

Let $B_\varepsilon(x) = B(\rho_\varepsilon * x)$ a regularization of the non-linearity.

Can show that for a controlled path y this limit exists:

$$\lim_{\varepsilon \to 0} \int_0^t \langle \varphi, \partial_\xi B_\varepsilon(y_s) \rangle \mathrm{d}s = \mathcal{B}_t(\varphi)$$

(independently of regularization) and we can use it to define the drift in the Burgers equation.

A solution u of the Burgers equation is a good process such that

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u_s(\partial_\xi^2 \varphi) \mathrm{d}s + \mathcal{B}_t(\varphi) + W_t(\partial_\xi \varphi)$$

The controlled path approach provides compactness estimates for Galerkin approximation. Uniqueness seems difficult in this approach.

The process $\mathcal{B}_t(\varphi)$ is only $3/2-$ Hölder in time.

Schrödinger equation with random dispersion

Consider the (Stratonovich-) stochastic Schrödinger equation

$$\mathrm{d}\phi_t = i\Delta\phi_t \circ \mathrm{d}B_t + |\phi_t|^2\phi_t \mathrm{d}t$$

for $\phi : [0, T] \times \mathbb{T} \to \mathbb{C}$.

[Debussche–De Bouard]

Let $U_t = e^{i\Delta B_t}$ so that

$$\mathrm{d}U_t = i\Delta U_t \circ \mathrm{d}B_t$$

then

$$\phi_t = U_t(\phi_0 + \int_0^t U_s^{-1}(|\phi_s|^2\phi_s)\mathrm{d}s).$$

Formulation as a controlled path problem

The path ϕ is controlled if

$$\phi_t = U_t \psi_t$$

with $\psi_t \in C^\rho(\mathbb{R}_+; L^2(\mathbb{T}))$ for some $\rho > 1/2$. Then it is possible to show that

$$t \mapsto \int_0^t U_s^{-1}(|\phi_s|^2 \phi_s) \mathrm{d}s$$

exists, coincide with the following limit

$$\lim_{n\to\infty} \int_0^t U_s^{-1}(|P_n \phi_s|^2 P_n \phi_s) \mathrm{d}s$$

(P_n is the projector on the Fourier modes $|k| \leqslant n$) and is γ-Hölder in time for some $\gamma > 1/2$ and locally Lipshitz in ϕ in the controlled path norm.

By standard fixed-point argument we get a (unique) local solution to the NSE and the L^2 conservation law allows to extend it to a global one.

1d periodic KdV equation

$$\begin{cases} \partial_t u(t,\xi) + \partial_\xi^3 u(t,\xi) + \frac{1}{2}\partial_\xi u(t,\xi)^2 = 0 \\ u(0,\xi) = u_0(\xi) \end{cases} \qquad (t,\xi) \in \mathbb{R} \times \mathbb{T}$$

with initial condition $u_0 \in H^\alpha(\mathbb{T})$, $\mathbb{T} = [-\pi, \pi]$.

We look for solutions for any $\alpha > -1/2$.

Airy group

$$\mathcal{F}(U_t \varphi)(k) = e^{-ik^3 t} \hat{\varphi}(k), \qquad k \in \mathbb{Z}.$$

Mild form

$$u_t = U_t(u_0 + \int_0^t U_{-s}(\partial_\xi u_s^2) \mathrm{d}s)$$

Series solution

Formally the series expansion of the solution looks like

$$u_t = U_{t-s}(u_s + \int_s^t U_{-r}(\partial_\xi (U_{r-s} u_s)^2) \mathrm{d}r + \cdots)$$

A computation show that the bilinear operator

$$X^1_{s,t}(\varphi_1, \varphi_2) = \int_s^t U_{-r}(\partial_\xi [(U_r \varphi_1)(\partial_\xi (U_r \varphi_2)]) \mathrm{d}r$$

is bounded from $H^\alpha \times H^\alpha$ to H^α for $\alpha > -1/2$ and that the norm is of order $|t-s|^\gamma$ for some $\gamma > 1/3$.

Uniqueness of weak solutions

Using rough paths theory we can prove that the nonlinear term is defined of every controlled path:

Let $\mathcal{N}(\varphi)(t,\xi) = \partial_\xi(\varphi(t,\xi)^2)/2$ for smooth functions φ. Any path u in H^α such that

$$u_t = U_{t-s}u_s + U_t \int_s^t U_{-r}(\partial_\xi(U_{r-s}v_s)^2)\mathrm{d}r + U_t O(|t-s|^{2\gamma})$$

for some $v \in C^\gamma(\mathbb{R}, H^\alpha)$ enjoy the property that

$$\mathcal{N}(P_N u) \to \mathcal{N}(u)$$

as space-time distribution. The non-linear term is then well-defined.

There exists a unique local controlled solutions to the distributional equation

$$\partial_t u + \partial_\xi^3 u + \mathcal{N}(u) = 0$$