NATIONAL INSTITUTE OF TECHNOLOGY, TIRUCHIRAPPALLI
Department Of Computer Science And Engineering

Second Class Test

Combinatorics and Graph Theory

Marks: 20 Course Code: CSPE32 Time: 45 mins

Instructions to the Students: Answer all questions.

1. In an undirected connected planar graph G, there are eight vertices and five faces. The number of edges in G is ______ [1]

   a. 8 c. 11
   b. b. 10 d. 13

Answer: **c. 11**

In a connected planar graph f = e - n + 2. Here, n = 8, f= 5, ∴ 5 = e - 8 + 2 => e = 11

2. A graph is planar if and only if, [1]
   a. it does not contain subgraphs homeomorphic to K_5 and $K_{3,3}$.
   b. it does not contain subgraphs isomorphic to K_5 or $K_{3,3}$.
   c. it does not contain subgraphs isomorphic to K_5 and $K_{3,3}$.
   d. it does not contain subgraphs homeomorphic to K_5 or $K_{3,3}$.

Answer: **d. it does not contain subgraphs homeomorphic to K_5 or $K_{3,3}$.**

3. Let G be the non-planar graph with minimum possible number of edges. Then G has
   a. 9 edges and 5 vertices
   b. 9 edges and 6 vertices
   c. 10 edges and 5 vertices
   d. 10 edges and 6 vertices [1]

Answer: **b. 9 edges and 6 vertices**

K_5 and $K_{3,3}$ are the smallest non planar graphs. K_5 has 5 vertices and 10 edges and $K_{3,3}$ has 6 vertices and 3 × 3 = 9 edges. So, the non-planar graph with minimum number of edges $K_{3,3}$ with 9 edges and 6 vertices.

Note: K_5 is the non planar graph with minimum number of vertices.

4. G is a simple undirected graph. Some vertices of G are of odd degree. Add a node v to G and make it adjacent to each odd degree vertex of G. The resultant graph is sure to be [1]
   a. Regular b. Complete c. Hamiltonian d. Euler

Answer: **d. Euler.**

After the transformation, all vertices in the graph are of even degree and graph is connected, so it is an Euler graph.

5. If G is a forest with n vertices and k connected components, how many edges does G have? [1]
   a. $\lfloor n/k \rfloor$ b. $\lceil n/k \rceil$ c. n-k d. n-k+1

Answer: **c. n-k**

The number of edges in a forest G with n vertices and k components = Rank of G = n - k.

Alternatively, each component will have n/k vertices (pigeonhole principle). Hence, for each component there will be (n/k)-1 edges. Since there are k components, total number of edges= k*((n/k)-1) = n-k.

6. Let G be a simple, finite, undirected graph with vertex set $\{v_1, v_2, \ldots, v_n\}$.
   Let $\Delta(G)$ denote the maximum degree of G and let $C=\{1,2,\ldots\}$ denote the set of all possible colors. Color the vertices of G using the following greedy strategy:
   for i=1,…,n color(v_i)←min{i∈C : no neighbour of v_i is colored j}
   Which of the following statements is/are TRUE? [2]

   a. This procedure results in a proper vertex coloring of G.
   b. The number of colors used is at most $\Delta(G)+1$.
   c. The number of colors used is at most $\Delta(G)$.
   d. The number of colors used is equal to the chromatic number of G.

Answer: **a & b**

Explanations:

a. The greedy coloring strategy ensures that each vertex is assigned the smallest possible color that is not already used by its neighbors. Therefore, this procedure does result in a proper vertex coloring of G.

b. Since the graph G is finite and simple, the maximum degree Δ(G) represents the highest number of neighbors any vertex can have. In the worst-case scenario, when all neighbors of

a vertex have distinct colors, the vertex itself will need one additional color not used by its neighbors. Therefore, the number of colors used by this procedure is at most Δ(G) + 1.

7. An articulation point in a connected graph is a vertex such that removing the vertex and its incident edges disconnects the graph into two or more connected components. Let T be a DFS tree obtained by doing DFS in a connected undirected graph G. Which of the following option is/are correct? [2]

   a. If u is an articulation point in G such that x is an ancestor of u in T and y is a descendent of u in T, then all paths from x to y in G must pass through u.
   b. Root of T is an articulation point in G if and only if it has 2 or more children.
   c. Root of T can never be an articulation point in G.
   d. A leaf of T can be an articulation point in G.

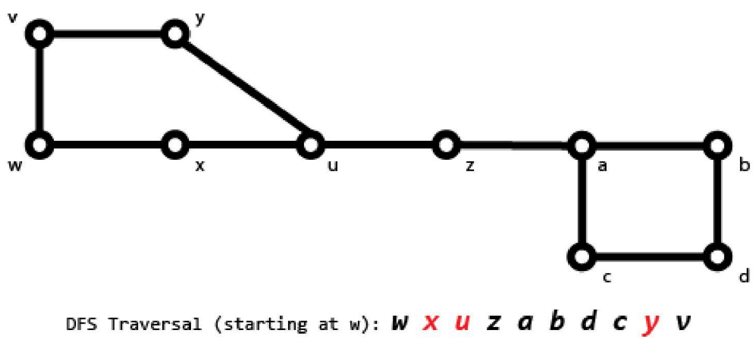
Answer: **b. Root of T is an articulation point in G if and only if it has 2 or more children.**

Explanation:

We check all the options one by one:

(a): If u is an articulation point in G such that x is an ancestor of u in T and y is a descendant of u in T, then all paths from x to y in G must pass through u.This option is **FALSE**.

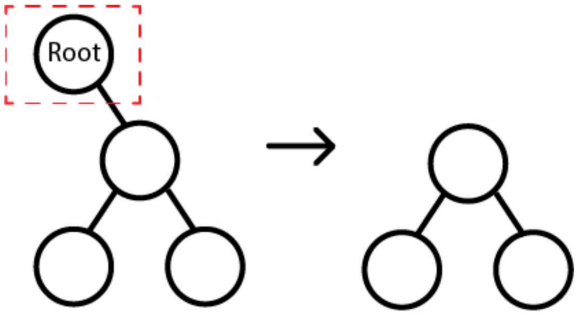
Let us take an example of a graph G:



For this graph G, we can verify that in T (obtained by doing DFS): **u** is an **articulation point** in G, **x** is an **ancestor** of **u** and **y** is a **descendent** of **u**. All the conditions are satisfied, yet we have a path from x to y in G (x->w->v->y) that **does not** pass through u.

(b): Root of T is an articulation point in G if and only if it has 2 or more children. This option is **TRUE**.

From the above example and the following tree:

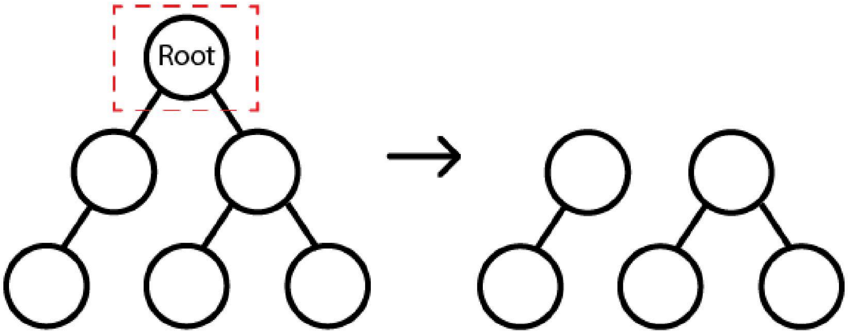


Root can be an articulation iff there are 2 or more children

The root of T has 1 child, and as such removing its corresponding vertex in G does not disconnect the graph into two (or more) components. Thus the root can not be an articulation point in this scenario. Therefore having **at least 2 children** is a compulsory condition for the root of T to be an articulation point.

c) <u>Root of T can never be an articulation point in G.</u> This option is **FALSE**.

Look at the following DFS tree T of a graph G:

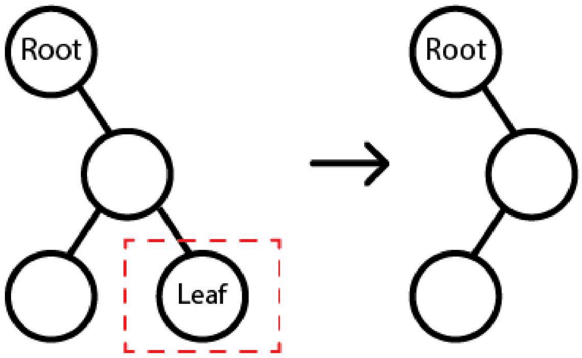


When root can be an articulation point

We can clearly see that removing the vertex corresponding to the root of T, will disconnect the graph into two components. Thus the root of T **can** be an articulation point in G.

(d): <u>A leaf of T can be an articulation point in G.</u> This option is **FALSE**.

From the tree T below:

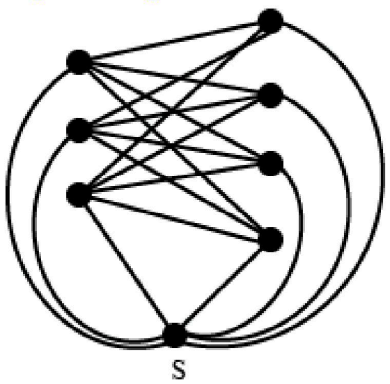


We can understand that any leaf of T is connected to only one node, i.e. its parent. Thus removing the vertex corresponding to T in the graph will not disconnect the graph into two or more components. So, a leaf of T can **never** be an articulation point in G.

8. Graph G is obtained by adding vertex s to $K_{3,4}$ and making s adjacent to every vertex of $K_{3,4}$. The minimum number of colours required to edge-colour G is _____. [2]
   a. 5
   b. 6
   c. 7
   d. 8

Answer: c. 7

Edge Coloring of a graph: the least number of colors needed to color the edges of so that any two edges that share a vertex have different colors.

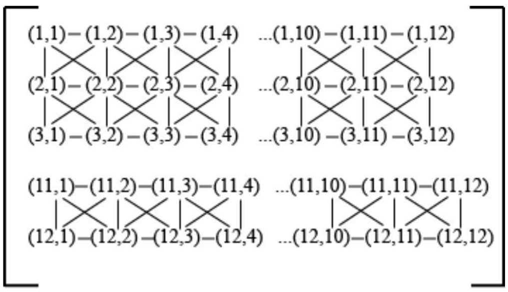


7 edges are touching at S each of which needs to be colored by different colors. So a minimum of 7 colors are required. All other vertices have degrees less than 7. So 7 colors are required for edge coloring.

9. Consider an bidirectional graph G where self-loops are not allowed. The vertex set of G is $\{(i,j):1\leq i\leq 12, 1\leq j\leq 12\}$.
There is an edge between (a,b) and (c,d) if $|a-c|\leq 1$ and $|b-d|\leq 1$. The number of edges in this graph is _____. [2]
    a. 605 b. 506 c. 144 d. 132

Answer: b. 506

The given condition translates into the graph shown here where every vertex is connected only with its neighbours. From this diagram:



(i) The four corner vertices each have 3 degrees which gives 4 × 3 = 12 degrees.
(ii) The 40 side vertices have 5 degrees each contributing a total of 40 × 5 = 200 degrees.
(iii) The 100 interior vertices each have 8 degrees contributing a total of 100 × 8 = 800 degrees.
So total degree of the graph = 12 + 200 + 800 = 1012 degrees

Now the number of edges in any undirected graph = Total degrees/2 = 1012/2 = 506.

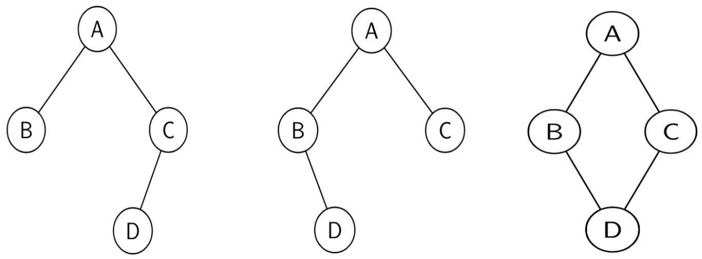
10. Let $G_1=(V,E_1)$ and $G_2=(V,E_2)$ be connected graphs on the same vertex set V with more than two vertices. If $G_1\cap G_2=(V,E_1\cap E_2)$ is not a connected graph, then the graph $G_1 \cup G_2=(V, E_1 \cup E_2)$ [2]
    a. cannot have a cut vertex
    b. must have a cycle
    c. must have a cut-edge (bridge)
    d. has chromatic number strictly greater than those of G_1 and G_2

Answer: b. must have a cycle

We are given that $G_1=(V,E_1)$ and $G_2=(V,E_2)$ are connected. So, if we take any two vertices, there must be a path between them in both $G_1=(V,E_1)$ and $G_2=(V,E_2)$. Now, it is given that $G_1\cap G_2=(V,E_1\cap E_2)$ is disconnected. That is, we have at least two vertices v_i and v_j such that there is no path between them in $G_1\cap G_2$. This means the paths between them in G_1 and in

G_2 are distinct. So, in $G_1 \cup G_2$, we have two distinct paths between a pair of vertices, so, it forms a cycle.

For other options, take the following trees:



G_2 G_1 $G_1 \cup G_2$

a. False. Every vertex of a tree (other than leaves) is a cut vertex.

c. False. There is no cut-edge (an edge whose removal increases the number of connected components in graph) in

d. False, all three graphs have the same chromatic number.

11. The degree sequence of a simple graph is the sequence of the degrees of the nodes in the graph in decreasing order. Which of the following sequences cannot be the degree sequence of any graph? [3]
    a. 7,6,5,4,4,3,2,1
    b. 7,6,6,4,4,3,2,2
    c. 6,6,6,6,3,3,2,2
    d. 8, 7, 7, 6, 4, 2, 1, 1

Answer: c and d

Explanation:
Apply Havel–Hakimi Algorithm to given sequences:

a) 7,6,5,4,4,3,2,1 → 5,4,3,3,2,1,0 → 3,2,2,1,0,0 → 1,1,0,0,0 → 0,0,0,0. So it is graphical.

b) 7,6,6,4,4,3,2,2 → 5,5,3,3,2,1,1 → 4,2,2,1,0,1 (arrange in descending order) → 4,2,2,1,1,0 → 1,1,0,0,0 → 0,0,0,0. So it is graphical.

c) 6, 6, 6, 6, 3, 3, 2, 2 → 5, 5, 5, 2, 2, 1, 2 (arrange in descending order) → 5, 5, 5, 2, 2, 2, 1 → 4, 4, 1, 1, 1, 1 → 3, 0, 0, 0, 1 (arrange in descending order) → 3, 1, 0, 0, 0 → 0, -1, -1, 0 (arrange in descending order) → 0, 0, -1, -1

Degree of a vertex can not be negative, so the sequence is not graphical.

d) 8,7,7,6,4,2,1,1 , here the degree of a vertex is 8 and total number of vertices are 8 , so it's impossible, hence it's not graphical.

12. Let G be a connected undirected weighted graph. Consider the following two statements.
S_1: There exists a minimum weight edge in G which is present in every minimum spanning tree of G.
S_2: If every edge in G has distinct weight, then G has a unique minimum spanning tree.
Which one of the following options is correct? [2]
    a. S_1 is false and S_2 is true.
    b. S_1: is true and S_2 is false.
    c. Both S_1: and S_2 are true.
    d. Both S_1: and S_2 are false.

Answer: (a) S_1 is false and S_2 is true.

Explanation:

We can think of running Kruskal's algorithm for finding the Minimum Spanning Tree on a graph. While doing that, we sort the edges based on their weight and start selecting edges from the smallest weight).

Problem with S_1: If we have multiple edges with the same minimum weight, then a specific weighted edge is not guaranteed to be selected for MST.

S_2 is Correct: If the sorted order of the edges contains only distinct values, Kruskal's algorithm will always select a unique set of edges resulting in a unique minimum spanning tree.