(1) a) Schrödinger Eqⁿ :- $i\hbar \frac{d}{dt}|\Psi_n(t)\rangle = \hat{H}|\Psi_n(t)\rangle$.

Hamiltonian is given as $\hat{H} = \frac{\Omega}{2}\hat{\sigma}_x$

Initial State of Qubit is given as, $|\Psi(t=0)\rangle = |\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$.

- Differential Equation for Basis coefficients in $\hat{\sigma}_z$ EigenBasis.

$\Rightarrow |\Psi(t)\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$. } Substituting this in S. Eqⁿ.

$$i\hbar \frac{d}{dt}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \frac{\Omega}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}.$$

Plank unit $\hbar = 1$

$$= i\frac{d}{dt}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \frac{\Omega}{2}\begin{pmatrix}\beta\\\alpha\end{pmatrix}.$$

$$\Rightarrow i\frac{d(\alpha)}{dt} = \frac{\Omega\beta}{2} \rightarrow \text{(i)} \quad \& \quad i\frac{d}{dt}(\beta) = \frac{\Omega\alpha}{2} \rightarrow \text{(ii)}.$$

$$\Downarrow$$

Differentiate this again.

$$\frac{d(\beta)}{dt} = \frac{\Omega\alpha}{2i}.$$

$$i\frac{d^2}{dt^2}\alpha = \frac{\Omega}{2}\frac{d(\beta)}{dt}.$$

$$i\frac{d^2}{dt^2}\alpha = \frac{\Omega}{2}\cdot\frac{\Omega\alpha}{2i}$$

$$-\frac{d^2}{dt^2}\alpha = \frac{\Omega^2}{4}\cdot\alpha \Rightarrow \boxed{\frac{d^2}{dt^2}\alpha = -\left(\frac{\Omega}{2}\right)^2\alpha}$$

$$\frac{d^2}{dt^2}\alpha = -\left(\frac{\Omega}{2}\right)^2\alpha.$$

$$\frac{d^2}{dt^2}\alpha + \left(\frac{\Omega}{2}\right)^2\alpha = 0$$ } Simple Harmonic Oscillator Eq^n.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

$$\omega = \frac{\Omega}{2}.$$

Ω = Rabi-frequency

and α = Amplitude of State $|\uparrow\rangle$

Solution to SHO is.

$$\alpha = A\cos\left(\frac{\Omega t}{2}\right) + B\sin\left(\frac{\Omega t}{2}\right).$$

for initial State $(t=0)$.

$$\alpha(t=0) \Rightarrow A\cos(0) + B\sin(0)$$

$$\alpha(t=0) = A(1).$$

for normalised state $|\uparrow\rangle$.
$\alpha(0) = 1$

$$\boxed{\therefore A = 1}$$

Putting "α" in Eq (i) we get.

$$\Rightarrow \frac{i\,d\left(A\cos\left(\frac{\Omega t}{2}\right) + B\sin\left(\frac{\Omega t}{2}\right)\right)}{dt} = \frac{\Omega}{2}\beta.$$

$$\Rightarrow i \quad -\frac{\Omega A}{2}\sin\left(\frac{\Omega t}{2}\right) + \frac{i\Omega B}{2}\cos\left(\frac{\Omega t}{2}\right). = \frac{\Omega}{2}\beta.$$

at $t=0$, (initial State is $|\uparrow\rangle$) $P.|\downarrow\rangle = 0$).

$$\frac{iB\Omega}{2} = \frac{\Omega}{2}\beta = 0.$$

$$\boxed{\therefore B = 0}.$$

Hence at initial State, the [A & B] values are Satisfied.
and hence we can say that.

$\alpha = \cos\left(\frac{\Omega t}{2}\right)$ & $\beta = i\sin\left(\frac{\Omega t}{2}\right)$.

β = Amplitude of Quantum State $|\downarrow\rangle$ in Z-Basis.

$$\therefore \; P_{|\downarrow\rangle}(t) = |\beta|^2 = \left| i^2 \sin^2\left(\frac{\Omega t}{2}\right)\right|.$$

$$\boxed{P_{|\downarrow\rangle}(t) = \sin^2\left(\frac{\Omega t}{2}\right)}$$

- Eigen States & Eigen Values of $\left[\hat{H} = \frac{\Omega}{2}\hat{\sigma}_x\right]$.

Eigen States for $\hat{\sigma}_x$ are,

for Eigenvalue $+1 \Rightarrow |+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$.

for Eigenvalue $-1 \Rightarrow |-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$.

and Initial State $|\uparrow\rangle$ can be written as,

$$|\uparrow\rangle = \frac{(|+\rangle - |-\rangle)}{\sqrt{2}}.$$

WKT, time Evolution is given by,

$$|\psi(t)\rangle = e^{(-iHt)}|\uparrow\rangle = \left(e^{-iE_+ t}|+\rangle + e^{-iE_- t}|-\rangle\right)1/\sqrt{2}.$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\frac{\Omega t}{2}}|+\rangle \; e^{+i\frac{\Omega}{2}t}|-\rangle\right)\Big\}.$$ writing this Back to σ_z-Basis.

$$|\psi(t)\rangle = \left(e^{\left(-i\frac{\Omega t}{2}\right)}|+\rangle + e^{\left(+i\frac{\Omega t}{2}\right)}|-\rangle\right).1/\sqrt{2}.$$

$$= e^{-i\frac{\Omega}{2}t}(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2} + e^{+i\frac{\Omega}{2}t}(|\uparrow\rangle - |\downarrow\rangle)/\sqrt{2}.$$

$$= \frac{e^{-i\Omega/2t}}{\sqrt{2}}|\uparrow\rangle + \frac{e^{-i\Omega/2t}|\downarrow\rangle}{\sqrt{2}}.$$

$$+ \frac{e^{+i\frac{\Omega}{2}t}}{\sqrt{2}}|\uparrow\rangle - \frac{e^{+i\frac{\Omega}{2}t}|\downarrow\rangle}{\sqrt{2}}.$$

$$= \frac{\left(e^{-i\Omega/2t} + e^{+i\Omega/2t}\right)}{\sqrt{2}}|\uparrow\rangle + \frac{\left(e^{-i\Omega/2t} - e^{-i\Omega/2t}\right)}{\sqrt{2}}|\downarrow\rangle$$

$$|\psi(t)\rangle = \cos\left(\frac{\Omega t}{2}\right)|\uparrow\rangle + (-i)\sin\left(\frac{\Omega t}{2}\right)|\downarrow\rangle.$$

$$\boxed{|\psi(t)\rangle = \cos\left(\frac{\Omega t}{2}\right)|\uparrow\rangle + (-i)\sin\left(\frac{\Omega t}{2}\right)|\downarrow\rangle}.$$

$$\boxed{P_{|\downarrow\rangle}(t) = \left|(-i)^2 \sin^2\left(\frac{\Omega t}{2}\right)\right| = \sin^2\left(\frac{\Omega t}{2}\right)}.$$

- Using Time Evolution operator $\left[e^{-i\hat{H}t}\right]$.

$$\Rightarrow e^{-i\hat{H}t} = e^{-i\left(\frac{\Omega}{2}\sigma_x\right)t}$$

$$\Rightarrow \quad |\psi(t)\rangle = e^{-i\hat{H}t}|\psi_0\rangle.$$

Writing the above Expression by Expanding the previous Eq^n in terms of Pauli-operator

$$|\psi(t)\rangle = e^{-i\hat{H}t} |\psi_0\rangle.$$

$$= e^{-i\frac{\Omega}{2}\sigma_x t} |\psi_0\rangle = \sum_{n=0}^{\infty} \frac{\left(i\frac{\Omega}{2}\sigma_x t\right)^n}{n!}.$$

$$= \cancel{\text{By Taylor Series Expansion}}.$$

$$= \left[\underbrace{\left[\sum \frac{\left(+i\frac{\Omega}{2}\sigma_x t\right)^{2n+1}}{(2n+1)!}\right]}_{\text{odd}} + \underbrace{\left[\sum \frac{\left(+i\,\Omega/2\,\sigma_x t\right)^{2n}}{2n!}\right]}_{\text{Even}} \right] |\psi_0\rangle.$$

$\Rightarrow$ Pauli operator

for odd function $\sigma_x^1 = \sigma_x^3 = [\sigma_x^{2n+1}] = \sigma_x$.

$\Rightarrow$ — " —

for Even function. $\sigma_x^2 = \sigma_x^4 = \{\sigma_x^{2n}\} = \mathbb{I}$

The above Eq^n can therefore be re-written as,

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \frac{\left(i\frac{\Omega}{2}t\right)^1 \left(i\frac{\Omega}{2}t\right)^{2n} \mathbb{I}}{(2n+1)!} + \sum_{n=0}^{\infty} \left(\frac{\left(i\frac{\Omega}{2}t\right)^{2n}}{2n!}\right)$$

By taylor Series Expansion again.

$$= \cos\left(\frac{\Omega}{2}t\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \left(\frac{\Omega}{2}t\right)^{2n}.$$

& $\sin\left(\frac{\Omega}{2}t\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\Omega}{2}t\right)^{2n+1}$

$$\therefore \quad |\psi(t)\rangle = \left(\mathbb{I}.\cos\left(\frac{\Omega}{2}t\right) + (-i)\,\sigma_x.\sin\left(\frac{\Omega}{2}t\right)\right)|\psi_0\rangle$$

for $|\psi_0\rangle = |\uparrow\rangle$ at $t=0$.

$$|\psi(t)\rangle = \left(\mathbb{I}.\cos\left(\frac{\Omega}{2}t\right)|\uparrow\rangle - i\sin\left(\frac{\Omega}{2}t\right)\sigma_x|\uparrow\rangle\right).$$

$$|\psi(t)\rangle = \cos\left(\frac{\Omega}{2}t\right)|\uparrow\rangle - i\sin\left(\frac{\Omega}{2}t\right)|\downarrow\rangle.$$

$\sigma_x|\uparrow\rangle = |\downarrow\rangle$

x-Gate flips the State.

$$P_{|\downarrow\rangle}(t) = |\langle\downarrow|\psi(t)\rangle|^2.$$

$$= \left|\cos\left(\frac{\Omega}{2}t\right)\langle\downarrow|\uparrow\rangle - i\sin\left(\frac{\Omega}{2}t\right)\langle\downarrow|\downarrow\rangle\right|^2$$

$$P_{|\downarrow\rangle}(t) = \left|0 - i^2\sin^2\left(\frac{\Omega}{2}t\right)(1)\right|$$

$$= \sin^2\left(\frac{\Omega}{2}t\right).$$

$$\boxed{P_{|\downarrow\rangle}(t) = \sin^2\left(\frac{\Omega}{2}t\right)}$$

b) Given Bloch Sphere representation is done using,

$$|\psi\rangle = \cos(\theta/2)|\uparrow\rangle + e^{i\phi}\sin(\theta/2)|\downarrow\rangle. \rightarrow (i)$$

from previous,

$$|\psi(t)\rangle = \mathbb{I}.\cos\left(\frac{\Omega}{2}t\right)|\uparrow\rangle + (-i)\,\sigma_x \sin\left(\frac{\Omega}{2}t\right)|\downarrow\rangle.$$

$$|\psi(t)\rangle = \mathbb{I}.\cos\left(\frac{\Omega}{2}t\right)|\uparrow\rangle - i\,\sigma_x \sin\left(\frac{\Omega}{2}t\right)|\downarrow\rangle - (ii).$$

from Eq (i) & (ii)

we can say that,

$$\theta = \Omega t$$

$e^{i\phi} = -i$, this is true for $\varphi = \frac{3\pi}{2} / \frac{-\pi}{2}$.

$$\therefore \phi = -\pi/2.$$

θ varies from 0 to 2π. [for one rotation] to return to its initial position.

$\theta = 2\pi n$ (n = no of rotation) $n \in \mathbb{Z}$.

The initial state $|\uparrow\rangle$ corresponds to $\theta = 0$.

$$\therefore \Omega t = 2\pi n.$$

$t = \frac{2\pi n}{\Omega}$ for one rotation $n = 1$.

$t = \frac{2\pi}{\Omega}$ } time for first return.

at time $t = \frac{2\pi}{\Omega}$.

$$|\psi(t = 2\pi/\Omega)\rangle = \cos\left(\frac{\cancel{\Omega}}{\cancel{2}} \times \frac{\cancel{2}\pi}{\cancel{\Omega}}\right)|\uparrow\rangle - i \sin\left(\frac{\cancel{\Omega}}{\cancel{2}} \times \frac{\cancel{2}\pi}{\cancel{\Omega}}\right)|\downarrow\rangle$$

$$= \cos(\pi)|\uparrow\rangle - i\sin(\pi)|\downarrow\rangle$$

$$= -|\uparrow\rangle \Big\}$$ a Global phase of $e^{i\pi}$ is acquired to the initial state once it returns to its initial state.

To negate this ($e^{i\pi}$ phase) the state goes for one more rotation ($n=2$).

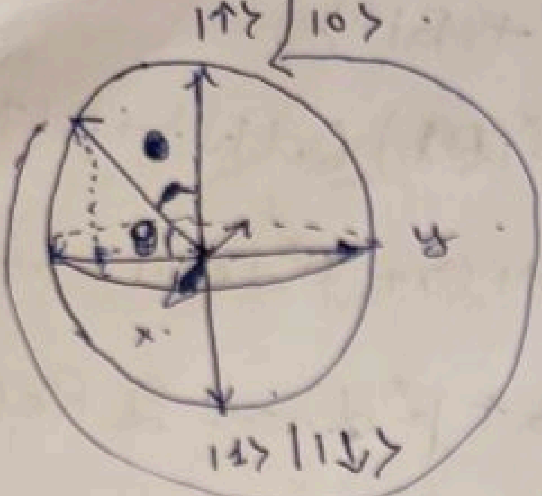
$$t = \frac{2\pi \times 2}{\Omega} = \frac{4\pi}{\Omega}.$$

$$|\psi(t = 4\pi/\Omega)\rangle = \cos\left(\frac{\cancel{\Omega}}{\cancel{2}} \times \frac{\overset{2}{\cancel{4}}\pi}{\cancel{\Omega}}\right)|\uparrow\rangle - i\sin\left(\frac{\cancel{\Omega}}{\cancel{2}} \times \frac{\overset{2}{\cancel{4}}\pi}{\cancel{\Omega}}\right)|\downarrow\rangle$$

$$|\psi\rangle = \cos(2\pi)|\uparrow\rangle - i\sin(2\pi)|\downarrow\rangle.$$

$$|\psi\rangle = |\uparrow\rangle.$$

at $t = \frac{4\pi}{\Omega}$ time the state truly returns to its initial state. (for $n=2$).



θ = 2 times full rotation for constant $\varphi = \frac{3\pi}{2}$. } Will give us the initial state.

c). $\vec{r} = [\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos\theta]$.
Corresponds to Pauli operators.
$[\langle\hat{\sigma}_x\rangle, \langle\hat{\sigma}_y\rangle, \langle\hat{\sigma}_z\rangle]$. } To Show.

from Question 1(b) we know that,

$|\psi\rangle = \cos(\theta/2)|\uparrow\rangle + e^{i\phi}\sin(\theta/2)|\downarrow\rangle \rightarrow$ (i).

also true.

$\underbrace{|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle, \ \& \ \langle\psi| = \alpha^*\langle\uparrow| + \beta^*\langle\downarrow|}_{(ii)}$.

$\Rightarrow \langle\psi|\sigma_x|\psi\rangle = \langle\hat{\sigma}_x\rangle$.

$\langle\hat{\sigma}_x\rangle = \langle\psi|\sigma_x|\psi\rangle = \langle\psi|\sigma_x|(\alpha|\uparrow\rangle + \beta|\downarrow\rangle)$

$= \langle\psi|(\alpha\sigma_x|\uparrow\rangle + \beta\sigma_x|\downarrow\rangle)$

$= \langle\psi|(\alpha|\downarrow\rangle + \beta|\uparrow\rangle)$.

$$\langle\psi|\hat{\sigma}_x|\psi\rangle = \langle\psi|(\alpha|\downarrow\rangle + \beta|\uparrow\rangle)$$

$$= (\alpha^*\langle\uparrow| + \beta^*\langle\downarrow|)(\alpha|\downarrow\rangle + \beta|\uparrow\rangle)$$

$$= \alpha^*\beta\langle\uparrow|\uparrow\rangle + \beta^*\alpha\langle\downarrow|\downarrow\rangle$$

$$= \alpha^*\beta + \beta^*\alpha = 2\,\mathrm{Re}(\alpha^*\beta)$$

from Eq (i) & (ii) we can say.

$$\alpha = \cos(\theta/2) \;\&\; \beta = e^{i\phi}\sin(\theta/2).$$

$$\therefore = 2\,\mathrm{Re}\left(\left(\cos(\theta/2)\right)\left(\sin(\theta/2)\right)e^{i\phi}\right)$$

Euler's relation

$$= 2\cdot\cos(\theta/2)\cdot\sin(\theta/2)\cos\phi$$

$$\left[e^{i\phi} = \cos\phi + i\sin\phi.\right]$$

$$= 2\cdot\cos\phi\cdot\cos(\theta/2)\cdot\sin(\theta/2)$$

$$= \boxed{\sin\theta\cdot\cos\phi = \langle\hat{\sigma}_x\rangle}$$

for. $\langle\hat{\sigma}_y\rangle$.

$$\langle\hat{\sigma}_y\rangle = \langle\psi|\hat{\sigma}_y|\psi\rangle = \langle\psi|\left[\alpha\hat{\sigma}_y|\uparrow\rangle + \beta\hat{\sigma}_y|\downarrow\rangle\right]$$

$$= \langle\psi|\left[-i\alpha|\downarrow\rangle + i\beta|\uparrow\rangle\right].$$

$$\langle\hat{\sigma}_y\rangle = (\alpha^*\langle\uparrow| + \beta^*\langle\downarrow|)\left[-i\alpha|\downarrow\rangle + i\beta|\uparrow\rangle\right].$$

$\langle \hat{\sigma}_y \rangle = i\alpha^* \beta - i\beta^* \alpha = 2i\, Im(\alpha^* \beta)$.

$\langle \hat{\sigma}_y \rangle = Im\left(\cancel{2}\cos(\theta/2) \cdot \sin(\theta/2)\, e^{i\phi}\right)$

$= 2i \cos(\theta/2) \sin(\theta/2) (-i \sin\phi)$.

$= 2(-i)^2 \cdot \cos(\theta/2) \sin(\theta/2) \sin\phi$.

$= 2\cos\theta/2 \cdot \sin\theta/2 \cdot \sin\phi$.

$$\boxed{\langle \hat{\sigma}_y \rangle = \sin\theta \cdot \sin\phi.}$$

for $\langle \hat{\sigma}_z \rangle = \langle \psi | \hat{\sigma}_z | \psi \rangle$

$= \langle \psi | [\alpha \hat{\sigma}_z |\uparrow\rangle + \beta \hat{\sigma}_z |\downarrow\rangle]$.

$= \langle \psi | [\alpha |\uparrow\rangle - \beta |\downarrow\rangle]$.

$= |\alpha|^2 - |\beta|^2$

$$\boxed{\langle \hat{\sigma}_z \rangle = \cos^2(\theta/2) - \sin^2(\theta/2) = \cos(\theta)}$$

2 (a). To verify $[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\hat{\sigma}_z$.

WKT,

$[\hat{\sigma}_x, \hat{\sigma}_y] = \hat{\sigma}_x \cdot \hat{\sigma}_y - \hat{\sigma}_y \cdot \hat{\sigma}_x$.

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} i+i & 0 \\ 0 & -i-i \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\,\hat{\sigma}_z.$$

Hence proved.

(b) $$\underbrace{(\Delta\hat{\sigma}_x)^2(\Delta\hat{\sigma}_y)^2 \geq 1/4\,|\langle[\hat{\sigma}_x,\hat{\sigma}_y]\rangle|^2}$$

Robertson's uncertainity principle for Qubits.

$$\left.\begin{aligned}(\Delta\hat{\sigma}_x)^2 &= \langle\hat{\sigma}_x^2\rangle - \langle\hat{\sigma}_x\rangle^2 = 1 - \langle\hat{\sigma}_x\rangle^2 \\ (\Delta\hat{\sigma}_y)^2 &= \langle\hat{\sigma}_y^2\rangle - \langle\hat{\sigma}_y\rangle^2 = 1 - \langle\hat{\sigma}_y\rangle^2.\end{aligned}\right\} \text{LHS.}$$

RHS, $1/4\,|\langle[\hat{\sigma}_x,\hat{\sigma}_y]\rangle|^2$

$$\Rightarrow 1/4\,|\langle\hat{\sigma}_z\rangle|^2 = 1/4\,\langle\hat{\sigma}_z\rangle^2$$

from 1(c), $\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = 1.$

$$\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 = 1 - \langle\sigma_z\rangle^2$$

$$\Rightarrow (1 - \langle\hat{\sigma}_x\rangle^2)(1 - \langle\hat{\sigma}_y\rangle^2) \geq 1/4\,\langle\hat{\sigma}_z\rangle^2$$

$$\Rightarrow \underbrace{1 - \langle\hat{\sigma}_x\rangle^2 - \langle\hat{\sigma}_y\rangle^2} + \langle\hat{\sigma}_x\rangle^2\langle\hat{\sigma}_y\rangle^2$$

$$\Rightarrow \langle\hat{\sigma}_z\rangle^2 + \langle\hat{\sigma}_x\rangle^2\langle\hat{\sigma}_y\rangle^2 \geq 1/4\,\langle\hat{\sigma}_z\rangle^2$$

$\{ \langle \hat{\sigma}_z \rangle^2 \cdot \langle \hat{\sigma}_y \rangle^2 \geq 0 \}$.

$\therefore \quad \{ \langle \hat{\sigma}_z \rangle^2 + \langle \hat{\sigma}_x \rangle^2 \langle \hat{\sigma}_y \rangle^2 \} > 0.$

$$\underbrace{\{ \langle \hat{\sigma}_z \rangle^2 + \langle \hat{\sigma}_x \rangle^2 \langle \hat{\sigma}_y \rangle^2 \}}_{} > 1/4 \langle \hat{\sigma}_z \rangle^2.$$

Sum ~~this~~ has to be greater than Single Pauli-operator.

$$\boxed{\langle \hat{\sigma}_z \rangle^2 + \langle \hat{\sigma}_x \rangle^2 \langle \hat{\sigma}_y \rangle^2 \geq 1/4 \langle \hat{\sigma}_z \rangle^2}.$$

c) for $\langle \sigma_x \rangle^2$ (or) $\langle \sigma_y \rangle$ (or) both $= 0$.

The Inequality is Saturated.

3 a). Hadamard $(H) = 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Rotation operators,

$$R_y(\theta) = R_y(\pi/2) = e^{-i(\pi/4)\sigma_y}$$

$$= \cos(\pi/4)\, \mathbb{I} - i \sin(\pi/4) \hat{\sigma}_y$$

$$= \begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} = 1/\sqrt{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$R_z(\theta) = R_z(\pi) = \cos(\pi/2)\,\mathbb{I} - i\sin(\pi/2)\,\sigma_z$.

$$= \begin{pmatrix} \cos\pi/2 & 0 \\ 0 & \cos\pi/2 \end{pmatrix} - \begin{pmatrix} i\sin\pi/2 & 0 \\ 0 & -i\sin\pi/2 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\pi/2} & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}.$$

$$H = R_y(\pi/2)\, R_{\cancel{z}}(\pi) = 1/\sqrt{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-i\pi/2} & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}$$

~~Ignoring~~

$$= 1/\sqrt{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

$$= -i \cdot 1/\sqrt{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$= -i \cdot H.$$

Ignoring global phase.

we have $\boxed{H = R_y(\pi/2)\, R_z(\pi)}$.

Now for General Rotation $R_n(\pi)$. for $n = (1/\sqrt{2}, 0, 1/\sqrt{2})$

WKT, $R_n(\pi) = \cos\left(\frac{\pi}{2}\right)\mathbb{1} - i\sin\left(\frac{\pi}{2}\right)\left(\frac{1}{\sqrt{2}}\sigma_x + \frac{1}{\sqrt{2}}\sigma_z\right)$

$$= -i\, 1/\sqrt{2}\left(\sigma_x + \sigma_z\right).$$

$(\sigma_x + \sigma_z) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

$$R_n(\pi) = -i \; 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -i H$$

Ignoring global phase (i) we get.

$$\boxed{H = H = R_n(\pi).}$$

ⓑ $H \times H = Z$ .ly To show.

$$1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$1/2 \begin{pmatrix} 1 & +1 \\ -1 & +1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 1/2 \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

$$\boxed{H \times H = 1/2 \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z.}$$

$H Y H = -Y$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 1/2 \begin{pmatrix} i & -i \\ -i & -i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$\frac{1}{2} \begin{pmatrix} 0 & 2i \\ -2i & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -Y.}$$

$HZH = X.$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix} = \boxed{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = X}$$

ⓒ. $R_x(\pi/4) = \cos\left(\frac{\pi}{8}\right)\mathbb{1} - i\sin\left(\frac{\pi}{8}\right)\sigma_x.$

$$= \begin{pmatrix}\cos(\pi/8) & 0\\ 0 & \cos(\pi/8)\end{pmatrix} - \begin{pmatrix}0 & i\sin(\frac{\pi}{8})\\ i\sin(\frac{\pi}{8}) & 0\end{pmatrix}$$

$$= \begin{pmatrix}\cos(\pi/8) & -i\sin(\pi/8)\\ i\sin(\pi/8) & \cos(\pi/8)\end{pmatrix}.$$

$= e$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = HTH.$$

$$\frac{1}{2}\begin{pmatrix}1 & e^{i\pi/4}\\ 1 & -e^{+i\pi/4}\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.$$

$$\frac{1}{2}\begin{pmatrix}1+e^{i\pi/4} & 1-e^{i\pi/4}\\ 1 - e^{-i\pi/4} & 1+e^{-i\pi/4}\end{pmatrix}.$$

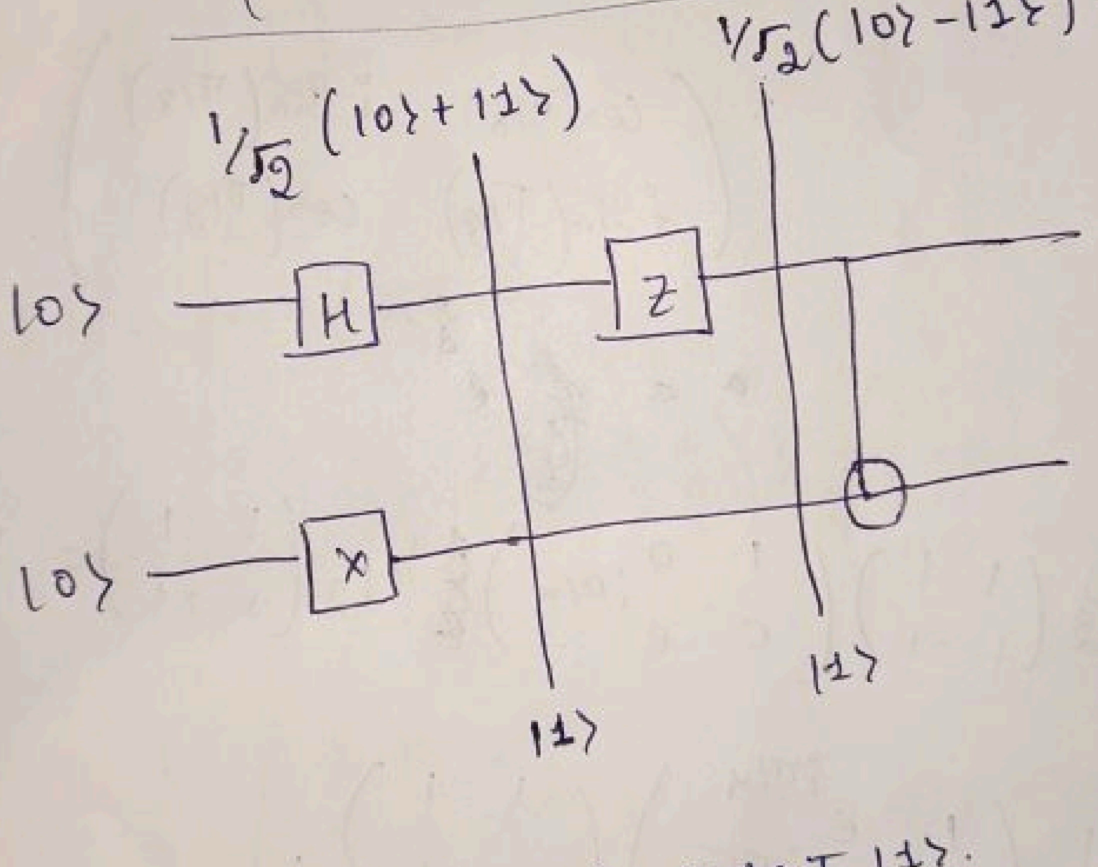
$$HTH = \frac{1}{2}\begin{pmatrix} \frac{\sqrt{2}+1+i}{\sqrt{2}} & \frac{\sqrt{2}-1-i}{\sqrt{2}} \\ \frac{\sqrt{2}-1-i}{\sqrt{2}} & \frac{\sqrt{2}+1+i}{\sqrt{2}} \end{pmatrix}$$

$$R_x(\pi/4) = \begin{pmatrix} \cos(\pi/8) & -i\sin\left(\frac{\pi}{8}\right) \\ i\sin(\pi/8) & \cos(\pi/8) \end{pmatrix}$$

$$\boxed{R_x(\pi/4) = HTH}.$$

4.

a)

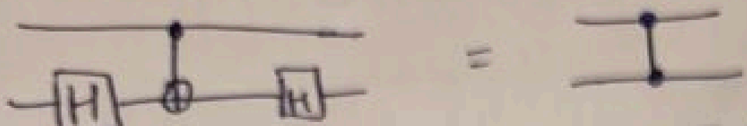


$$|\psi\rangle = 1/\sqrt{2}\,(|0\rangle - |1\rangle)\,.\,CNOT\,|1\rangle.$$

$$= 1/\sqrt{2}\,(|0\rangle . CNOT\,|1\rangle - |1\rangle CNOT\,|1\rangle).$$

$$= \boxed{1/\sqrt{2}\,(|01\rangle - |10\rangle) = |\phi^-\rangle}.$$

(b) · To prove,



C-Z Gate.

$\Downarrow$.

CNOT-X

H ~~.~~ . H.

H . CNOT-X . H.

$(\mathbb{I} \otimes H) . CNOT_X (\mathbb{I} \otimes H)$.

$$\mathbb{I} \otimes H = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

$(I \otimes H)\ CNOT\text{-}X\ (\mathbb{I} \otimes H) =$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

$$\frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = (I \otimes H)\ CNOT\text{-}X\ (I \otimes H)$$

$$\text{C-Z Gate} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ldots$$

Hence proved,

$$\boxed{(\mathbb{I} \otimes H \cdot \text{CNOT-}x \cdot \mathbb{I} \otimes H) = C\text{-}Z}.$$