from the departure of the log and exponential characteristics of the individual stages and the loading of the amplifier with increasing input. The slight discrepancy of the calculated values of the transfer gradient obtained from the experimental curves from those of the linear amplifier gain γ set for the respective cases may be due to some difference of the values of p between the logarithmic and the exponential stages. There is an inherent tendency for self-compensation of the thermal instability and nonlinear distortion due to the use of two nonlinearities of inverse characteristics, the cancellation being maximum for unity gain of the linear stage.

The system provides a basic operation necessary in computers, in nonlinear feedback amplifiers for volume compression and expansion of input signals, and for nonlinear amplitude transformation of a transmission system. The frequency response of this system was found to be within 3 db from dc to about 5 Mc with OC44 transistors. The complete details of this system will be published elsewhere.

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Elimination of Slip and Instability Effects in Certain *M*-Type Electron Beams*

Several of the difficulties common in crossed field electron beams due to space charge effects can be eliminated by a slight change in geometry.

The space charge field gives rise to a variation in the dc drift velocity across the linear M-type beam. It is well known that the growing and decaying nature of the pair of synchronous waves is intimately associated with the slip in the velocity. In a simplified model, the two synchronous waves can be identified with two surface waves on either side of the beam where one wave carries positive, and the other, negative energy. If there is a velocity difference between the two surface waves, they couple actively and a pair of growing and decaying waves result. Random disturbances introduced at the gun end of the tube are amplified and cause the beam to spread. This effect, known as the diocotron effect, limits the operational range of conventional M-type tubes. In the M-type parametric amplifier,[1] the velocity slip introduces noise in the form of an imperfect cancellation of the noise orbits in the output coupler. In the M-type delay line,[2] the slip causes signal loss at very low beam velocities.

A way to eliminate the detrimental effects of the slip is to make all electrons move with the same angular velocity in a circular interaction region. Except for having a curvature, the interaction region is the same as in the linear device as shown in Fig. 1. It can be shown, in general, that a beam between two conducting cylinders in which all electrons have the same dc angular velocity supports four unattenuated waves. Furthermore, there are no overtaking effects as in the linear beam.

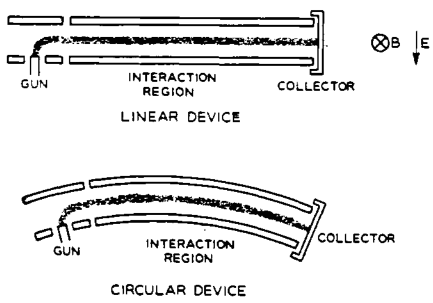

Fig. 1.

One physically realizable case of this arrangement would be when the angular velocity is given by

$$\omega_r = \frac{\omega_c}{2} - \sqrt{\frac{\omega_c^2}{4} - \frac{\omega_p^2}{2}} \tag{1}$$

where ω_c and ω_p are the cyclotron and plasma frequencies. Such an electron flow is similar to that which is described as "space-charge-balanced" flow for O-type beams. For the M-type version, the aixal velocity is zero and only a section for a circular strip of the cross section of the O-type beam is used. In order to support the beam with the angular velocity given by (1), the potentials on the position and negative cylindrical electrodes must be such that the radial electric field inside the beam is

$$E_r = -\frac{\omega_p^2 r}{2\eta}$$

where r is the radial coordinate and η is the charge to mass ratio of the electrons. The required voltage across the interaction region becomes

$$V = \frac{\omega_p^2}{2\eta}\left\{ r_2^2\left[\log_e \frac{r_2}{r_-} - \frac{1}{2}\right] - r_1^2\left[\log_e \frac{r_1}{r_+} - \frac{1}{2}\right]\right\} \tag{2}$$

where r_+ and $r_-(r_+ > r_-)$ are the radii of the positive and the negative electrodes and r_1 and $r_2(r_1 > r_2)$ are the radii of the outer and inner surfaces of the beam. When $r_+ - r_- = d \ll r_-$ the voltage in (2) reduces approximately to $V = v_a B d$ where v_a is the average beam velocity and B is the magnetic field.

Returning to (1) we obtain, when $\omega_c^2 \gg \omega_p^2$,

$$\omega_r = \frac{\omega_p^2}{2\omega_c}. \tag{3}$$

If

$$R = \frac{r_1 + r_2}{2}$$

is the mean radius of the beam, $r_1 - r_2$ is the beam thickness and v_a is the velocity of the electrons at the radius R, we get from (3)

$$R = \frac{2\epsilon_0 B A v_a^2}{I},$$

where I is the beam current, and A is the cross-sectional area of the beam. For a current density of $10^{-4} A/\text{cm}^2$, an equivalent beam voltage of 25 v and a magnetic field of 500 gauss, $R = 8$m. If the beam current is increased to $10^{-1} A/\text{cm}^2$, the beam voltage is 250 v and the magnetic field 500 gauss, $R = 8$ cm.

It appears that this method would be best suited for low current, high magnetic field devices, such as the parametric amplifier[1] or the delay line.[2] It should be noticed that any variation of the voltages of the focusing electrodes would destroy the optimum conditions and introduce a slip.

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* Received December 18, 1962.

[1] J. W. Klüver, "A low noise M-type parametric amplifier," to be published.

[2] J. W. Klüver, "An electronically variable delay line," Proc. IRE (*Correspondence*), vol. 50, p. 2487; December, 1962.

A Product Theorem for Hilbert Transforms*

In engineering analysis, the need for the Hilbert transform of a product of functions occasionally arises. Also, forming products is a useful method for extending tables of Hilbert transforms. The following theorem provides a simple method for handling such products under certain conditions.

Theorem: Let $f(x)$ and $g(x)$ denote generally complex functions in $L^2(-\infty, \infty)$ of the real variable x. If

1) the Fourier transform $F(u)$ of $f(x)$ vanishes for $|u| > a$ and the Fourier transform $G(u)$ of $g(x)$ vanishes for $|u| < a$, where a is an arbitrary positive constant, or
2) $f(x)$ and $g(x)$ are analytic (*i.e.*, their real and imaginary parts are Hilbert pairs),

then the Hilbert transform of the product of $f(x)$ and $g(x)$ is given by

$$H[f(x)g(x)] = f(x)H[g(x)]. \tag{1}$$

Proof: In terms of their Fourier transforms, the product $f(x)g(x)$ can be written

$$f(x)g(x) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv F(u)G(v)e^{i(u+v)x}. \tag{2}$$

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Now [2]

$$H\left[e^{ibx}\right] = i \operatorname{sgn}(b) e^{ibx} \qquad (3)$$

so

$$H[f(x)g(x)] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv F(u)G(v) i \cdot \operatorname{sgn}(u+v) e^{i(u+v)x}. \qquad (4)$$

The shaded regions in Fig. 1 are those in which the product $F(u)G(v)$ is nonvanishing for the conditions of the theorem. In Fig. 1(a) the nonoverlapping Fourier transforms yield two semi-infinite strips in which the product is nonvanishing; in Fig. 1(b), for the analytic functions the Fourier transforms vanish for negative arguments [4] so that the product is nonvanishing only in the first quadrant. In both cases

$$\operatorname{sgn}(u+v) = \operatorname{sgn} v \qquad (5)$$

over the regions of integration in which the integrand is nonvanishing. Thus, (4) can be written

$$H[f(x)g(x)] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv F(u)G(v) i \operatorname{sgn} v e^{i(u+v)x}$$

$$= f(x) \frac{1}{2\pi} \int_{-\infty}^{\infty} dv G(v) i \operatorname{sgn} v e^{ivx}. \qquad (6)$$

But

$$H[g(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} dv G(v) H\left[e^{ivx}\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} dv G(v) i \operatorname{sgn} v e^{ivx}, \qquad (7)$$

so (6) finally becomes

$$H[f(x)g(x)] = f(x)H[g(x)], \quad \text{Q.E.D.} \qquad (8)$$

In circuit or communication theory terms, part (1) of the theorem states that the Hilbert transform of the product of a low-pass and a high-pass signal with nonoverlapping spectra is given by the product of the low-pass signal and the Hilbert transform of the high-pass signal.

Examples of product transforms derivable from the foregoing are given by Erdélyi [1]. For instance, his entry 15.3 (18) gives the Hilbert transform

$$H[\sin ax J_n(bx)] = \cos ax J_n(bx), \quad 0 < b < a, \; n = 0, 1, 2, \cdots$$

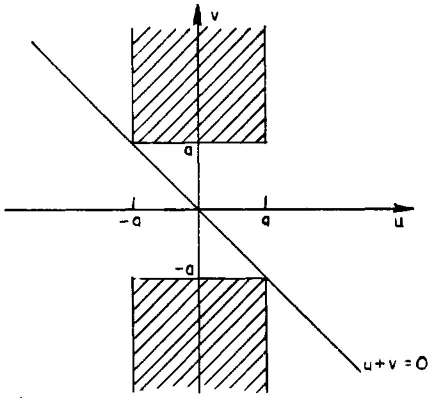
The Fourier transform of the Bessel function vanishes for values of the transform variable above b, so that it constitutes the low-pass function. Consequently, the Hilbert transform of the product is given by the product of the Bessel function and the Hilbert transform of the circular function.

Other suitable functions which can be used to build up further entries can be found in Erdélyi [5] and Campbell and Foster [7]. For example, the Fourier transform of

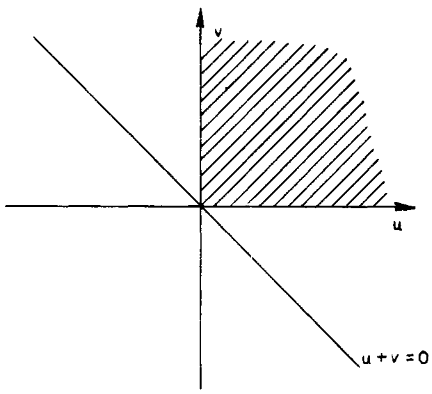
$$Ci(ax) = -\int_{ax}^{\infty} \frac{\cos t}{t} dt$$

vanishes for values of the transform variable below a [6] so it can serve as a high-pass function. Since its Hilbert transform is given by [3]

$$H[Ci(ax)] = -\operatorname{sgn} x \, si(a|x|)$$



(a) $F(u) = 0, |u| > a$
$G(v) = 0, |v| < a$



(b) $F(u) = 0, u < 0$
$G(v) = 0, v < 0$

Fig. 1—Regions of integration.

it follows that the Hilbert transform of $\sin bx \, Ci(ax)$ is given by

$$H[\sin bx Ci(ax)] = -\operatorname{sgn} x \sin bx \, si(a|x|), \quad 0 < b < a,$$

where

$$si(x) = -\int_{x}^{\infty} \frac{\sin t}{t} dt.$$

A product of interest in communication theory is the form $r(t) \cos(\omega_0 t + \phi)$ which represents a general double-sideband amplitude-modulated signal. As indicated by Urkowitz [8], and as can be seen readily from the product theorem, its Hilbert transform is given by $r(t) \sin(\omega_0 t + \phi)$, $\omega_0 > 0$, provided that the highest frequency component in $r(t)$ is less than ω_0. It is interesting to contrast this apparently simple form, which requires a spectral restriction, with the more general form $r(t) \cos[\omega_0 t + \phi(t)]$ whose Hilbert transform is given by $r(t) \sin[\omega_0 t + \phi(t)]$ with no such restriction.[1]

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[1] Kelly, Reed and Root [9] show that the representation $r(t) \cos[\omega_0 t + \phi(t)]$ is actually the real part of a complex signal $r(t)e^{i[\omega_0 t + \phi(t)]}$ which is analytic so that its real and imaginary parts form a Hilbert pair. Note that $r(t)$ and $\phi(t)$ are uniquely specified for a given real wave form. It should be mentioned in this connection that the result given by Lerner [10] indicating an error term related to the signal bandwidth is incorrect.

References

[1] A. Erdélyi, Ed., "Tables of Integral Transforms," McGraw-Hill Book Co., Inc., New York, N. Y., vol. 2; 1954.
[2] *Ibid.* 15.1 (5) and 15.2 (38).
[3] *Ibid.*, 15.3 (3).
[4] E. C. Titchmarsh, "Introduction to the Theory of Fourier Integrals," Oxford, University Press, New York, N. Y., 2nd ed., theorem 95, p. 128; 1948.
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[6] *Ibid.*, 1.11 (13).
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Correction to "Function Generator for $Y = AX^3 + BX^2 + CX + D$, Employing the Galvanomagnetic Effects in Semiconductors"*

The authors of the above communication would like to make the following correction.[1] Fig. 1, shown here, should be substituted for Fig. 3 since this new figure will give readers a better understanding of the performance of this device.

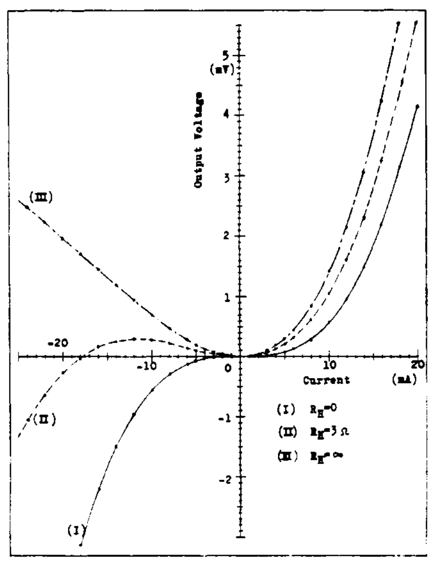

Fig. 1—Characteristics of an analog device for $Y = AX^3 + BX^2$.

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* Received December 21, 1962.
[1] S. Kataoka and H. Yamada, Proc. IRE (*Correspondence*), vol. 50, pp. 2522–2523; December, 1962