

ON INSTANTANEOUS FREQUENCY

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Instantaneous frequency (IF) is necessary for understanding the detailed mechanisms for nonlinear and nonstationary processes. Historically, IF was computed from analytic signal (AS) through the Hilbert transform. This paper offers an overview of the difficulties involved in using AS, and two new methods to overcome the difficulties for computing IF. The first approach is to compute the quadrature (defined here as a simple 90° shift of phase angle) directly. The second approach is designated as the normalized Hilbert transform (NHT), which consists of applying the Hilbert transform to the empirically determined FM signals. Additionally, we have also introduced alternative methods to compute local frequency, the generalized zero-crossing (GZC), and the teager energy operator (TEO) methods. Through careful comparisons, we found that the NHT and direct quadrature gave the best overall performance. While the TEO method is the most

localized, it is limited to data from linear processes, the GZC method is the most robust and accurate although limited to the mean frequency over a quarter wavelength of temporal resolution. With these results, we believe most of the problems associated with the IF determination are resolved, and a true time–frequency analysis is thus taking another step toward maturity.

Keywords: Instantaneous frequency; Hilbert transform; quadrature; empirical mode decomposition; normalized intrinsic mode function; empirical AM/FM decomposition.

1. Introduction

The term “instantaneous frequency” (IF) has always elicited strong opinions in the data analysis and communication engineering communities, covering the range from “banishing it forever from the dictionary of the communication engineer,”¹ to being a “conceptual innovation in assigning physical significance to the nonlinearly distorted waveforms.”² In between these extremes, there are plenty of more moderate opinions stressing the need for and also airing the frustration of finding an acceptable definition and workable way to compute its values.

Before discussing any methods for computing the IF, we have to justify the concept of an instantaneous value for the frequency. After all, the traditional frequency analysis method is mostly based on the Fourier transform, which gives time-invariant amplitude and frequency values. Furthermore, the inherited uncertainty principle associated with the Fourier transform pair has prompted Gröchenig³ to say, “The uncertainty principle makes the concept of an Instantaneous Frequency impossible.” As Fourier analysis is a well-established subject in mathematics and the most popular method in time–frequency transform, this verdict against IF is a serious one indeed. This seemingly rigorous objection, however, could be easily resolved, for the uncertainty principle is a consequence of the Fourier transform (or any other type of integral transform) pair; therefore, its limitation could only be applied to such integral transforms, in which time would be smeared over the integral interval. Consequently, if we eschew an integral transform in the frequency computation, we would not be bounded by the uncertainty principle. Fourier analysis is only one of the mathematical methods for time–frequency transform; we have to look beyond Fourier analysis to find a solution. Indeed, the need of the frequency as a function of time and the fact that the frequency should be a function of time, and having an instantaneous value, can be justified from both mathematical and physical grounds.

Mathematically, the commonly accepted definition of frequency in the classical wave theory is based on the existence of a phase function (see, e.g., Refs. 4, 5). Here, starting with the assumption that the wave surface is represented by a “slowly” varying function consisting of time-varying amplitude $a(x, t)$, and phase, $\theta(x, t)$, functions, such that the wave profile is the real part of the complex valued function, we have

$$\varsigma(x, t) = R(a(x, t)e^{i\theta(x, t)}). \quad (1)$$

Then, the frequency, ω , and the wave-number, k , are defined as

$$\omega = -\frac{\partial\theta}{\partial t} \quad \text{and} \quad k = \frac{\partial\theta}{\partial x}. \quad (2)$$

Cross-differentiating the frequency and wave-number, one immediately obtains the wave conservation equation,

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0. \quad (3)$$

This is one of the fundamental laws governing all wave motions. The assumption of the classic wave theory is very general: that there exists a “slowly” varying function such that we can write the complex representation of the wave motion given in Eq. (1). If frequency and wave-number can be defined as in Eq. (2), they have to be differentiable functions of the temporal and the spatial variables for Eq. (3) to hold. Thus, for any wave motion, other than the trivial kind with constant frequency sinusoidal motion, the frequency representation should have instantaneous values. Therefore, there should not be any doubt of the mathematical meaning of or the justification for the existence of IF. Based on the simple “slowly varying” assumption, the classical wave theory is founded on rigorous mathematic grounds and with many of the theoretical results confirmed by observations.⁵ This model can be generalized to all kinds of wave phenomena such as in surface water waves, acoustics, and electromagnetics. The pressing questions are how to define the phase function and the IF for a given wave data set.

Physically, there is also a real need for IF in a faithful representation of underlying mechanisms for data from nonstationary and nonlinear processes. Obviously, the nonstationarity is one of the key features here, but, as explained by Huang *et al.*,² the concept of IF is also essential for a physically meaningful interpretation of nonlinear processes: for a nonstationary process, the frequency should be ever changing. Consequently, we need a time–frequency representation for the data, or that the frequency value has to be a function of time. For nonlinear processes, the frequency variation as a function of time is even more drastic. To illustrate the need for IF in the nonlinear cases, let us examine a typical nonlinear system as given by the Duffing equation:

$$\frac{d^2x}{dt^2} + x + \varepsilon x^3 = \gamma \cos \omega t, \quad (4)$$

in which ε is a parameter not necessarily small, and the right-hand term is the forcing function of magnitude γ and frequency ω . This cubic nonlinear equation can be rewritten as

$$\frac{d^2x}{dt^2} + x(1 + \varepsilon x^2) = \gamma \cos \omega t, \quad (5)$$

where the term in the parenthesis can be regarded as a single quantity representing the spring constant of the nonlinear oscillator, or the pendulum length of a nonlinearly constructed pendulum. As this quantity is a function of position, the frequency

of this oscillator is also ever changing, even within one oscillation. This intrawave frequency modulation is the singular most unique characteristic of a nonlinear oscillator as proposed by Huang *et al.*^{2,6} The geometric consequences of this intrawave frequency modulation are the waveform distortion. Traditionally, such nonlinear phenomena are represented by harmonics. As the waveform distortion can be fitted by harmonics of the fundamental wave in Fourier analysis, it is viewed as harmonic distortions. This traditional view, however, is the consequence of imposing a linear structure on a nonlinear system: the superposition of simple harmonic functions with each as a solution for a linear oscillator. One can only assume that the sum and the total of the linear superposition would give an accurate representation of the full nonlinear system. The difficulty here is this: we would need infinitely many terms to represent a caustic point. But large number of terms would be impractical. Even if we could obtain the Fourier expansion, all the individual harmonic terms, however, are mathematic artifacts and have no physical meaning. For example, in the case of water surface waves, the harmonics are not a physical wave train, for they do not satisfy the dispersive relationship.⁶ Although, the perturbation approach seems to have worked well for systems with infinitesimal nonlinearity, the approach fails when the nonlinearity is finite and the motion becomes chaotic (see, e.g., Ref 7). A natural and logical approach should be one that can capture the physical meaning: the physical essence of this nonlinear system is an oscillator with variable intrawave-modulated frequency assuming different values at different times even within one single period. To describe such a motion, we should use IF to represent this essential physical characteristic of nonlinear oscillators. In fact, the intrawave frequency modulation is a physical meaningful and effective way to describe the waveform distortions.

In real-world experimental and theoretic studies, the conditions of ever-changing frequency are common, if not prevailing. Chirp signal is one class of the signals used by bats as well as in radar. The frequency content in speech, though not exactly a chirp, is also ever changing, and many of the consonants are produced through highly nonlinear mechanisms such as explosion or friction. Furthermore, for any nonlinear system, the frequency is definitely modulating not only among different oscillation periods, but also within one period as discussed above. To understand the underlying mechanisms of these processes, we can no longer rely on the traditional Fourier analysis with components of constant frequency. We have to examine the true physical processes through instantaneous frequency from non-Fourier based methods.

There have been copious publications in the past on the IF, for example: Refs. 8–16. Most of these publications, however, were concentrated on Wigner–Ville distribution (WVD) and its variations, where the IF is defined through the mean moment of different components at a given time. But the WVD is essentially Fourier based analysis. Other than the WVD, IF obtained through the analytic signal (AS) produced by the Hilbert transform (HT) has also received a lot attention. Boashash,^{9,10} in particular, gave a summary history of the evolution of the IF definition. Most of

the discussions given by Boashash^{8–10} were on monocomponent signals. For more complicated signals, he again suggested utilizing the moments of the WVD. But there are no *a priori* reasons to assume that the multicomponent signal should have a single-valued instantaneous frequency at any given time and still be retaining its full physical significance. Even for a monocomponent signal, the Wigner–Ville method still relies on the moment approach. Boashash also suggested crossing the WVD of the signal with a reference signal. This method will be seriously compromised when the signal to noise ratio is high. We will return to these points and discuss them in more detail later.

One of the most basic yet confusing points concerning IF stems from the erroneous idea that for each IF value there must be a corresponding frequency in the Fourier spectrum of the signal. In fact, IF of a signal when properly defined should have a very different meaning when compared with the frequency in the Fourier spectrum, as discussed in Huang *et al.*² But the divergent and confused viewpoints on IF indicate that the erroneous view is a deeply rooted one associated with and responsible for some of the current misconceptions and fundamental difficulties in computing IF. Some of the traditional objections on IF actually can be traced to the mistaken assumption that a single-valued IF exists for any function at any given instant. Obviously, a complicated signal could consist of many different frequencies at any given time, such as a recorded music of a symphonic orchestra performance.

The IF witnessed two major advances recently. The first one is through the introduction of the empirical mode decomposition (EMD) method and the intrinsic mode function (IMF) introduced by Huang *et al.*² for data from nonlinear and nonstationary processes. The second one is through wavelet based decomposition introduced by Olhede and Walden¹⁷ for data from linear nonstationary processes. Huang *et al.*⁶ have also introduced the Hilbert view on nonlinearly distorted waveforms, which provided explanations to many of the paradoxes raised by Cohen¹³ on the validity of IF, which will be discussed in detail later. Indeed, the introduction of EMD or the wavelet decomposition resolved one key obstacle for computing a meaningful IF from a multicomponent signal by reducing it to a collection of monocomponent functions. Once we have the monocomponent functions, there are still limitations on applying AS for physically meaningful IF as stipulated by the well-known Bedrosian¹⁸ and Nuttall¹⁹ theorems. Some of the mathematic problems associated with the HT of IMFs have also been addressed by Vatchev.²⁰

In this paper, we propose an empiric AM–FM decomposition²¹ based on a spline fitted normalization scheme to produce a unique and smooth empiric envelope (AM) and a unity-valued carrier (FM). Our experience indicates that the empiric envelope so produced is identical to the theoretic one when explicit expressions exist, and it provides a smoother envelope than any other method including the one based on AS when there is no explicit expression for the data. The spline based empiric AM–FM

decomposition will not only remove most of the difficulties associated with AS, but also enable us to compute the quadrature directly, and then compute IF through a direct quadrature function without any approximation. The normalization also resolves many of the traditional difficulties associated with the IF computed through AS: it makes AS satisfy the limitation imposed by the Bedrosian theorem.¹⁸ At the same time, it provides a sharper and easily computable error index than the one proposed by the Nuttall theorem,¹⁹ which governs the case when the HT of a function is different from its quadrature. Additionally, we will also introduce alternative methods based on a generalized zero-crossing (GZC) and an energy operator to define frequency locally for cross comparisons.

The paper is organized as follows: This introduction will be followed by a discussion of the definitions of frequency. Then we will introduce the empiric AM-FM decomposition, or the spline normalization scheme, and all the different instantaneous frequency computation methods in Sec. 3. In Sec. 4, we will present the comparisons of the IF values defined from various alternative methods to establish the validity, advantages and disadvantages of each method through testing on model functions and real data. We will also introduce the frequency-modulated (FM) and amplitude-modulated (AM) representations of the data in Sec. 5. Finally, we will discuss the merits of the different methods and make a recommendation for general applications, and offer a short conclusion. To start, however, we will first present the various definitions of frequency in the next section as a motivation for the subsequent discussions.

2. Definitions of Frequency

Frequency is an essential quantity in the study of any oscillatory motion. The most fundamental and direct definition of frequency, ω , is simply the inverse of period, T ; that is

$$\omega = \frac{1}{T}. \quad (6)$$

Following this definition, the frequency exists only if there is a whole cycle of wave motion. And the frequency would be constant over this length, with no finer temporal resolution. In fact, a substantial number of investigators still hold the view that frequency cannot be defined without a whole wave profile.

Based on the definition of frequency given in Eq. (6), the obvious way of determining the frequency is to measure the time intervals between consecutive zero-crossings or the corresponding points of the phase on successive waves. This is very easily implemented for a simple sinusoidal wave train, where the period is a well-defined constant. For real data, this restrictive view presents several difficulties: to begin with, those holding this view obviously are oblivious to the fundamental wave conservation law, which requires the wave-number and frequency to be differentiable. How can the frequency be differentiable if its value is constant over a whole wavelength? Secondly, this view cannot reveal the detailed frequency modulations

with ever-changing frequency in nonstationary and, especially, nonlinear processes with intrawave frequency modulations. And finally, in a complicated vibration, there might be multi-extrema between two consecutive zero-crossings, a problem treated extensively by Rice,²²⁻²⁵ who restricted the applications of the zero-crossing method to narrow-band signals, where the signal must have equal numbers of extrema and zero-crossings. Therefore, without something like the EMD method to decompose the data into IMFs, this simple zero-crossing method has only been used for the band-passed data (see, e.g., Ref. 26), or more recently by the more sophisticated wavelet based filtering proposed by Olhede and Walden.¹⁷ As the bandpass filters used all work in frequency space, they tend to separate the fundamental from its harmonics; the filtered data will lose most, if not all, of the nonlinear characteristics. With these difficulties, the zero-crossing method has seldom been used in serious research work.

Another definition of frequency is through the dynamic system. This elegant method determines the frequency through the variation of the Hamiltonian, $H(q, p)$, where q is the generalized coordinate, and p , the generalized momentum (see, e.g., Refs. 27, 28) as,

$$\omega(A) = \frac{\partial H(A)}{\partial A}, \quad (7)$$

in which A is the action variable defined as

$$A = \oint p dp, \quad (8)$$

where the integration is over a complete period of a rotation. The frequency so defined is varying with time, but the resolution is no finer than the averaging over one period, for the action variable is an integrated quantity as given in Eq. (8). Thus, the frequency defined by Eq. (7) is equivalent to the inverse of the period, the classical definition of frequency. This method is elegant theoretically, but its utility is limited to relatively simple low-dimensional dynamic systems, linear or nonlinear, whenever integrable solutions describing the full process exist. It cannot be used for data analysis routinely.

In practical data analysis, the data consist of a string of real numbers, which may have multi-extrema between consecutive zero-crossings. Then, there can be many coexisting frequency values at any given time. Traditionally, the only way to define frequency is to compute through the Fourier transform. Thus, for a time series, $x(t)$, we have

$$x(t) = R \sum_{j=1}^N a_j e^{-i\omega_j t}, \quad (9)$$

where

$$a_j = \int_0^T x(t) e^{i\omega_j t} dt, \quad (10)$$

with R indicating the real part of the quantity. With classic Fourier analysis, the frequency values are constant over the whole time span covering the range of the integration. As the Fourier definition of frequency is not a function of time, we can easily see that the frequency content would be physically meaningful only if the data represent a linear (to allow superposition) and stationary (to allow a time-independent frequency representation) process.

A slight generalization of the classic Fourier transform is to break the data into short subspans. Thus the frequency value can still vary globally, but is assumed to be constant within each subintegral span. Nevertheless, the integrating operation leads to the fundamental limitation on this Fourier type of analysis by the uncertainty principle, which states that the product of the frequency resolution, $\Delta\omega$, and the time span over which the frequency value is defined, ΔT , shall not be less than $1/2$. As Fourier transform theory is established over an infinite time span, then the uncertainty principle dictates that this time interval cannot be too short related to the period of the oscillation. At any rate, the uncertainty principle dictates that, for the Fourier-type methods, it is impossible to resolve the signal with the frequency varying faster than the integration time scale, certainly not within one period.³ This seemingly weak restriction has in fact limited the Fourier spectral analysis to linear and stationary processes only.

A further generation of the Fourier transform is the wavelet analysis, a very popular data analysis method (see, e.g., Refs. 29, 30), which is also extremely useful for data compression, and image edge definitions, for example. True, the wavelet approach offers some time–frequency information with an adjustable window. The most serious weakness of wavelet analysis is again the limitation imposed by the uncertainty principle: to be local, a base wavelet cannot contain too many waves; yet to have fine frequency resolution, a base wavelet will have to contain many waves. As the numerous examples have shown, the uniformly poor frequency resolution renders wavelet results only as a qualitative tool for time–frequency analysis. The frequency resolution problem is mitigated greatly through the Hilbert spectral representation based on the wavelet projection.¹⁷ Nevertheless, this improved method is still burdened by harmonics; therefore, their result can only be physical meaningful when the data are from nonstationary but linear processes.

Still another extension of the classical Fourier analysis is the WVD (see, e.g., Ref. 13), which is defined as

$$V(t, \omega) = \int_{-\infty}^{\infty} x\left(t + \frac{\tau}{2}\right) x^*\left(t - \frac{\tau}{2}\right) e^{-i\omega\tau} d\tau. \quad (11)$$

By definition, the marginal distribution, by integrating the time variable out, is identical to the Fourier power density spectrum. Even though the full distribution does offer some time–frequency properties, it inherits many of the shortcomings of Fourier analysis. The additional time variable, however, provides a center of gravity

type of weighted mean local frequency as

$$\overline{\omega(t)} = \frac{\left(\int_{-\infty}^{\infty} \omega V(t, \omega) d\omega\right)}{\left(\int_{-\infty}^{\infty} V(t, \omega) d\omega\right)}. \quad (12)$$

Here we have only a single value as a mean for all the different components. This mean value lacks the necessary details to describe the complexity imbedded in a multicomponent data set. For example, if we have a recording of a symphonic music piece, when many instruments are playing at the same time, Wigner–Ville would allow one single frequency at any given time. This is certainly unreasonable and unrealistic.

As our emphasis is on analyzing data from nonstationary and nonlinear processes, we have to examine the frequency content of the data in detail at any given time with the subperiod temporal resolution. We have proposed solutions: time–frequency analysis based on AS functions and direct quadrature, which will be the subject of the next section. It should be noted that all the above methods work for any data, while the methods to be discussed in the next section work only for monocomponent functions.

3. Instantaneous Frequency

Ideally, the IF for any monocomponent data should be through its quadrature, defined as a simple 90° phase shift of the carrier phase function only. Thus from any monocomponent data, we have to find its envelope, $a(t)$, and carrier, $\cos \phi(t)$, as,

$$x(t) = a(t) \cos \phi(t), \quad (13)$$

where $\varphi(t)$ is the phase function to represent the AM and FM parts of the signal respectively. Its quadrature then is

$$xq(t) = a(t) \sin \phi(t), \quad (14)$$

where the change comparing to the original data is limited only to the phase. With these expressions, the instantaneous frequency can be computed as in the classical wave theory given in Eq. (2). These seemingly trivial steps have been impossible to implement in the past. To begin with, not all the data are monocomponent. Even though decomposing the data into a collection of monocomponent functions is now available by wavelet decomposition¹⁷ or the empirical mode decomposition,² there are other daunting difficulties: to find the unique pair of $[a(t), \varphi(t)]$ to representing the data, and to find a general method to compute the quadrature directly. Traditionally, the accepted way is to use the AS through the HT as a proxy for the quadrature. This has made the AS approach the most popular method to define IF.

The approach of using AS, however, is not without its difficulties. The most fundamental one is that AS is only an approximation to the quadrature, except

for some very simple cases. And due to this and other difficulties associated with this approach, it has also contributed to all the controversies related to IF. To fully appreciate the subtlety of the IF defined through HT, a brief history of IF is necessary. A more detailed one can be found in Boashash,^{8,9} for example. For the sake of completeness and to facilitate our discussions, we will trace certain essential historic milestones of the approach from its beginning to its present state as follows:

The first important step for defining IF was due to Van der Pol,³¹ a pioneer in nonlinear system studies, who first explored the idea of IF seriously. He proposed the correct expression of the phase angle as an integral of the IF. The next important step was made by Gabor,³² who introduced the HT to generate a unique AS from real data, thus removing the ambiguity of the infinitely many possible amplitude and phase pair combinations to represent the data. Gabor's approach is summarized as follows: for the variable $x(t)$, its HT, $y(t)$, is defined as

$$y(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau} d\tau, \quad (15)$$

with P indicating the Cauchy principal value of the complex integral. The HT provides the imaginary part of the analytic pair, $y(t)$, of the real data. Thus, we have a unique AS given by

$$z(t) = x(t) + iy(t) = A(t)e^{i\theta(t)}, \quad (16)$$

in which

$$A(t) = \{x^2(t) + y^2(t)\}^{1/2} \quad \text{and} \quad \theta(t) = \tan^{-1} \frac{y(t)}{x(t)}, \quad (17)$$

form the canonical pair, $[A(t), \theta(t)]$, associated with $x(t)$. Gabor even proposed a direct method to obtain AS through two Fourier transforms:

$$z(t) = 2 \int_0^{\infty} F(\omega) e^{i\omega t} d\omega, \quad (18)$$

where $F(\omega)$ is the Fourier transform of $x(t)$. In this representation, the original data $x(t)$ becomes

$$x(t) = R\{A(t)e^{i\theta(t)}\} = A(t) \cos \theta(t). \quad (19)$$

It should be pointed out that this canonical pair, $[A(t), \theta(t)]$, is in general different from the complex number defined by the quadrature, $[a(t), \varphi(t)]$, though their real parts are identical. For the analytic pair, the IF can be defined as the derivative of the phase function of this analytic pair given by

$$\omega(t) = \frac{d\theta(t)}{dt} = \frac{1}{A^2} (xy' - yx'). \quad (20)$$

In general, for stochastic data, the phase function is a function of time; therefore, the IF is also a function of time. This definition of frequency bears a striking similarity with that of the classical wave theory.

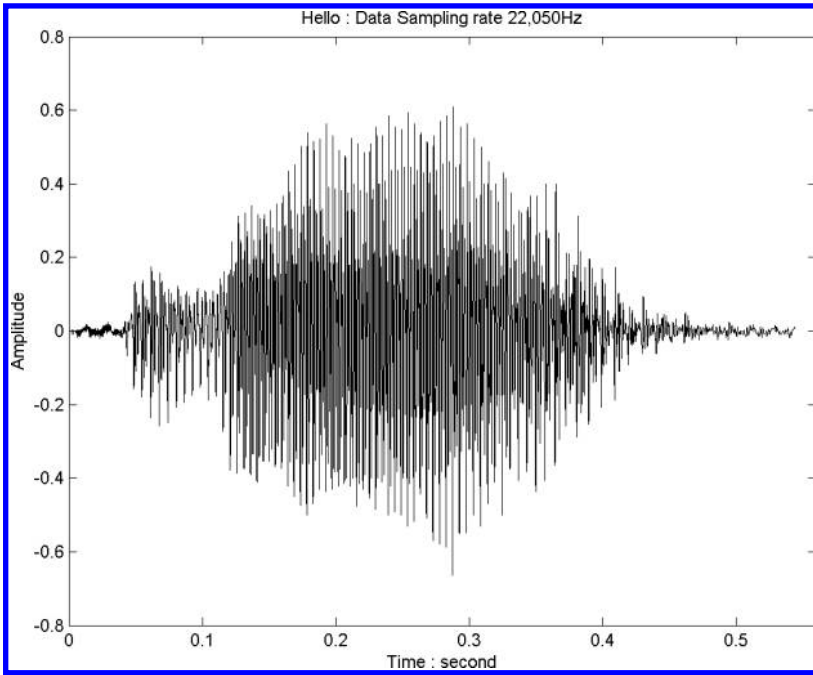


Fig. 1. Data of the recorded word, “Hello”, digitized at a rate of 22,050 Hz.

As the HT exists for any function of L^p class, there is a misconception that one can put any function through the above operation and obtain a physically meaningful IF as advocated by Hahn.³³ Such an approach has created great confusion for the meaning of the IF in general, and tarnished the approach of using the HT for computing the IF in particular. Let us take the data recording of a voice saying “Hello”, given in Fig. 1, as an example. Through HT, we have the AS plotted in the complex phase plane given in Fig. 2, which just shows a collection of random loops. If we designate the derivative of the phase function as the IF according to Eq. (20), the result is shown in Fig. 3. Clearly, the frequency values are scattered over a wide range with both positive and negative values. Furthermore, any speech could have multicomponent sounds, but this representation gives only a single frequency at any given time and ignores the multiplicity of the coexisting components. Consequently, these values are not physically meaningful at all, instantaneously or otherwise. The difficulties encountered here actually can be illustrated by a much simpler example using the simple function employed by Huang *et al.*²:

$$x(t) = a + \cos \alpha t, \quad (21)$$

with a as an arbitrary constant. Its HT is simply

$$y(t) = \sin \alpha t; \quad (22)$$

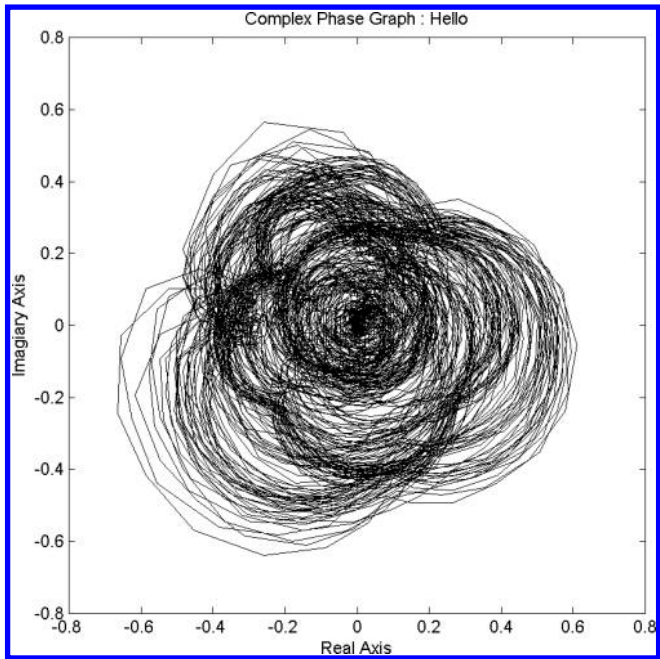


Fig. 2. Complex phase graph of the analytic signal derived from data in Fig. 1 through HT.

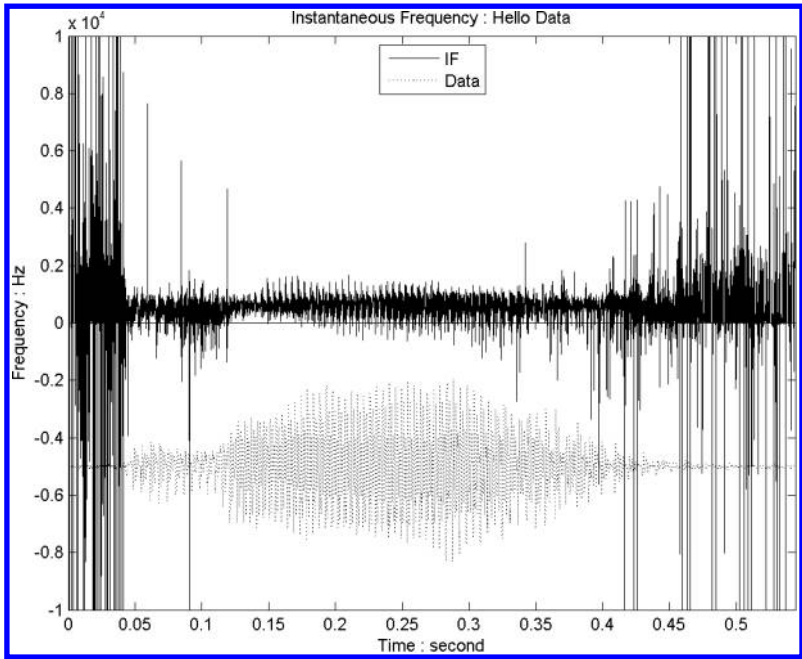


Fig. 3. The IF derivative through the AS given in Fig. 2, with the original data plotted in arbitrary unit and a magnitude shift.

therefore, the IF according to Eq. (20) is

$$\omega = \frac{\alpha(1 + a \sin \alpha t)}{1 + 2a \cos \alpha t + a^2}. \quad (23)$$

Equation (23) can give any value for the IF, depending on the value of a . To recover the frequency of the input sinusoidal signal, the constant, a , has to be zero. This simple example illustrates some crucial necessary conditions for the AS approach to give a physical meaningful IF: the function will have to be monocomponent, zero mean locally, and the wave will have to be symmetric with respect to the zero mean. All these conditions are satisfied by either EMD or the wavelet projection methods mentioned above. But these are only the necessary conditions. There are other more subtle and stringent conditions for the AS approach to produce a meaningful IF. For example, Loughlin and Tracer¹⁵ proposed physical conditions for the AM and FM of a signal for IF to be physically meaningful, and Picinbono¹⁶ proposed spectral properties of the envelope and carrier in order to have a valid AS representation. Indeed, the unsettling state of the AM, the FM decomposition, and the associated instantaneous frequency have created great misunderstanding, which has prompted Cohen¹³ to list a number of “paradoxes” concerning instantaneous frequency. Some of the paradoxes concerning negative frequency are a direct consequence of these necessary conditions given by the IMFs. All the paradoxes will be discussed later.

In fact, the most general conditions are already summarized most succinctly by the Bedrosian¹⁸ and Nuttall¹⁹ theorems. Bedrosian¹⁸ established another general necessary condition for obtaining a meaningful AS for IF computation, which set a limitation of separating the HT of the carrier from its envelope as

$$H\{a(t) \cos \theta(t)\} = a(t)H\{\cos \theta(t)\}, \quad (24)$$

provided that the Fourier spectra of the envelope and the carrier are non-overlapping. This is a much sharper condition on the data: the data has to be not only monocomponent, but also narrow band; otherwise the AM variations will contaminate the FM part. The IMF produced by EMD does not satisfy this requirement automatically. With the spectra from amplitude and carrier not clearly separated, the IF will be influenced by the AM variations. As a result, the applications of the HT as used by Huang *et al.*^{2,6} are still plagued by occasional negative frequency values. Strictly speaking, unless one uses bandpass filters, any local AM variation will violate the restriction of the Bedrosian theorem. If any data violate the condition set forth in Eq. (24), the operations given in Eqs. (16) to (20) will not be valid anymore. Although we can still obtain an AS with the real part identical to the data, but the imaginary part would not be the same through the effect on the phase function contaminated by the AM. The result will be meaningless as the example given in Fig. 3.

It can be further shown that the Bedrosian condition is not the only problem. More fundamentally, Nuttall¹⁹ questions the condition under which we can

write

$$H\{\cos \phi(t)\} = \sin \phi(t), \quad (25)$$

for an arbitrary function of $\varphi(t)$. This difficulty has been ignored by most investigators using the HT to compute IF. Picinbono¹⁶ stated that it would be impossible to justify Eq. (25) from only the spectral properties. He then entered an extensive discussion on the specific properties of the phase function under which Eq. (25) is true. The conditions were recently generalized by Qian *et al.*³⁴ But such discussions would be of very limited practical use in data analysis, for we cannot force our data to satisfy the conditions prescribed. Picinbono finally concluded that “the only scientific procedure would require the calculation of the error coming from the approximation”, which resides only in the imaginary part of the AS. He also correctly pointed out that there is no general procedure to calculate this error from the spectrum of the amplitude function, for the error depends on the structure of the phase function rather than on spectral properties of the amplitude function. With the difficulties presented by Picinbono,¹⁶ we can only settle on the partial solution provided by the Nuttall¹⁹ theorem.

Nuttall¹⁹ first established the following theoretic result: for any given function

$$x(t) = a(t) \cos \phi(t), \quad (26)$$

for arbitrary $a(t)$ and $\varphi(t)$, not necessarily narrow band functions, and if the HT of $x(t)$ is given by $xh(t)$, and the quadrature of $x(t)$ is $xq(t)$, then

$$E = \int_{t=-\infty}^{\infty} [xh(t) - xq(t)]^2 dt = 2 \int_{-\infty}^{\omega_0} F_q(\omega) d\omega, \quad (27)$$

where

$$F_q(\omega) = F(\omega) + i \int_{-\infty}^{\infty} a(t) \sin \theta(t) e^{-i\omega t} dt, \quad (28)$$

in which $F(\omega)$ is the spectrum of the signal, and $F_q(\omega)$ is the spectrum of the quadrature of the signal. Therefore, the necessary and sufficient conditions for the HT and the quadrature to be identical is $E = 0$. This is an important and brilliant, yet not very practical and useful, result. The difficulties are due to the following three deficiencies. First, the result is expressed in terms of the quadrature spectrum of the signal, which is an unknown quantity, if quadrature is unknown. Therefore, the error bound could not be evaluated easily. Second, the result is given as an overall integral, which provides a global measure of the discrepancy. Therefore, we would not know which part of the data causes the error in a nonstationary time series. Finally, the error index is energy based; it only states that the $xh(t)$ and $xq(t)$ are different, but does not offer an error index on the frequency.¹⁶ Therefore, the Nuttall theorem offers only a proxy for the error index of IF; it is again a necessary condition for the AS approach to yield the exact IF. These difficulties, however, do not diminish the significance of Nuttall’s result: it points out a serious problem and

limitation on equating the HT and the quadrature of a signal; therefore, there is a serious problem on using the HT to compute physically valid IF values.

All these important results were known by the late sixties. For lack of a satisfactory method to decompose the data into the monocomponent functions other than the traditional bandpass filters, the limitations set by Bedrosian and Nuttall were irrelevant, for the bandpassed signal is linear and narrow band and satisfies the limitation automatically. As it is also well known that a bandpassed signal would eliminate many interesting nonlinear properties from the data, the band-pass approach could not make the Hilbert transform generated AS into a general tool for physically valid IF computation. Consequently, the HT method remains as an impractical method for data analysis. The solutions to these difficulties are presented in the next section.

3.1. *The normalization scheme: an empirical AM and FM decomposition*

Both limitations stated by the Bedrosian and Nuttall theorems have firm theoretic foundations, and must be satisfied. To this end, we propose a new normalization scheme, which is an empirical AM and FM decomposition method enabling us to separate any IMF empirically and uniquely into envelope (AM) and carrier (FM) parts. This normalization decomposition scheme has three important consequences: first, and most importantly, the normalized carrier also enables us to directly compute quadrature (DQ). Second, the normalized carrier has unity amplitude; therefore, it satisfies the Bedrosian theorem automatically. Finally, the normalized carrier enables us to provide a ready and sharper local energy based measure of error than that given by the Nuttall theorem. The method using the empirical AM–FM decomposition and the normalization scheme is designated as the normalized Hilbert transform (NHT).

Other than direct quadrature and NHT based IF computations, we will also introduce two additional methods for determining the local frequency independent of the HT, each based on different assumptions, and each giving slightly different values for IF from the same data. For all of these methods to work, the data will have to be reduced to an IMF first. In this section, we shall present these different approaches and the most crucial step, the normalization scheme.

The normalization scheme is designed to separate the AM and FM parts of the IMF signal uniquely but empirically; it is based on iterative applications of cubic spline fitting through the data. As this empirical AM–FM decomposition is of great importance to the subsequent discussions, we will present it first as follows:

First, from the given IMF data in Fig. 4, identify all the local maxima of the absolute value of the data as in Fig. 5. By using the absolute value fitting, we are guaranteed that the normalized data are symmetric with respect to the zero axis. Next, we connect all these maxima points with a cubic spline curve. This spline curve is designated as the empiric envelope of the data, $e_1(t)$, also shown in

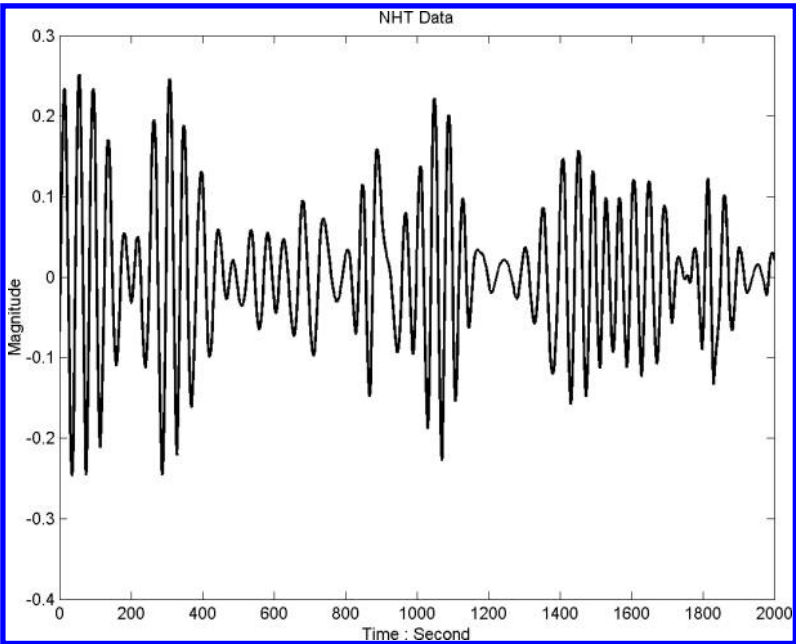


Fig. 4. Arbitrary sample data used here as example to illustrate the empirical AM–FM decomposition through the spline fitted normalization scheme.

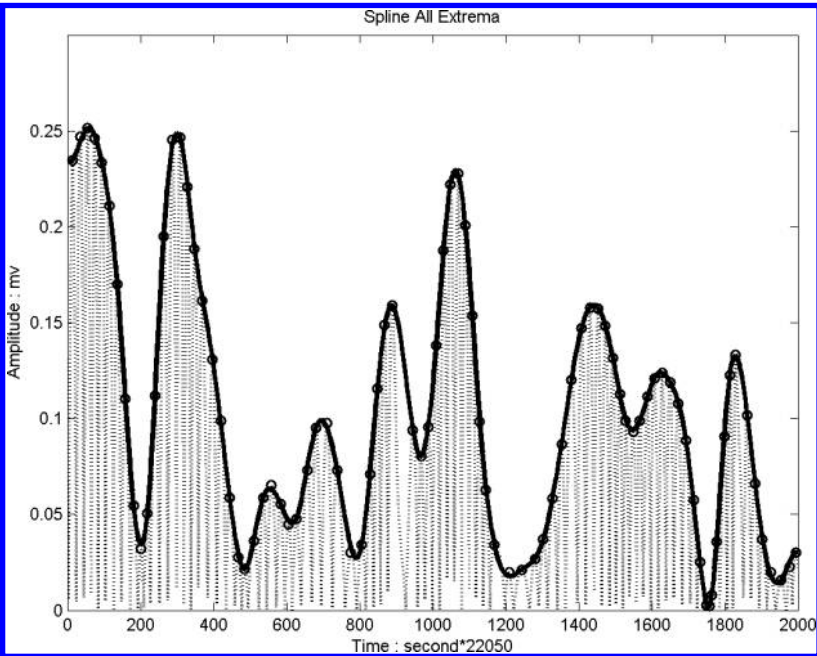


Fig. 5. The maxima of the absolute values (dot) of the data (dash line) given in Fig. 4 and the spline fitting (solid line) through those values. The spline line is defined as the envelope (instantaneous amplitude) to be used as the base for normalizing the data.

Fig. 5. In general, this envelope is different from the modulus of the AS. For any given real data, the extrema are fixed; therefore, this empiric envelope should be fixed and uniquely defined, with respect to a given spline function, without any ambiguity. Having obtained the empiric envelope through spline fitting, we can use this envelope to normalize the data, $x(t)$, by

$$y_1(t) = \frac{x(t)}{e_1(t)}, \quad (29)$$

with $y_1(t)$ as the normalized data. Ideally, $y_1(t)$ should have all the extrema with unity value. Unfortunately, Fig. 6 shows that the normalized data still have amplitudes higher than unity occasionally. This is due to the fact that the spline is fitted through the maximum points only. At the locations of fast changing amplitudes, the envelope spline line, passing through the maxima, can go below some data points. Even with these occasional flaws, the normalization scheme has effectively separated the amplitude from the carrier oscillation. To remove any flaws of this type, the normalization procedure can be implemented repeatedly with $e_2(t)$ defined as the empiric envelope for $y_1(t)$, and so on as,

$$\begin{aligned} y_2(t) &= \frac{y_1(t)}{e_2(t)}, \\ &\vdots \\ y_n(t) &= \frac{y_{n-1}(t)}{e_n(t)}, \end{aligned} \quad (30)$$

after n th iteration. When all of the values of $y_n(t)$ are less than or equal to unity, the normalization is complete; it is then designated as the empirical FM part of the data, $F(t)$,

$$y_n(t) = \cos \phi(t) = F(t), \quad (31)$$

where $F(t)$ is a purely FM function with unity amplitude. With the FM part determined, the AM part, $A(t)$, is defined simply as,

$$A(t) = \frac{x(t)}{F(t)} = e_1(t)e_2(t) \cdots e_n(t). \quad (32)$$

Therefore, from Eq. (32), we have

$$x(t) = A(t) * F(t) = A(t) \cos \phi(t). \quad (33)$$

Thus, we accomplished the empiric AM–FM decomposition through repeated normalization. Typically the converging is very fast; two or three rounds of iterations would be sufficient to make all data points equal to or less than unity. The thrice normalized result is given in Fig. 7 as an example, where no point is greater than unity. The empiric AM and the modulus of AS from the data were all plotted in Fig. 8. It is clear that the empiric AM is smoother and devoid of the higher-frequency fluctuation and overshoots as in the modulus of AS. Our experience also

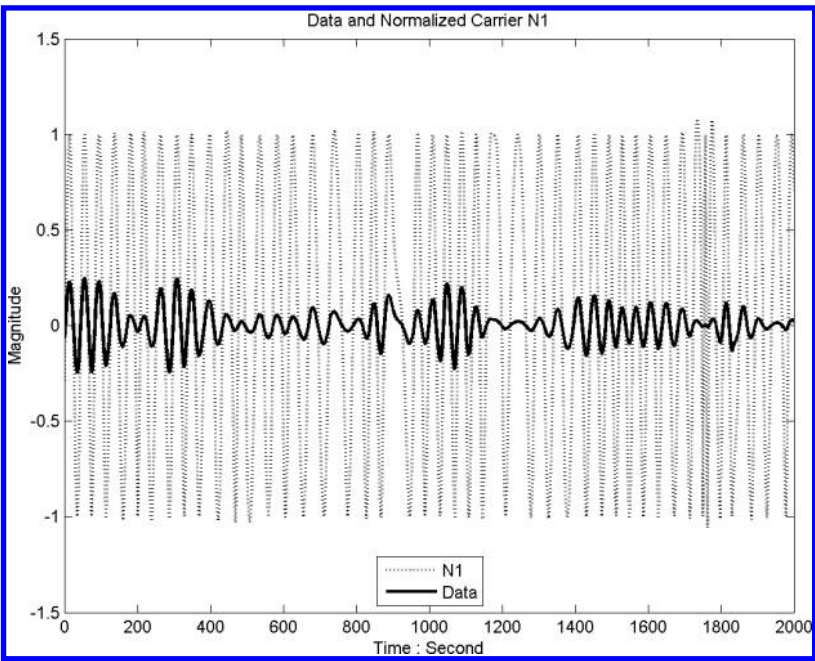


Fig. 6. The one time normalized data (dotted line) compared with the original data (solid line). Notice that the normalized data still have values greater than unity.

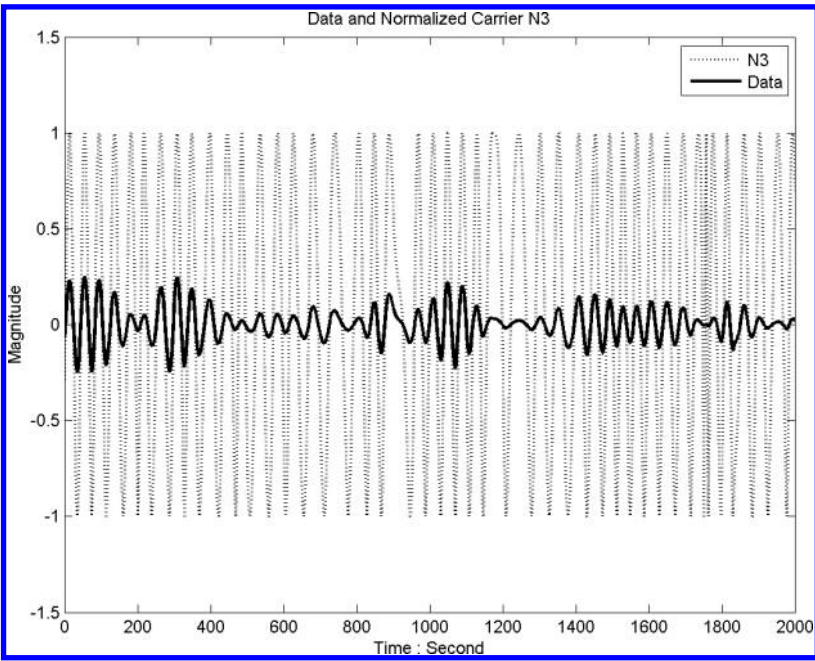


Fig. 7. The three times normalized data (dotted line) compared with the original data (solid line). Notice that the normalized data have no value greater than unity.

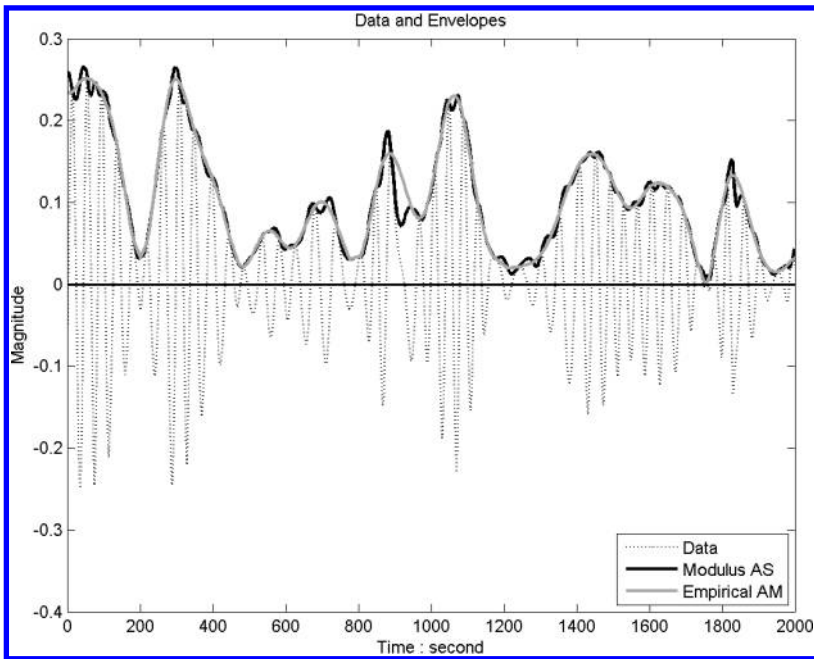


Fig. 8. The data (dotted line) and various envelopes: although both envelopes agree in general, the modulus of AS (black line) shows high-frequency intrawave modulations, while the empiric envelope (gray line) fitted by spline is smooth.

shows that the spline fitted envelopes serve as a much better base for the normalization operations.

As in the EMD, this approach lacks analytic expressions for the operation and the final results, which might hamper the formulation of a theoretical proof. This approach, just like the EMD, is direct and simple to implement. Analyticity, however, is not a requirement for computing the IF. As we have shown that the resulting empiric envelope is unique and even smoother than the modulus of the AS obtained through HT. We will also show that the IF values determined are exactly based on the phase function without any approximation. These advantages, in our judgment, have far out-weighted the deficiency of lacking an analytic expression. After all, in most cases there is no analytic expression for the data anyway.

It should be noted that the normalization process could cause some deformation of the original data, but the amount of the deformation is negligible, for there are rigid controlling points for the periodicity provided by the zero-crossing points in addition to the extrema. The zero-crossing points are totally unaltered by the normalization process. As we discussed above, an alternative method to normalizing an IMF is to use the modulus of the AS instead of the spline envelope in the normalization scheme. This will certainly avoid the envelope dipping under the data, but any nonlinear distorted wave form will give a jagged AS modulus envelope, which could cause even worse deformation of the waveforms in the normalized data.

Still another consideration in favor of the empiric envelope is in the application of computing the damping of a dynamic system. Salvino and Cawley³⁵ used the modulus of AS as the envelope, which worked for simple nearly linear systems. In more complicated vibrations, the intrawave amplitude variations cause the time derivative of the amplitude to be highly oscillatory and thus made the damping computation impossible. Huang *et al.*³⁶ had used the empirical envelope and resolved the difficulties. Based on this consideration, we decided against using the modulus of AS as the base for normalization.

3.2. Direct quadrature

Having proposed the empiric AM–FM decomposition, we can use the normalized IMF as a base to compute its quadrature directly. This approach will eschew the HT totally, and enable us to get an exact IF. After the normalization, the empirical FM signal, $F(t)$, is the carrier part of the data. Assuming the data to be a cosine function, we have its quadrature simply as,

$$\sin \phi(t) = \sqrt{1 - F^2(t)}. \quad (34)$$

The complex pair formed by the data and its direct quadrature is not necessarily analytic. They are computed solely to define the correct phase function, for they preserve the phase function of the real data without the kind of distortion caused by the AS. There seems to be many advantages for this direct quadrature approach: it bypasses HT totally; therefore, it involves no integral interval. Its value is not influenced by any neighboring points and the frequency computation is based only on differentiation; therefore, it is as local as any method can be. Furthermore, without any integral transform, it preserves the phase function of any data exactly for an arbitrary phase function.

Once we have the quadrature, there are two possible ways to compute the phase from the FM signal: one possibility is to compute the phase angle by simply taking the *arc-cosine* of the empiric FM signal as given in Eq. (31) directly. But the computation is occasionally unstable near the local extrema. To improve the computation stability, we propose a slightly modified approach through computing the phase angle by

$$\phi(t) = \arctan \frac{F(t)}{\sqrt{1 - F^2(t)}}. \quad (35)$$

Here $F(t)$ has to be a perfectly normalized IMF after repeated rounds of normalization. This is very critical, for any value of the normalized data that goes beyond unity will cause the formula given in Eq. (34) to become imaginary, and Eq. (35) to breakdown.

Although the *arccosine* and *arctangent* approaches are mathematically equivalent, they are computationally different. The *arctangent* approach enables us to use the four-quadrant inverse *tangent* to uniquely determine the specific quadrant of

the phase function, which is essential for the proper unwrapping (from 2π cycles to free running). Furthermore, the computational stability in the *arctangent* is also much improved, as will be demonstrated later. Using *arctangent*, we could still experience unstable results occasionally, when the data contain some irregularities such as jumps or sharp slope changes. Yet the most serious problem is especially sparse data points near the extrema. This occurs frequently for the high-frequency components. Sparse data would also cause difficulties in the normalization, for the maxima might not locate exactly on one of the available data points. Therefore, using any available point would cause the waveform to deform. The computation stability could be much improved with a three- or five-point medium filter, which will not degrade the answer noticeably, for the derivative had already involved two points in the computation. A three-point medium covers only a slightly wider region. In all the subsequent computations, we have used the *arctangent* approach and a three-point medium filter as the default operation in the Direct Quadrature method unless otherwise noted.

The IF computed from the DQ is given in Fig. 9 together with the NHT method to be discussed later. Here the improvement of the DQ is clearly shown: the initial negative IF values, from the simple non-normalized AS at the location of an amplitude minimum region, totally disappeared. These negative frequency values, near the neighborhood of this minimum amplitude location, are the consequence of violating the condition stipulated by the Bedrosian theorem.

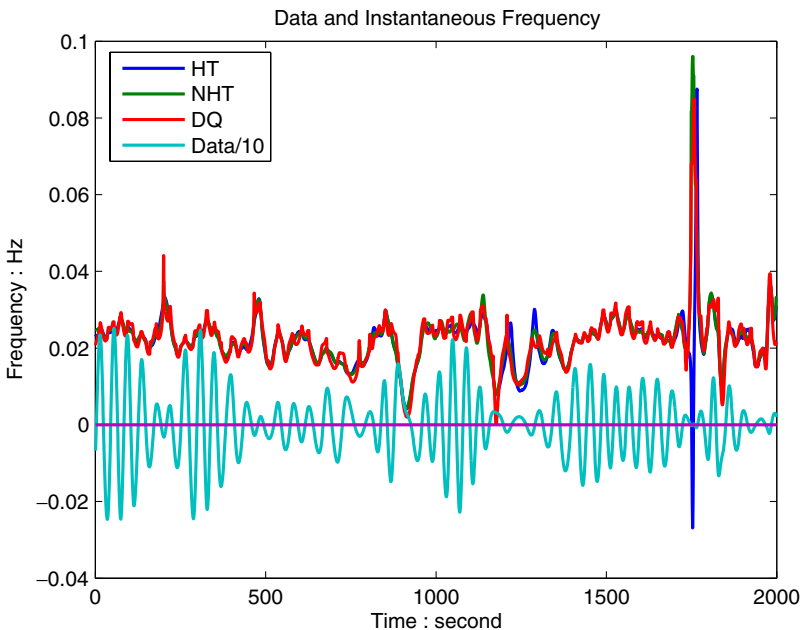


Fig. 9. The IF of the sample data based on various methods: direct quadrature (DQ), HT, and NHT, with the data plotted at one-tenth scale.

By definition, the energy based error index as defined by Nuttall would be zero identically for the quadrature method. DQ gives the correct phase functions even for extremely complicated phase functions. In most locations, however, the numerical difference between the quadrature and HT is small as shown in Fig. 9.

3.3. Normalized Hilbert transform

As the amplitude of the empirical FM signal is identically unity, the limitation of the Bedrosian theorem is no longer a concern in computing AS through the HT. The IF computed from the normalized data is also given in Fig. 9 marked as NHT case. Here the improvement of the normalization scheme is also clearly seen: the initial negative IF values, from non-normalized data near the amplitude minimum locations, were eliminated, for the condition stipulated by the Bedrosian theorem is satisfied automatically. The only noticeable differences between NHT and DQ all occur near where the waveform suffers some distortion. Such distortions are due to complicated phase function changes, the condition stipulated by the Nuttall theorem. At such locations NHT can only give an approximate answer anyway.

Next, we can define a sharper error bound than given by the Nuttall theorem. The principle is very simple: if the HT indeed produces the quadrature, then the modulus of AS from the empiric envelope should be unity. Any deviation of the modulus of AS from unity is the error; thus we have an energy based indicator of the difference between the quadrature and the HT, which can be defined simply as

$$E(t) = [abs(analytic\ signal\ (z(t))) - 1]^2. \quad (36)$$

This error indicator is a function of time as shown in Fig. 10. It gives a local measure of the error incurred in the amplitude, but not of the IF computation directly.¹⁶ Nevertheless, this surrogate measure of error is both logically and practically superior to the constant error bound established by the Nuttall theorem. If the quadrature pair and the AS are identical, the error should be zero. They usually are not identical. Based on our experience, the majority of the error comes from the following two sources: the first source is due to data distortion in the normalization at a location near drastic changes of amplitude, where the envelope spline fitting will not be able to turn sharply enough to cover all, but goes under some data points. Repeated normalization will remove this imperfection in normalization, but it would inevitably distort the wave profile, for the original location of the extrema could be shifted in the process. This difficulty is even more severe when the amplitude is also locally small, where any error will be amplified by the smallness of the amplitude used in the normalization process in Eq. (30). The error index from this condition is usually extremely large. Some of the errors in this category could be alleviated by using different spline function in the normalization process. For example, the Hermite cubic spline with the monotonic condition would not cause overshoot or undershooting. In our implementation, there are occasion that this approach indeed improves the results considerably. We, however, do not use this

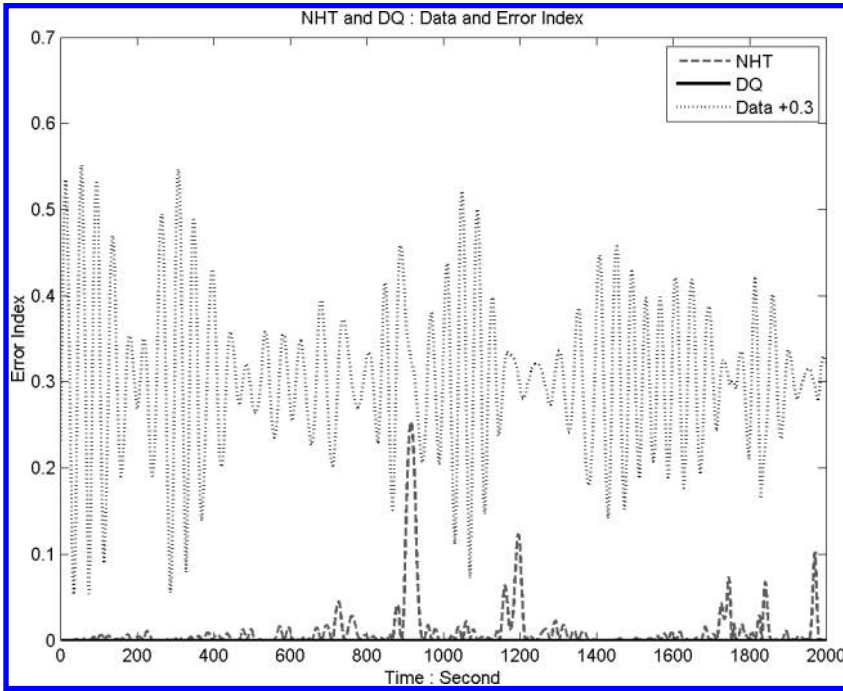


Fig. 10. The energy based error index values for the sample data. Notice the error index is high whenever the waveform of the data is highly distorted from the regular sinusoidal form.

approach because the Hermite cubic spline is not continuous in slope, which would give unsmooth IF values. The second source is due to the nonlinear waveform distortion of the waveform, which will cause a corresponding variation of the phase function, $\phi(t)$, as stipulated by the Nuttall theorem. As discussed in Hahn³³ and Huang *et al.*,² when the phase function is not an elementary function, the phase function from AS and that from the DQ approach would not be the same. This is the condition stipulated by the Nuttall theorem. The error index from this condition is usually small.

Based on our experience, both the NHT and the DQ can be used routinely to give valid IF. The advantage of NHT is that it has a slightly better computational stability than the DQ method, but DQ certainly gives a more accurate IF, if the data is dense enough.

3.4. Teager energy operator

The Teager energy operator (TEO; see, e.g., Refs. 37, 38) has been proposed as a method to compute IF without involving integral transforms; it is totally based on differentiations. The idea is based on a signal of the form,

$$x(t) = a \sin \omega t, \quad (37)$$

then, an energy operator is defined as

$$\psi(x) = \dot{x}^2 - x\ddot{x} \quad (38)$$

where the overdots represent first and second derivatives of $x(t)$ with respect to time. Physically, if x represents displacement, the operator, $\psi(x)$, is the sum of kinetic and potential energy, hence the method is designed as TEO. For this simple oscillator with constant amplitude and frequency, we will have

$$\psi(x) = a^2\omega^2 \quad \text{and} \quad \psi(\dot{x}) = a^2\omega^4. \quad (39)$$

By simply manipulating the two terms in Eq. (39), we have

$$\omega = \sqrt{\frac{\psi(\dot{x})}{\psi(x)}} \quad \text{and} \quad a = \frac{\psi(x)}{\sqrt{\psi(\dot{x})}}. \quad (40)$$

Thus one can obtain both the amplitude and frequency with the energy operator. Kaiser³⁷ and Maragos *et al.*^{39,40} have proposed to extend the energy operator approach to the continuous functions of AM-FM signals, where both the amplitude and the frequency are functions of time. In those cases, the energy operator will offer only an approximation, a rather poor one as will be shown presently. A distinct advantage of the energy operator is its superb localization property, a property unsurpassed by any other approach, except DQ. This localization property is the consequence of the differentiation based method; therefore, it involves at most five neighboring data points to evaluate the frequency at the central point. No integral transform is needed as in Hilbert or Fourier transforms. The shortcomings of the method are also obvious: from the very definition of the frequency and amplitude, we can see that the method only works for monocomponent functions; therefore, before an effective decomposition method is available, the application of the method is limited to bandpass data only. Even more fundamentally, the method is based on a linear model for a single harmonic component only; therefore, the approximation produced by the energy operator method will deteriorate and even break down when either the amplitude is a function of time or the wave profiles have any intrawave modulations or harmonic distortions. Mathematically, Eqs. (39) and (40) could only be true if amplitude and frequency are constant. Therefore, the existence of either amplitude modulation or harmonics distortion violates the basic assumptions of TEO. In comparisons, we found the nonlinear waveform distortion present a more serious problem for TEO than the amplitude fluctuations, for the derivatives from the amplitude fluctuations are general mild, while the derivative values could vary widely from the nonlinear phase deformations. These difficulties put a severe limitation on the application of TEO. In the past, TEO has only been applied to the Fourier bandpassed signals. As a result, the difficulty with the nonlinear distorted waveform was not assessed at all. Having employed EMD to produce IMF, we are able to test TEO on nonstationary and nonlinear data for the first

time. These very shortcomings and breakdown caused by nonlinear waveform distortions, however, make TEO a very nice nonlinearity detector, which will be discussed presently.

3.5. Generalized zero-crossing

Finally, we will present the GZC method. As discussed above, zero-crossing method is the most fundamental method for computing local frequency, and it has long been used to compute the mean period or frequency for narrow band signals.^{22–25} Of course, this approach is again only meaningful for monocomponent functions, where the numbers of zero-crossings and extrema must be equal in the data. Unfortunately, the results are relatively crude, for the frequency so defined would be constant over the period between zero-crossings. In GZC, we will improve the temporal resolution to a quarter wave period by taking all zero-crossings and local extrema as the critical control points.

In the present generalization, the time intervals between all the combinations of critical control points are considered as a whole or partial wave period. For example, the period between two consecutive up (or down) zero-crossings or two consecutive maxima (or minima) can be counted as one whole period. Each given point along the time axis will have four different values from this class of period, designed as T_{4j} , where $j = 1$ to 4. Next, the period between consecutive zero-crossings (from up to the next down zero-crossing, or from down to the next up zero-crossing), or consecutive extrema (from maximum to the next minimum, or from minimum to the next maximum) can be counted as a half period. Each given point along the time axis will have two different values from this class of period, designed as T_{2j} , where $j = 1$ to 2. Finally, the period between one kind of extrema to the next zero-crossings, or from one kind of zero-crossings to the next extrema can be counted as a quarter period. Each given point along the time axis will have only one value from this class of period, designed as T_1 . Clearly, the quarter period class, T_1 , is the most local, so we give it a weight factor of 4. The half period class, T_2 , is the less local, so we give it a weight factor of 2. And finally, the full period class, T_4 , is the least local, so we give it a weight factor of 1. In total, at any point along the time axis, we will have seven different period values, each weighted by their properties of localness. By the same argument, each place will also have seven corresponding different amplitude values. The mean frequency at each point along the time axis can be computed as

$$\bar{\omega} = \frac{1}{12} \left(\frac{1}{T_1} + \sum_{j=1}^2 \frac{1}{T_{2j}} + \sum_{j=1}^4 \frac{1}{T_{4j}} \right), \quad (41)$$

and the standard deviation can also be computed accordingly. This approach is based on the fundamental definition of frequency given in Eq. (6); it is the most direct, and also gives the most accurate and physically meaningful *mean* local frequency: it is local down to a quarter period (or wavelength); it is direct and robust

and involves no transforms or differentiations. Furthermore, this approach will also give a statistic measure of the scattering of the frequency value. The weakness is its crude localization, only down to a quarter wavelength at most. Another drawback is its inability to represent the detailed waveform distortion, for it admits no harmonics and no intrafrequency modulations. Unless the waveform contains asymmetries (either up and down, or left and right), the GZC will give it the same frequency as a sinusoidal wave. With all these advantages and limitations, for most of the practical applications, however, this mean frequency localized down to a quarter wave period is already better than the widely used Fourier spectrogram, say. This method is also extremely easy to implement, once the data is reduced to a collection of IMFs. As this method physically measures the periods, or part of them thereof, the values obtained can serve as the most stable local mean frequency over the time span to which it is applied. In the subsequent comparisons, we will use the GZC results as the baseline reference. Any method producing a frequency or amplitude grossly different from the GZC result in the mean simply cannot be correct. Therefore, GZC offers a standard reference in the mean for us to validate the other methods.

Having presented all these IF or local frequency computing methods, we will present some intercomparisons of the results in the following sections. In all these cases, the data will have to be reduced to IMF components. For arbitrary data, we have used the EMD method² and ensemble EMD (EEMD, Ref. 41) to decompose the data into the IMF components before applying any of the above methods to compute the instantaneous frequency.

4. Intercomparisons of Results from Different Methods and Discussions

For the intercomparisons, we will use two examples: one from a model and the other from a real physical phenomenon, to illustrate the difference in the IFs produced by the different methods. The first example is a model function to illustrate details and the potential problems of the methods; the second example is a real speech signal, which will give us an illustration of how the various methods perform in practical applications. The methods used to compute the IF are the TEO, the GZC, the NHT, the DQ, and sometimes also the simple HT methods.

4.1. Validation

The first example is the modeled damped Duffing wave with chirp frequency. The explicit expression of the model gives us the truth and enables us to calibrate and validate the methods quantitatively. The model is given by

$$x(t) = \exp\left(-\frac{t}{256}\right) \cos\left(\frac{\pi}{64} \left(\frac{t^2}{512} + 32\right) + 0.3 \sin \frac{\pi}{32} \left(\frac{t^2}{512} + 32\right)\right), \quad (42)$$

with $t = 0 : 1024$.

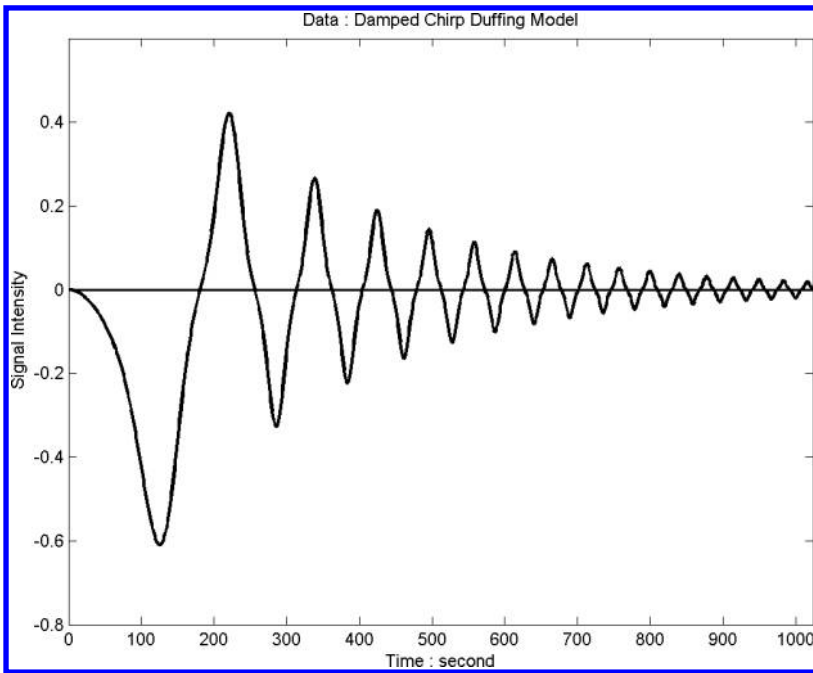


Fig. 11. The modeled damped chirp Duffing waves based on Eq. (42).

Assuming the sampling rate to be 1 Hz, the numeric values of the signal are plotted in Fig. 11. As the amplitude decays exponentially, we have to normalize the data using the method described in Eq. (30). From the normalized data, we can also compute the quadrature. The complex phase graphs of all different methods are given in Fig. 12, each representing the AS from the original un-normalized data and the normalized empiric envelope, and also from the quadrature. As the HT is implemented through Gabor³² method, the effect of the jump in values at the beginning and the end is clearly visible from the imaginary part shown in Fig. 13, where the amplitude of the imaginary part of the data deviates widely from the data towards the end. The amplitude from the normalized data has corrected the effect of the jump condition between the beginning and end, and gives a major improvement of the result.

It is important to point out that, for this Duffing model, the quadrature and the AS are not identical as shown vividly in this complex phase graph. Therefore, we should anticipate problems for the IF computed from the AS methods. The phase function of the quadrature is given by a perfect unity circle, for the modulus of the complex number formed by the data and its quadrature is identically unity. But the amplitude of AS from the HT deviates from the unity circle systematically. This deviation results in an energy measure of the error in using the AS as an approximation for the quadrature. In fact, the computed quadrature and the imaginary part of the AS were shown in Fig. 13. The computed quadrature is exactly the same

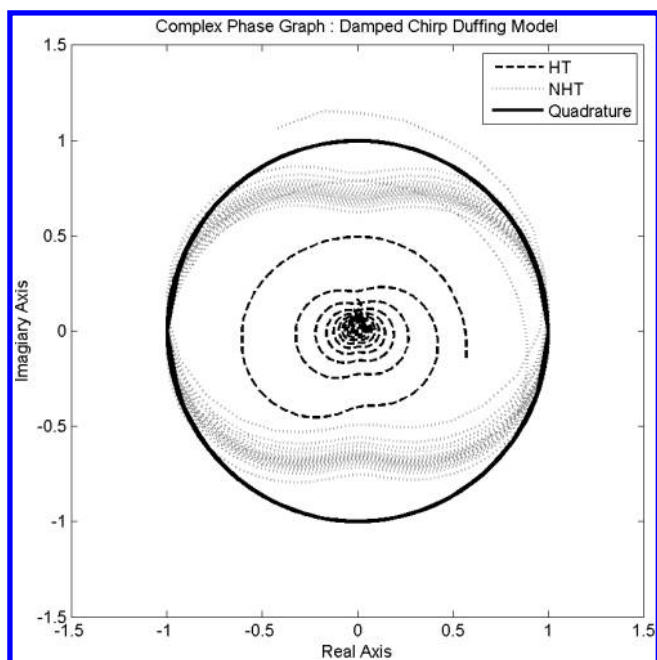


Fig. 12. The complex phase graph for the damped chirp Duffing wave model based on AS from the original data (dash line), normalized data (dotted line), and directly computed quadrature (thick solid line). Normalization has certainly improved the phase graph, but the phase graph is still not a unit circle except for the quadrature.

as the one given by the theoretical expression. This offers a clear validation of the direct quadrature computation method (DQ).

The amplitudes determined from the various methods are given in Fig. 14. The empiric envelope determined through spline fitting agrees almost exactly with the theoretic values, except near the beginning, where the end effects have caused the empiric envelope to dip slightly. Again the empiric envelope is also the only one agreeing well with the envelope determined from the GZC method over practically the whole range. The stepped values from the GZC method show the limit of localization of the method. The amplitude from AS is influenced strongly by the complicated phase function as stipulated by the Nuttall theorem. The worse overall performance among all the methods is the TEO. Whenever the waveform is distorted, the values of the amplitude drop even to zero, which bear no similarity to the reality. The amplitude from HT performs poorly near the end of the data span caused by the jump condition between the beginning and the end. Here the limitations of both the Bedrosian and Nuttall theorems are visible.

To examine the effect of the Bedrosian theorem in detail, we computed the Fourier power spectra for both the AM and FM signals as given in Fig. 15. Although the AM signal is a monotonic, exponentially decaying function, the power spectral density would treat it as a “saw-tooth” function, and have a wide spectrum.

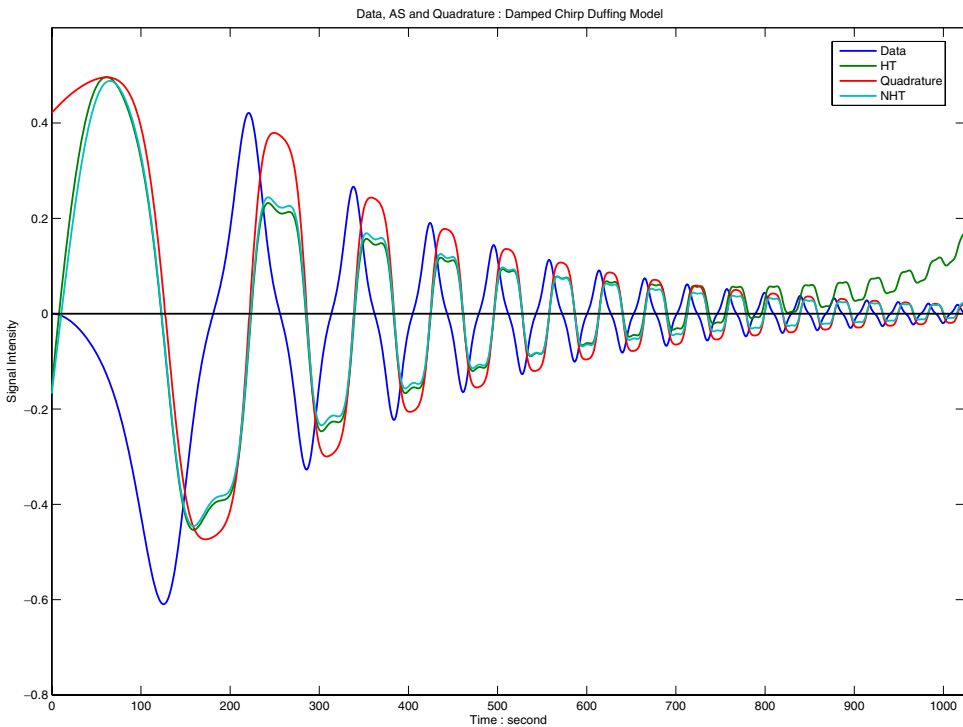


Fig. 13. Comparison of the imaginary part from AS based on the simple HT, NHT, and DQ. While the quadrature is identical with the theoretic result, the ASs are visibly different from the theoretic results especially the one without normalization, where the jump condition at the ends forced the AS to diverge from the data.

Therefore, the spectra of the AM and FM signal are not disjoint at all. Consequently, the phase function of the AS will be contaminated by the amplitude variations, for they violate the Bedrosian theorem. As we will see later, the spectra from the AM and FM signals of an IMF would never be disjoint in general, unless the signals were separated specifically by a bandpass filter. The small but nonzero overlapping indicates that the FM part of the AS generated through the HT is always an approximation contaminated by the AM variations.

Now, let us examine the IF values given in Fig. 16. Here again, the IF from the DQ method coincides with the theoretic values with no visible discrepancy in the figure on this scale. To examine the discrepancy in detail, a selected section is expanded in Fig. 17. Here, the IF values from DQ show some deviation from the true theoretic ones near the peaks of each wave toward the end, where the sparse data points become a problem. Nevertheless, the overall performance is still very good. The HT without normalization performs poorly with many negative values, especially toward the end, where the jump condition between the beginning and the end had caused the imaginary to deviate too far from the true value and hence the poor IF performance. The IF values from either HT (even when it performs

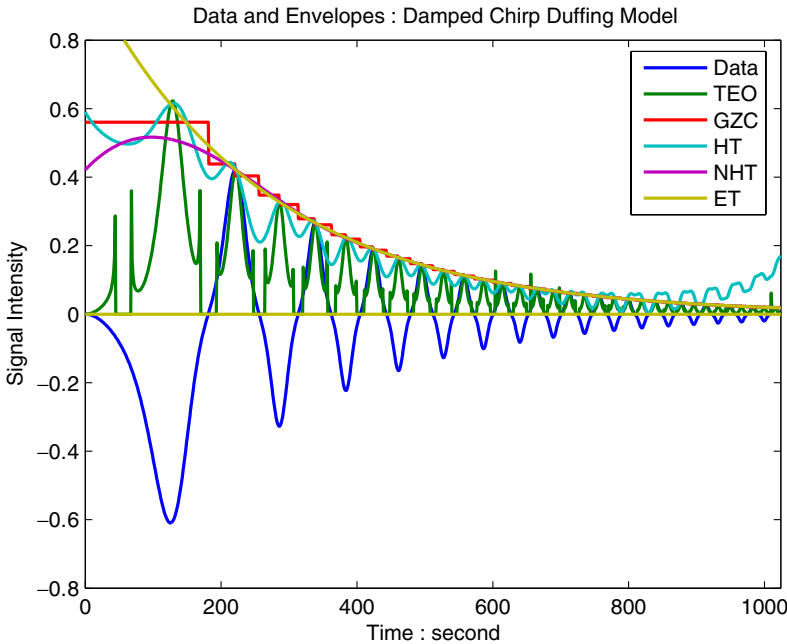


Fig. 14. The amplitude determined from various methods: TEO, GZC, and HT, spline fitting used in DQ and NHT. The spline line is identical to the theoretic value; therefore, it is defined as the envelope in the direct quadrature method.

well) or NHT show insufficient modulation than the theoretic values as reported by Huang *et al.*² The normalization step indeed has provided more stable, albeit still insufficiently modulated, IF values. The GZC gives a slightly stepped constant sloped mean value as expected. The result of TEO is again plagued by the nonlinear distortion of the waveform to the degree that the frequency is totally useless.

A crucial criterion for judging the viability of the different methods is to examine the error from two points of view. First, we will use the energy based criterion as given in Fig. 18. Based on the energy considerations, we can only compute the errors from the normalized signals. The errors from the DQ and GZC are negligible, for the envelope determined from the GZC is nearly perfect. The error is relatively low for NHT compared to the much larger values from TEO. Second, we will examine the error by direct comparisons of the ratios of IF values from the various methods to the truth as the base. The overall results are given in Fig. 19, and the details are given in Fig. 20. Here the DQ result is almost exactly as computed from the theoretic expression. The only large discrepancies occur at the beginning due to the end effect from the spline envelope fittings. Toward the end, the insufficient digitalization rate has caused some small discrepancy as discussed before. At most of the time, the IF values are identical to the theoretic true values computed from the model. NHT shows great improvement over HT, because of the removal of the effects stipulated in the Bedrosian theorem through normalization. The effect of

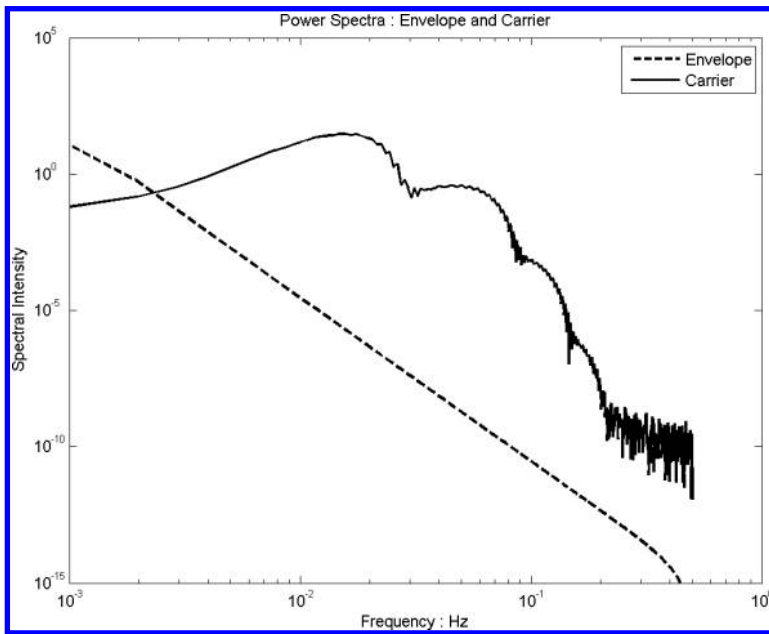


Fig. 15. Fourier power spectral density for the normalized carrier (FM, the solid line) and envelope (AM, the dash line). They are overlapping each other indicating that the data violated the limitation of the Bedrosian theorem; therefore, the contamination of the FM part defined from AS by the AM variations is to be expected.

the Nuttall theorem is visible from the insufficient intrawave frequency modulation. The ratio from GZC is pretty poor, for it totally missed the intrawave frequency modulation, but it still is correct in the mean. This result offers a good proof of the claim by Picinbono¹⁶: error of IF cannot be measured by the envelope spectra alone as given by Nuttall theorem. It should be pointed out, however, that the TEO still offers the poorest agreement. The IF values could even reach zero at many locations near the amplitude equal to zero line, and never give reasonable value almost everywhere except at one point in every wave near the local maximum or minimum.

Finally, we will give a comparison between the *arccosine* and *arctangent* approaches. The difficult in using *arccosine* is in the unwrapping, for the typical *arccosine* will only give phase function between 0 and π . As a result, to have a proper phase will involve an additional step of flipping the directly computed phase before the unwrapping. Without such a step, the result will be erroneous. Though such a step could be implemented, the four-quadrant inverse *tangent* is much easier. Figure 21 gives IF values obtained from the *arccosine*, *arctangent*, and the truth. Without proper unwrapping, the result based on direct *arccosine* approach contains frequency in the negative range, while the result based on four-quadrant inverse *tangent* are in good agreement with the truth. A detailed view shows this comparison in Fig. 22, in which the improvement of the three-point medium filter

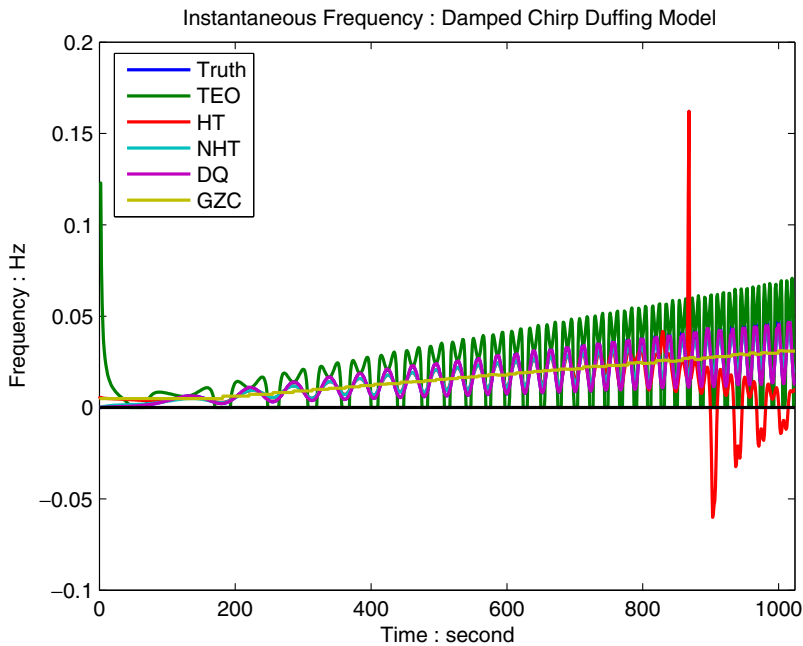


Fig. 16. The IF computed from the various methods: TEO, GZC, HT, NHT, DQ, and the theoretic value (truth). IF from HT (without normalization) and TEO perform poorly.

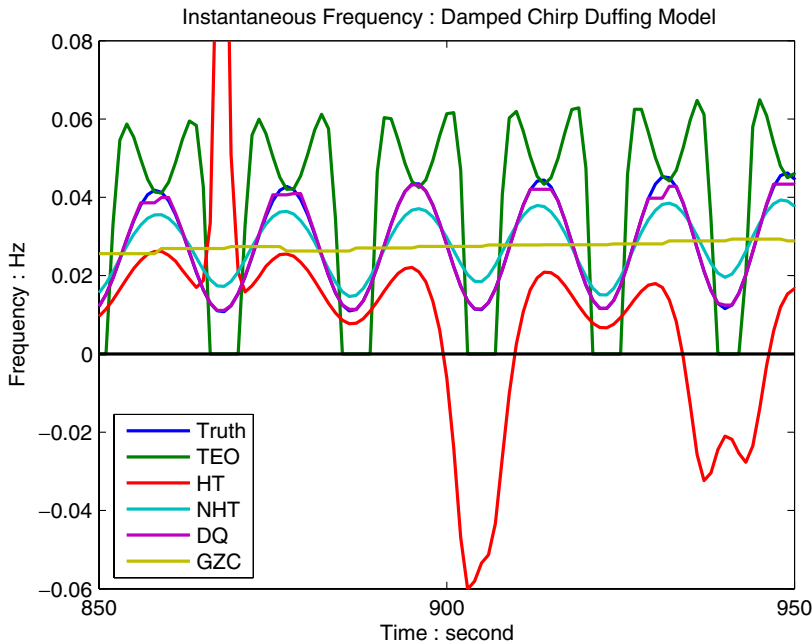


Fig. 17. Details of Fig. 16 near the end, where the slight deviation of DQ from the truth due to the data rate at this high frequency is visible. Even with this discrepancy, DQ still outperform all the other methods.

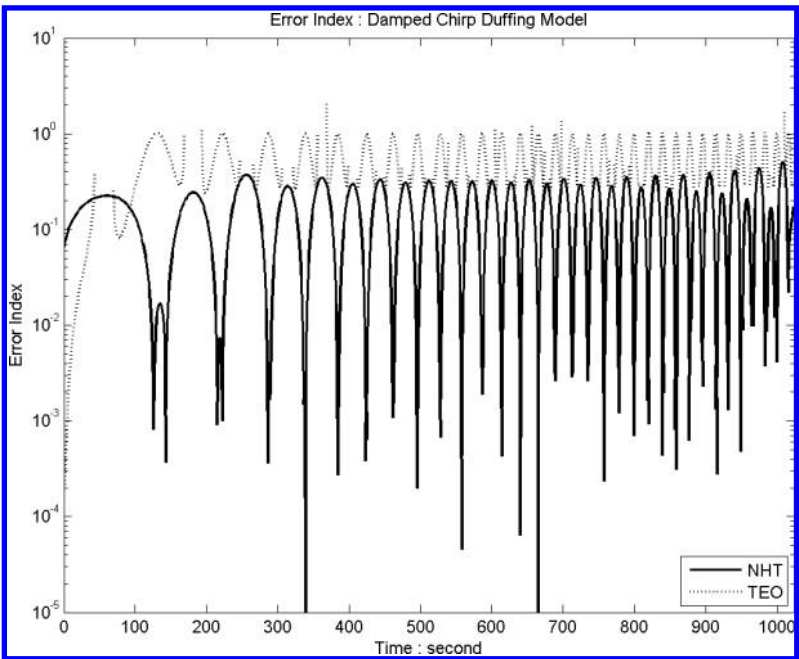


Fig. 18. The energy based error index of the damped chirp duffing model. The error of NHT is nonzero everywhere as a consequence of Nuttall theorem, but it is consistently smaller than the TEO method.

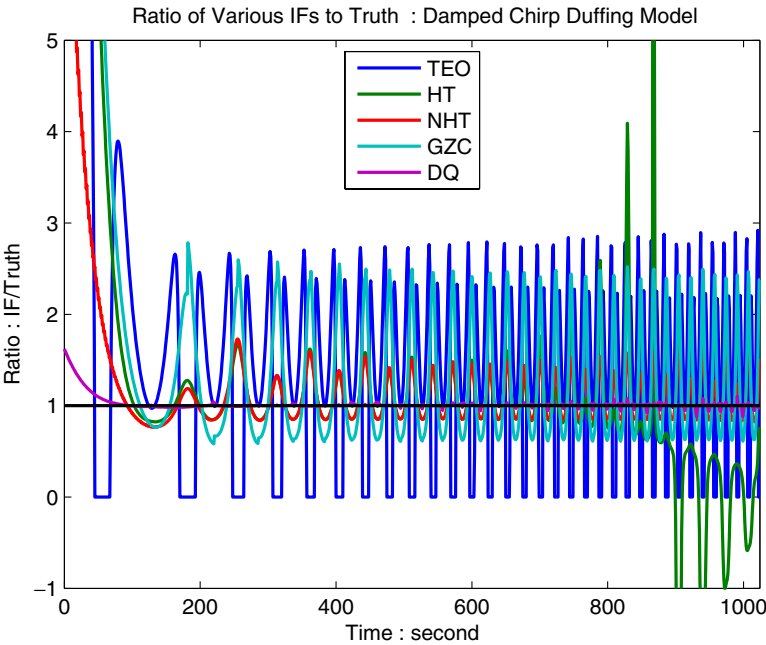


Fig. 19. Using the ratios of IF values give in Fig. 16 to the truth as a measure of error directly. Other than at the very beginning, the DQ gives nearly a perfect ratio of one.

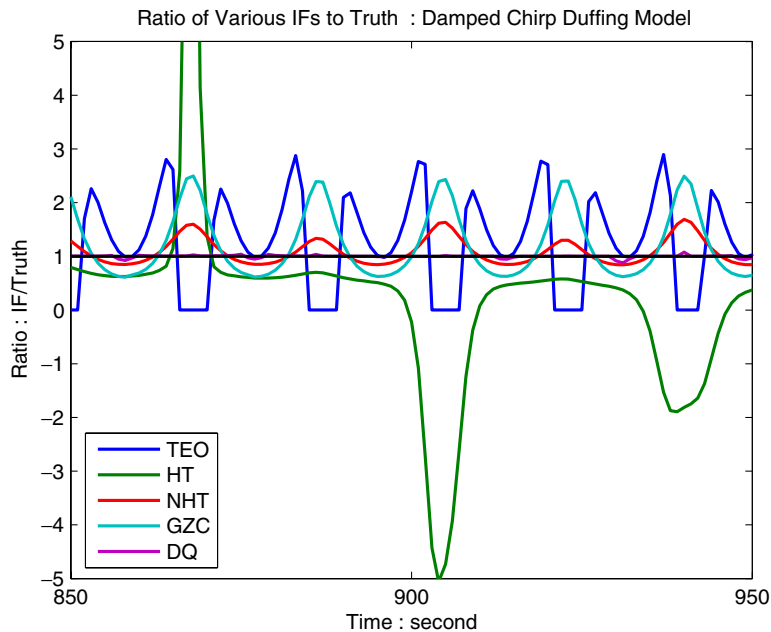


Fig. 20. Details of Fig. 19, where the deviations of the DQ values are barely discernable, where the discrepancies from all the other method are clearly visible.

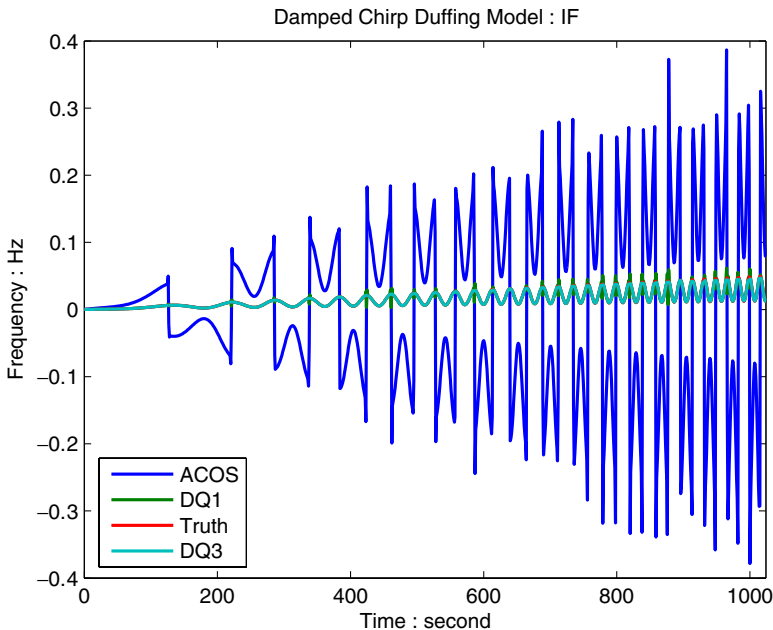


Fig. 21. Comparison of IF values obtained from the *arccosine* and *arctangent* approaches. The four-quadrant inverse tangent method employed in the DQ gives correct IF values. The difficulty in the unwrapping for the phase function obtained from direct *arccosine* approach results in erroneous IF values.

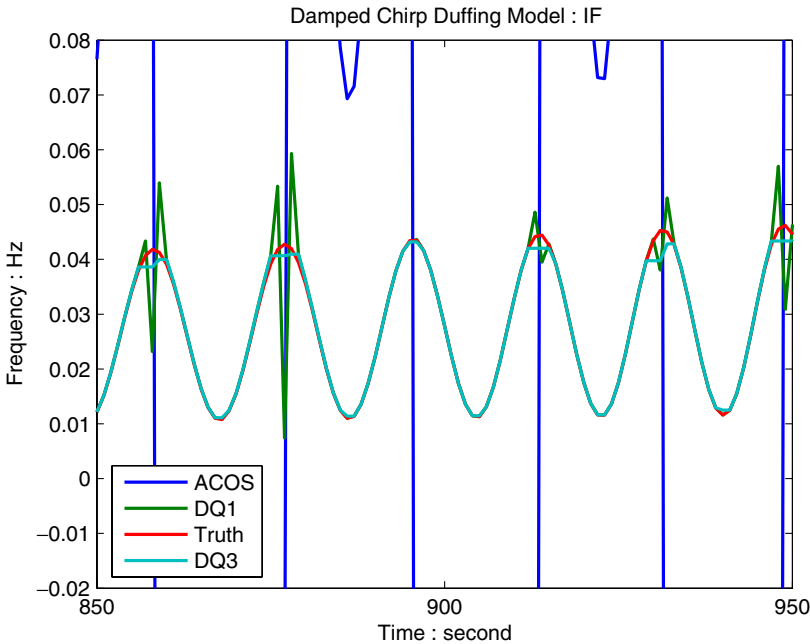


Fig. 22. Details of Fig. 21, where the DQ values from the *arctangent* approach always gives the correct results, except when the data become sparse.

values (DQ3) over the unfiltered results (DQ1) can be seen clearly. The effect of sparse data points near the maxima is also shown as discussed above.

As mentioned above, the energy based index is only sensitive to the envelope fitting by definition. As such, it only serves as a surrogate for the IF values. In analyzing real data, the true frequency is unknown anyway; therefore, the energy based error index is the only error measure available. If it is large, there certainly should be problems. Even if the error index is small, as in the case of GZC, the IF values still might not be small. Thus, the energy based error index might not be a foolproof indication of error in the IF values; it thus provides only a guidance and a necessary condition for small errors.

There are several lessons learned here through this validation exercise: the TEO is for linear signals, and the GZC can only give a *mean* value, but not the intrawave frequency modulation, which is a sure sign of signal from nonlinear processes. This exercise has clearly validated the NHT method and, more importantly, the superiority of the direct quadrature approach is clearly demonstrated.

4.2. An application

Having examined the different methods with the model equation and validated both DQ and NHT, we will proceed to test the methods on an example of practical application, an IMF extracted from the voice record of “Hello”. The original signal

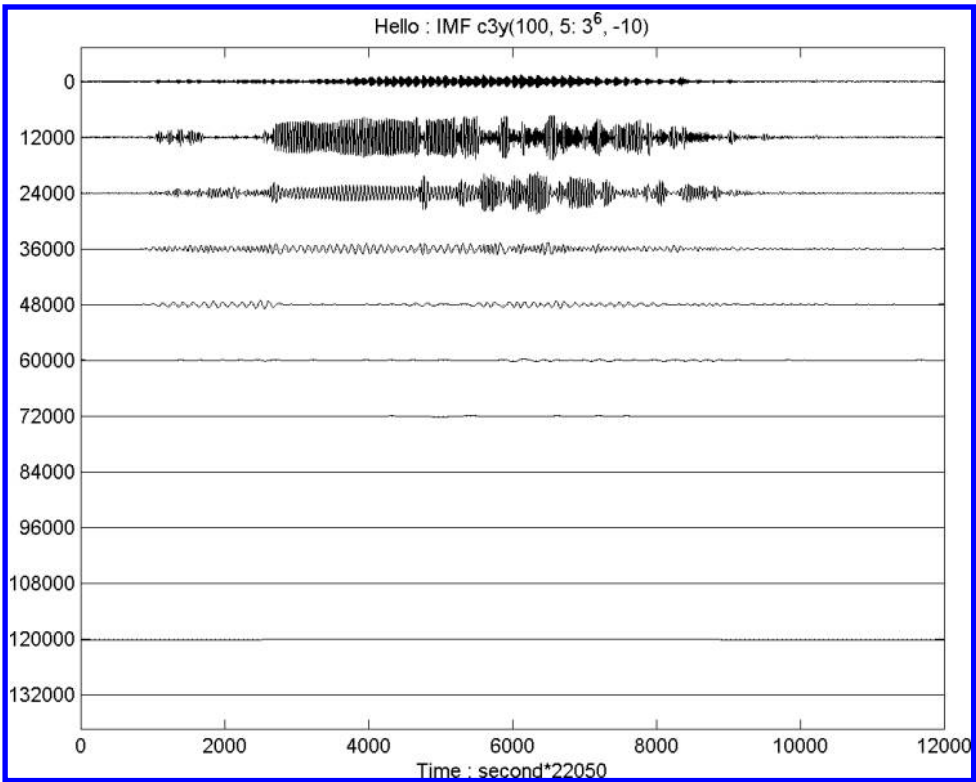


Fig. 23. The intrinsic mode functions (IMF) derived from the data given in Fig. 1 plotted on the same scale. The second IMF is clearly the most energetic, and is used as sample data for this study.

of the sound for “Hello” is given in Fig. 1, with the IMF components extracted from EMD given in Fig. 23, where the most energetic component, given in Fig. 23, is selected as the test data, shown in Fig. 24, to make comparisons among the different methods. It should be note that we used only EMD rather than EEMD to accentuate the potential problems of large amplitude fluctuation of amplitude and IF values caused by mode mixing. Such case is very common in everyday applications. This signal could be decomposed much more reasonably with EEMD as demonstrated by Wu and Huang,⁴¹ in which the mode mixing is totally removed and the large amplitude fluctuation absent.

Now, let us proceed with the comparisons of different methods. First, we will present the different envelopes obtained from spline and HT fittings and all the other methods as in Fig. 25. From this global view, the amplitude from the TEO fitting has already shown serious problems, with the magnitude exceeding the data by a large margin. All the other methods (HT, spline, and GZC) seem to have collated in a narrow-band symmetric to the silhouette of the data, indicating all envelopes having been reasonable.

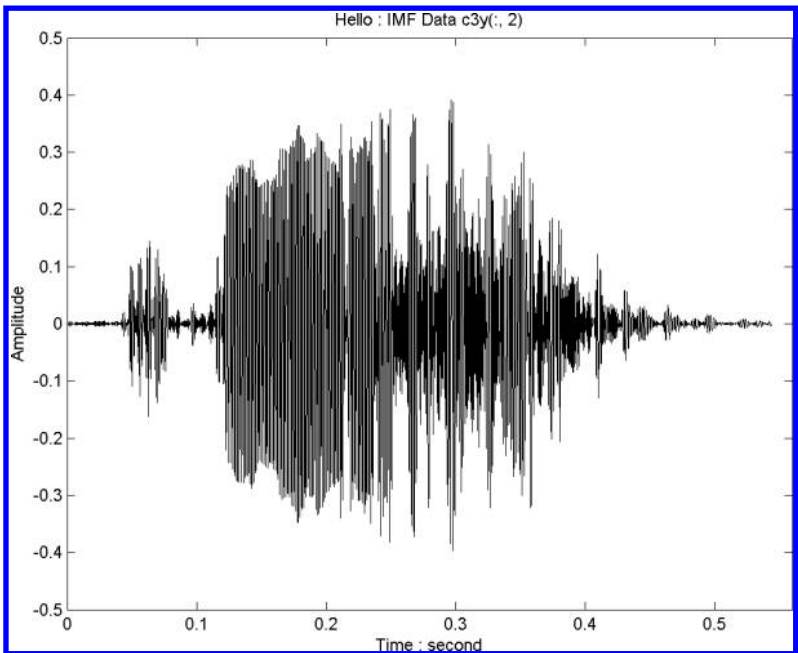


Fig. 24. The sample data of the second IMF from Fig. 23.

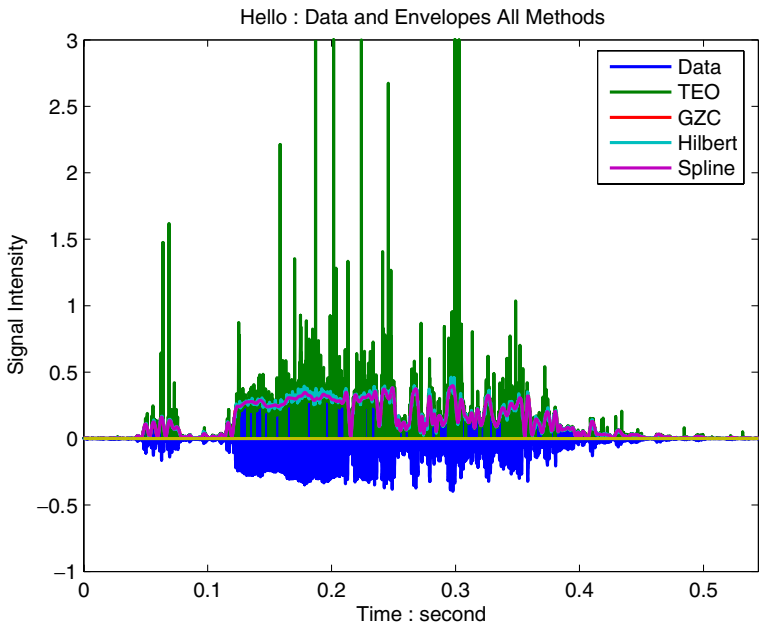


Fig. 25. The envelopes determined from various methods: TEO, GZC, HT (Hilbert), and spline, together with the data plotted as a reference. As the data is an IMF, it is symmetric; therefore, any envelope deviation from the symmetric image of the data, such as the one from TEO, could not be right. All the other methods performed reasonably well.

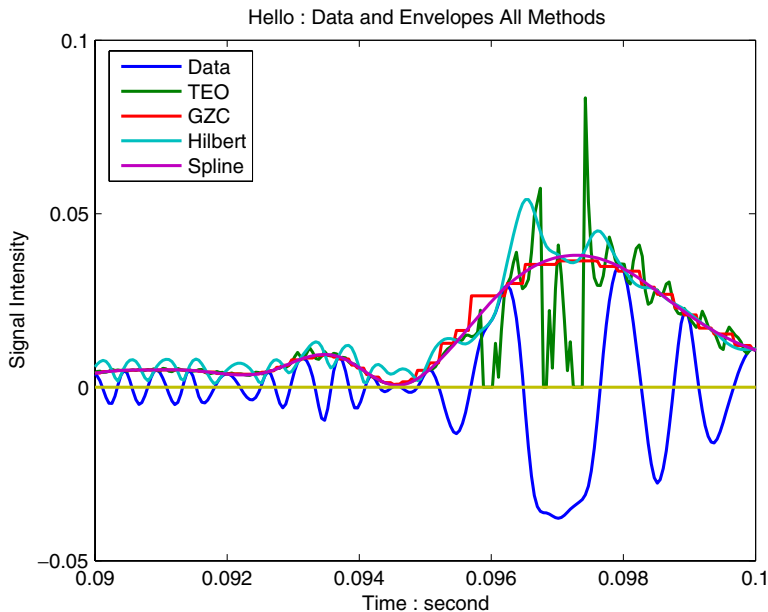


Fig. 26. Details of a section with small and highly variable amplitude values near 0.1 s. TEO is still erroneous; the HT also shows visible variations from the spline, which follows the stepped GZC closely.

To examine the details, we have selected and enlarged two subsections near 0.1 and 0.2 s given in Figs. 26 and 27. From these figures, we can immediately see that the envelope from the TEO becomes problematic whenever the waveform is distorted. This is to be expected from the model functions discussed above. It is a clear indication that the TEO is based on a stationary and linear model. As the error occurs most seriously at the location of nonlinear wave form distortion as discussed above, this shortcoming can be turned into an advantage: to use it as an indicator for nonlinear wave deformation. Other than the TEO envelope, all the envelopes are much closer to each other. Indeed, from the performance of TEO here we can surmise that the speech signal is highly nonlinear with high confidence. The empiric spline envelope is the smoothest one, while the Hilbert envelope shows some subperiod intrawave fluctuations due to the nonlinear waveform distortion. As the GZC is the most robust, albeit coarsely localized, definition for envelope and period, it is reassuring to see that the empiric envelope follows the one from GZC closely almost everywhere. This is especially true when the amplitude variation is small as in Fig. 27, yet the empiric envelope still retains the smoothness that is lacking in the GZC envelope.

As before, the misfits by the HT method are indicating the limitations stipulated by the Bedrosian and Nuttall theorems. To examine the condition of the Bedrosian theorem, we again plot the spectra of the FM (normalized carrier) and AM (the empiric envelope) signals in Fig. 28. Clearly, they are not disjoint, just as in the case

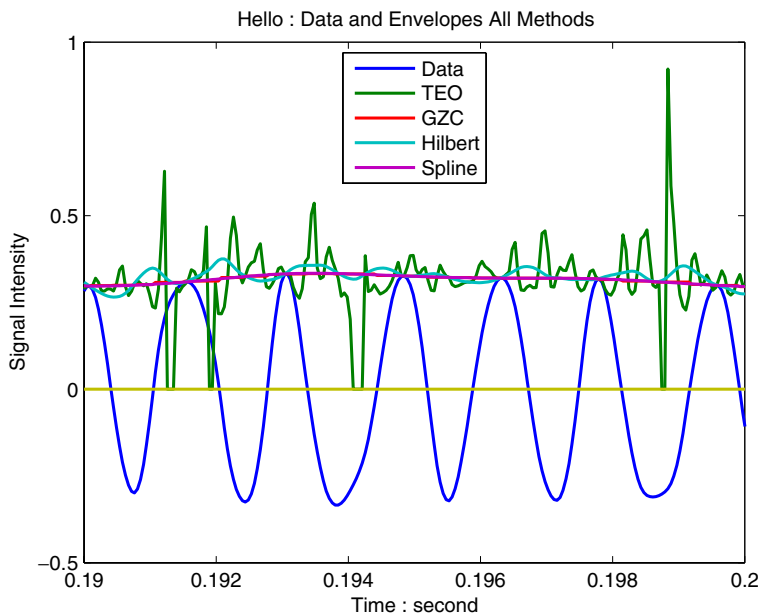


Fig. 27. Details of a section with large and uniform amplitude values near 0.2s. TEO is still erroneous; the HT also shows visible variation from the spline, which is almost identical with GZC.

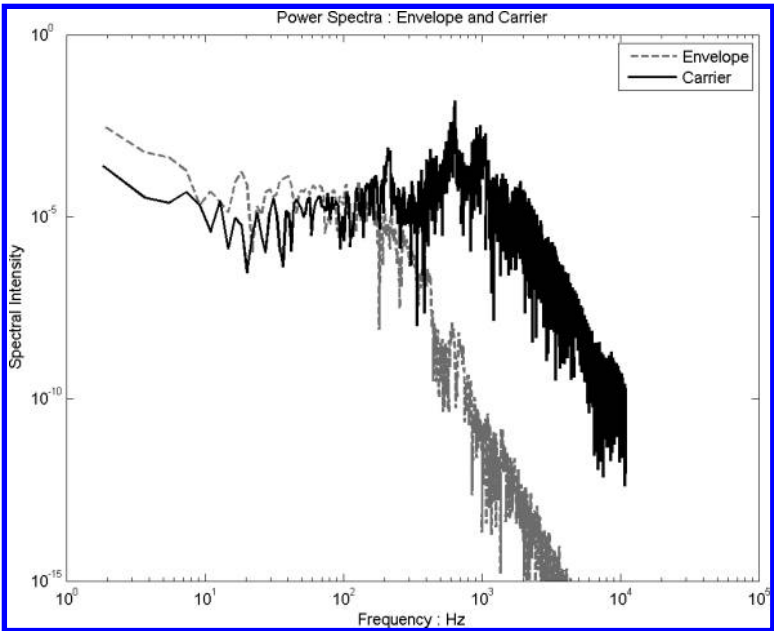


Fig. 28. Similar to Fig. 15, Fourier power spectral density for the normalized carrier (FM, the solid line) and envelope (AM the dotted line). They are overlapping each other again, indicating that the data violated the limitation of the Bedrosian theorem, and thus a problem in IF values through AS is expected.

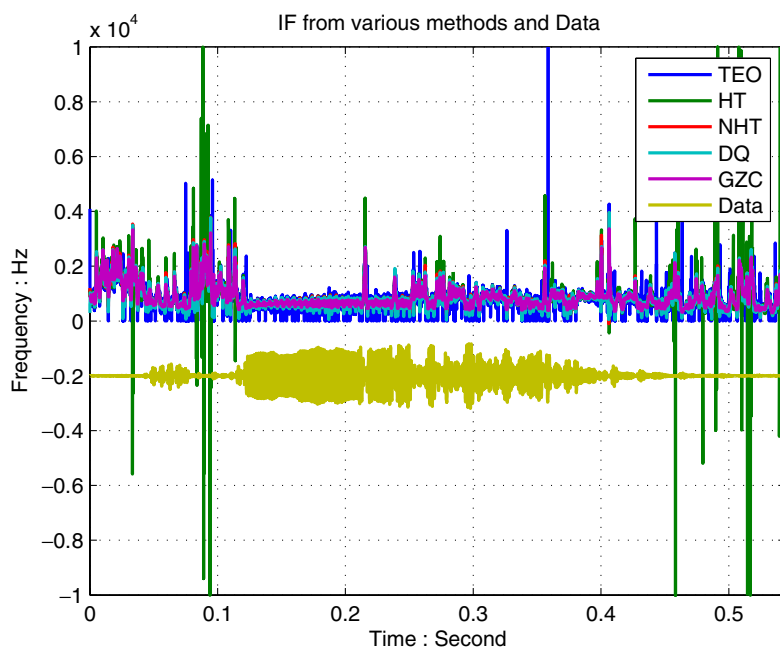


Fig. 29. Similar to Fig. 16, the IF computed from the various methods: TEO, GZC, HT, NHT, and DQ. No theoretic value is available here for comparison. Again, IF from HT (without normalization) and TEO perform poorly with large fluctuations for HT and numerous zero-frequency values from TEO.

of the Duffing model. With this in mind, we should expect the IF derived from the AS signal to be problematic. An independent way to confirm the limitation imposed by the Bedrosian theorem, we have used a narrow window to exclude the large amplitude variation within the window and obtained positive frequency locally. This window is very difficult to implement. Normalization of the IMF resolves the problem of amplitude variation; it also drastically reduces the error index.

Now, we will examine the IF, with the global view given in Fig. 29. Two features become obvious. First, there are numerous instances when the TEO values drop to zero level, an indication of nonlinear processes. Second, there are also large fluctuations of the HT determined IF values dipping even into the negative range. These wild fluctuations are the consequence mostly of the limitation stipulated by the Bedrosian theorem. As all the problem areas occur at the locations when the signal has very small amplitude values and in the neighborhood of large amplitude fluctuations, we can also assume that the IF value calculations are unstable at small amplitude values. Therefore, the poor performance of HT could be the combination of both small local amplitude values and the large fluctuation of the amplitude in the neighborhood. The IF values obtained from the NHT, having removed the AM fluctuations, certainly satisfy the Bedrosian theorem and eliminate these anomalies from the simple AS approach. In fact, the results from the NHT have never dipped

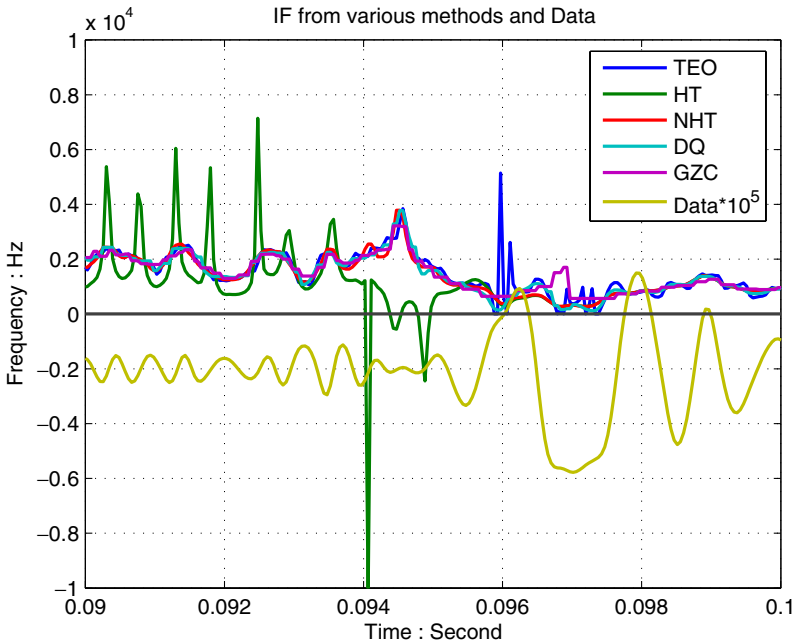


Fig. 30. Details of a section with small and highly variable amplitude values near 0.1 s. TEO performs well except near the location when amplitude values have a problem. The IF values hit zero value there as well. HT shows visible deviations from the GZC values, mostly from the violation of the Bedrosian theorem, an error corrected successfully with normalization. Quadrature again outperforms all the other methods.

to zero or a negative range. Again, we will use the identical two sections as in the amplitude study to examine the details of IF shown in Figs. 30 and 31. From Fig. 30, we can see that the IF values from HT are very unstable whenever the amplitude values are small or the amplitude of the signal experiences large magnitude changes. The IF values for TEO are relatively stable with respect to the amplitude changes as we argued before, but they suffer at the locations of waveform distortions. The best results again come from the DQ and NHT methods, which are closely intertwined with each other and both matched with the GZC values. DQ values show some deeper modulations especially at the locations with large waveform distortions as expected. Although we do not have the truth here to make an absolute comparison, based on the validation study of the Duffing model above we believe that the deep modulating DQ values offer the best answer.

Finally, let us examine the errors from the various methods. The error indices of the HT, NHT, and DQ are given in Fig. 32, where HT here stands for the normalization using the modulus from un-normalized AS. The results indicate that the error index for NHT is consistently smaller than that from HT except for one location near 0.4 s. The results of this particular location of large NHT error is enlarged and examined in Fig. 33. We found that the envelope from NHT had cut

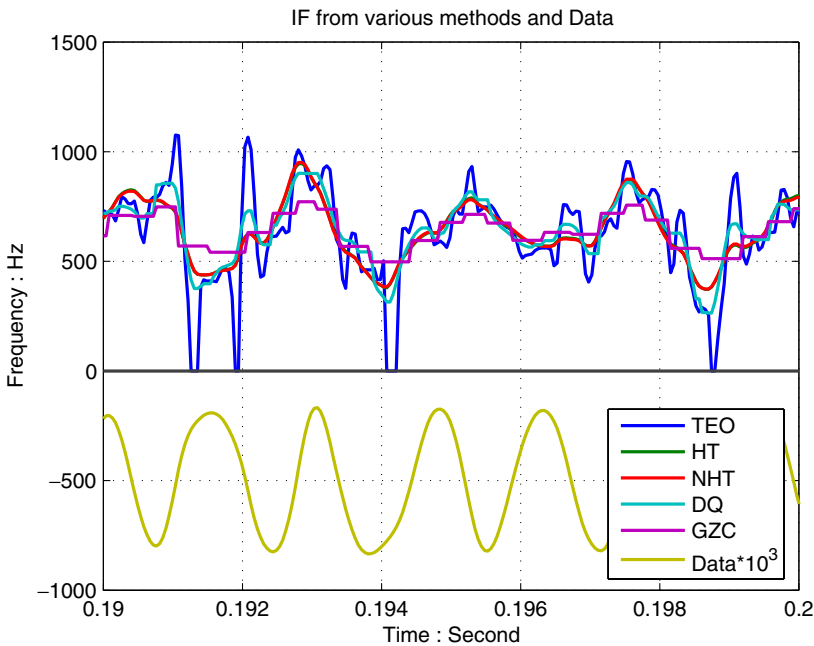


Fig. 31. Details of a section with large and uniform amplitude values near 0.2s. TEO is still erroneous; the HT shows no visible discrepancy from the normalized, the quadrature and GZC, except DQ shows some deeper modulation especially at the location where wave form distortion is strong, which is also the locations where the TEO perform the poorest.

through the data near the location of very low amplitude and caused waveform distortions in poor normalization. If the modulus of AS is used as the base for normalization, there will not be any case with the envelope below the data. It is tempting to adopt this alternative for a normalization scheme. But as discuss above, and from the overall performance of HT and NHT, we believe the empirical spline envelope based normalization is still the preferred option. All the failures of NHT are at the locations of low amplitude with negligible energy density; therefore, even with large error, the impact on energy–time–frequency spectral distribution is insignificant. The overall performance of NHT is far better than HT. By definition, the energy based error index from DQ is identically zero. As discussed above, this might not be an indication of zero error in IF, but it is a necessary condition for small errors.

5. AM and FM Representation of the Data

Until now, the main result of Hilbert spectral analysis has always emphasized the FM part of the data: the data is decomposed into IMFs first. The IF and amplitude of each IMF is then computed through AS to form the Hilbert spectrum with the frequency calculated from the carriers. Through the empirical AM–FM decomposing, we have uniquely separated the AM from the FM part of the data. Though the

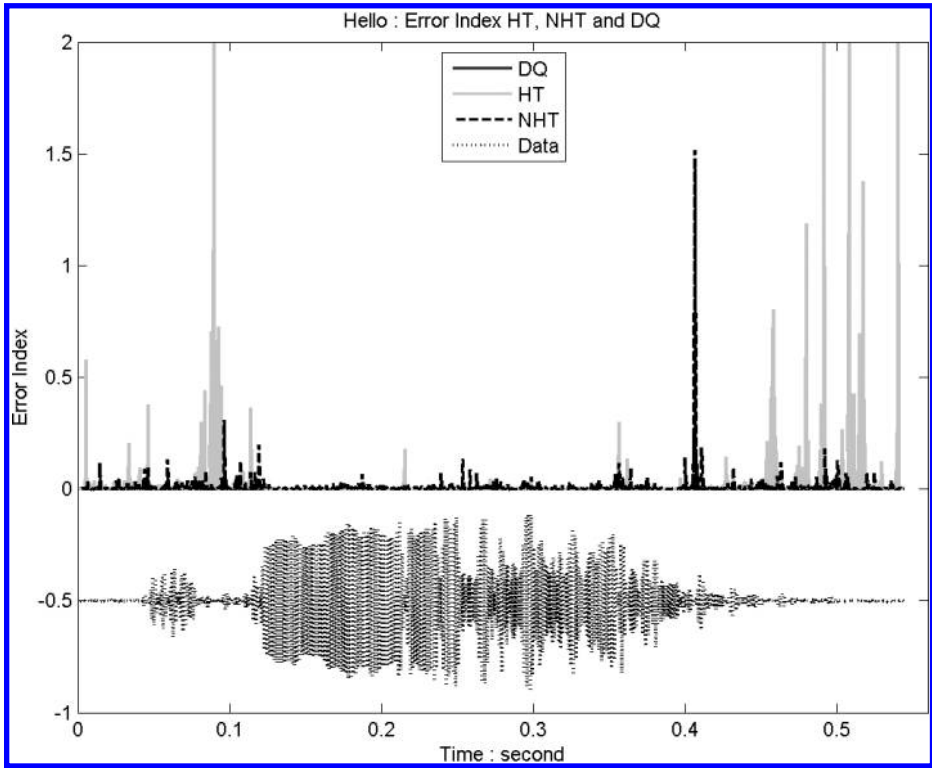


Fig. 32. The energy based error index. Normalization has improved the error drastically.

variations on the AM or envelope variations (FM) are included in the Hilbert spectrum, they have not been examined in detail. In the EMD and Hilbert spectral analysis approach, it is justifiable not to put too much attention on the amplitude variations, for if there is mode mixing, the amplitude variation from such mixed mode IMF really does not reveal any true underlying physical processes. Mode mixing can be alleviated through an intermittence test⁶ or by the EEMD.⁴¹ Either way, we can obtain IMF sets without mode mixing. Then, the envelope variation would contain additional information, for example, when there is a beating signal representing the sum of two co-existing sinusoidal waves. In an earlier paper, Huang *et al.*² tried to extract individual components out of the sum of two linear trigonometric functions such as

$$x(t) = \cos at + \cos bt. \quad (43)$$

Through 3000 repeated siftings, two seemingly separate components were recovered. But the intrinsic mode functions (IMFs) were not clean trigonometric functions anymore, and there were obvious aliases in the resulting IMF components and the residue. The approach proposed then is both unnecessary and unsatisfactory. The problem, in fact, has a much simpler solution: treating the envelope as an amplitude

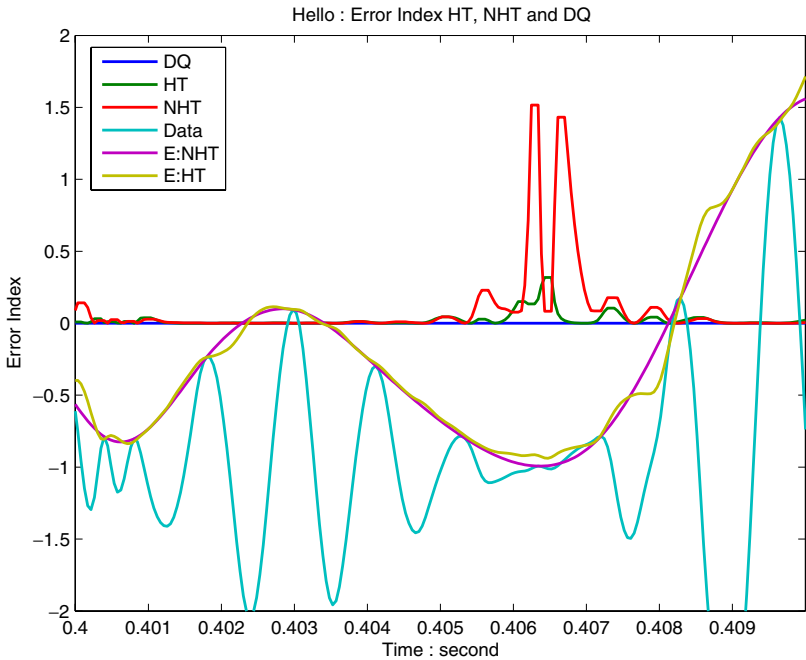


Fig. 33. Detail of Fig. 32, at the location where NHT shows large error. This is caused by the spline as it went under the peak of an extremely low amplitude wave, located near 0.406 to 0.407 s, that distorted the normalized profile.

modulation and just processing the envelope data. Thus the function $x(t)$ given in Eq. (43) can be rewritten as

$$x(t) = \cos at + \cos bt = 2 \cos \left(\frac{a+b}{2} t \right) \cos \left(\frac{a-b}{2} t \right). \quad (44)$$

There is no difference between the sum of the individual components and the modulating envelope form; they are trigonometric identities. If one can obtain both the frequency of the carrier wave, $(a+b)/2$, and the frequency of the envelope, $(a-b)/2$, then one has indeed extracted all the information in the data. The problem of whether one could or could not separate two components forming a beating signal have been studied by Rilling and Flandrin⁴² thoroughly. Their conclusion is both rigorous and thoughtful. To separate the signal is to conform to the mathematic requirement under the Fourier scheme of reason. In fact, if one would match one's results with physics and/or perception, to treat the signal as one component is more reasonable. For example, if one is to make a monotonal sound with slightly variable amplitude, should we label such a sound as two tone? Most likely and more reasonably we would not do so. Yet, mathematically, once the amplitude is fluctuating regularly, the signal would be represented as two tones under the strict stationary and linear Fourier model. For complicated AM, the modulation itself contains valuable information that is worthy exploring.

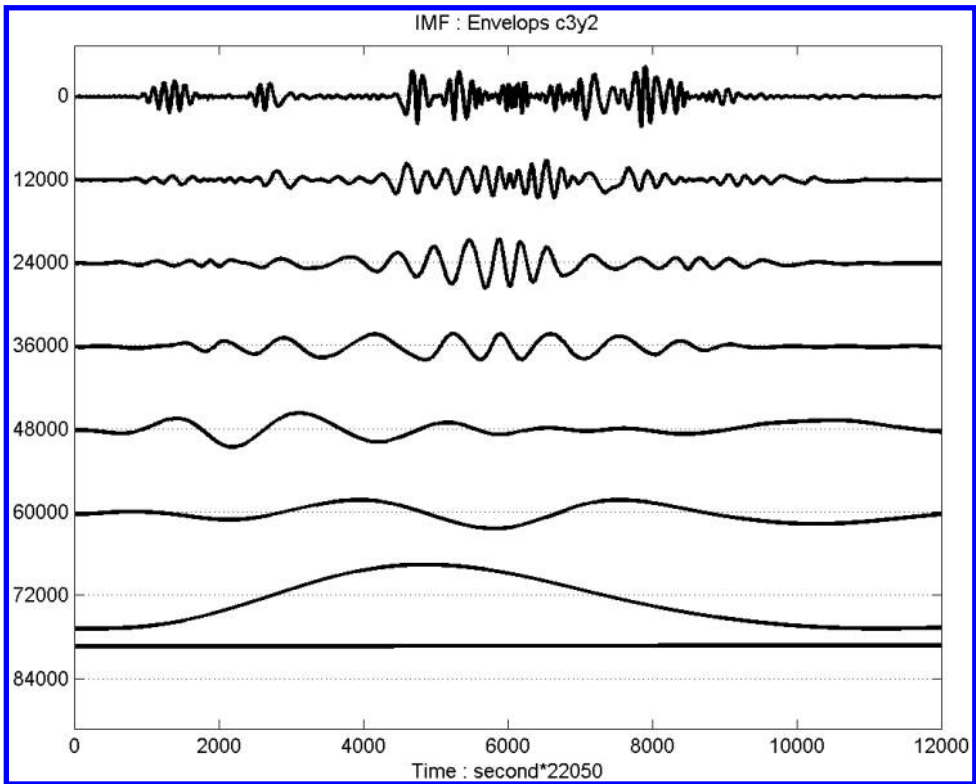


Fig. 34. The intrinsic mode functions for the AM signal of the data (given in Fig. 24), which is defined as the spline fitted envelope (given in Fig. 25).

This discussion gives us the reason to look for a new approach to extract additional information from the envelope. In the simple example given in Eq. (44), the envelope becomes a rectified cosine wave; the frequency would be easier to determine from the simple period counting than from the Hilbert spectral result. For a more general case, when the amplitudes of the two sinusoidal functions are not equal, the modulation is no longer so simple. For even more complicated cases, when there are more than two co-existing sinusoidal components with different amplitudes and frequencies, there is no general expression for the envelope and carrier anymore. The final result could be represented as more than one frequency-modulated band in the Hilbert spectrum. It is, then, impossible to describe the individual components under this situation. In these cases, to represent the signal as an FM and AM variation is more meaningful, for the dual representations of frequency arise from the different definitions of frequency: in the Hilbert view, amplitude and frequency modulations still render a correct representation of the signal, but this view is very different from that of the Fourier analysis. In such cases, if the process is indeed stationary and linear, Fourier analysis could be used, as suggested by Huang *et al.*,² which will give the more familiar results. In this case, the judgment for method

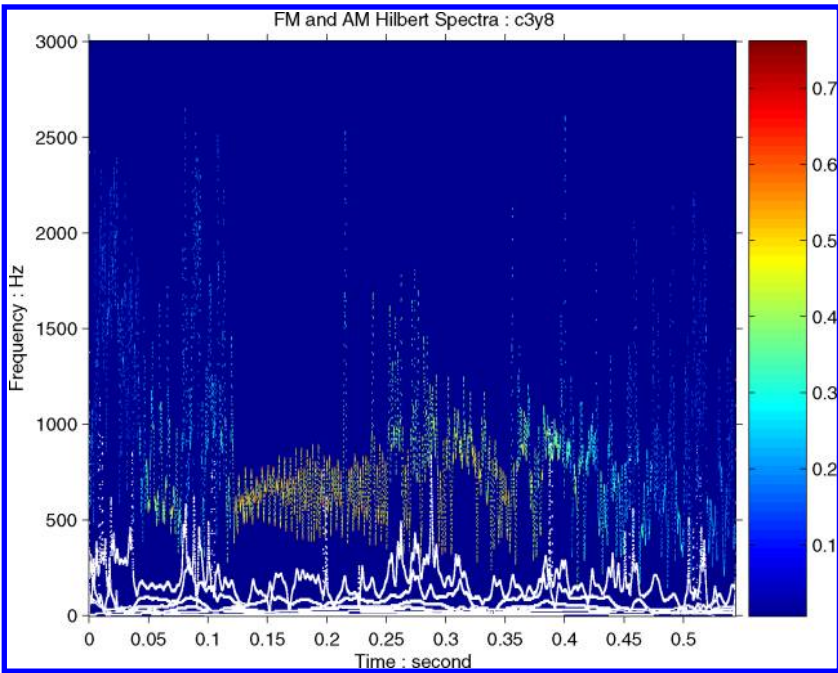


Fig. 35. The Hilbert spectrum for the AM signal (white contours) compared to the FM signal (given in color contours). The frequency content of the AM signal in general is lower than the FM counterpart.

selection is not based on which one is correct, for they both are; rather, it is on which one is more familiar and more revealing.

When we have a more complicated data as in the case of the speech signal, we can also find the amplitude variation information by processing the envelope or the AM part of the data. When the empiric envelope of the data given in Fig. 24 is decomposed through EMD, we have the IMF given in Fig. 34. Using these IMFs, we can construct the Hilbert spectrum of the AM part as given in Fig. 35, where the FM counterpart is also given as a reference. The physical meaning of the AM spectrum is not as clearly defined in this case as in the FM signal, for obvious intermittence is still present. The methodology introduced here, however, serves to indicate the AM contribution to the variability of the local frequency. In speech analysis, the AM spectrum would give additional information on the pause and the speech pattern. For most of the problems with AM and FM variations, the total information of the data is fully represented if we process both the AM and FM spectra.

6. Discussions

Having presented all different methods for computing the IF, we would like to emphasize that the IF is a very different concept from the frequency content of the

data derived from Fourier based methods as discussed in great details by Huang *et al.*² IF introduced here is based on the instantaneous variation of the phase function from the direct quadrature or AS through the HT on adaptively decomposed monocomponent functions, while the traditional Fourier-type frequency content is an averaged frequency based on integral transform of the data with an *a priori* basis. Therefore, when the basis is changed, the frequency content will also change. Similarly, when we change the decomposition method, the IF will also change. The IF and the integral transform computed mean frequency do not have the same physical meaning, and would not have an exact one-to-one correspondent relationship. Before getting into the discussions of the results from different methods presented here, a few words to dispel some of the common misconceptions (or paradoxes as given in, e.g., Ref. 13) on IF are necessary.

One of the most prevailing misconceptions of IF is that, for a data with discrete line spectrum, how can IF be a continuous function? A variation of this misconception is that IF can give frequency values that are not even one of the discrete spectral lines. Both of these paradoxes can be resolved easily. In the case of signals from nonlinear processes, IF methods treat the harmonic distortions as continuous intrawave frequency modulations; on the other hand, Fourier based methods treat the frequency content as discrete harmonic spectral lines. In case of a two or more beating waves, IF methods treat the data as AM and FM modulation, while Fourier based methods treat each constituent wave as a separate discrete spectral line. Although they appear perplexingly different, they are representing the same data from two different viewpoints. The discrete spectral lines are the product of Fourier analysis based on linear stationary assumption. While frequency defined as the derivative of phase function is mathematically and physically sound definition. There should be no mystery.

Another misconception is on the negative IF values from AS. Cohen¹³ again contended that, according to Gabor's approach,³² the AS is computed through two Fourier transforms: first transform the data into frequency space, then use an inverse Fourier transform after discarding all the negative frequency parts (see, e.g., Ref. 13). Therefore, because all the negative frequency content has been discarded, how can there still be negative IF values? This is a total misunderstanding of the nature of negative IF computed based on AS. The direct cause of negative frequency in AS is the consequence of multi-extrema between two zero-crossings, which will cause local loops in the complex phase plane not centered at the origin of the coordinate system as discussed by Huang *et al.*² Negative frequency values can also occur even if there is no multi-extrema but with large amplitude fluctuations, which can also make the AS phase loop miss the origin, a consequence of violating the Bedrosian theorem as discussed above. At any rate, the negative IF has nothing to do with Gabor's implementation of the AS computation; they are from the data violating known theoretic limitations, the Bedrosian theorem. The negative IF values can be successfully removed by the direct quadrature and NHT methods presented here, or by a window method to be discussed presently.

Next, we have to address a subtler question: how local is the IF, if the HT is evaluated over the real axis theoretically? Unfortunately, no theoretic value can be offered here. Empirically, however, the HT indeed gives a very local representation as discussed in Huang *et al.*² albeit with some distortion in both amplitude and frequency values within the integration window. DQ does not involve integration, so it should be extremely local. The HT is a singular integral with the integrand decaying as $1/t$, a very narrow window. Limiting the amplitude variation implies that anytime a $1/t$ decay of a large amplitude wave can overpower the neighboring low amplitude wave; there would be a violation of the Bedrosian theorem. Indeed we have tried a windowed approach where short data piecewisely limited the amplitude variation within the window to a certain preassigned value, and this has improved the answer. But this is only a patched remedy; it does not offer a true solution. Normalization is a way to resolve the difficult of amplitude variation. With these discussions, the common misconceptions could be settled easily.

Now, let us discuss the results obtained above. From these examples, we can see that most of the methods presented here perform reasonably well. Yet all the methods that produce IF can only be applied to the IMFs, a necessary condition not just for HT based methods, but also for all the additional methods discussed here. The strength and weakness of each method are summarized as follows:

TEO is by far a very local method, for it is totally based on differentiation operations. But, as it is based on and derived from a linear assumption; therefore, whenever there is pronounced nonlinear (harmonic) waveform distortion, the TEO result breaks down and gives zero for IF even after the five-point medium filtering. In the past, TEO is used in conjunction with band pass filtering of the signal. As the traditional filtering is based on the linear Fourier analysis, it has thus destroyed the nonlinear characteristics of the signal. Therefore, the problem of TEO with nonlinearity was not revealed, and the filtered data will give the impression of well-behaved IF from the TEO. The examples here, however, clearly indicate the problems of applying TEO to data from nonlinear processes. It should be emphasized that, though TEO involved some nonlinear manipulation of the data, it is not a nonlinear method as suggested by Quatieri,³⁸ for the method is based on a linear wave model, and can only be applied to data from such processes. The shortcoming of TEO can be turned into a useful tool: to detect nonlinear distortion of the waveform, as used by Huang *et al.*³⁶

Next, we will examine the GZC method. This method, based on the fundamental definition of the frequency, is physically the most direct but mathematically the least elegant. As the method relies only on the intervals of the critical points consisting of extrema and zero-crossings, it cannot resolve the subtlety of intrawave frequency modulation from nonlinear waveform distortions. Through different combinations of the intervals between the critical points, we can get a mean frequency and the standard deviation from the mean, but all the values are smoothed with a temporal resolution of a quarter wave period at most. The combination of this mean and standard deviation, however, offer the most stable and accurate overall

smoothed *mean* frequency over a whole or part of the wave. Because of this stable property, the IF determined from GZC has been used as a standard in the above comparisons, for no method can offer any IF totally different from this method in the mean and still be correct. For many engineering applications where information on the detailed waveform distortion is not of critical importance, the GZC should be the method of choice for its stability, directness, and simplicity.

The HT (based on AS from un-normalized data) is mathematically the most simple, elegant, and intuitively pleasing one. Yet in detailed examination, we find that the AS approach has certain limitations. Mathematically, the only signal that will give physically meaningful results has to be monocomponent and also obeys the limitation stipulated in the Bedrosian¹⁸ and Nuttall¹⁹ theorems. When the data violate these conditions, HT will give physically meaningless IF results. The problem is usually aggravated when the amplitude is locally small. The Bedrosian theorem also indicates that the HT is not perfectly local, for the frequency is not determined by the carrier wave only as discussed above. Furthermore, a more fundamental limitation on the AS approach is given by the Nuttall theorem, which states that not all analytic pairs from HTs are identical to the quadrature. The consequence is that the frequency would, in most cases, be only an approximation. Even with these limitations, our experience indicates that the results provided by the HT are consistently better than most of the other methods.

If we use the normalized data as in NHT, the results are drastically improved over the HT whenever the amplitude variation is large. When the amplitude varies gradually, the difference between NHT and HT is negligible. But when the amplitude is small and the variation large, NHT consistently gives more stable IF values. Therefore, NHT should be the preferred method, if AS is used. It satisfies the limitation set by the Bedrosian theorem, and offers a local measure of error sharper than the Nuttall theorem. As the normalization procedure is introduced specifically for ameliorating the difficulties of obtaining meaningful AS, the smooth spline fitting empirical AM-FM decomposing method offers the best results.

Finally, the direct quadrature as implemented here offers an easy and direct method to compute IF from any IMF. With the quadrature, we can eschew the HT generated AS; theoretically, all the problems associated with the AS and IF could be eliminated. Once the IMF is normalized properly, the quadrature can be computed without any integration or transform; it gives the exact local IF without any limitations and approximations. The IF is even more local than the TEO, for DQ depends only on the first derivative. It should be the method of choice. The occasional computational instability, however, is an annoying problem, which prevents the method to give the exact IF. Using a medium value filter could alleviate the problem, but the problem could not be totally eliminated by filters alone. This problem is especially severe when the sampling rate is low that leave the high frequency wave with a few data points per oscillation. Under such condition, local spline of the data to add additional points could improve the result. Otherwise, the

HT method actually performs better. Until the computation instability problem is resolved completely, HT is still very useful.

Ever since the introduction of the HMD and Hilbert spectral analysis method by Huang *et al.*,^{2,6,43} the method has attracted increasing attention. Recently, Flandrin *et al.*⁴⁴ and Wu and Huang⁴⁵ have established the EMD as a bank of dyadic filters. But Flandrin has refrained from using the HT based method to compute the IF. In our present analysis, we conclude that the caution of not invoking the AS through the HT is fully justified. The limitations imposed by Bedrosian and Nuttall, on using AS to compute a physically meaningful IF, certainly have a solid theoretic foundation. We hope the normalization procedure proposed here, the NHT and the DQ computations have removed the difficulties of computing IF for an adaptive time–frequency analysis.

Before ending this discussion, a few words have to be added in regard to the latest development of EMD, the ensemble approach or EEMD.⁴¹ In EEMD, noises are introduced to help eliminating mode mixing, which lead to a robust and physically meaningful decomposition. The resulting IMFs, however, would always contain contribution from noise. Theoretically, this noise contribution could be reduced to zero, but that would call for an ensemble of infinite many trials, a practical impossibility. As a result, all results from EEMD would contain noise of small but non-zero noise elements. Under such condition, the normalization scheme induced here would encounter some difficulties, for the small noise could introduce large undesirable overshooting in the spline fitting during the normalization processes. Our preliminary finding at this time is as follows. The best normalization scheme for EEMD results is to use the analytic signal through regular HT, in which the HT could effectively control the undesirable overshooting. The next best choices would be cubic Hermite spline, which also limits the overshooting, or even linear spline. The resulting IF would be less regular compared to the values produced by the analytic signal approach. This whole problem of computing IF for EEMD will be investigated thoroughly and reported in the future.

7. Conclusion

Based on extensive comparisons and detailed theoretic considerations of all the instantaneous or local frequency computations, it is determined that the DQ and NHT are the better methods to determine the IF. Depending on the applications, data characteristics and data analysis goals, one or the other of these methods can also provide quite satisfactory answers. But the DQ and NHT are clearly the best overall methods for determining the IF for nonlinear and nonstationary data.

With the introduction of the empirical AM–FM decomposition through spline fitted normalization procedure, we have removed one of the major obstacles for Hilbert spectral analysis and made a true time–frequency analysis feasible. Together with the establishment of the confidence limit^{41,46} through the variation of the stoppage criterion and EEMD,⁴¹ and the statistically significant test of

the information content for IMF^{44,45} the EMD and Hilbert spectral analysis has indeed taken another step toward being a mature time–frequency analysis method. Thus EMD has provided a solid base for time–frequency analysis empirically if not theoretically.

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