Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} \cdots\cdots$$

Ex $\zeta(2) = \frac{\pi^2}{6}$; $\zeta(4) = \frac{\pi^4}{90}$;

$\zeta(s)$ is defined for only $s \geq 1$; s can be complex $a+ib$

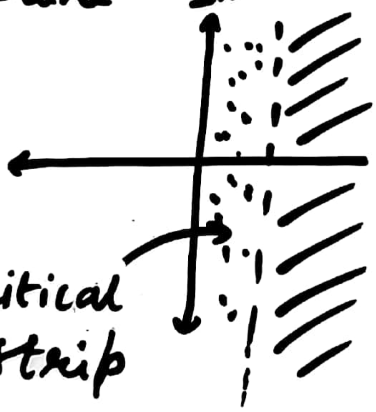
Ex $s = 2+i$

$$\zeta(2+i) = \frac{1}{1^{2+i}} + \frac{1}{2^{2+i}} + \frac{1}{3^{2+i}} + \cdots\cdots$$

$$\left(\frac{1}{2}\right)^{2+i} = \underbrace{\left(\frac{1}{2}\right)^2}_{\text{Size}} \cdot \underbrace{\left(\frac{1}{2}\right)^i}_{\text{rotation}}$$

s-plane

Im



Re

Critical strip

$Re(s) \geq 1$

$\zeta(s)$ is analytic function on RHP → Angle preserving

- Analytic continuation of $\zeta(s)$ in LHP

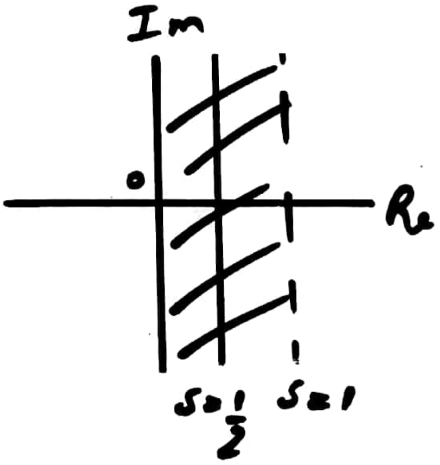
$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s)$$

For what s, $\zeta(s) = 0$? ⟹ −ve even no. gets mapped to zero → Trivial solution/zero

Rest of the zeroes lies in critical strip

Riemann Hypothesis:

All of non trivial zeroes of $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ lies in the middle of critical strip



All zeroes lies on $s = \frac{1}{2}$

↳ critical line

Zeta fnc $\zeta(s)$ can also be defined as:

$$\zeta(s) = \frac{1}{\underbrace{\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\cdots}_{\text{All primes}}}$$

$s = \overset{1/2}{x} + iy$

first zero

$\zeta\left(\frac{1}{2} + 14.134725i\right) = 0$

Riemann zeta fnc encodes the information about the prime numbers.

We know that 10^{36} th zero lies on $s = 1/2$

Alan turing used the first electronic computer to calculate the zeroes of $\zeta(s)$

$$\zeta(s) = \frac{1}{\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\cdots}$$

L - function

Keep $1-\frac{1}{2^s}$ aside ; & divide other primes by 4 & we get remainder either 1 or 3. The terms where remainder is 3 lets change their sign from -ve to +ve

$$L(s) = \frac{1}{\left(1+\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1+\frac{1}{7^s}\right)\left(1+\frac{1}{11^s}\right)\left(1-\frac{1}{13^s}\right)}$$

L-func can be written as sum of integers, product of primes, they have a symmetry line. * They have a Riemann hypothesis.

Ramajun:

$$x(1-x)^{24}(1-x^2)^{24}(1-x^3)^{24}(1-x^4)^{24}\cdots$$
$$= 1.x - 24x^2 + 252x^3 - 1472x^4 + 4830x^5 - 6048x^6 \cdots$$

$-24 \times 252 = -6048$

$2 \times 3 = 6$

also $3 \times 5 = 15$

& Coeff of x^{15} is $252 \times 4830 = 1217160$

$$1.x - 24x^2 + 252x^3 - 1472x^4 + 4830x^5 - 6048x^6 \ldots..$$

Can be written as L-fnc

$$1 - \frac{24}{2^s} + \frac{252}{3^s} - \frac{1472}{4^s} + \frac{4836}{5^6} \ldots.$$

Functions of this type are Called Modular Forms

→ They have symmetry properties

Andrew wile used connection b/w Modular Forms & L-fncs to prove Fermat's Last Theorem

$a^n + b^n = c^n$; No integal solutions for $n \geq 3$

For all the L-fncs, it is believed that there is Riemann Hypothesis. Can we find any patterns which gives us clue about why Riemann hypothesis is true?

www.lmfdb.org has database of millions of L-fncs.

Can you find that pattern?

Prize $ 1000000 by Clay Math Institute.