Testing the Estimated Noise Sequence

After transforming a time series to remove trends and seasonality, the resulting residuals must be rigorously evaluated to determine if they exhibit properties of independent and identically distributed (iid) random variables. If the residuals show no significant dependence, they can be treated as unstructured noise, requiring only estimation of their mean and variance. However, if dependence is detected, further modeling of the residual structure such as autoregressive or moving average components, becomes necessary to leverage historical patterns for improved forecasting. Statistical tests for this purpose include analyzing autocorrelation patterns to identify serial dependence, evaluating fluctuations via **turning point counts** to detect excess volatility or rigidity, and applying trend-specific tests like **difference-sign or rank tests** to uncover monotonic or linear trends. **Normality checks**, such as quantile-quantile plots or squared correlation tests, assess whether residuals conform to a Gaussian distribution. If these tests collectively reject the iid hypothesis, the residuals warrant modeling as a stationary process to account for their inherent dependence, enabling more accurate predictions. Conversely, confirmation of iid behavior simplifies the analysis, indicating no further temporal structure to exploit.

Basic Properties of Stationary Processes

Non-Negative Definite Functions

A function $\kappa : \mathbb{Z} \to \mathbb{R}$ is nonnegative definite if for every positive integer n and every real vector $a = (a_1, \cdots, a_n)'$, the quadratic form satisfies:

$$\sum_{i,j=1}^{n} a_i \kappa(i-j) a_j \geq 0$$

This ensures that the matrix $[\kappa(i-j)]_{i,j=1}^{n}$ is positive semi-definite for all n.

Autocovariance Characterization

A real-valued function $\gamma : \mathbb{Z} \to \mathbb{R}$ is the autocovariance function of a stationary time series if and only if it satisfies:

1) Evenness: $\gamma(h) = \gamma(-h) \forall h \in \mathbb{Z}$.

2) Nonnegative Definiteness: γ is nonnegative definite.

Remarks:

1) $\rho(\cdot)$ is the ACF of a stationary process if and only if it is a normalized ACVF (i.e., $\rho(\cdot)$ is even, nonnegative definite, and $\rho(0) = 1$.

2) To verify that a given function is nonnegative definite it is often simpler to find a stationary process that has the given function as its ACVF.

Properties of Strictly Stationary Series $\{X_t\}$

1) The random variables X_t are identically distributed.

2) For all integers t and h, the joint distribution satisfies:

$$(X_t, X_{t+h}) \stackrel{d}{=} (X_1, X_{1+h})$$

3) **Weak Stationarity Under Finite Variance:** If $E(X_t^2) < \infty$ for all t, the series is weakly stationary.

4) Weak stationarity does not imply strict stationarity.

5) An independent and identically distributed **(iid)** sequence is strictly stationary.

Weak stationarity preserves means and covariances under time shifts while **strict stationarity** preserves entire joint distributions under time shifts.

Constructing Strictly Stationary Time Series via Filtering:

To build a strictly stationary time series $\{X_t\}$, one can apply a filter to an iid sequence $\{Z_t\}$. Define:

$$X_t = g(Z_t, Z_{t-1}, \cdots, Z_{t-q})$$

where g is a fixed real-valued function. Since $\{Z_t\}$ is iid (and thus strictly stationary), shifting the time indices by any h preserves the distribution of $\{X_t\}$, ensuring strict stationarity. We can consider following properties as well:

1) $\{X_t\}$ is q-Dependence, when observations X_s and X_t are independent if $| t-s |> q$. As then both of the X_t and X_s won't come in the definition of each other defined through $g(\cdot)$.

2) A stationary series is q-correlated if its autocovariance $\gamma(h) = 0$ for $| h |> q$.

3) Every q-correlated process can be represented as a **moving-average process of order** q **(MA(q))**.

A white noise is 0-correlated.

The MA(q) Process

A **moving-average process** of order q (MA(q)) is defined as:

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ (white noise with mean 0 and variance σ^2) and $\theta_1, \cdots, \theta_q$ are constants. This model expresses X_t as a weighted sum of the current and past q noise terms. **An MA(q) process is a linear combination of current and past white noise terms.**

Proposition 1:

If a stationary time series $\{X_t\}$ has mean zero and is q-correlated (i.e., its autocovariance $\gamma(h) = 0$ for all $\mid h \mid > q$), then it can be represented as an MA(q) process.

Every stationary q-correlated series with mean zero is structurally equivalent to an MA(q) model. This allows modeling such series using the MA(q) framework, simplifying forecasting and analysis.

Linear Processes

A time series $\{X_t\}$ is termed a linear process if it can be expressed as a doubly infinite weighted sum of white noise terms:

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

where $\{\psi_j\}$ are coefficients satisfying $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$.

Properties of Linear Processes

1.) Using the operator B (where $B^k Z_t = Z_{t-k}$), the process is compactly written as:

$$X_t = \psi(B) Z_t, \quad \textbf{with} \;\; \psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$$

This operator $\psi(B)$ can be thought of as a **linear filter** as well.