The MA(q) Process

A **moving-average process** of order q (MA(q)) is defined as:

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ (white noise with mean 0 and variance σ^2) and $\theta_1, \cdots, \theta_q$ are constants. This model expresses X_t as a weighted sum of the current and past q noise terms. **An MA(q) process is a linear combination of current and past white noise terms.**

Proposition 1:

If a stationary time series $\{X_t\}$ has mean zero and is q-correlated (i.e., its autocovariance $\gamma(h) = 0$ for all $\mid h \mid > q$), then it can be represented as an MA(q) process.

Every stationary q-correlated series with mean zero is structurally equivalent to an MA(q) model. This allows modeling such series using the MA(q) framework, simplifying forecasting and analysis.

Linear Processes

A time series $\{X_t\}$ is termed a linear process if it can be expressed as a doubly infinite weighted sum of white noise terms:

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

where $\{\psi_j\}$ are coefficients satisfying $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$.

Properties of Linear Processes

1.) Using the operator B (where $B^k Z_t = Z_{t-k}$), the process is compactly written as:

$$X_t = \psi(B) Z_t, \quad \textbf{with} \;\; \psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$$

This operator $\psi(B)$ can be thought of as a **linear filter** as well.

2.) If $\psi_j = 0$ for all $j < 0$, the process becomes a moving average of infinite order (MA(∞)):

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

Since, $\{Z_t\}$ comes from white noise, we can see the convergence conditions fulfilled for absolute summability and square summability. $\sum |\psi_j| < \infty$ and $\sum \psi_j^2 < \infty$ ensures the series converges almost surely and in mean square (as $E|Z_t| < \sigma \implies E|X_t| < \infty$).

Proposition 2

Let $\{Y_t\}$ be a stationary time series with mean 0 and autocovariance function γ_Y. If the coefficients $\{\psi_j\}$ satisfy $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, then the filtered series:

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j} = \psi(B) Y_t$$

is also stationary with mean 0 and autocovariance function:

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h+k-j)$$

Special Case (Linear Process): If $\{Y_t\}$ is white noise ($\{Z_t\} \sim WN(0, \sigma^2)$, the autocovariance simplifies to:

$$\gamma_X(h) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}$$

Proof:

It is given to us that $\{Y_t\}$ is stationary, so $E[Y_t] \leq \sqrt{\gamma_Y(0)}$ and also the coefficients satisfy: $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ so

$$E|X_t| = E|\sum_{j=-\infty}^{\infty} \psi_j Y_{t-j}| \leq \sum_{j=-\infty}^{\infty} |\psi_j||E[Y_{t-j}| \leq \left(\sum_{j=-\infty}^{\infty} |\psi_j|\right) \sqrt{\gamma_Y(0)} < \infty$$

So, this ensures that the infinite sum in X_t converges with probability 1. Since $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$ will also hold, this will ensure that the series converges in mean square. We can also say that X_t is infact the mean square limit of the partial sums $\sum_{j=-\infty}^{\infty} \psi_j Y_{t-j}$.

Since $E[Y_t] = 0$, so

$$E[X_t] = E[\sum_{j=-\infty}^{\infty} \psi_j Y_{t-j}] = \sum_{j=-\infty}^{\infty} \psi_j E[Y_{t-j}] = 0$$

Since $E[X_{t+h}] = E[X_t] = 0$, the autocovariance $\gamma_X(h) = Cov(X_{t+h}, X_t)$ is computed as:

$$\gamma_X(h) = E\left[\left(\sum_{j=-\infty}^{\infty} \psi_j Y_{t+h-j}\right), \left(\sum_{k=-\infty}^{\infty} \psi_k Y_{t-k}\right)\right]$$

$$= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k E(Y_{t+h-j}, Y_{t-k})$$

Since autocovariance of $\{Y_t\}$ is γ_Y, $E(Y_{t+h-j}, Y_{t-k}) = \gamma_Y(h - j + k)$ and so:

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h - j + k)$$

The autocovariance of $\{X_t\}$ depends only on the lag h, confirming stationarity.

Now, for **special case**:

If $\{Y_t\}$ is white noise ($\{Y_t\} \sim WN(0, \sigma^2)$, then $\gamma_Y(h - j + k) = \sigma^2$, if $k = j - h$ and 0 otherwise.

So by symmetry of the summation over all the integers:

$$\gamma_X(h) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}$$

Filtering a stationary series with an absolutely summable linear operator preserves stationarity.The autocovariance of the output is a convolution of the filter weights and input autocovariance.

Remark:

When applying linear filters to a stationary time series $\{Y_t\}$, the combined effect of two filters can be expressed as a single equivalent filter. Let$\alpha(B) = \sum_{j=-\infty}^{\infty} \alpha_j B^j$ and $\beta(B) = \sum_{j=-\infty}^{\infty} \beta_j B^j$ be linear filters where $\sum |\alpha_j| < \infty$ and

$\sum |\beta_j| < \infty$. Applying $\alpha(B)$ and $\beta(B)$ successively to $\{Y_t\}$ generates a new stationary series:

$$W_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j}$$

where the coefficients ψ_j are the convolution of α_j and β_j:

$$\psi_j = \sum_{k=-\infty}^{\infty} \alpha_k \beta_{j-k} = \sum_{k=-\infty}^{\infty} \beta_k \alpha_{j-k}$$

The combined filter is equivalently written as:

$$\psi(B) = \alpha(B)\beta(B) = \beta(B)\alpha(B)$$

Combining linear filters via convolution preserves stationarity,and the order of filtering is irrelevant due to commutativity.

AR(1) Process

We define AR(1) process as a solution of:

$$X_t - \phi X_{t-1} = Z_t, \quad \{Z_t\} \sim WN(0, \sigma^2) \tag{1}$$

We will now see for what value of ϕ, we can expect a stationary solution of Equation 1.

Case $|\phi| < 1$**:** On iterating the AR(1) equation backward will yield:

$$X_t = Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \cdots = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$$

The coefficients ϕ^j decay geometrically (since $|\phi| < 1$), ensuring the series converges absolutely and in mean square. Using proposition 2, $\{X_t\}$ is stationary with mean 0 and autocovariance:

$$\gamma_X(h) = \sum_{j=0}^{\infty} \phi^j \phi^{j+h} \sigma^2 = \frac{\sigma^2 \phi^{|h|}}{1 - \phi^2}, \quad h \in \mathbb{Z}$$

Let $\{Y_t\}$ be any stationary solution, then:

$$Y_t = \phi Y_{t-1} + Z_t = Z_t + \phi Z_{t-1} + \phi^2 Y_{t-2}$$

$$= Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \phi^3 Y_{t-3}$$

$$= \sum_{j=0}^{k} \phi^j Z_{t-j} + \phi^{k+1} Y_{t-k-1}$$

$$Y_t - \sum_{j=0}^{k} \phi^j Z_{t-j} = \phi^k Y_{t-k-1}$$

$$E[Y_t - \sum_{j=0}^{k} \phi^j Z_{t-j}]^2 = E[\phi^k Y_{t-k-1}]^2$$

Since Y_t is stationary, hence $E[Y]^2$ is finite and independent of t, so

$$E[Y_t - \sum_{j=0}^{k} \phi^j Z_{t-j}]^2 = E[\phi^k Y_{t-k-1}]^2 \to 0 \ \textbf{as} \ k \to 0$$

Hence, Y_t is equal to the mean square limit $\sum_{j=0}^{\infty} \phi^j Z_{t-j}$ and hence $X_t \equiv Y_t$ and so, the unique stationary solution of the equation Equation 1 for $|\phi| < 1$ is given by:

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$$

Case $|\phi| > 1$:

Rewriting the equation Equation 1 as:

$$X_t = \phi^{-1} Z_{t+1} + \phi^{-1} X_{t+1}$$

Iterating forward will give:

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}$$

Coefficients ϕ^{-j} decay as $j \to \infty$ (since $|\phi| > 1$), ensuring convergence. The solution is stationary but not physically meaningful for forecasting or modeling because X_t defined here depends on the future values of noise. It is customary therefore in modeling stationary time series to restrict attention to AR(1) processes with $|\phi| < 1$.

Case $\phi = \pm 1$:

Equation Equation 1 will become:

$$X_t - (\pm 1)X_{t-1} = Z_t$$

Iterating the equation will lead to cumulative noise terms:

$$X_t = X_0 + \sum_{j=1}^{t} Z_j$$

resulting in unbounded variance as $t \to \infty$.

AR(1) processes are stationary and causal only if $|\phi| < 1$. For $|\phi| > 1$, solutions exist but are non-causal and impractical. No stationary solution exists if $|\phi| = 1$.

Remark

We can also derive the stationary solution using operator algebra:

For $|\phi| < 1$ and backward shift operator B:

$$\sum_{j=0}^{\infty} \phi^j B^j = \frac{1}{1 - \phi B}$$

So, if we set: \$ (B) = 1- B and $\pi(B) = \sum_{j=0}^{\infty} \phi^j B^j$, then

$$\phi(B)\pi(B) = 1$$

Now, using backward shift operator B in Equation 1 ,

$$\phi(B)X_t = (1 - \phi B)X_t = Z_t$$

applying $\pi(B)$ both sides:

$$\pi(B)\phi(B)X_t = \pi(B)Z_t$$

$$X_t = \pi(B)Z_t = \sum_{j=0}^{\infty} \phi^j B^j Z_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$$

This method **leverages the invertibility** of $\phi(B)$ (guaranteed when $|\phi| < 1$) to directly express X_t as a causal linear process of past and present white noise terms.