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LAA :

Unit 1 :-

Consistency, rank & Solutions :-

rank (A) = rank (A|b) = r : consistent 1/∞ sol^n

rank(A) = rank (A|b) : $\boxed{n = r}$ consistent 1 solution

rank(A) = rank (A|b) : $\boxed{r < n}$: consistent ∞ sol^n

rank(A) ≠ rank(A|b) : inconsistent × sol^n

Gaussian Elimination :

- Find echelon form & check for conditions & find sol^n.

→ Breakdown of elimina^n : pivots become zero

⇒ Try to switch rows & then solve

Elementary Matrices :

$$R_2 \rightarrow R_2 - \frac{x}{y} R_1$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{x}{y} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{-\frac{x}{y}}$: multiplier

Triangular Factorisation :

→ LU :-

L : All transformations together

Ex: $R_2 \rightarrow R_2 - \frac{x}{y} R_1$, $\boxed{E_{21} \rightarrow -\frac{x}{y}}$

$\boxed{l_{21} \rightarrow \frac{x}{y}}$

$\boxed{Lz = b}$

find using E

solve for z

given

Then equate $\boxed{Ux = z}$ & solve for x now

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→ LDU :-

Square matrix.

L : from E → lower triangular matrix.

D : diagonal elements as pivots

U : echelon form but make pivots/diags. = 1

In case of zero valued pivots,

- exchange rows
- multiply the RHS by P^T or $PA = LU / LDU$

P is the corresponding permutation matrix

$P^T = P^{-1}$ & if $P_{n \times n}$ then $(n!)$ matrices possible.

PA : A but with rows exchanged.

Gauss Jordans Method:

$[A : I] \longrightarrow [I : A^{-1}]$

- find echelon form
- convert the upper triangle to 0 to form a diagonal matrix
- Finally make diagonals one by dividing each row.

LAA:

Unit 2:

Speacial Solutions :-

- Find echelon form, swap rows if needed.
- Find pivot variables, free variables

Find value of pov. in terms of free vars.

Ex: if: x = p.v ; y, z, t = fv. (f.v)

then, $x = -y - 2z - t$

$$\begin{bmatrix} -y-2z-t \\ y \\ z \\ t \end{bmatrix} = y\begin{bmatrix} \\ \end{bmatrix} + z\begin{bmatrix} \\ \end{bmatrix} + t\begin{bmatrix} \\ \end{bmatrix}$$

p.v : the vars. corresponding to columns that have a pivot.

f.v : the vars. that correspond to the cols. w/o a pivot.

Independence of vectors:

→ Vectors are independent if "null space = zero vector"

→ pivot exists in every column

∴ rank (A) = n

- Columns of invertible square matrix : always independent
- $A_{m \times n}$ if $m < n$ (rows < columns) : always dependent

Basis:

- Dimension of the vector space = no. of basis vectors

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rank nullity theorem →

$$\boxed{\dim(C(A)) + \dim(N(A)) = n}$$

$$\boxed{\dim(C(A^T)) + \dim(N(A^T)) = m}$$

4 Fundamental Subspaces:-

Matrix: $A_{m \times n}$

1) $C(A)$: col. space
 → subspace of $\mathbb{R}^m$ → combin^n of cols of A
 → $\text{rank}(A) = \dim(C(A))$
 → basis : columns having pivots

2) $C(A^T)$: row space
 → subspace of $\mathbb{R}^m$ → combin^n of rows of A
 → ~~rank~~ $\text{rank}(A^T) = \dim(C(A^T))$
 → basis is set of rows having pivots.

3) $N(A)$: null space
 → subspace of $\mathbb{R}^n$
 → $A \cdot x = 0$ all sol^{n}s of this
 → if $\text{rank}(A) = k$ then $\underbrace{\dim(N(A))}_{\text{NULLITY}} = n - k$

4) $N(A^T)$: left null space
 → $A^T x = 0$ → subspace of $\mathbb{R}^m$
 → $\dim(N(A^T)) = m - k$
 → linear comb. of rows that gives 0

Inverse:

Right inverse : $C = A^T(AA^T)^{-1}$
(~~Step~~ horizontal matrix)

Left inverse : $B = (A^TA)^{-1}A^T$
(verticle matrix)

LAA :-

Unit 3

Rotation :-

$Q_\theta : \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ rotate by θ anticlockwise.

Projection:- (onto a θ line)

$P = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$; → singular
→ non-invertible ($|P| = 0$)

$P^n = P$

$P^T = P$

Reflection :-

$H = 2P - I$

$|H| = -1$; non singular, invertible.

$H^{2n} = I$

Rule of linearity:

→ $T(cx + dy) = c\,T(x) + d\,T(y)$

→ Preserves origin

Polynomial space:

$P_n = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \dots c_n t^n$

Basis: $[1 + t^2 \dots t^n]$

dim. = n+1

Differentiaⁿ:

$P_{n+1} \to P_n$

$C(A)$ = all of P_n

$N(A) = P_0$ (1D space of constants)

Integration:

$P_n \to P_{n+1}$

$C(A)$: subspace of P_{n+1}

$N(A)$: $Z / \vec{0}$

Orthogonal Complements : V & W $\Rightarrow$ $\dim(V) + \dim(W) = n$

Orthonormal : $\|a\| = 1$

Schwarz inequality : $|a^T b| \leq \|a\| \, \|b\|$

diff. matrix : $P_3 \to P_2$ $A_{n \times n+1}$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & 4 \end{bmatrix}$$

intg. matrix : $P_2 \to P_3$ $A_{n+1 \times n}$

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1/2 & 0 & \cdots \\ 0 & 0 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

☆ ◦ find $Ax - b$
◦ find norm.
◦ diff wrt each vale in x
◦ solve the $\hat{a}$'s

OR ◦ $A^T A x = A^T b$
◦ And multiply & find the values of x.

Proj : $P = A\hat{x} = b$
b is in the col. space of A.

Orthogonal vectors :

$a^T b = 0$ (inner product) $\langle a, b \rangle$

$\|a\|^2 = a^T a$

$\langle a, b \rangle > 0$: acute
< 0 : obtuse
$= 0$: orthogonal ($90°$)

$\Big\}$ $\langle a, b \rangle = \cos\theta$

Orthogonal Subspaces: $\dim(V) = n$

S & $T \in V \Rightarrow \dim(S) + \dim(T) \leq n$

$x \in S$ $y \in T$ $\Rightarrow x^T y = 0$

$C(A^T)$ & $N(A)$: orthogonal in R^n

$C(A)$ & $N(A^T)$: " in R^m

Projections: $P^T = P$

$$p = a\hat{x} = a \cdot \frac{a^T b}{a^T a} = \underbrace{a(a^T a)^{-1} a^T}_{P \text{ : proj. matrix}} . b$$

$C(P)$: line through a, $r(P) = 1$

☆ Least Square Fit : $\hat{x} = \dfrac{a^T b}{a^T a}$

$\|E\|^2 = \|Ax - b\|^2$ solve.

$A^T A x = A^T b$
normal eq^n

LAA:-

Unit 4:

Orthogonal Matrix:- Matrix w/ "orthonormal" columns (Q)

$m \geq n$; Orthogonal if $m = n$

$Q^T Q = I$ (square or rect. matrix)

$Q^T = Q^{-1}$ (if square matrix)

$\|Qx\| = \|x\|$ (length preserved)

$(Qx)^T (Qy) = x^T y$ ($<x, y>$ & angles preserved)

$q_1^T b = x_1 \, q_1^T q_1$

$$x_1 = \frac{q_1^T b}{\underbrace{q_1^T q_1}_{1}} = q_1^T b$$

$x_n = q_n^T b$: projection of b onto q_n

$$\boxed{x = Q^T b}$$

GS Process:-

$q_1 = \frac{a}{\|a\|}$ $q_2 = \frac{B}{\|B\|}$; $B = b - (q_1^T b) q_1$

$q_3 = \frac{C}{\|C\|}$; $C = c - (q_1^T c) q_1 - (q_2^T c) q_2$

$Q = [q_1 \; q_2 \; q_3]$

QR Factorisation:

$Q_{m \times n} = [q_1 \; q_2 \; \ldots \; q_n]$ $R_{n \times n} = \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix}$

upto $q_m^T a_n$

Least squares for inconsistent systems:-

Replace A by $QR \Rightarrow Rx = Q^T b$

$R = Q^T A$ ↳ given

Q^T: can find from A; b: given

Eigen Values & vectors:-

- For a vector → value unique, ~~←~~
- $\lambda(A) = \lambda(A^T)$
- $\lambda(A^2) = \lambda^2(A) \Rightarrow \lambda(A^n) = \lambda^n(A)$ ← vectors are same for A & A^2
- λ of idempotent matrix (ie $A = A^2 = A^n$) : 0 or 1
- $Tr(A) = \Sigma \lambda_i$
- $|A| = \prod \lambda_i$
- λ(diag. or triangular matrix) = diag. of matrix
- λ(orthogonal matrix) $\Rightarrow \frac{1}{\lambda}$ is also an eigen value

Diagonalisation:

$AS = S\Lambda$

$A = S\Lambda S^{-1}$ / $\Lambda = S^{-1}AS$

CH Theorem for very big n → $A^n = S\Lambda^n S^{-1}$; $A^n \to 0$ as $n \to \infty$ if $|\lambda_i| < 1$

S: matrix of eigen vectors

Λ: diagonal matrix with λs as elements

All λ_i must be unique only then matrix A is diagonalisable

OR else for A^{-1} → get equation in λ, replace λ with A

Ex: $\lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$

$A^3 - 7A^2 + 15A - 9I = 0$

$9I = -A^3 + 7A^2 + 15A$

$9A^{-1} = -A^2 + 7A + 15I$

↓ can calc. manually.

LAA:- Unit 5 Symmetric Matrices.

$A = S \Lambda S^{-1} = Q \Lambda Q^{-1} = Q \Lambda Q^{T}$ ($\because Q^T = Q^{-1}$)

$\because$ A is symm. & $\therefore$ square

Positive definiteness:-

- all pivots > 0
- all det. & det. of sub matrices > 0   always. $A_{11} > 0$
- all $\lambda > 0$
- sum of square terms : $a > 0$, $\frac{ac - b^2}{a} > 0$
- $x^T A x > 0$

$A = LDL^T$ (LDU) from

↳ Cholesky Decomposi°n

$A = R^T R$

$R = \sqrt{D} L^T$

$A = Q \Lambda Q^T$

$R = \sqrt{\Lambda} Q^T$

→ find U, find L & L^T, find D

Find $L^T x$ & $(L^T x)^T$

$(L^T x)^T D (L^T x)$

SVD:-

$A_{m\times n} = U_{m\times m} \Sigma_{m\times n} V^T_{n\times n}$

U & V : orthogonal (orthonormal)

Σ : diag. w/ $\sqrt{\text{eigen values}}$ singular values.

→ start from highest eigen value.

→ • horizontal × vertical

• then find λ & vectors

if A: horizontal then matrix from vectors is U (should be normalised)

• Then find Σ : same dim. as A

• Then we find V ($v_i = \frac{A^T u_i}{\sqrt{\lambda_i}}$) after normalising

Then for last v_i → $\begin{bmatrix} v_1 & - \\ -v_2 & - \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → v_3

$\begin{bmatrix} - v_1 - \\ - v_2 - \end{bmatrix}$ → convert to echelon & then solve normally. to get x, y, z

Finally. $V^T = \begin{bmatrix} - v_1 - \\ - v_2 - \\ - v_3 - \end{bmatrix}$ ~~$\therefore V^T = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$~~

if A is vertical matrix

we get V from eigen vectors & hence V^T
so we find U_8 using

$$U_i = \frac{A v_i}{\sqrt{\lambda_i}}$$

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If one of the λ is 0 then we use orthogonality & GS process to calc. the rest of U_2 & U_3

$AV = U\Sigma$

If A is square matrix then $A = Q \Lambda Q^T$ ← orthonormal.

- Check if A is positive definite.
  - YES
  - NO → Use normal procedure.

If A:
Symmetric: UDU^T (U: eig vectors; D: λ values.)

Covariance matrix:
- find mean (M): add all matrices divide by N
- B = sub. M from all vectors. combined
- find B^T
- $\boxed{S = \frac{1}{N-1} B \cdot B^T}$

from SVD :- U & V give fundamental spaces

C(A) : first r columns of U
$C(A^T)$: first r columns of V
$N(A^T)$: last m−r " of U
N(A) : last n−r " of V