

On the Mechanical Advantage of Compliant Mechanisms

M2D2 Group Meeting

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Overview

1. Introduction

2. Mechanical advantage based on kinetoelastics

- 2.1 For large deformations
- 2.2 For small deformations

Outline

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Introduction

Compliant mechanisms are defined as those that gain some or all of their mobility from the flexibility of their members.

For a single-input single-output compliant mechanism,

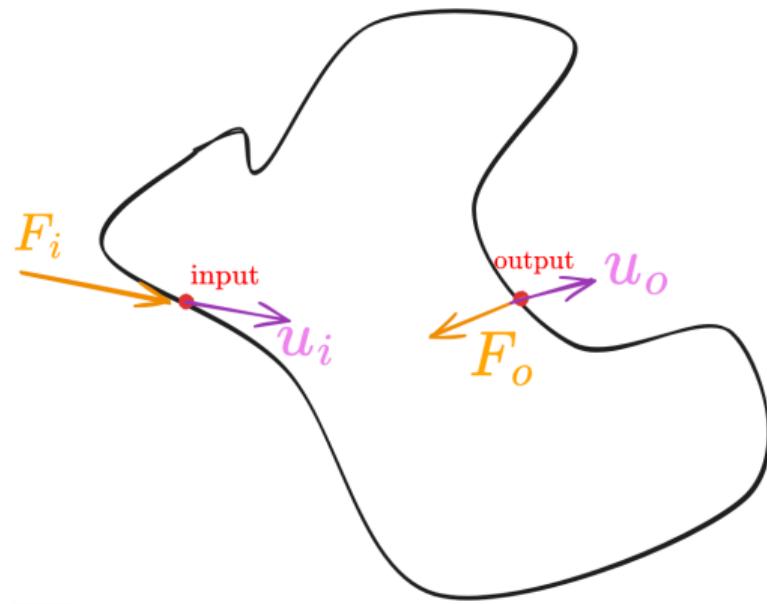
- **Displacement advantage** (DA) refers to the ratio of the displacement of the designated output and input degrees of freedom,
- **Mechanical advantage** (MA) refers to the ratio of force experienced along the designated output and input degrees of freedom.

$$DA = \frac{u_o}{u_i},$$

$$MA = \frac{f_o}{f_i}$$

Introduction

We are interested in mechanisms which perform work on the surroundings, i.e. output force and output displacement are considered in opposite directions.



Revisiting MA in compliant mechanisms

$$MA = \frac{f_0}{f_i} = \frac{1}{\Delta d_o} \left(\Delta d_i - \frac{\Delta U}{f_i} \right)$$

$$MA = \frac{f_0}{f_i} = \frac{\Delta d_i}{\Delta d_o} - \frac{\Delta U}{f_i \Delta d_o} = MA_r - MA_c$$

Excerpt from Salamon and Midha (1998)*: “*This general relation is valid for any mechanism, compliant or otherwise, provided it has a single-input and a single-output port*”

*B. A. Salamon and A. Midha. “An Introduction to Mechanical Advantage in Compliant Mechanisms”. In: *Journal of Mechanical Design* 120.2 (June 1998), pp. 311–315.

Revisiting MA in compliant mechanisms

$$MA = \frac{F_0}{F_i} = \frac{\Delta d_i}{\Delta d_o} - \frac{\Delta U}{F_i \Delta d_o} = MA_r - MA_c$$

Excerpt from Salamon and Midha (1998):

"The first term in Eq. (14) takes the form of a rigid-body mechanical advantage. This term would result if an instant center analysis for mechanical advantage (Shigley and Uicker, 1980) could be applied to the compliant mechanism in any instantaneous position. It would be a function of several parameters including those defining the original mechanism geometry as well as the externally applied loads. The effective link lengths thus change with the load, and the "rigid-body" mechanical advantage of the compliant mechanism (MA_r) cannot be represented by a single rigid-body counterpart for the entire range of operation of the compliant mechanism."

Revisiting MA in compliant mechanisms

$$MA = \frac{f_0}{f_i} = \frac{\Delta d_i}{\Delta d_o} - \frac{\Delta U}{f_i \Delta d_o} = MA_r - MA_c$$

"The second term in Eq. (14) also resembles a mechanical advantage term. It is referred to as the compliant component of the mechanical advantage (MA_c), and it accounts for the energy stored in the mechanism. The single-input and single-output port compliant mechanism may be considered to have two output ports, the actual physical output port and an internal port which performs work by elastically deforming the mechanism members. The mechanical advantage is thus maximized at a given instant when the compliant component of mechanical advantage (MA_c) becomes zero. When this occurs, the compliant mechanism behaves identically as a representative rigidbody mechanism."

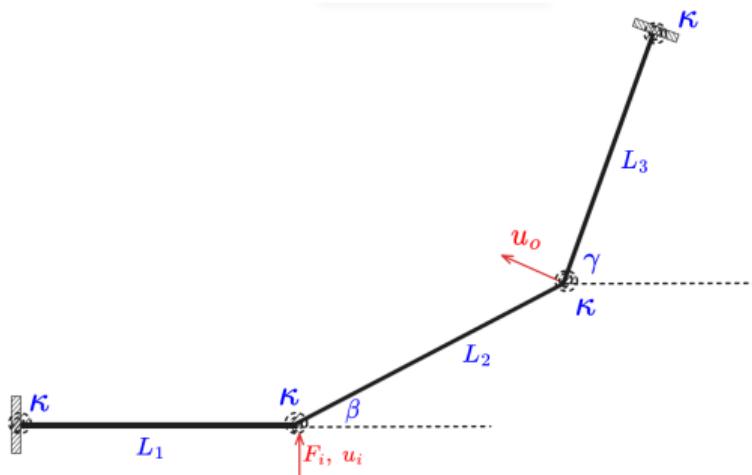
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Motivation



On κ

$\kappa \rightarrow 0 \implies$ 4-bar rigid body linkage

$\kappa \rightarrow \infty \implies$ compliant 4-bar linkage

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MA based on kinetoelastics

$$[\mathbf{K}_t(\mathbf{u})]\{\Delta \mathbf{u}\} = \{\Delta \mathbf{f}\}$$

For a single-input single-output mechanism:

$$\Delta u_i = [\mathbf{K}_t^{-1}(\mathbf{u})]_{ii} \Delta f_i - [\mathbf{K}_t^{-1}(\mathbf{u})]_{io} \Delta f_o$$

$$\Delta u_o = [\mathbf{K}_t^{-1}(\mathbf{u})]_{oi} \Delta f_i - [\mathbf{K}_t^{-1}(\mathbf{u})]_{oo} \Delta f_o$$

Integrating the last expression from the initial stress-free state

(1 $\equiv \{u_i = 0, u_o = 0, f_i = 0, f_o = 0\}$) to its final state

(2 $\equiv \{u_i = u_i^*, u_o = u_o^*, f_i = f_i^*, f_o = f_o^*\}$):

$$\begin{aligned} \Delta f_o &= \frac{[\mathbf{K}_t^{-1}(\mathbf{u})]_{oi} \Delta f_i}{[\mathbf{K}_t^{-1}(\mathbf{u})]_{oo}} - \frac{\Delta u_o}{[\mathbf{K}_t^{-1}(\mathbf{u})]_{oo}} \\ \implies \frac{f_o^*}{f_i^*} &= \frac{1}{f_i^*} \int_1^2 \frac{[\mathbf{K}_t^{-1}(\mathbf{u})]_{oi} df_i}{[\mathbf{K}_t^{-1}(\mathbf{u})]_{oo}} - \frac{1}{f_i^*} \int_1^2 \frac{du_o}{[\mathbf{K}_t^{-1}(\mathbf{u})]_{oo}} \end{aligned}$$

MA based on kinetoelastics

The last expression can be rewritten in the following form:

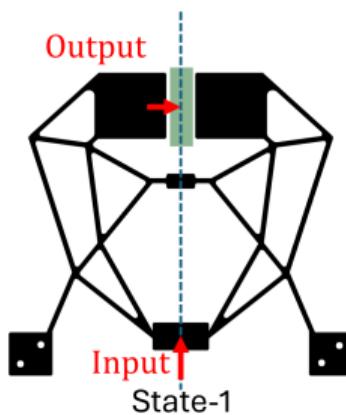
$$\frac{f_o^*}{f_i^*} = \text{MA}_s \left(1 - \frac{f_n}{f_i^*} \right)$$
$$f_n \equiv \left(\frac{\int_1^2 \frac{du_o}{[\mathcal{K}_t^{-1}(\mathbf{u})]_{oo}}}{\frac{1}{f_i^*} \int_1^2 \frac{[\mathcal{K}_t^{-1}(\mathbf{u})]_{oi} df_i}{[\mathcal{K}_t^{-1}(\mathbf{u})]_{oo}}} \right), \quad \text{MA}_s = \frac{1}{f_i^*} \int_1^2 \frac{[\mathcal{K}_t^{-1}(\mathbf{u})]_{oi} df_i}{[\mathcal{K}_t^{-1}(\mathbf{u})]_{oo}}$$

MA based on kinetoelastics

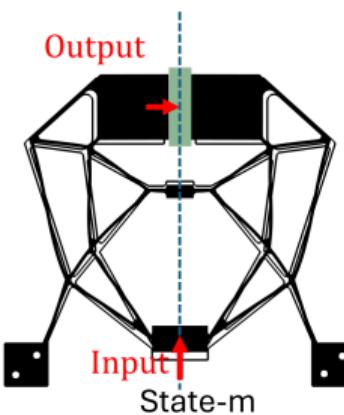
Let us define the intermediate stage where the mechanism begins to engage the workpiece as state $m \equiv \{u_i^m, u_o^*, f_i^m, f_o = 0\}$.

$$\frac{f_o^*}{f_i^*} = \frac{1}{f_i^*} \int_1^2 \frac{[\mathbf{K}_t^{-1}(\mathbf{u})]_{oi}}{[\mathbf{K}_t^{-1}(\mathbf{u})]_{oo}} df_i - \frac{1}{f_i^*} \int_1^m \frac{[\mathbf{K}_t^{-1}(\mathbf{u})]_{oi}}{[\mathbf{K}_t^{-1}(\mathbf{u})]_{oo}} df_i = \frac{1}{f_i^*} \int_{f_i^m}^{f_i^*} \frac{[\mathbf{K}_t^{-1}(\mathbf{u})]_{oi}}{[\mathbf{K}_t^{-1}(\mathbf{u})]_{oo}} dF_i,$$

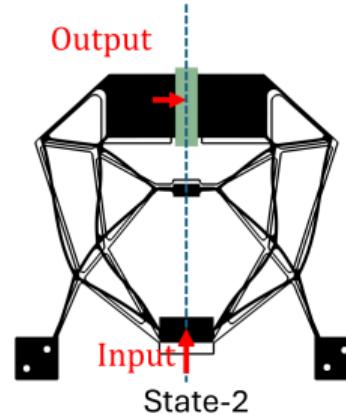
where f_i^m is the input force at which the mechanism engages the workpiece.



$$\{f_i = 0, u_i = 0, f_o = 0, u_o = 0\}$$



$$\{f_i = f_i^m, u_i = u_i^m, f_o = 0, u_o = u_o^*\}$$



$$\{f_i = f_i^*, u_i = u_i^2, f_o = f_o^2, u_o = u_o^*\}$$

MA based on kinetoelastics

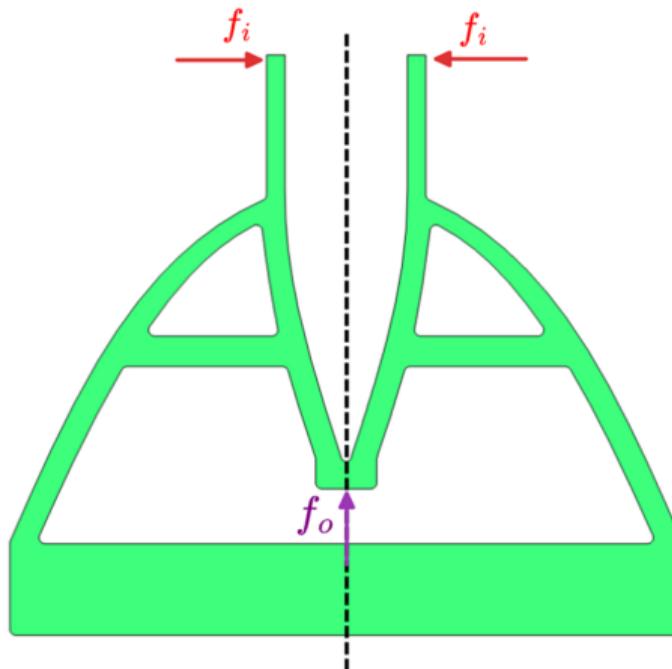
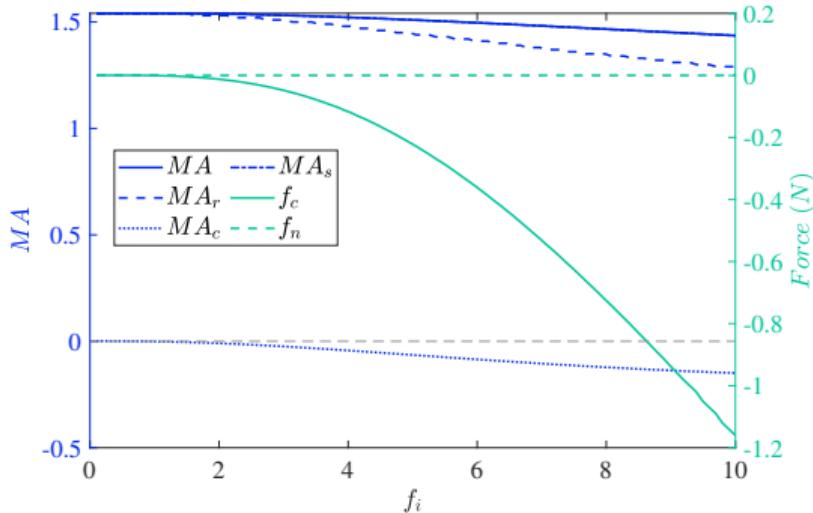
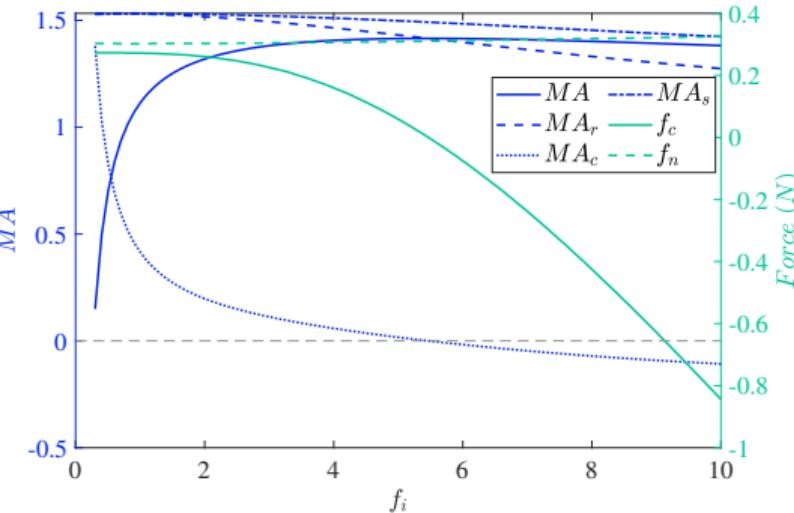


Figure: A compliant crimping mechanism

MA based on kinetoelastics



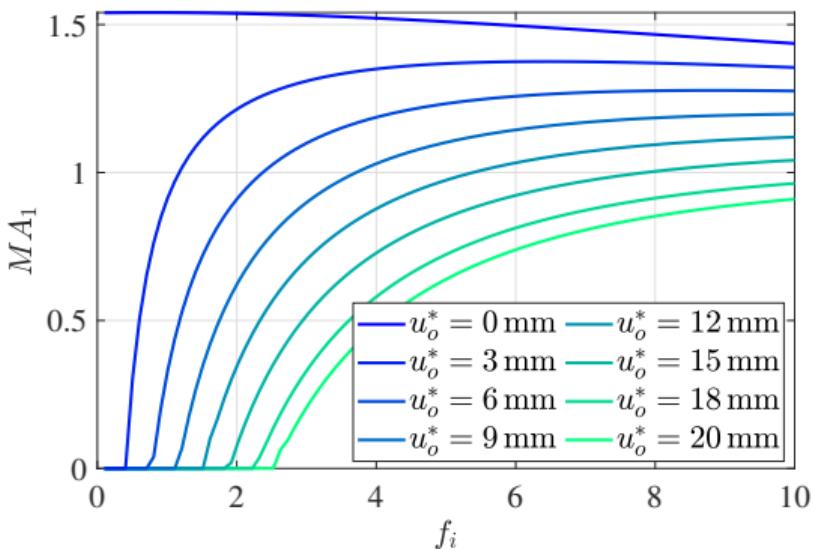
(a) Variability of the different MA terms and compliance forces with input force for $u_o^* = 0$



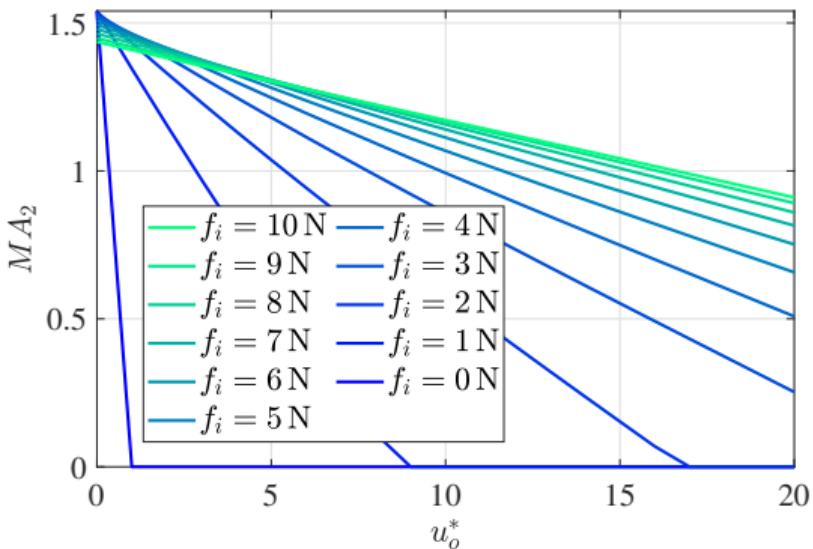
(b) Variability of the different MA terms and compliance forces with input force for $u_o^* = 2 \text{ mm}$

MA based on kinetoelastics

$$\frac{f_o^*}{f_i^*} = \frac{1}{f_i^*} \int_1^2 \frac{[\mathcal{K}_t^{-1}(\mathbf{u})]_{oi} df_i}{[\mathcal{K}_t^{-1}(\mathbf{u})]_{oo}} - \frac{1}{f_i^*} \int_1^2 \frac{du_o}{[\mathcal{K}_t^{-1}(\mathbf{u})]_{oo}}$$



(a) MA_1



(b) MA_2

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Eigenvalue based decomposition of DA and MA

$$[K]\{u\} = \{f\}$$

eigenvalues: $(\lambda_k \forall k \in \{1, 2..n\})$

orthonormal eigenvectors: $(a_k \forall k \in \{1, 2..n\})$

$$\{u\} = \sum_{k=1}^n \frac{(a_k \otimes a_k)\{f\}}{\lambda_k}$$

Eigenvalue based decomposition of DA and MA

For single-input single-output mechanisms:

$$u_i = \sum_{k=1}^n \frac{f_i(\mathbf{a}_k \otimes \mathbf{a}_k)_{ii}}{\lambda_k} - \frac{f_o(\mathbf{a}_k \otimes \mathbf{a}_k)_{io}}{\lambda_k}$$
$$u_o = \sum_{k=1}^n \frac{f_i(\mathbf{a}_k \otimes \mathbf{a}_k)_{io}}{\lambda_k} - \frac{f_o(\mathbf{a}_k \otimes \mathbf{a}_k)_{oo}}{\lambda_k}$$

The last relation can be simplified to:

$$\frac{f_o}{f_i} = \text{MA}_s \left(1 - \frac{f_n}{f_i} \right)$$
$$f_n \equiv \frac{u_o}{\sum_{k=1}^n \frac{(\mathbf{a}_k \otimes \mathbf{a}_k)_{io}}{\lambda_k}}, \quad \text{MA}_s = \frac{\sum_{k=1}^n \frac{(\mathbf{a}_k \otimes \mathbf{a}_k)_{io}}{\lambda_k}}{\sum_{k=1}^n \frac{(\mathbf{a}_k \otimes \mathbf{a}_k)_{oo}}{\lambda_k}}$$

Eigenvalue based decomposition of DA and MA

Similarly, it is possible to represent the displacement advantage as:

$$\frac{u_o}{u_i} = \frac{\sum_{k=1}^n \frac{(\mathbf{a}_k \otimes \mathbf{a}_k)_{io}}{\lambda_k}}{\sum_{k=1}^n \frac{(\mathbf{a}_k \otimes \mathbf{a}_k)_{ii}}{\lambda_k}} \left(1 - \frac{u_c}{u_i} \right)$$
$$u_c \equiv \frac{f_o \left(\left(\sum_{k=1}^n \frac{(\mathbf{a}_k \otimes \mathbf{a}_k)_{ii}}{\lambda_k} \right) \left(\sum_{k=1}^n \frac{(\mathbf{a}_k \otimes \mathbf{a}_k)_{oo}}{\lambda_k} \right) - \left(\sum_{k=1}^n \frac{(\mathbf{a}_k \otimes \mathbf{a}_k)_{io}}{\lambda_k} \right)^2 \right)}{\left(\sum_{k=1}^n \frac{(\mathbf{a}_k \otimes \mathbf{a}_k)_{ii}}{\lambda_k} \right)}$$

Eigenvalue based decomposition of DA and MA

DA_f can be rewritten as:

$$\text{DA}_f = \frac{\sum_{k=1}^n \frac{(\mathbf{a}_k \otimes \mathbf{a}_k)_{io}}{\lambda_k}}{\sum_{k=1}^n \frac{(\mathbf{a}_k \otimes \mathbf{a}_k)_{ii}}{\lambda_k}} = \frac{\sum_{k=1}^n \frac{(\mathbf{a}_k)_i \times (\mathbf{a}_k)_o}{\lambda_k}}{\sum_{k=1}^n \frac{(\mathbf{a}_k)_i \times (\mathbf{a}_k)_i}{\lambda_k}} = \frac{\sum_{k=1}^n \frac{(\mathbf{a}_k)_i^2}{\lambda_k} \text{DA}_k}{\sum_{k=1}^n \frac{(\mathbf{a}_k)_i^2}{\lambda_k}},$$

where DA_k is the ratio $(\mathbf{a}_k)_o / (\mathbf{a}_k)_i$ for k^{th} eigenvector. Also,

$$\text{MA}_s = \frac{\sum_{k=1}^n \frac{(\mathbf{a}_k)_i \times (\mathbf{a}_k)_o}{\lambda_k}}{\sum_{k=1}^n \frac{(\mathbf{a}_k)_o \times (\mathbf{a}_k)_o}{\lambda_k}} = \frac{\sum_{k=1}^n \frac{(\mathbf{a}_k)_o^2}{\lambda_k} \left(\frac{1}{\text{DA}_k} \right)}{\sum_{k=1}^n \frac{(\mathbf{a}_k)_o^2}{\lambda_k}} = \frac{\sum_{k=1}^n \frac{(\mathbf{a}_k)_i^2}{\lambda_k} \text{DA}_k}{\sum_{k=1}^n \frac{(\mathbf{a}_k)_i^2}{\lambda_k} \text{DA}_k^2}$$

Eigenvalue based decomposition of DA and MA

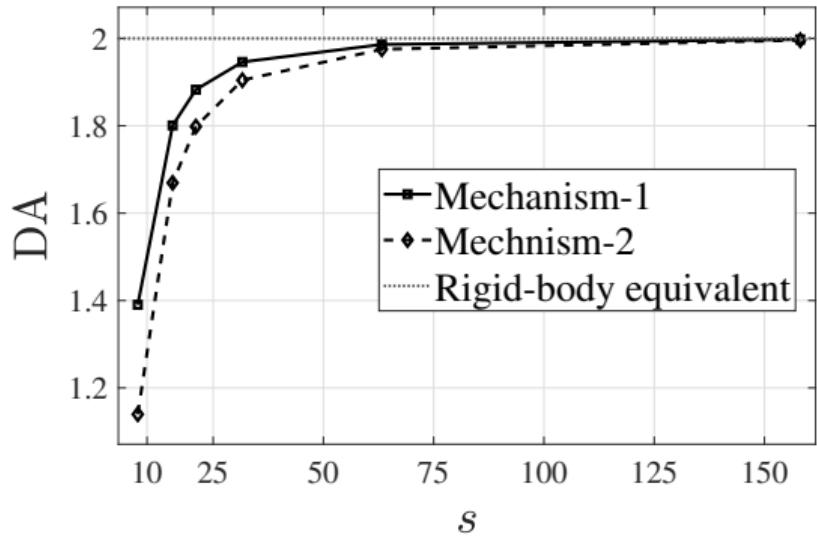
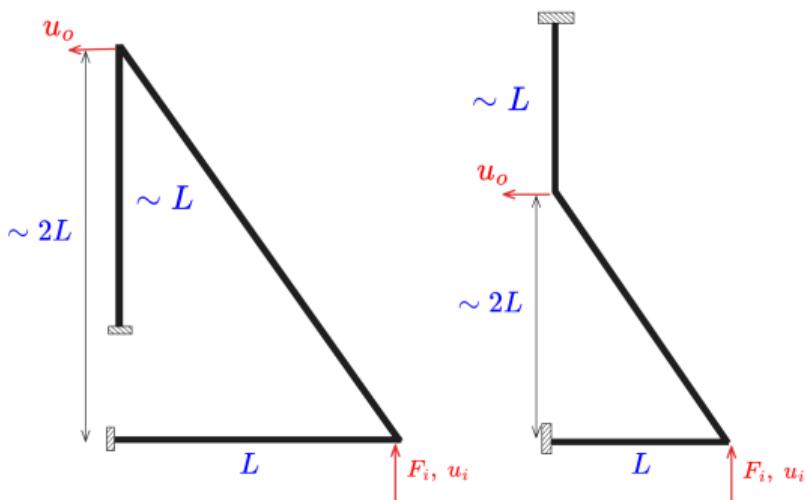
Consequently,

$$\text{DA} \times \text{MA} \leq \text{MA}_s \times \text{DA}_f = \frac{\left(\sum_{k=1}^n \frac{(a_k)_i^2}{\lambda_k} \text{DA}_k \right)^2}{\left(\sum_{k=1}^n \frac{(a_k)_i^2}{\lambda_k} \text{DA}_k^2 \right) \left(\sum_{k=1}^n \frac{(a_k)_i^2}{\lambda_k} \right)} \leq 1$$

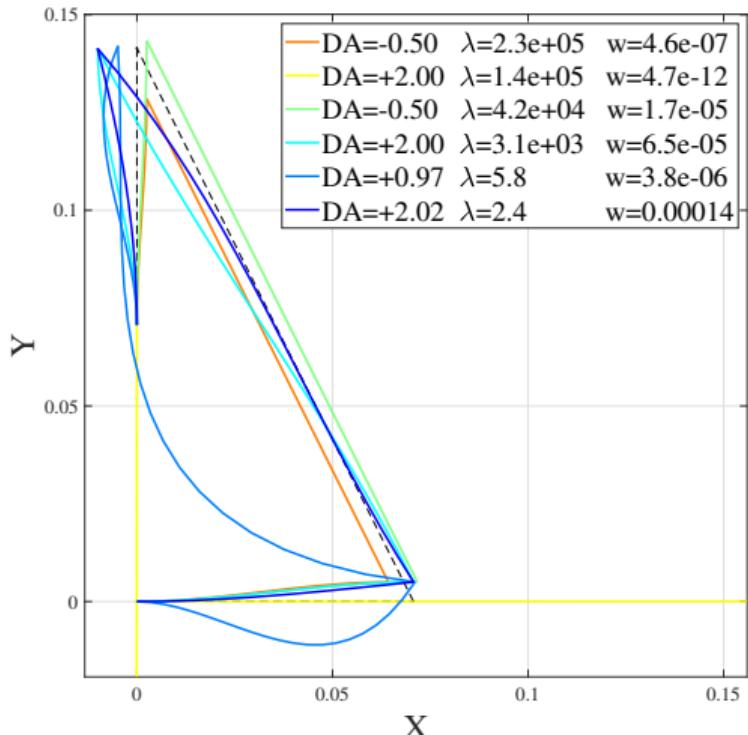
$$\text{MA}_s \times \text{DA}_f = 1$$

The equality is only satisfied when $\lambda_1 = 0$!

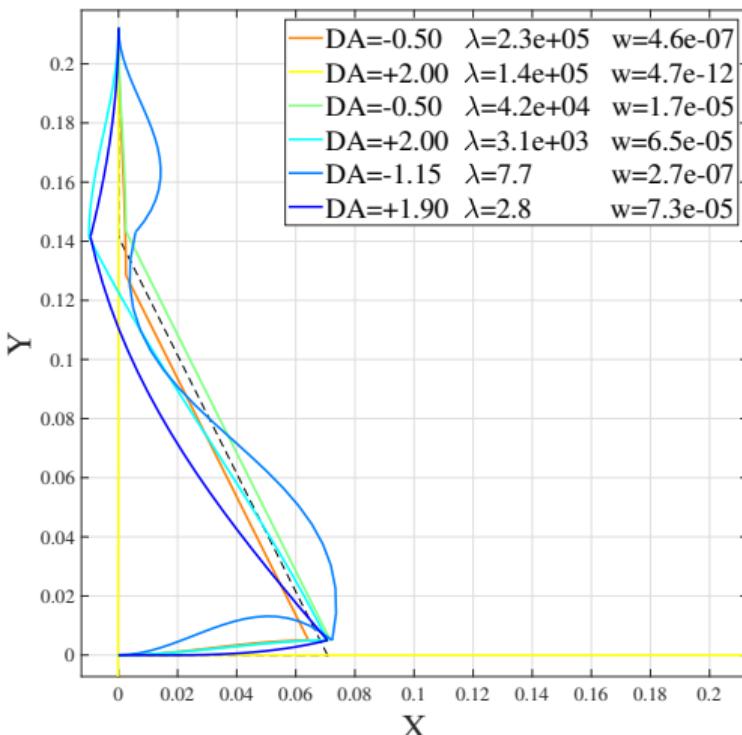
Eigenvalue based decomposition of DA and MA



Eigenvalue based decomposition of DA and MA

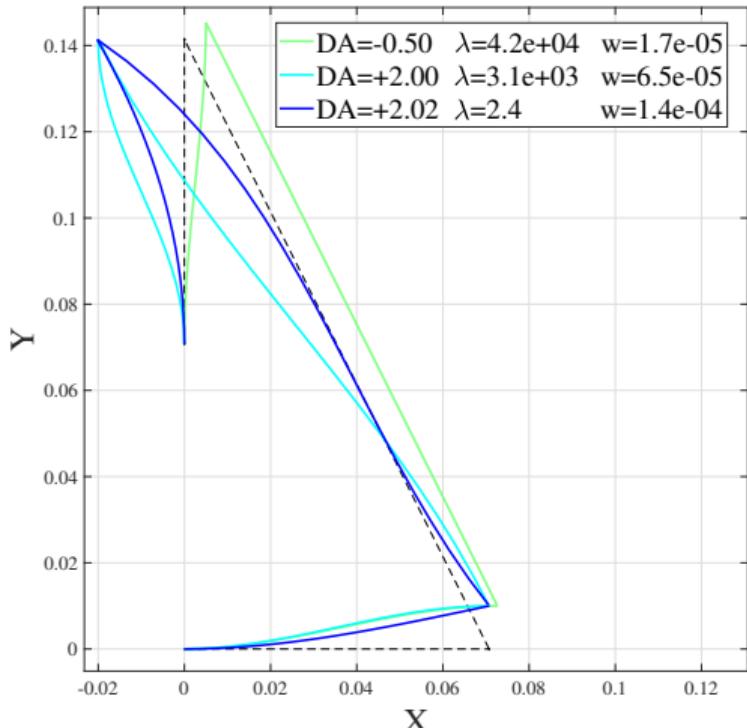


(a) Eigenvectors of mechanism-1
($s = 15.8$, $DA_f = 1.80$)

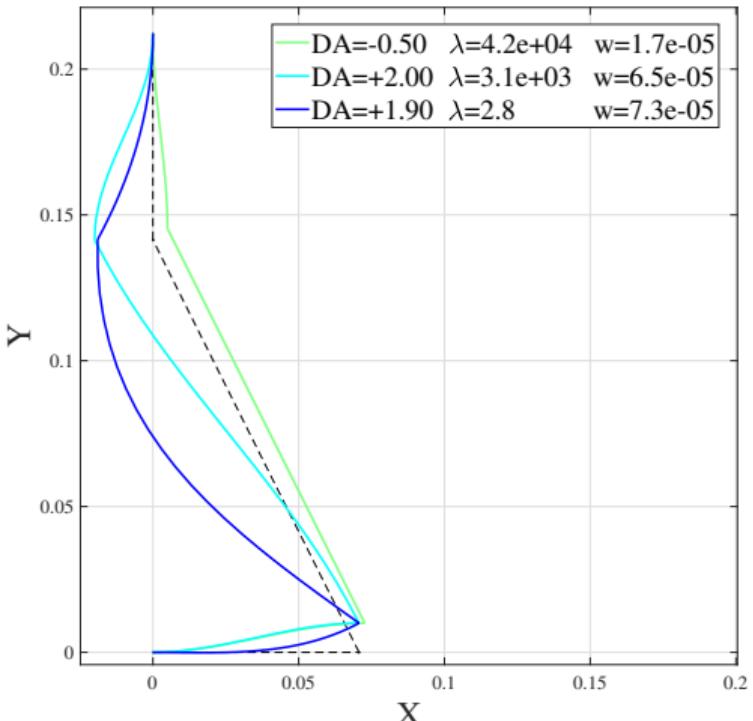


(b) Eigenvectors of mechanism-2
($s = 15.8$, $DA_f = 1.67$)

Eigenvalue based decomposition of DA and MA

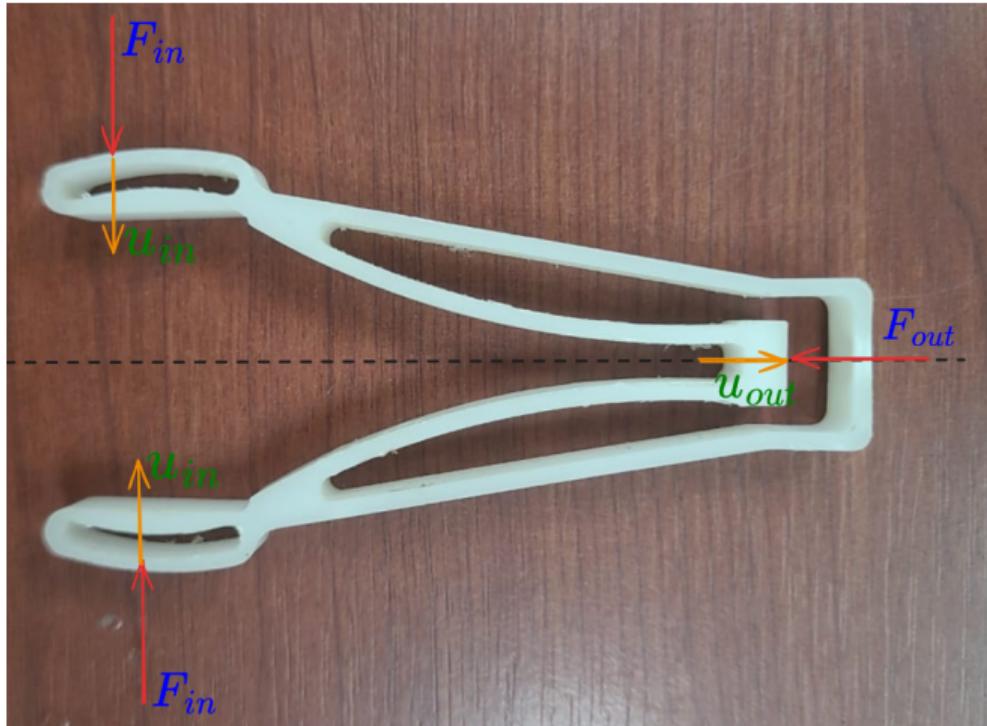


(a) Salient eigenvectors of mechanism-1
($s = 15.8$, $DA_f = 1.80$)



(b) Salient eigenvectors of mechanism-2
($s = 15.8$, $DA_f = 1.67$)

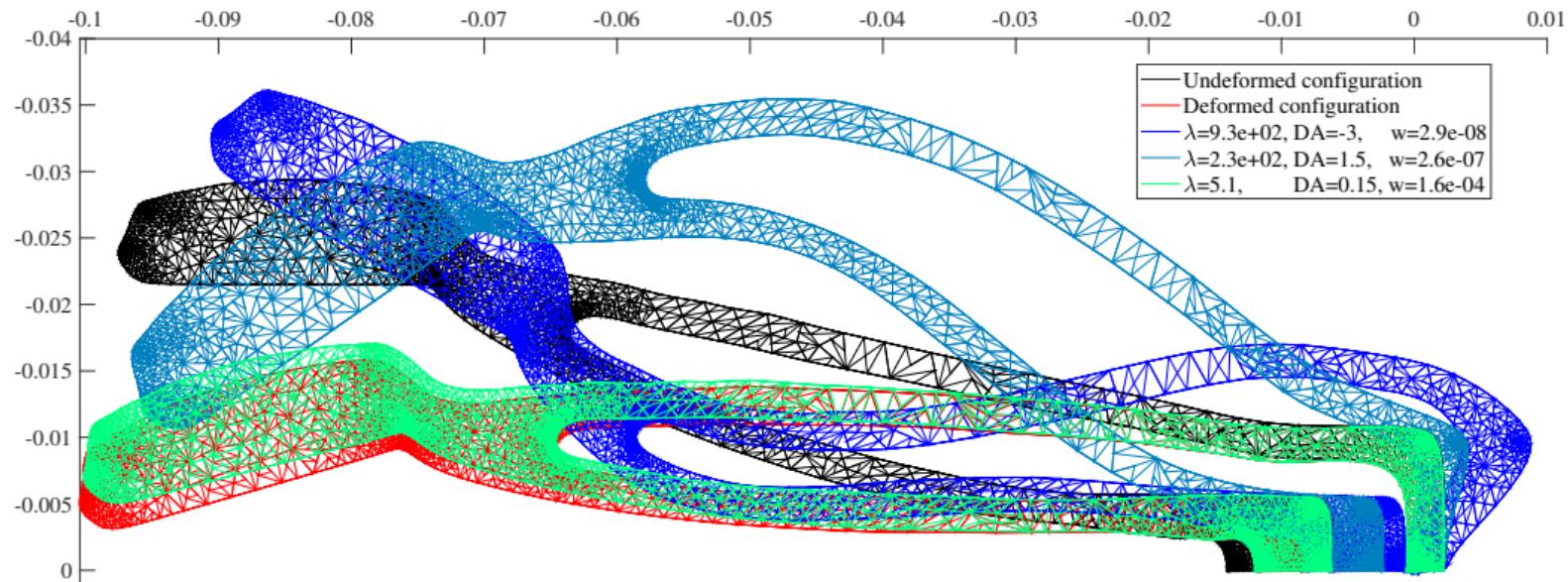
Eigenvalue based decomposition of DA and MA



$$\begin{aligned} \text{MA}_s &= 5.35, \\ \text{DA}_f &= 0.15, \\ \text{MA}_s \times \text{DA}_f &= 0.80 \end{aligned}$$

Figure: The first compliant mechanism designed by Prof. Ashok Midha

Eigenvalue based decomposition of DA and MA



Combining compliant mechanisms

Imagine two compliant mechanisms with spring lever parameters k_i^1 , k_o^1 , n_1 and k_i^2 , k_o^2 , n_2 .

$$\begin{bmatrix} k_i^1 + (n_1)^2 k_o^1 & -n_1 k_o^1 & 0 \\ -n_1 k_o^1 & +k_i^2 + (n_2)^2 k_o^2 + k_o^1 & -n_2 k_o^2 \\ 0 & -n_2 k_o^2 & k_o^2 \end{bmatrix} \begin{Bmatrix} u_i^1 \\ u_o^1 \text{ or } u_i^2 \\ u_o^2 \end{Bmatrix} = \begin{Bmatrix} f_i^1 \\ f_o^1 \text{ or } f_i^2 \\ f_o^2 \end{Bmatrix}$$

Now, eliminating the intermediate degree of freedom (no load at this point):

$$\frac{1}{k_o^2 n_2^2 + k_i^2 + k_o^1} \begin{bmatrix} k_o^1 k_o^2 n_1^2 n_2^2 + k_i^2 k_o^1 n_1^2 + k_i^2 k_o^1 n_2^2 + k_i^1 k_i^2 + k_i^1 k_o^1 & -k_o^1 k_o^2 n_1 n_2 \\ -k_o^1 k_o^2 n_1 n_2 & k_o^2 (k_i^2 + k_o^1) \end{bmatrix} \begin{Bmatrix} u_i^1 \\ u_o^2 \end{Bmatrix} = \begin{Bmatrix} f_i^1 \\ f_o^2 \end{Bmatrix}$$

The equivalent DA for this mechanism is:

$$n^* = \left(\frac{k_o^1}{k_i^2 + k_o^1} \right) n_1 n_2$$

Thank you!