

Report: Verification of Analytical Sensitivity Relations through Finite Difference Scheme

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1 Objective

Analytically derive relations for a particular optimization problem and verify it through the finite-difference scheme, and note the range of perturbation for which the finite difference scheme matches the analytical result.

2 Topology optimization using a mixed formulation: An alternative way to solve pressure load problems¹

2.1 Brief overview

Surface loads depend on the structure's geometry (surface), which can vary in a structural optimization problem. For example, a pressure load (hydrostatic) acting on a dam and its contribution to the structure response would not remain identical as the geometry of the surface changes. In the case of pressure load, it must act normally on the new surface. The boundary of the problem and the load on the boundary are both dynamic variables. To account for this, the paper introduces an additional fluid phase, which is used to transmit the pressure to the boundary of the surface and due to the incompressible nature of the assumed fluid phase, the pressure load gets transmitted normally to the dynamic boundary of the problem. To mitigate numerical difficulties with the incorporation of material with high bulk modulus a mixed formulation (both strain-based and pressure-based calculation)² is recommended, as is done in the paper.

2.2 Optimization problem

The optimization problem³ is stated below:

$$\begin{aligned} & \underset{\mu_1, \mu_2}{\text{minimize}} \quad \frac{1}{2} \int_{\Omega} \sigma_{ij} \epsilon_{ij} d\Omega \equiv \int_{\Omega} G \epsilon_{ij} \epsilon_{ij} - \frac{1}{2} \int_{\Omega} p \delta_{ij} \epsilon_{ij} d\Omega - \frac{1}{3} \int_{\Omega} G \epsilon_{jj} \epsilon_{kk} d\Omega \\ & \text{subject to} \quad \Gamma_1 : \int_{\Omega} \mu_1 (1 - \mu_2) d\Omega / \int_{\Omega} d\Omega \leq f_0, \\ & \quad \Gamma_2 : \int_{\Omega} \mu_1 \mu_2 d\Omega / \int_{\Omega} d\Omega \geq f_{fluid}, \\ & \quad \Lambda_1 : \int_{\Omega} (\epsilon_{ij}^v 2G \epsilon_{ij} - \epsilon_{ij}^v \delta_{ij} p - v_i F_i - \frac{2}{3} \epsilon_{ii}^v \epsilon_{jj} G) d\Omega - \int_{\partial\Omega_T} v_i T_i ds - \int_{\partial\Omega_u} v_i (u_i - u_i^*) ds = 0 \\ & \quad \Lambda_2 : \int_{\Omega} p^v (p/K + \epsilon_{kk}) d\Omega = 0 \\ & \quad \alpha_1 : -\mu_1 \leq 0, \\ & \quad \alpha_2 : \mu_1 - 1 \leq 0, \\ & \quad \beta_1 : -\mu_2 \leq 0, \\ & \quad \beta_2 : \mu_2 - 1 \leq 0, \\ & \text{Data} \quad G(\mu_1, \mu_2), K(\mu_1, \mu_2), \rho(\mu_1, \mu_2), \Omega, \Omega_T, \Omega_u, \bar{F}, \bar{T} \end{aligned} \tag{1}$$

The scheme involves three phases: Void ($\mu_1 = 0$) and Material phase ($\mu_1 = 1$) the material phase can be imagined to consist of two sub-species: Solid ($\mu_2 = 0$) and Fluid ($\mu_2 = 1$).

¹O. Sigmund, P.M. Clausen, Topology optimization using a mixed formulation: An alternative way to solve pressure load problems, Computer Methods in Applied Mechanics and Engineering, 2007.

²For more info refer, COMSOL-help: Mixed-Formulation, https://doc.comsol.com/5.5/doc/com.comsol.help.sme/sme_ug_theory.06.24.html.

³It is to be noted that there is a typo in constraint corresponding to Γ_2 in the above-mentioned paper, the inequality should be reversed, the change has been made in this section to maintain uniformity with the equations given in the paper but is corrected in section 2.5.

2.2.1 Interpolation scheme

The relevant interpolation scheme used in this work is shown below:

$$K(\mu_1, \mu_2) = K_{void} + \mu_1^n [\mu_2^n K_{fluid} + (1 - \mu_2^n) K_0 - K_{void}] \quad (2a)$$

$$G(\mu_1, \mu_2) = G_{void} + \mu_1^n (1 - \mu_2^n) (G_0 - G_{void}) \quad (2b)$$

2.3 Derivation of analytical formulae for sensitivity analysis

The derivation of the analytical formula for sensitive analysis is shown in this section. The final boundary condition terms are highlighted in red and final relations and governing equations are shown in blue.

$$\begin{aligned} L = & \int_{\Omega} G \epsilon_{ij} \epsilon_{ij} - \frac{1}{2} p \delta_{ij} \epsilon_{ij} - \frac{1}{3} G \epsilon_{jj} \epsilon_{kk} + \Gamma_1 (\mu_1 (1 - \mu_2)) / \Omega^* - \Gamma_2 \mu_1 \mu_2 / \Omega^* + \Lambda_1 (\epsilon_{ij}^v 2 G \epsilon_{ij} - \epsilon_{ij}^v \delta_{ij} p - v_i F_i - \frac{2}{3} \epsilon_{ii}^v \epsilon_{jj} G) \\ & + \Lambda_2 (p^v (p / K + \epsilon_{kk})) d\Omega - \Gamma_1 (f_0) + \Gamma_2 (f_{fluid}) - \Lambda_1 \int_{\partial\Omega_T} v_i T_i ds + \alpha_1 (-\mu_1) + \alpha_2 (\mu_1 - 1) + \beta_1 (-\mu_2) + \beta_2 (\mu_2 - 1) \end{aligned} \quad (3)$$

$$\begin{aligned} \delta_{\mu_1} L = & \int_{\Omega} \delta_{\mu_1} G \epsilon_{ij} \epsilon_{ij} - \frac{1}{3} \delta_{\mu_1} G \epsilon_{ii} \epsilon_{jj} + \Gamma_1 \delta \mu_1 ((1 - \mu_2)) / \Omega^* - \Gamma_2 \delta \mu_1 \mu_2 / \Omega^* + \Lambda_1 (\epsilon_{ij}^v 2 \delta_{\mu_1} G \epsilon_{ij} - \frac{2}{3} \delta_{\mu_1} G \epsilon_{ii}^v \epsilon_{jj}) \\ & + \Lambda_2 (p^v (-p / K^2 \delta_{\mu_1} K)) d\Omega + \alpha_1 (-\delta \mu_1) + \alpha_2 (\delta \mu_1) d\Omega \end{aligned} \quad (4)$$

$$\begin{aligned} \delta_{\mu_2} L = & \int_{\Omega} \delta_{\mu_2} G \epsilon_{ij} \epsilon_{ij} - \frac{1}{3} \delta_{\mu_2} G \epsilon_{ii} \epsilon_{jj} - \Gamma_1 \delta \mu_2 (\mu_1) / \Omega^* - \Gamma_2 \delta \mu_2 \mu_1 / \Omega^* + \Lambda_1 (\epsilon_{ij}^v 2 \delta_{\mu_2} G \epsilon_{ij} - \frac{2}{3} \delta_{\mu_2} G \epsilon_{ii}^v \epsilon_{jj}) \\ & + \Lambda_2 (p^v (-p / K^2 \delta_{\mu_2} K)) d\Omega + \beta_1 (-\delta \mu_2) + \beta_2 (\delta \mu_2) d\Omega \end{aligned} \quad (5)$$

$$\delta_u L = \int_{\Omega} 2 G \delta_u (\epsilon) : \epsilon - \frac{1}{2} p \text{trace}(\delta_u \epsilon) - \frac{2}{3} G \epsilon_{ii} \delta_u \epsilon_{jj} + \Lambda_1 (2 G \epsilon^v : \delta_u \epsilon - \frac{2}{3} G \epsilon_{ii}^v \delta_u \epsilon_{jj}) + \Lambda_2 (p^v \text{trace}(\delta_u \epsilon)) d\Omega = 0 \quad (6)$$

$$\begin{aligned} \delta_u L = & \int_{\Omega} \text{div}(2 G \epsilon \delta u) - \text{div}(2 G \epsilon) \cdot \delta u - (\frac{1}{2} p + \frac{2}{3} G \epsilon_{kk}) (\nabla \cdot \delta u) + \Lambda_1 \text{div}(2 G \epsilon^v \delta u) - \Lambda_1 \text{div}(2 G \epsilon^v) \cdot \delta u \\ & - \frac{2}{3} \Lambda_1 G \epsilon_{ii}^v (\nabla \cdot \delta u) + \Lambda_2 (p^v (\nabla \cdot \delta u)) d\Omega = 0 \end{aligned} \quad (7)$$

$$\begin{aligned} \delta_u L = & \int_{\Omega} \text{div}(2 G \epsilon \delta u - (\frac{1}{2} p + \frac{2}{3} G \epsilon_{kk} - \Lambda_2 p^v + \frac{2}{3} \Lambda_1 G \epsilon_{ii}^v) \delta u + \Lambda_1 2 G \epsilon^v \delta u) + (-\Lambda_1 \text{div}(2 G \epsilon^v) - \text{div}(2 G \epsilon) \\ & + \nabla (\frac{1}{2} p + \frac{2}{3} G \epsilon_{kk} - \Lambda_2 p^v + \frac{2}{3} \Lambda_1 G \epsilon_{ii}^v)) \cdot \delta u d\Omega = 0 \end{aligned} \quad (8)$$

$$\begin{aligned} \delta_u L = & \int_{\partial\Omega} [(2 G \epsilon - (\frac{1}{2} p + \frac{2}{3} G \epsilon_{ii} - \Lambda_2 p^v) + \Lambda_1 2 G \epsilon^v - \frac{2}{3} \Lambda_1 G \epsilon_{ii}^v) \delta u] \cdot \bar{n} ds \\ & + \int_{\Omega} (-\Lambda_1 \text{div}(2 G \epsilon^v) - \text{div}(2 G \epsilon) + \nabla (\frac{1}{2} p + \frac{2}{3} G \epsilon_{ii} + \Lambda_1 \frac{2}{3} G \epsilon_{ii}^v - \Lambda_2 p^v)) \cdot \delta u d\Omega = 0 \end{aligned} \quad (9)$$

$$\Rightarrow -\Lambda_1 \text{div}(2 G \epsilon^v) - \text{div}(2 G \epsilon) + \nabla (\frac{1}{2} p + \frac{2}{3} G \epsilon_{ii} + \Lambda_1 \frac{2}{3} G \epsilon_{ii}^v - \Lambda_2 p^v) = 0 \quad (10)$$

$$\delta_v L = \int_{\Omega} \Lambda_1 (2 G \epsilon : \delta_v \epsilon^v - \text{trace}(\delta_v (\epsilon^v))) (p + \frac{2}{3} \epsilon_{ii} G) - F \cdot \delta v d\Omega - \Lambda_1 \int_{\partial\Omega_T} T \cdot \delta v ds = 0 \quad (11)$$

$$\delta_v L = \int_{\Omega} \Lambda_1 (2G\epsilon : \nabla \delta v - (\nabla \cdot \delta v)(p + \frac{2}{3}\epsilon_{ii}G) - F \cdot \delta v) d\Omega - \Lambda_1 \int_{\partial\Omega_T} T \cdot \delta v ds = 0 \quad (12)$$

$$\delta_v L = \int_{\Omega} \operatorname{div}(\Lambda_1(2G\epsilon\delta v - (p + \frac{2}{3}\epsilon_{ii}G)\delta v)) d\Omega - \int_{\Omega} \Lambda_1(\operatorname{div}(2G\epsilon) - \nabla(p + \frac{2}{3}\epsilon_{ii}G) + F) \cdot \delta v d\Omega - \Lambda_1 \int_{\partial\Omega_T} T \cdot \delta v ds = 0 \quad (13)$$

$$\delta_v L = \Lambda_1 \left[\int_{\partial\Omega} ((2G\epsilon\delta v - (p + \frac{2}{3}\epsilon_{ii}G)\delta v)) \cdot \bar{n} d\Omega - \int_{\Omega} (\operatorname{div}(2G\epsilon) - \nabla(p + \frac{2}{3}\epsilon_{ii}G) + F) \cdot \delta v d\Omega - \int_{\partial\Omega_T} T \cdot \delta v ds \right] = 0 \quad (14)$$

$$\operatorname{div}(2G\epsilon) - \nabla(p + \frac{2}{3}\epsilon_{ii}G) + F = 0 \quad (15)$$

$$\delta_p L = \int_{\Omega} [-\frac{1}{2}\epsilon_{kk} + \Lambda_1(-\epsilon_{kk}^v) + \Lambda_2(p^v/K)] \delta p d\Omega = 0 \quad (16)$$

$$-\frac{1}{2}\epsilon_{kk} + \Lambda_1(-\epsilon_{kk}^v) + \Lambda_2(p^v/K) = 0 \quad (17)$$

$$\delta_{p^v} L = \int_{\Omega} \Lambda_2(\delta p^v(p/K + \epsilon_{kk})) d\Omega = 0 \quad (18)$$

$$p/K + \epsilon_{kk} = 0 \quad (19)$$

2.3.1 Solving adjoint variables

The equations are simplified as shown below:

$$-\Lambda_1 \operatorname{div}(2G\epsilon^v) - \operatorname{div}(2G\epsilon) + \nabla(\frac{1}{2}p + \frac{2}{3}G\epsilon_{ii} + \Lambda_1 \frac{2}{3}G\epsilon_{ii}^v - \Lambda_2 p^v) = 0 \quad (20a)$$

$$\Rightarrow \operatorname{div}(-2G\Lambda_1\epsilon^v - 2G\epsilon + \frac{1}{2}pI + \frac{2}{3}G\epsilon_{ii}I + \Lambda_1 \frac{2}{3}G\epsilon_{ii}^v I - \Lambda_2 p^v I) = 0$$

$$-\frac{1}{2}\epsilon_{kk} + \Lambda_1(-\epsilon_{kk}^v) + \Lambda_2(p^v/K) = 0 \quad (20b)$$

$$\Rightarrow \Lambda_1(-\epsilon_{kk}^v) + \Lambda_2(p^v/K) = \frac{1}{2}\epsilon_{kk} \quad (20c)$$

$$\operatorname{div}(2G\epsilon) - \nabla(p + \frac{2}{3}\epsilon_{ii}G) + F = 0 \quad (20c)$$

$$\Rightarrow \operatorname{div}(2G\epsilon - pI - \frac{2}{3}\epsilon_{ii}GI) = -F \quad \& \quad \operatorname{div}(2G\epsilon^v - p^vI - \frac{2}{3}\epsilon_{ii}^vGI) = -F$$

$$p/K + \epsilon_{kk} = 0 \quad (20d)$$

$$\Rightarrow p/K + \epsilon_{kk} = 0 \quad \& \quad p^v/K + \epsilon_{kk}^v = 0$$

Simplifying:

$$(\Lambda_1 + \Lambda_2)(p^v/K) = -\frac{1}{2}p/K \quad (21a)$$

$$\operatorname{div}(-2G\Lambda_1\epsilon^v + \Lambda_1 \frac{2}{3}G\epsilon_{ii}^v I - \Lambda_2 p^v I) = 0 \quad (21b)$$

$$\Rightarrow \operatorname{div}(-(2G\epsilon - pI - \frac{2}{3}G\epsilon_{ii}I) - 2G\Lambda_1\epsilon^v + \Lambda_1 \frac{2}{3}G\epsilon_{ii}^v I + \Lambda_1 p^v I) = 0$$

$$\Rightarrow \operatorname{div}((2G\epsilon + K\epsilon_{kk} - \frac{2}{3}G\epsilon_{ii}I) + (2G\Lambda_1\epsilon^v - \Lambda_1 \frac{2}{3}G\epsilon_{ii}^v I - \Lambda_1 \epsilon_{kk}^v)) = 0$$

A trivial solution is:

$$\boxed{\Lambda_1 \epsilon^v = -\epsilon} \quad (22)$$

Then,

$$\boxed{\Lambda_2 \epsilon^v = \frac{1}{2} \epsilon} \quad (23)$$

$$\Lambda_2 p^v = \frac{1}{2} p \quad (24)$$

Substituting them in the design equation⁴:

$$\begin{aligned} \delta_{\mu_1} L = \int_{\Omega} & (-\epsilon_{ij}\epsilon_{ij}\delta_{\mu_1}G + \frac{1}{3}\epsilon_{ii}\epsilon_{jj}\delta_{\mu_1}G + \frac{1}{2}\epsilon_{kk}(p\frac{\delta_{\mu_1}K}{K}) + \Gamma_1\delta\mu_1(1-\mu_2)/\Omega^* \\ & - \Gamma_2\delta\mu_1\mu_2/\Omega^*) d\Omega + \alpha_1(-\delta\mu_1) + \alpha_2(\delta\mu_1) \end{aligned} \quad (25)$$

$$\begin{aligned} \delta_{\mu_2} L = \int_{\Omega} & (-\epsilon_{ij}\epsilon_{ij}\delta_{\mu_2}G + \frac{1}{3}\epsilon_{ii}\epsilon_{jj}\delta_{\mu_2}G + \frac{1}{2}\epsilon_{kk}(p\frac{\delta_{\mu_2}K}{K}) - \Gamma_1\delta\mu_2(\mu_1)/\Omega^* \\ & - \Gamma_2\delta\mu_2\mu_1/\Omega^*) d\Omega + \beta_1(-\delta\mu_2) + \beta_2(\delta\mu_2) \end{aligned} \quad (26)$$

$$\delta_{\mu_1} K(\mu_1, \mu_2) = n\mu_1^{n-1}[\mu_2^n K_{fluid} + (1 - \mu_2^n)K_0 - K_{void}] \delta\mu_1 \quad (27a)$$

$$\delta_{\mu_2} K(\mu_1, \mu_2) = n\mu_1^n[\mu_2^{n-1} K_{fluid} - \mu_2^{n-1} K_0] \delta\mu_2 \quad (27b)$$

$$\delta_{\mu_1} G(\mu_1, \mu_2) = n\mu_1^{n-1}(1 - \mu_2^n)(G_0 - G_{fluid}) \delta\mu_1 \quad (27c)$$

$$\delta_{\mu_2} G(\mu_1, \mu_2) = -n\mu_1^n(\mu_2^{n-1})(G_0 - G_{fluid}) \delta\mu_2 \quad (27d)$$

Now, we consider a very localized perturbation (mathematically, the variation is replaced by a Dirac-delta function at a point) at some particular point, we end up with the below stated relations. We notice the sensitivity of Strain energy at a particular point is:

Sensitivity w.r.t. μ_p ⁵ is given below:

$$\boxed{-\frac{1}{2}\epsilon_{ij}\frac{\partial\sigma_{ij}(\epsilon)}{\partial\mu_p}|_{\epsilon}} \quad (28)$$

and sensitivity of constraint eqn: Γ_1 is given below:

$$w.r.t. \mu_1 : (1 - \mu_2)/\Omega^*, \quad (29)$$

$$w.r.t. \mu_2 : -\mu_1/\Omega^*, \quad (30)$$

and sensitivity of constraint eqn: Γ_2 is given below:

$$w.r.t. \mu_1 : -\mu_2/\Omega^*, \quad (31a)$$

$$w.r.t. \mu_2 : (-\mu_1)/\Omega^*, \quad (31b)$$

2.4 Calculation of sensitivity through finite difference scheme

Sensitivity is analytically calculated by simulating the non-dimensional 2-D problem, and the boundary conditions of which are shown in fig. 1. The small rectangular part in the figure represents our region of interest where the parameters (μ_1 and μ_2) are varied, and the sensitivity is calculated at these regions. A simple case is considered for calculating the finite difference, where the design space is assigned a uniform concentration of design variables ($\mu_1 = 0.5$ and $\mu_2 = 0.5$). First, the value of μ_1 in the design space is set as $0.5 + \Delta$. The variation of sensitivity calculated through finite difference and analytical expression is presented in fig. 2a. Similarly, then the value of μ_1 is fixed as 0.5 and μ_2 is varied as $0.5 + \Delta$ and the plot presented in fig. 2b.

Based on the figure, we note that the range of delta between provides a roughly uniform gradient; thus, a finite difference scheme can be used in this perturbation range between 10^{-7} and 10^{-2} for the non-dimensionalised problem.

⁴Note: the values of α and β are identically zeros for intermediate values of μ_1 and μ_2

⁵where the partial derivative is evaluated for a constant value of ϵ

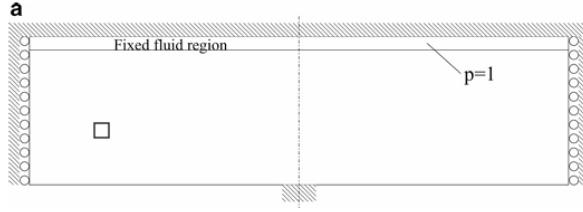


Figure 1: Boundary conditions for problem-1, Image credit¹

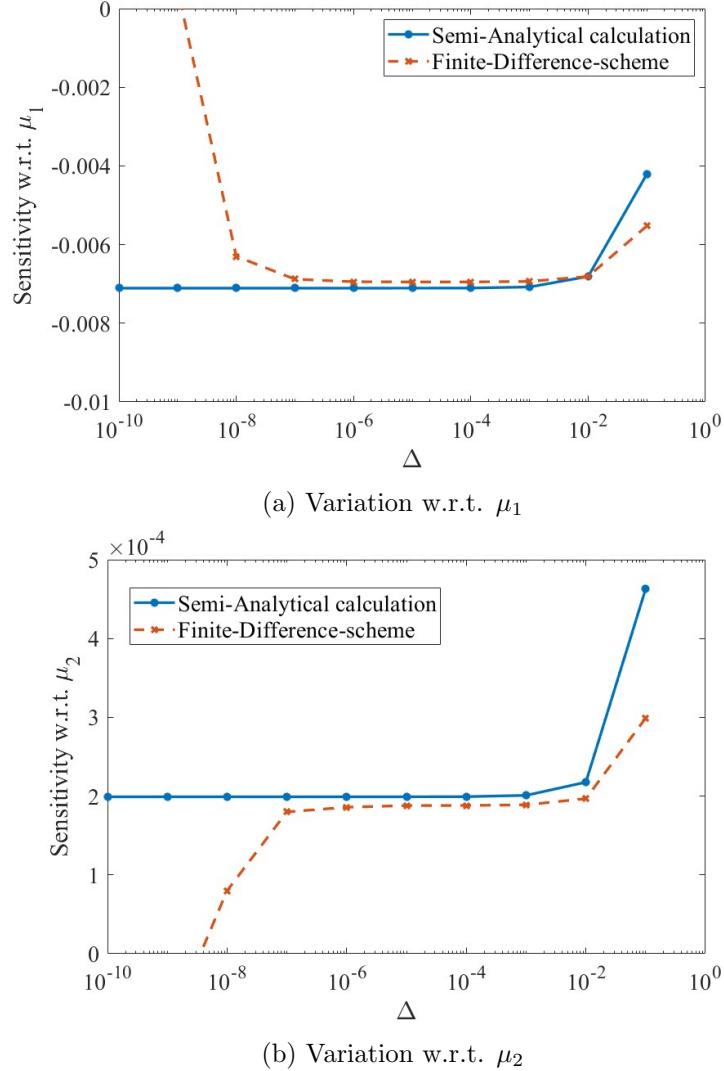


Figure 2: Validation of sensitivity analysis through finite-difference scheme by comparison with semi-analytical results

2.5 Optimized profile of problem-1

The optimized profile for problem 1 for different values of fluid volume constraint is shown in figs. 3 to 5. Where we notice that allowing for more fluid volume leads to a reduction in strain energy.⁶

⁶Schemes to improve filtering in three-phase problems can be considered to improve filtering but were not performed here (the obtained material concentrations are not ternary as would be expected in practice)

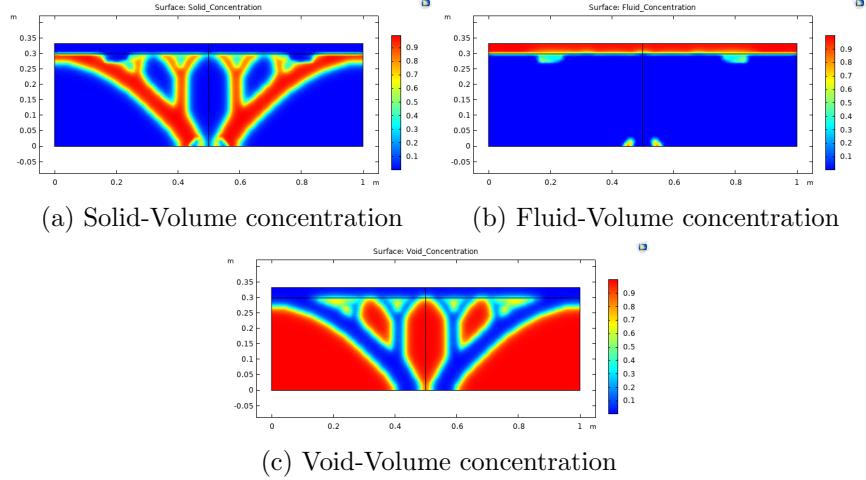


Figure 3: Optimization results of problem-1 for $f_0 = 0.3$ and $f_{fluid} = 0.1$, Objective function value=2.11 J

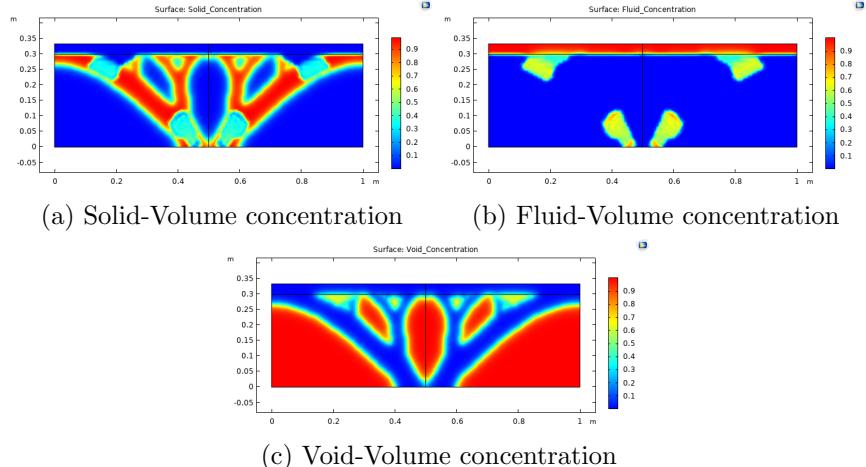


Figure 4: Optimization results of problem-1 for $f_0 = 0.3$ and $f_{fluid} = 0.145$, Objective function value=1.53 J

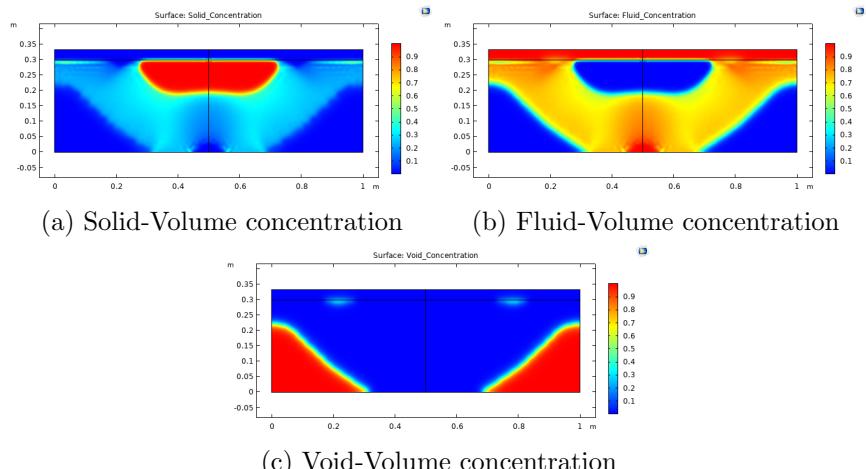


Figure 5: Optimization results of problem-1 for $f_0 = 0.3$ and $f_{fluid} = 0.5$, Objective function value=0.41 J