Report: Flexible truss optimization

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Contents

1 Objective **2**

2 Optimization of a truss for displacement at a particular point **2**

2.1 Problem statement . . . 2
2.2 Analytical solution . . . 2
2.3 Numerical implementation using optimality constraint method . . . 3
2.4 Example-1: Barnett truss . . . 4
2.5 Example-2: Mechanisms arising from flexibility optimization of 3×3 ground structure 4

3 Optimization of a truss for geometric advantage **6**

3.1 Problem statement . . . 6
3.2 Analytical solution . . . 6
3.3 Numerical implementation using optimality constraint method . . . 7
3.4 Example-1: Barnett truss . . . 7
3.5 Comments on sufficiency condition . . . 7

4 Conclusion **7**

1 Objective

The objective of this work is to numerically understand the nature of truss structures optimized for flexibility. Depending on the constraints, some of these optimized structures are known to resemble mechanisms. It is of interest in this work to understand the mathematical aspects of it along with the numerical implementation of a programme to optimize the structure. Different variations of the problem (optimization for flexibility) are studied, they are listed below:

1. Optimization of a truss for deflection at a particular node, subject to cumulative weight constraint.

2. Optimization of a truss for geometric advantage, subject to cumulative weight constraint.

3. Optimization of a truss for cumulative weight, subject to deflection constraints.

2 Optimization of a truss for displacement at a particular point

The sole objective of this program is to maximize the displacement of a particular node in a particular direction in response to a given load, subject to volume constraints.

2.1 Problem statement

Let us consider a 2-D truss of n nodes and m elements. Let the area, modulus of elasticity and length of element i be represented by A_i, E_i and L_i respectively. Vector notation will be adopted for these entities. In the equation below, $\mathbf{K}$, V^*, $\bar{A_{lb}}$ and $\bar{P}$ represent the global stiffness matrix, volume constraint, lower bound of area for each element and applied load respectively. Here $\bar{Q}$ represents the applied dummy load. Here, the desired direction of deflection in node i must be reflected in the sign of Q_i. The problem is stated below:

$$\begin{aligned} \underset{\bar{A}}{\text{minimize}} \quad & \bar{u} \cdot \mathbf{K}\bar{v} \\ \text{subject to} \quad & \mu: \ \bar{l} \cdot \bar{A} - V^* \leq 0, \\ & \bar{\lambda}: \ \mathbf{K} \cdot \bar{u} - \bar{P} = 0, \\ & \bar{\alpha}: \ \mathbf{K} \cdot \bar{v} - \bar{Q} = 0, \\ & \bar{\gamma}: \ \bar{A_{lb}} - \bar{A} \leq 0, \\ & \bar{\delta}: \ \bar{A} - \bar{A_{ub}} \geq 0 \end{aligned} \tag{1}$$

Here μ, $\bar{\lambda}$, $\bar{\alpha}$, $\bar{\gamma}$ and $\bar{\delta}$ are the Lagrange variables. It must be noted that the minimization would lead to the **maximization** of displacement opposite to the assigned direction (Such a convention is used so as to make the comparison with the special case v=u or Q=P).

2.2 Analytical solution

Solving the first order derivative condition of the Lagrange variable will results in the following equation:

$$\mu\bar{l} + \bar{u'} \cdot \mathbf{K}\bar{v} + \bar{u} \cdot \mathbf{K}\bar{v'} + \bar{u} \cdot \mathbf{K'}\bar{v} + \bar{\lambda} \cdot \mathbf{K'}\bar{u} + \bar{\lambda} \cdot \mathbf{K}\bar{u'} + \bar{\alpha} \cdot \mathbf{K'}\bar{v} + \bar{\alpha} \cdot \mathbf{K}\bar{v'} - \bar{\gamma} + \bar{\delta} = 0. \tag{2}$$

Here, ' denotes vector differentiation (Gradient) with respect to the area ($\bar{A}$) variable. From the adjoint equation (equating components with u' and v' to zero) and the invertibility of global stiffness matrix,

$$\begin{aligned} \bar{\lambda} &= -\bar{v}, \\ \bar{\alpha} &= -\bar{u}. \end{aligned} \tag{3}$$

This leads to the following solution,

$$\mu\bar{l} = \bar{u} \cdot \mathbf{K'}\bar{v} + \bar{\gamma} - \bar{\delta}. \tag{4}$$

To better understand it, let us consider the case of a statically determinate beam, where we know **K'** is constant and in the local stiffness matrix $\frac{\partial K_l}{A_i}$=K_l/A_i. This would suggest that whenever the area lower or upper bound constraint is inactive, the mutual energy density is uniform. Where mutual energy refers to displacement at the node for unit single component load $\bar{v}$ ($\bar{u} \cdot \mathbf{K}\bar{v}$) and the mutual energy density of each element is its contribution to the displacement of the node per volume of material.

Unlike in the case for optimization of mean compliance, which lead to similar uniformity of strain energy density. However, strain energy density is non-negative, which is not true for mutual energy density; thus, for the K.K.T condition, which forces μ to be non-negative, the lower bound constraints must be active for the case when the element's mutual energy density is negative.

To summarize,

1. If the upper bound for the area is not active, then
   - the area of each element with positive mutual strain energy density will have uniform strain energy density.
   - the area of each element with negative mutual strain energy density will have a lower bound value for the area.
2. If the upper bound constraint of an element is active, the area must be the upper bound value.

2.3 Numerical implementation using optimality constraint method

For the optimality constraint method, we assume the weight constraint to be active, and thus, this would lead to form for calculating the Lagrange variable mu in terms of $\bar{\gamma}$ and area at the current iteration.
The procedure of the numerical implementation is summarized below:

1. Initialize $\bar{\gamma}$, $\bar{\delta}$ with zeros and $\bar{A}^0$ with $V^*/\Sigma\bar{l}$ and μ as 1.
2. Solve for displacements in response to loads P and Q individually.
3. Calculate a temporary variable as shown below:
$$temp_i = \bar{u} \cdot \frac{\partial K}{\partial A_i}\bar{v}/l_i + \gamma_i - \delta_i. \tag{5}$$
4. If *temp* is less than $\mu A_{i,lb}/A_i$, then update gamma as
$$\gamma_i = \mu A_{i,lb}/A_i - \bar{u} \cdot \frac{\partial K}{\partial A_i}\bar{v}/l_i, \tag{6}$$
$$temp_i = \mu A_{i,lb}/A_i. \tag{7}$$
5. Calculate μ using the following relation:
$$\mu = (\Sigma temp_i A_i l_i) / (wV^*). \tag{8}$$
6. Update the Area of each element by following relation:
$$A_m^{i+1} = A_m^i \left((1-w) + w\ temp_m/\mu\right), \tag{9}$$
   where w is the relaxation parameter and superscript denotes iteration count.
7. Check that the area satisfies the upper and lower bound constraint; else, update the area with the respective values and repeat steps 5 and 6.

2.4 Example-1: Barnett truss

The schematic of a Barnett truss is shown below. The goal is to optimize for the deflection at the upper right node in the positive y direction for a load applied in the bottom middle node in the lower y direction. To better illustrate the problem, the response of a truss member with a uniform cross-sectional area is shown in Fig 1(b).
The important observations are listed below:

1. We note that the optimal structure is statically under determinate (almost as the area of the missing element is A_{lb}). Thus, the structure behaves like a mechanism.

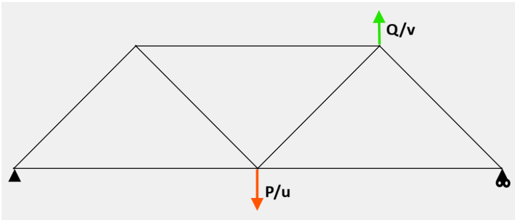
2. From Fig 1(e), we notice that the mutual strain energy of each element with an area not affected by constraints is uniform. In the figure, element 5 has an active lower bound constraint and thus, gamma is non-zero and only the sum of mutual strain energy and gamma amount to the uniform value.

3. From Fig 1(f), we observe that the solution has converged and a local minima has been found.

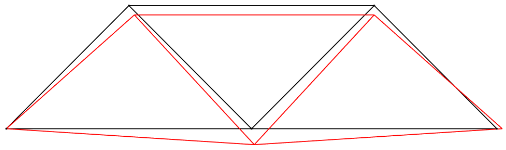
4. The geometric advantage (ratio of output displacement to displacement in the direction of load where load is applied) of the structure is found to be 0.488.

2.5 Example-2: Mechanisms arising from flexibility optimization of 3×3 ground structure

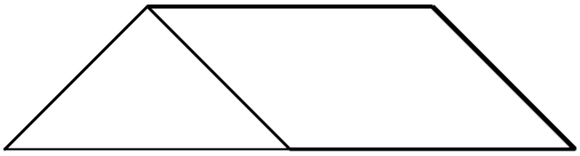
The mechanisms arising from flexibility optimization are presented in Fig 2.



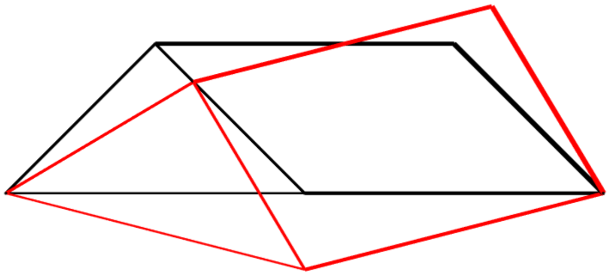
(a) Schematic of the problem (Optimization for deflection at the marked point)



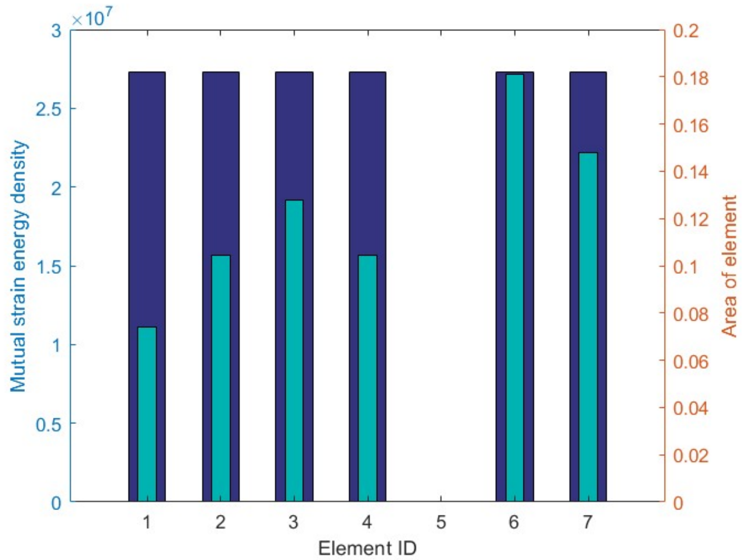
(b) Typical response when all members have uniform cross section



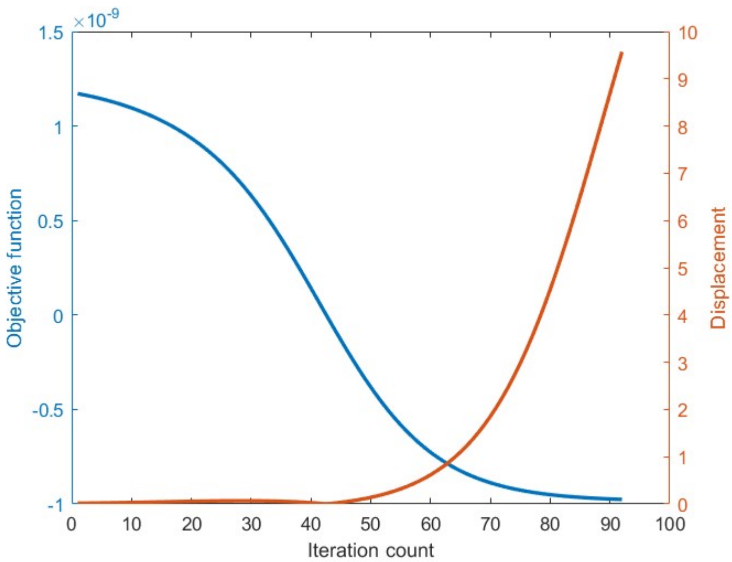
(c) Optimal structure



(d) Deformed structure (appropriately scaled)



(e) Bar chart of Mutual strain energy density of each element and respective area of element



(f) Convergence history

Figure 1: Barnett truss (thickness of each element is representative of its area)

3 Optimization of a truss for geometric advantage

The sole objective of this program is to maximize the ratio of displacement of a particular node in a particular direction in response to a given load and the displacement at the node where the load is applied, subject to volume constraints. In cases where only one load is applied and one one output displacement is desired, it maximizes the geometric advantage (ratio of output displacement to input displacement)

3.1 Problem statement

The notations listed below have same meaning as in section 2.

$$\begin{aligned} \underset{\bar{A}}{\text{minimize}} \quad & (\bar{u}\cdot\mathbf{K}\bar{v})/(\bar{u}\cdot\mathbf{K}\bar{u}) \\ \text{subject to} \quad & \mu:\ \bar{l}\cdot\bar{A}-V^{*}\leq 0, \\ & \bar{\lambda}:\ \mathbf{K}\cdot\bar{u}-\bar{P}=0, \\ & \bar{\alpha}:\ \mathbf{K}\cdot\bar{v}-\bar{Q}=0, \\ & \bar{\gamma}:\ \bar{A_{lb}}-\bar{A}\leq 0, \\ & \bar{\delta}:\ \bar{A}-\bar{A_{ub}}\geq 0 \end{aligned} \tag{10}$$

Here μ, $\bar{\lambda}$, $\bar{\alpha}$, $\bar{\gamma}$ and $\bar{\delta}$ are the Lagrange variables. It must be noted that the minimization would lead to the **maximization** of displacement opposite to the assigned direction (Such a convention is used so as to make the comparison with the special case v=u or Q=P).

3.2 Analytical solution

Solving the first order derivative condition of the Lagrange variable will results in the following equation along with the adjoint equation (equating components with u' and v' to zero) and the invertibility of global stiffness matrix,

$$\begin{aligned} &\bar{\lambda}=(\bar{v}(\bar{u}\cdot\mathbf{K}\bar{u})-2\bar{u}(\bar{u}\cdot\mathbf{K}\bar{v}))/(\bar{u}\cdot\mathbf{K}\bar{u})^{2}, \\ &\bar{\alpha}=-\bar{u}/(\bar{u}\cdot\mathbf{K}\bar{u}). \end{aligned} \tag{11}$$

This leads to the following solution,

$$\mu\bar{l}=\frac{(\bar{v}\cdot\mathbf{K'}\bar{u}(\bar{u}\cdot\mathbf{K}\bar{u})-\bar{u}\cdot\mathbf{K'}\bar{u}(\bar{u}\cdot\mathbf{K}\bar{v}))}{(\bar{u}\cdot\mathbf{K}\bar{u})^{2}}+\bar{\gamma}-\bar{\delta}. \tag{12}$$

To better understand it, let us consider the case of a statically determinate beam, where we know **K'** is constant and in the local stiffness matrix $\frac{\partial K_l}{A_i}=K_l/A_i$. This would suggest that whenever the area lower or upper bound constraint is inactive, the first term in equation 12 (referred to as β_i from now)is uniform. It can thus be inferred that it is the contribution of each element to the objective function (ratio of MSE and TSE) per unit volume.

Unlike in the case for optimization of mean compliance, which lead to similar uniformity of strain energy density. However, strain energy density is non-negative, which is not true for mutual energy density; thus, for the K.K.T condition, which forces μ to be non-negative, the lower bound constraints must be active for the case when the element's contribution term is negative.

To summarize,

1. If the upper bound for the area is not active, then
   - the area of each element with positive β_i will have uniform strain energy density.
   - the area of each element with negative β_i will have a lower bound value for the area.
2. If the upper bound constraint of an element is active, the area must be the upper bound value.

3.3 Numerical implementation using optimality constraint method

For the optimality constraint method, we assume the weight constraint to be active, and thus, this would lead to form for calculating the Lagrange variable mu in terms of $\bar{\gamma}$ and area at the current iteration.
The procedure of the numerical implementation is summarized below:

1. Initialize $\bar{\gamma}$, $\bar{\delta}$ with zeros and $\bar{A}^0$ with $V^*/\Sigma\bar{l}$ and μ as 1.

2. Solve for displacements in response to loads P and Q individually.

3. Calculate a temporary variable as shown below:

$$temp_i = \beta_i/l_i + \gamma_i - \delta_i. \tag{13}$$

4. If *temp* is less than $\mu A_{i,lb}/A_i$, then update gamma as

$$\gamma_i = \mu A_{i,lb}/A_i - \beta_i/l_i, \tag{14}$$

$$temp_i = \mu A_{i,lb}/A_i. \tag{15}$$

5. Calculate μ using the following relation:

$$\mu = (\Sigma temp_i A_i l_i)/\left(wV^*\right). \tag{16}$$

6. Update the Area of each element by following relation:

$$A_m^{i+1} = A_m^i\left((1-w) + w\ temp_m/\mu\right), \tag{17}$$

where w is the relaxation parameter and superscript denotes iteration count.

7. Check that the area satisfies the upper and lower bound constraint; else, update the area with the respective values and repeat steps 5 and 6.

3.4 Example-1: Barnett truss

The Barnett truss is once again optimized for displacement. The schematics are shown in Fig 3.
The important observations are listed below:

1. We note that the optimal structure is statically under determinate (almost as the area of the missing element is A_{lb}). Thus, the structure behaves like a mechanism.

2. The geometric advantage (ratio of output displacement to displacement in the direction of load where the load is applied) of the structure is found to be 0.494, which is noticeably higher than the geometric advantage calculated in section 2 (0.488). The difference can be visualized in Fig 3(c).

3. From Fig 3(d), we notice that the β per unit volume of each element with an area not affected by constraints is uniform. In the figure, element 5 has an active lower bound constraint and thus, gamma is non-zero and only the sum of mutual strain energy and gamma amount to the uniform value.

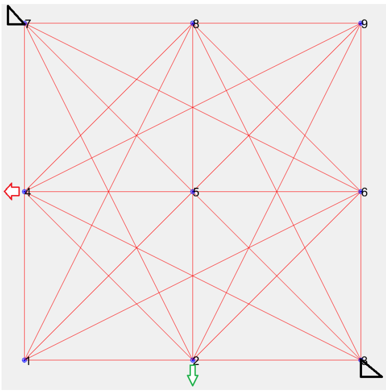
4. From Fig 3(e), we observe that the solution has converged and a local minima has been found.

3.5 Comments on sufficiency condition

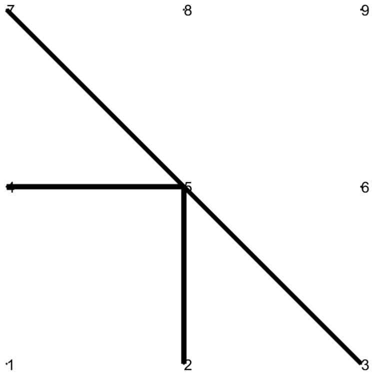
Here, from Fig 3(c), we noted that the obtained solution is a local minimum, but such a solution does not always exist, as in this case. As the ratio of two convex functions is not necessarily convex. The existence of a solution to this problem is not always guaranteed.

4 Conclusion

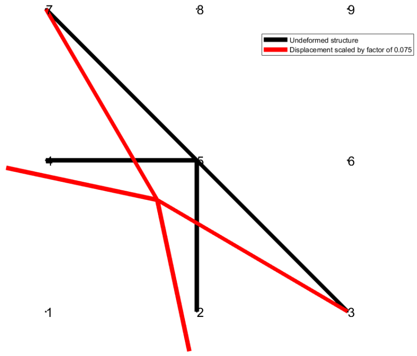
In this report, details of the programme for optimization of 2-D for desired flexibility are summarized along with a few examples.



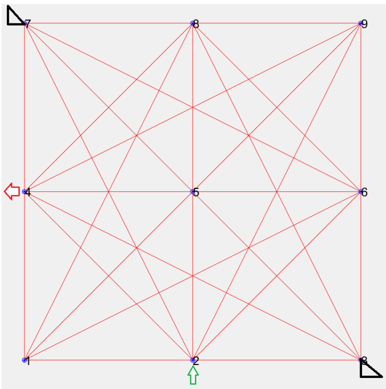
(a) Case-1: Loading diagram



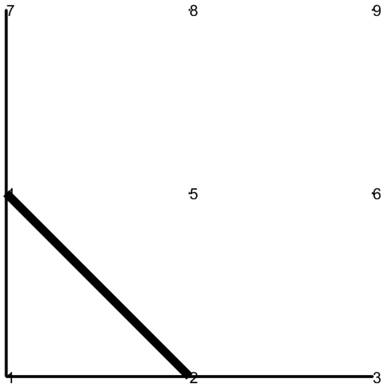
(b) Case-1: Optimal solution



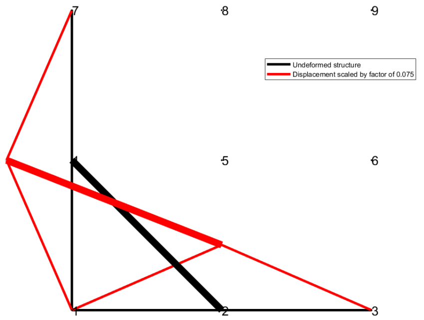
(c) Case-1: Displacement pattern (Geometric advantage: 0.996)



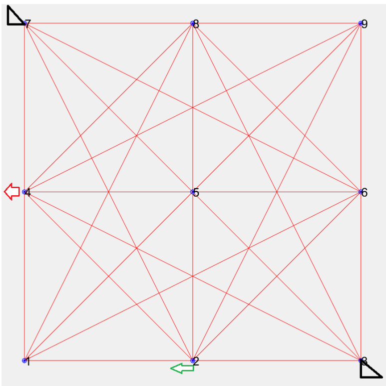
(d) Case-2: Loading diagram



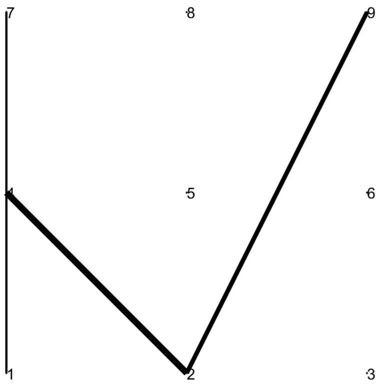
(e) Case-2: Optimal solution



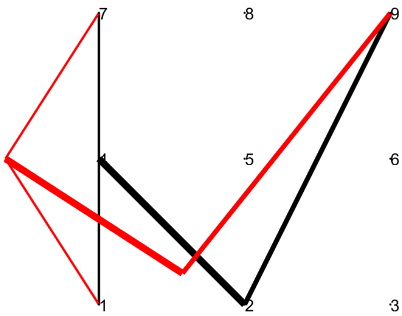
(f) Case-2: Displacement pattern (Geometric advantage: 0.997)



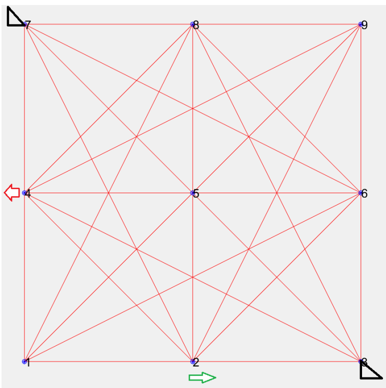
(g) Case-3: Loading diagram



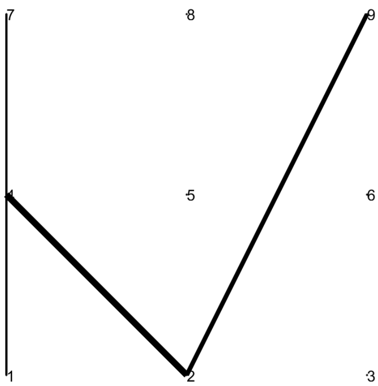
(h) Case-3: Optimal solution



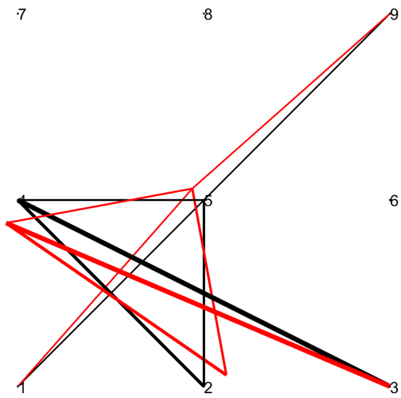
(i) Case-3: Displacement pattern (Geometric advantage: 0.665)



(j) Case-4: Loading diagram

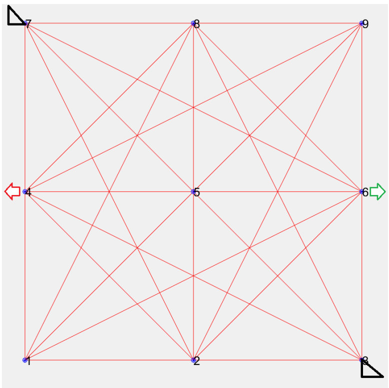


(k) Case-4: Optimal solution

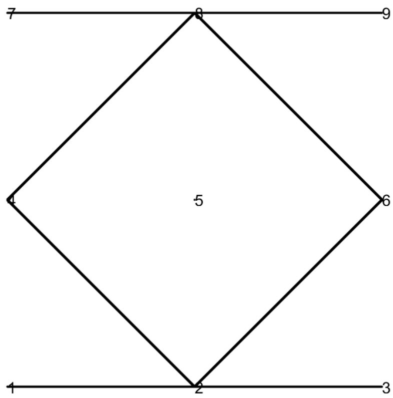


(l) Case-4: Displacement pattern (Geometric advantage: 1.91)

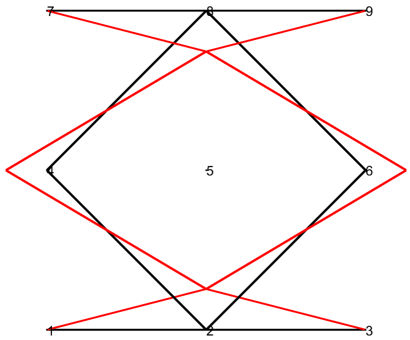
Figure 2: Mechanisms arising from flexibility optimization of 3×3 ground structure



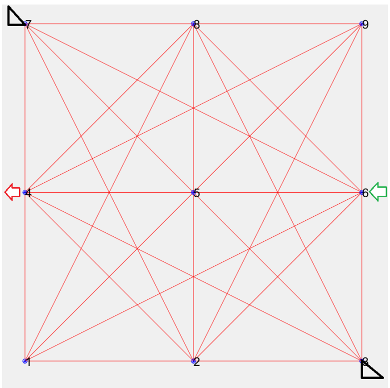
(m) Case-5: Loading diagram



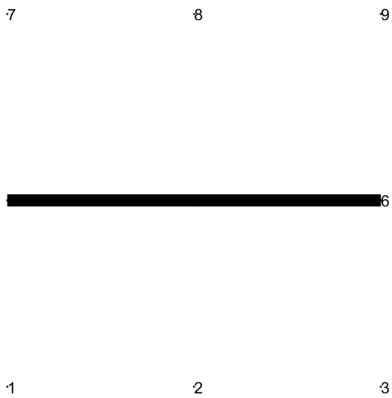
(n) Case-5: Optimal solution



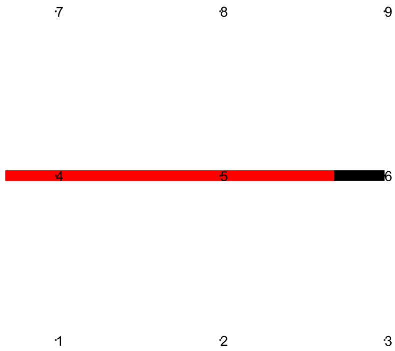
(o) Case-5: Displacement pattern (Geometric advantage: 0.99)



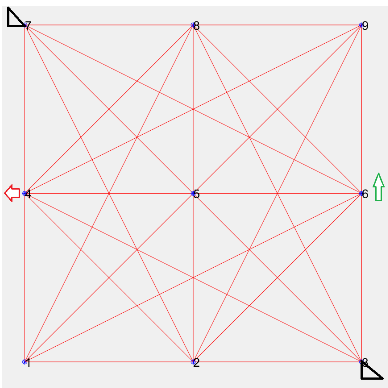
(p) Case-6: Loading diagram



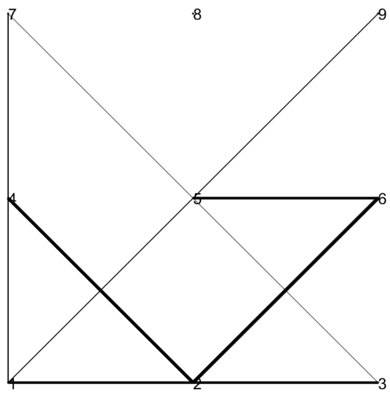
(q) Case-6: Optimal solution



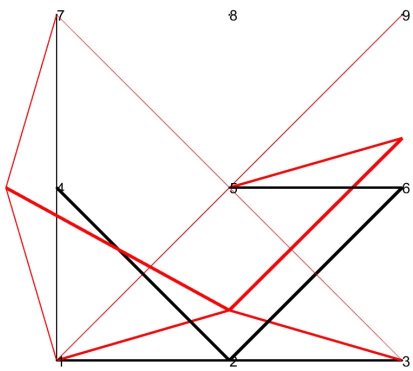
(r) Case-6: Displacement pattern (Geometric advantage: 1)



(s) Case-7: Loading diagram

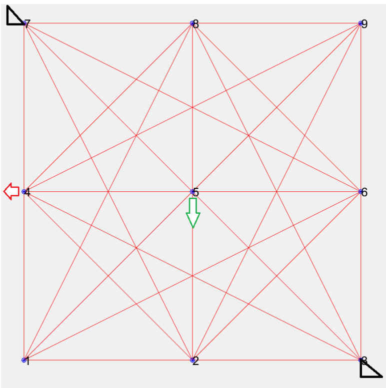


(t) Case-7: Optimal solution

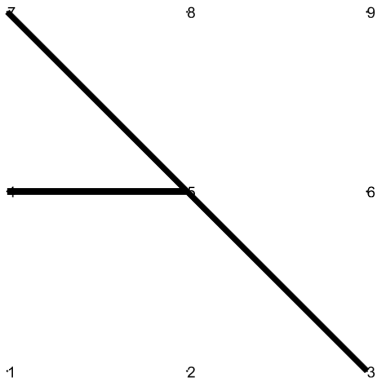


(u) Case-7: Displacement pattern (Geometric advantage: 0.977)

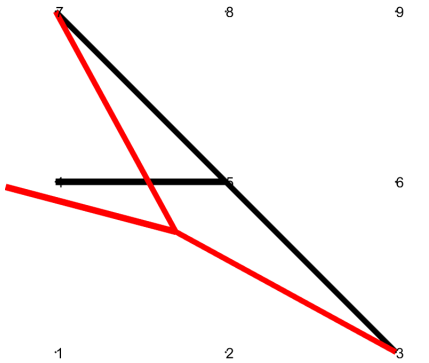
Figure 2: Mechanisms arising from flexibility optimization of 3×3 ground structure



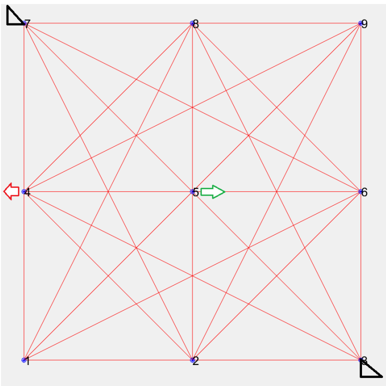
(v) Case-8: Loading diagram



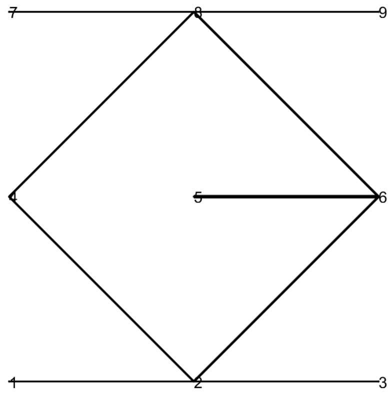
(w) Case-8: Optimal solution



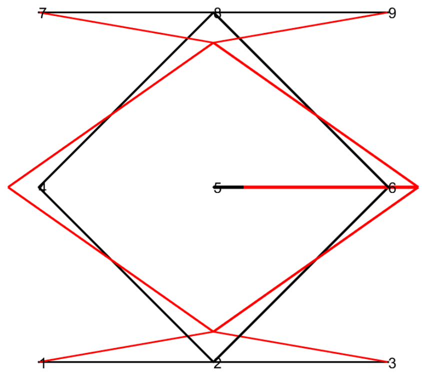
(x) Case-8: Displacement pattern (Geometric advantage: 0.997)



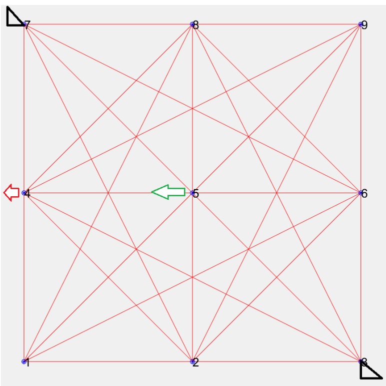
(y) Case-9: Loading diagram



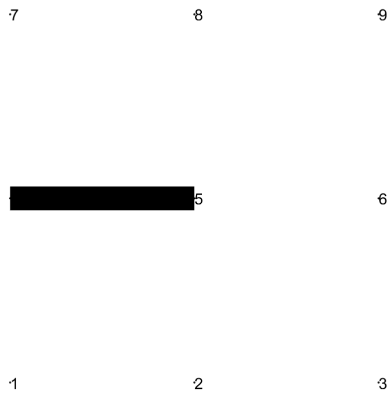
(z) Case-9: Optimal solution



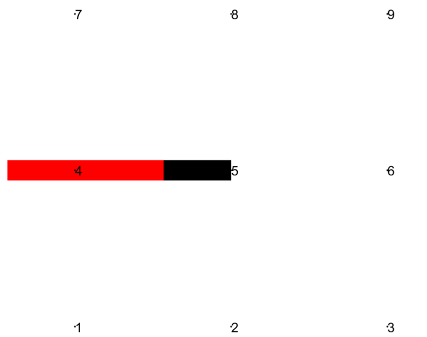
(aa) Case-9: Displacement pattern (Geometric advantage: 0.978)



(ab) Case-10: Loading diagram

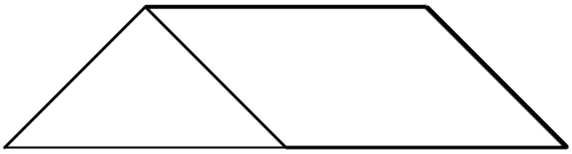


(ac) Case-10: Optimal solution

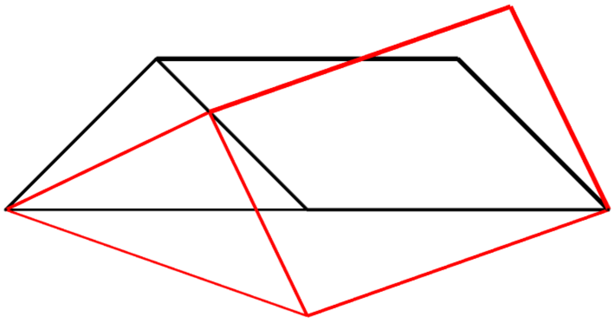


(ad) Case-10: Displacement pattern (Geometric advantage: 1)

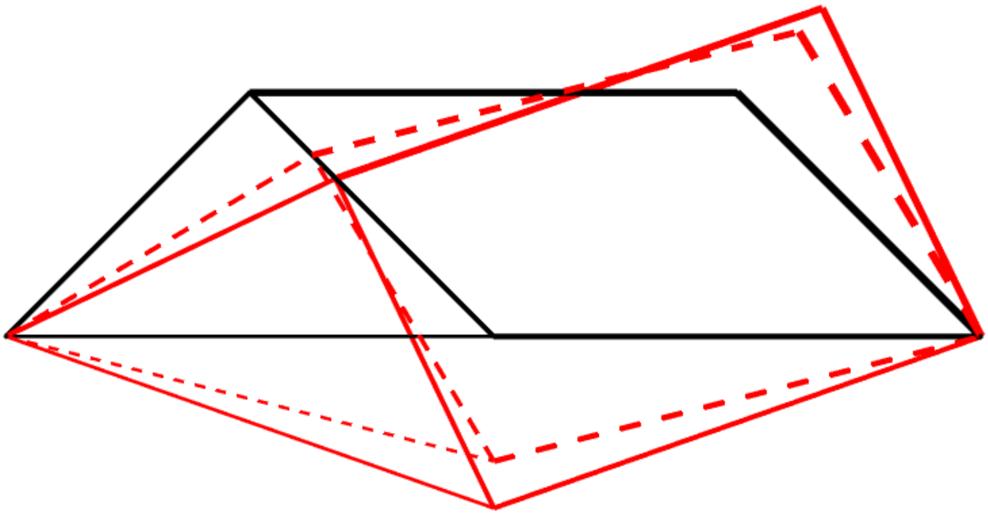
Figure 2: Mechanisms arising from flexibility optimization of 3×3 ground structure



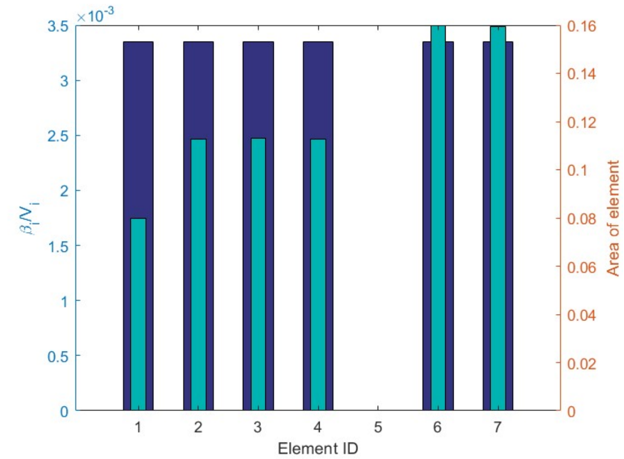
(a) Optimal structure



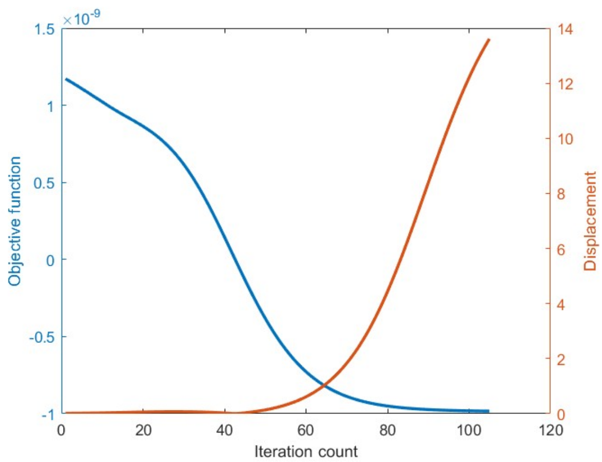
(b) Deformed structure (appropriately scaled)



(c) Comparison of solution from section 2 and section 3 (scaled appropriately for visualization); Dashed lines indicate results from section-2 (Geometric advantage=0.488), Solid lines indicate results from section-3 (Geometric advantage=0.494);



(d) Bar chart of β per unit volume of each element and respective area of element



(e) Convergence history

Figure 3: Barnett truss optimized for geometric advantage (thickness of each element is representative of its area)