Bifurcation in dynamical systems

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MA511L Ordinary Differential Equations

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Outline

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Overview

Bifurcation

- Bifurcations in a dynamical system (system of ODEs) describe the qualitative change in behavior of a system sensitive to change of some parameters of the system.
- The systems characteristics are non-extrapolate-able at the point of bifurcation.

Example

Consider the system $\dot{x} = f(t, x, \mu)$ with initial conditions $x(t = 0) = x_0.$, where μ is a system parameter and f is defined as

$$f(t, x, \mu) = \mu - x^2. \tag{1}$$

Depending on the value of the μ, the structure of the solution is highly different.

Saddle-node bifurcation

Let us consider the following cases,

Case-1: $\mu > 0$

$$\frac{1}{\sqrt{\mu}} log \frac{|\sqrt{\mu} + x|}{|\sqrt{\mu} - x|} = t + \frac{1}{\sqrt{\mu}} log \frac{|\sqrt{\mu} + x_0|}{|\sqrt{\mu} - x_0|}, \tag{2}$$

Case-2: $\mu < 0$

$$\lambda = -\mu \tag{3}$$

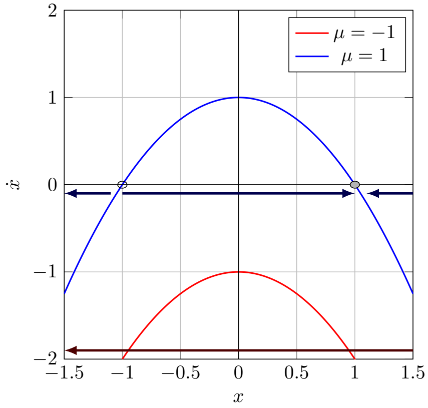
$$\frac{1}{\sqrt{\lambda}} \tan^{-1}\left(\frac{x}{\sqrt{\lambda}}\right) = -t + \frac{1}{\sqrt{\lambda}} \tan^{-1}\left(\frac{x_0}{\sqrt{\lambda}}\right), \tag{4}$$

Phase portraits

Fixed points

Fixed points of an ODE denote the value of the initial conditions such that the system (x) remains stationary, i.e. $\dot{x} = 0$.

- For $\mu > 0$, the fixed points are $\pm\sqrt{\mu}$,
  - $+\sqrt{\mu}$ corresponds to stable solution,
  - $-\sqrt{\mu}$ corresponds to unstable solution,
- For $\mu < 0$, the fixed points are $\pm i\sqrt{|\mu|}$, no fixed points in the real domain ($x_{fp} \notin R^1$).



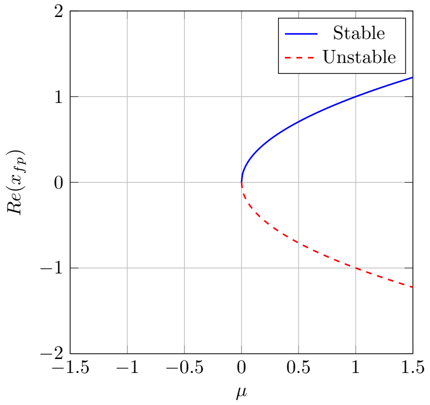
Bifurcation diagram

Bifurcation at $\mu = 0$

It could be observed that extrapolating results about the bifurcation point would provide unacceptable results.

Note

It is interesting to observe that the system is still continuous about the bifurcation point if the complex domain is considered, the fundamental change arises due to the inability to perform physical measurements of imaginary values.

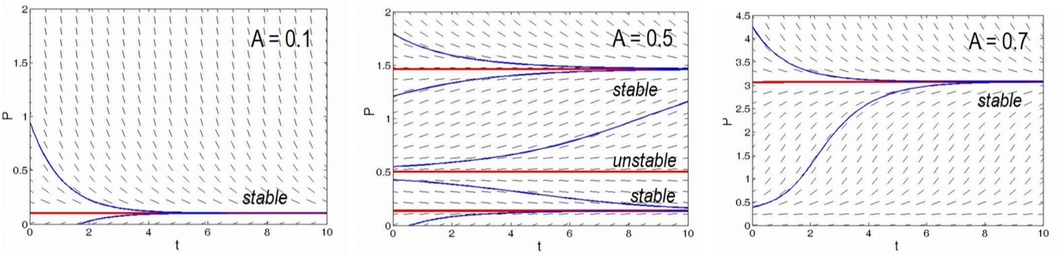


Outline

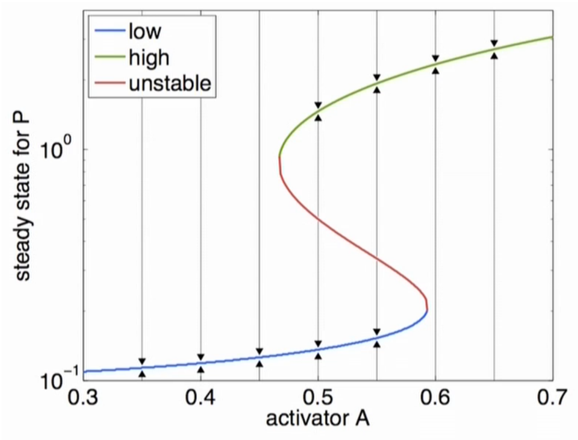
Protein stimuli interaction

Reaction dynamics

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.1 + 10\frac{A^2}{1+A^2} \cdot \frac{P^2}{1+P^2} - P \qquad (5)$$



Bifurcation Diagram[1]



[1] Image credit:- Introduction to Dynamical Models in Biology, NPTEL, IIT Guwhati

Outline

Random number generator

Logistic difference- differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \gamma x = \beta x(1-x) \tag{6}$$

At steady state $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$.

Logistic difference- discrete dynamical system

$$\frac{X_{n+1} - X_n}{\Delta t} + \gamma X_n \approx \beta X_n(1 - X_n) \tag{7}$$

At steady state $X_{n+1} = X_n$ and $n \in N$.

Random number generator (contd..)

Logistic difference- discrete dynamical system

$$\frac{X_{n+1} - X_n}{\Delta t} + \gamma X_n \approx \beta X_n (1 - X_n) \tag{8}$$

$$X_{n+1} = \beta^* X_n (1 - X_n) + (1 - \gamma^*) X_n \tag{9}$$

Let us consider $\gamma^* = 1$ and $\beta^* = \alpha$,

$$X_{n+1} = \alpha X_n (1 - X_n) \tag{10}$$

Random number generator (contd..)

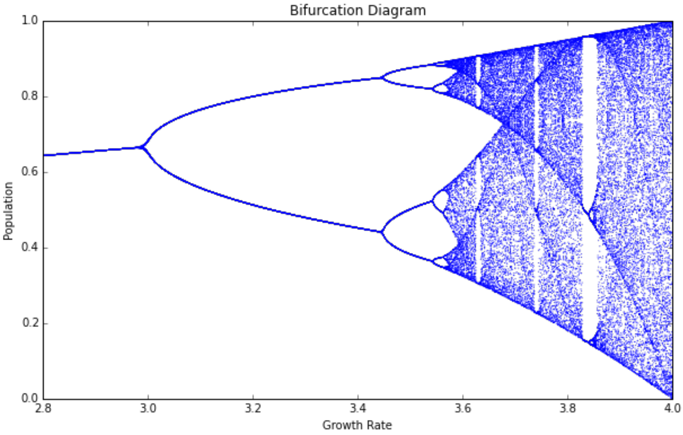
Logistic difference- discrete dynamical system

$$X_{n+1} = \alpha X_n(1 - X_n) \tag{11}$$

$$X^* = 0, 1 - \frac{1}{\alpha} \tag{12}$$

For stable soln, $|\frac{\mathrm{d}X_{n+1}}{\mathrm{d}X_n}|_{X_n=X^*} < 1$. Thus $X^* = 0$ a stable solution for $0 < \alpha < 1$, and $X^* = 1 - \frac{1}{\alpha}$, a stable solution for $1 < \alpha < 3$. For $\alpha > 3$, the chaotic nature of the system increases.

Random number generator (contd..)



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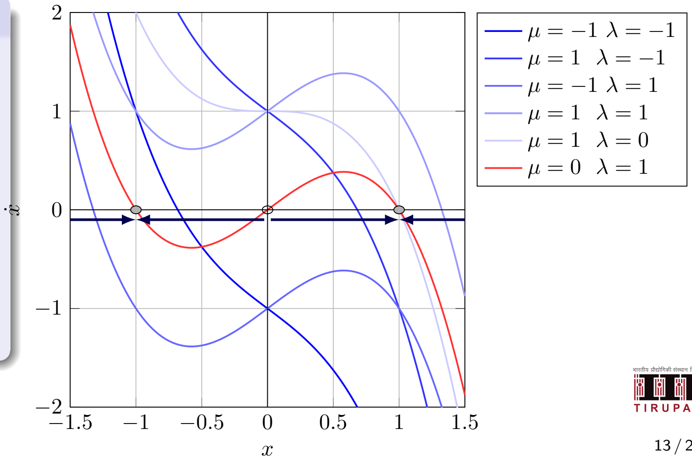
Cusp Bifurcation

Dynamical system

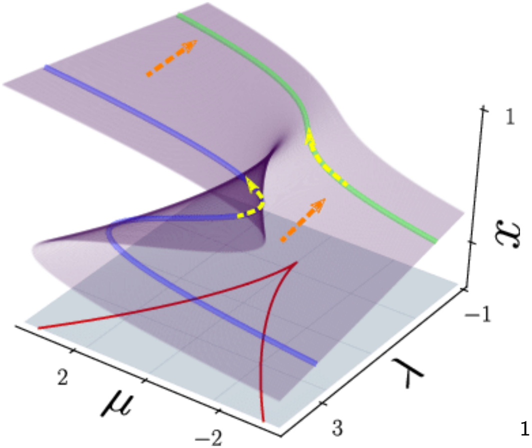
Cusp bifurcation is observed in some dynamical systems with two parameters, one particular example of such bifurcation in the following ODE,

$$\dot{x} = \mu + \lambda x - x^3, \qquad (13)$$

with initial condition $x(t=0) = x_0$.



Bifurcation diagram



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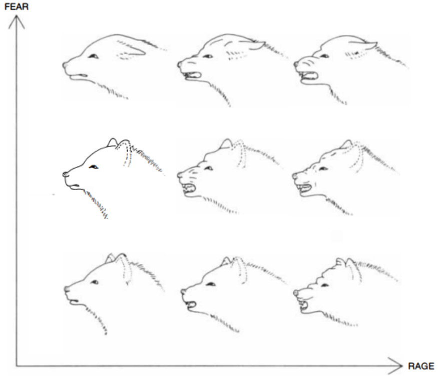
[1]Image source: Dietrich, et al, (2018). Manifold Learning for Bifurcation Diagram Observations.

Zeeman's Catastrophe theory

In his paper published by Zeeman in 1976, he attempted to explain sudden transitions in physical systems. He also illustrated the capability of cusp bifurcation to explain

1. human psychology
2. aggression in dogs
3. Coupled non-linear dynamical systems (e.g. catastrophe machines).

Aggression in dogs



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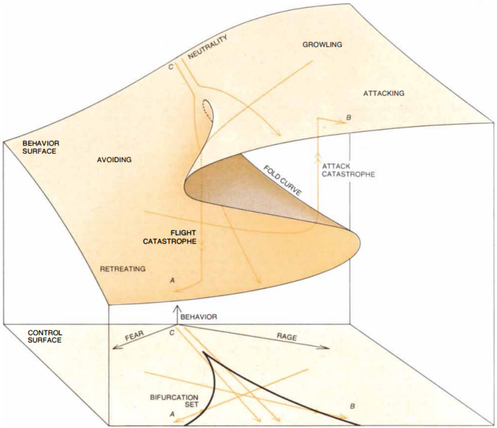
[1] Image credit: Zeeman, E. C. (1976). Catastrophe Theory. Scientific American, 234(4), 65–83.

Aggression in dogs (contd..)



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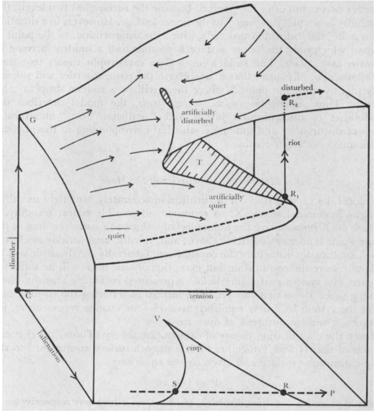
[1]Image credit: Zeeman, E. C. (1976). Catastrophe Theory. Scientific American, 234(4), 65–83.

Prediction of prison riots



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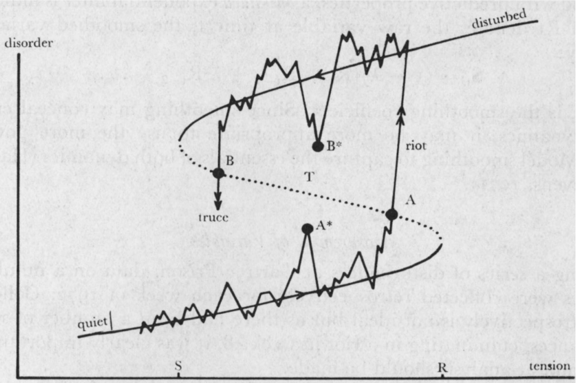
[2]Image credit: Zeeman, E. C., et al. "A MODEL FOR PRISON DISTURBANCES."

Bifurcation diagram



[3]

[3] Image credit: Zeeman, E. C., et al. "A MODEL FOR PRISON DISTURBANCES."

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Conclusion and perspectives

Conclusion

- This presentation gives an overview of bifurcation, with examples of bifurcation in biological, population dynamics and general behaviour dynamics.

Thank You!