Identifying Center of Elasticity of an Assembly of Beam Elements

ME 254: Compliant Mechanisms

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Overview

Outline

Stiffness matrix

Let us consider a solid (isotropic linear elastic) body subjected to transverse loads and moments at some point(s) in the body; then the displacement (translational and rotational) at some other point(s) can be represented as:

$$\{\boldsymbol{f}\} = [\boldsymbol{K}]\{\boldsymbol{u}\}$$

We know $[\boldsymbol{K}]$ is symmetric and positive semi-definite (in general). After restricting rigid body motion, the matrix is positive definite.

In mechanics, normal forms of symmetric tensors (such as Cauchy stress and Inertia matrix) are usually preferred for analysis and interpretation.

Stiffness matrix (contd..)

Consider the stiffness matrix for a cantilever beam:

$$[\boldsymbol{K}] = \begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c & d \end{bmatrix}$$

$$\implies [v1, v2, v3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2\,L}{3} - \frac{\sigma_2 + \sigma_1 + \mathrm{E\,I}\,L^2}{L\sigma_2} & \frac{2\,L}{3} - \frac{\sigma_2 - \sigma_1 + \mathrm{E\,I}\,L^2}{L\sigma_2} \\ 0 & 1 & 1 \end{bmatrix} \text{ and}$$

$$D = \begin{bmatrix} \frac{A\,\mathrm{E}}{L} & 0 & 0 \\ 0 & \frac{2\left(3\,\mathrm{E\,I} + \sigma_1 + \mathrm{E\,I}\,L^2\right)}{L^3} & 0 \\ 0 & 0 & \frac{2\left(3\,\mathrm{E\,I} - \sigma_1 + \mathrm{E\,I}\,L^2\right)}{L^3} \end{bmatrix}$$

where, $\sigma_1 = EI\sqrt{L^4 + 3L^2 + 9}^*$ and $\sigma_2 = 3EI$.

*Notice the incompatibility of dimensions!

Stiffness matrix (contd..)

As is usually done, let us attempt to do a rigid body transformation (only rotate the basis) to diagonalise the matrix.
Then

$$Q = [\hat{v}_1, \hat{v}_2, \hat{v}_3]$$

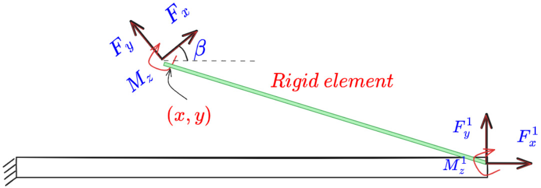
but this is not allowed!!!
As for frame elements, admissible Q for rigid body rotation in this case is :

$$Q = \begin{bmatrix} l & m & 0 \\ -m & l & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

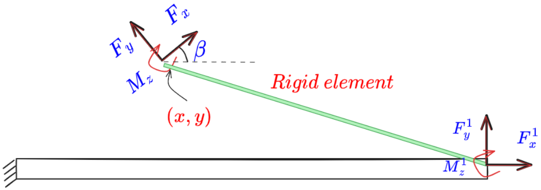
Stiffness matrix (contd..)

Thus, just rigid body rotation alone won't suffice here (unlike in the case of the inertia matrix). Thus, we also have to perform rigid body translation of our point of interest.

But what does that mean?

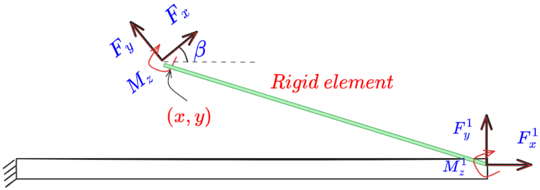


Stiffness matrix under RB transformation



$$F_y^1 = F_y \cos\beta + F_x \sin\beta$$
$$F_x^1 = F_x \cos\beta - F_y \sin\beta$$
$$M_z^1 = M_z - F_y^1(L - x) - F_x^1 y$$

Stiffness matrix under RB transformation (contd..)



$$\begin{Bmatrix} F_x^1 \\ F_y^1 \\ M_z^1 \end{Bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ -y\cos\beta - (L-x)\sin\beta & -(L-x)\cos\beta + y\sin\beta & 1 \end{bmatrix} \begin{Bmatrix} F_x \\ F_y \\ M_z \end{Bmatrix},$$

$$\begin{Bmatrix} u_x^1 \\ v_y^1 \\ \theta_z^1 \end{Bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta & l^*\sin\delta \\ \sin\beta & \cos\beta & l^*\cos\delta \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_x \\ v_y \\ \theta_z \end{Bmatrix},$$

where $\delta = \tan^{-1}(\frac{y}{L-x})$ and $l^* = \sqrt{(L-x)^2 + y^2}$.

Stiffness matrix under RB transformation (contd..)

$$\{\boldsymbol{f}^{\mathbf{1}}\} = [\boldsymbol{K}]\{\boldsymbol{u}^{\mathbf{1}}\}$$
$$\{\boldsymbol{f}\} = [\boldsymbol{K}']\{\boldsymbol{u}\}$$

where

$$[\boldsymbol{K}'] = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ -y\cos\beta - (L-x)\sin\beta & -(L-x)\cos\beta + y\sin\beta & 1 \end{bmatrix}^{-1} [\boldsymbol{K}] \begin{bmatrix} \cos\beta & -\sin\beta & l^*\sin\delta \\ \sin\beta & \cos\beta & l^*\cos\delta \\ 0 & 0 & 1 \end{bmatrix}$$

Stiffness matrix under RB transformation (contd..)

Let us consider the case $x = L/2$, $y = 0$, $\beta = 0$: where

$$[\boldsymbol{K}'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -L/2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & L/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\boldsymbol{K}'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & L/2 & 1 \end{bmatrix} \begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & L/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 0 \\ 0 & 0 & EI/L \end{bmatrix}$$

The stiffness matrix is diagonal!

Center of Elasticity

At this point, (x,y) rotation is decoupled from translation. This special point is known as *centre of elasticity*.

Comments on Center of Elasticity

- It need not always exist, and even if it exists is not necessarily unique[a].
- Center of elasticity may not always coincide with other geometric centers such as *center of stiffness* or *center of compliance* (whose existence is always guaranteed),
- But for the case of 2D geometries, CoE always exists[a] and coincides with these 2 centers.

[a] .*Functional Characterization of Compliant Building Blocks Utilizing Eigentwists and Eigenwrenches*. Vol. 32nd Mechanisms and Robotics Conference. Aug. 2008, pp. 175–182.

Center of Elasticity (contd..)

In 2D, 3 variables are needed to define a point (x, y, β). But how many equations do we have?

$$\{\boldsymbol{f}\}_{3\times1} = [\boldsymbol{K'}]_{3\times3}\{\boldsymbol{u}\}_{3\times1}$$

- **Case-1:** $f_x = 0,\ f_y = 0,\ m_z \implies u_x = 0,\ u_y = 0,\ \theta_z$ (2 equations)
- **Case-2:** $f_x = 0,\ f_y,\ m_z = 0 \implies u_x = 0,\ u_y,\ \theta_z = 0$ (2 equations)
- **Case-3:** $f_x,\ f_y = 0,\ m_z = 0 \implies u_x,\ u_y = 0,\ \theta_z = 0$ (2 equations)

We have six equations, of which 3 are degenerate (consequence of *Maxwell's Reciprocal theorem*). Thus, we have three equations and three unknowns; here the equations are nonlinear. But the existence of a solution is guaranteed.

$$\{\boldsymbol{f}\}_{6\times1} = [\boldsymbol{K'}]_{6\times6}\{\boldsymbol{u}\}_{6\times1}$$

The same can be extended for 3d too, but the solution is not guaranteed in this case.

Different kinds of geometric center

In 3d, consider the following stiffness matrix:

$$[\boldsymbol{K}]_{6\times 6} = \begin{bmatrix} \boldsymbol{A}_{3\times 3} & \boldsymbol{B}_{3\times 3} \\ \boldsymbol{B}^T_{3\times 3} & \boldsymbol{C}_{3\times 3} \end{bmatrix},$$

where $\boldsymbol{A}$ and $\boldsymbol{C}$ are symmetric.
If we can perform a rigid body transformation such that the off-diagonal matrix (B) in the new coordinate frame is symmetric (or equivalently diagonal), then that point is defined as *centre of stiffness*.

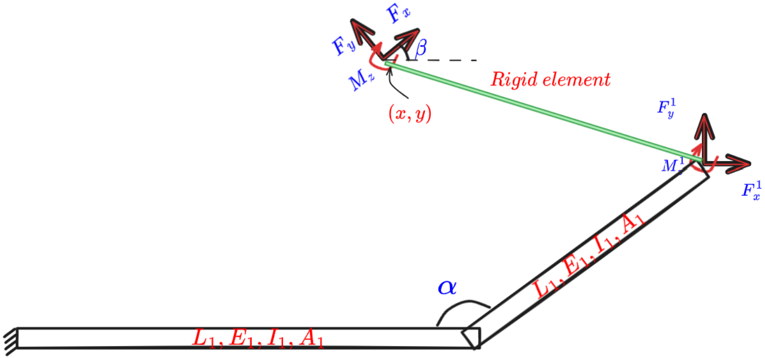
At this point, the rotation and translation components are said to be maximally decoupled[†] (if $\boldsymbol{B}$ can be made zero, this point is its centre of elasticity).

[†] J. Loncaric. "Normal forms of stiffness and compliance matrices". In: *IEEE Journal on Robotics and Automation* 3.6 (1987), pp. 567–572. DOI: 10.1109/JRA.1987.1087148.

Outline

Center of elasticity of two serial beams

Let us assume the except beam length, all other properties are identical.



Center of elasticity of two serial beams (contd..)

Procedure to find the new stiffness matrix

1. Perform rigid body rotation to represent the stiffness matrix of 2^{nd} beam element about global coordinates.
2. Assemble the global stiffness matrix and eliminate dofs with displacement constraints.
3. Invert the system to find the compliance matrix, now eliminate the dof corresponding to intermediate joints (having zero force).
4. Now, again invert the matrix to end up with the stiffness matrix relating the displacements and forces at the end point of the second beam.
5. Now follow the procedure earlier to perform rigid body transformation to diagnolise the stiffness matrix.

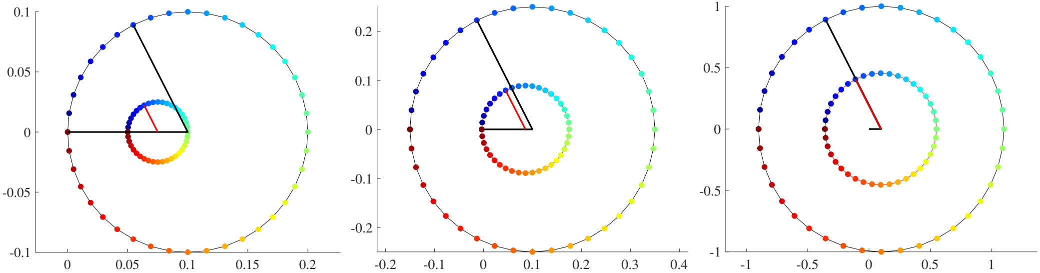
Center of elasticity of two serial beams (contd..)

Now we will assume E=1 Ga, A=100 mm^2, $I = 10^4/12$ mm^4,l_1=10 cm. The matrix relating forces in the original and new coordinate frame are:

$$\begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ -\Delta y\cos\beta - \Delta x\sin\beta & -\Delta x\cos\beta + \Delta y\sin\beta & 1 \end{bmatrix}^{-1} [\boldsymbol{K}] \begin{bmatrix} \cos\beta & -\sin\beta & l^*\sin\delta \\ \sin\beta & \cos\beta & l^*\cos\delta \\ 0 & 0 & 1 \end{bmatrix},$$

where $\Delta x = L_1 - L_2\cos(\alpha) - x$ and $\Delta y = y - L_2\sin\alpha$, $l^* = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ and $\delta = \tan^{-1}(\Delta y/\Delta x)$.

Center of elasticity of two serial beams (contd..)

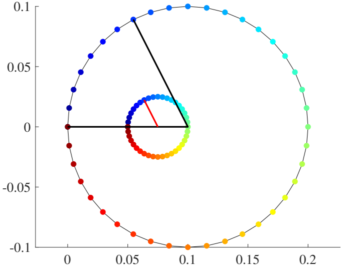


(a) $l_2/l_1 = 1$ (b) $l_2/l_1 = 2.5$ (c) $l_2/l_1 = 10$

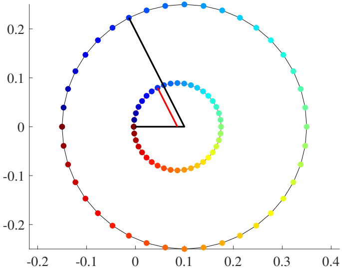
Figure: Inner circle: locus of center of elasticity, outer circle: locus of tip point

Center of elasticity of two serial beams (contd..)

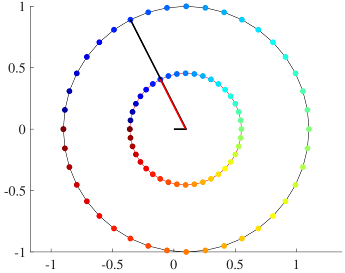
Notice anything interesting???



(a) $l_2/l_1 = 1$



(b) $l_2/l_1 = 2.5$



(c) $l_2/l_1 = 10$

Figure: Inner circle: locus of center of elasticity, outer circle: locus of tip point

Center of elasticity of two serial beams (contd..)

The circles are exact to the locus of *center of mass* of the beam (assumed uniform lengthwise distribution)!!!

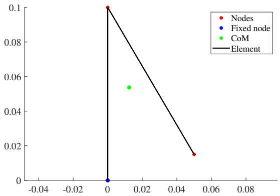
Both locus match because the center of mass and center of elasticity are equavilaent in this problem!

But, are they always identical?

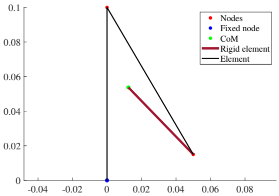
Center of elasticity of multiple serial beams

Proposition: For any mechanism which consists only of beams (of the same dimension) serially connected with one end fixed, the center of elasticity and center of mass are coincident.

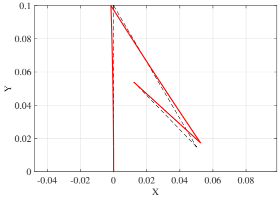
Center of elasticity of multiple serial beams (Example-1)



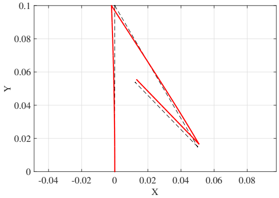
(a) Original mechanism



(b) Modified mechanism

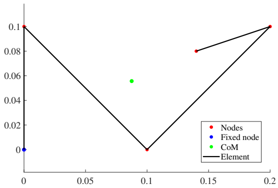


(c) Mechanism subject to pure moment (notice pure rotation about CoE)

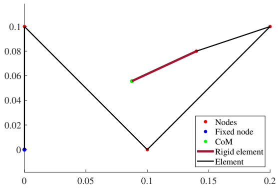


(d) Mechanism subject to pure transverse force (notice pure translation about CoE)

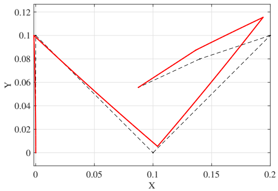
Center of elasticity of multiple serial beams (Example-2)



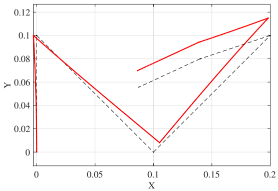
(a) Original mechanism



(b) Modified mechanism



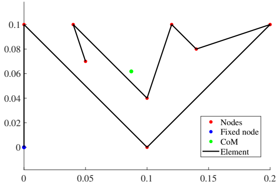
(c) Mechanism subject to pure moment (notice pure rotation about CoE)



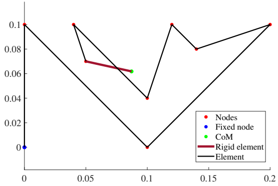
(d) Mechanism subject to pure transverse force (notice pure translation about CoE)

Center of elasticity of multiple serial beams (Example-3)

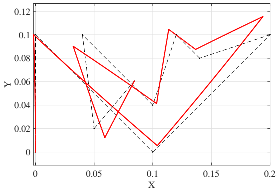
Let us overlook the intersection of beams (the intersection implies that it is not always possible to realise these structures only through serial assembly [uniqueness of CoE]).



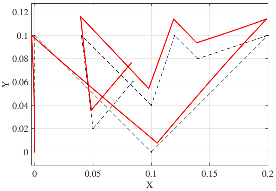
(a) Original mechanism



(b) Modified mechanism



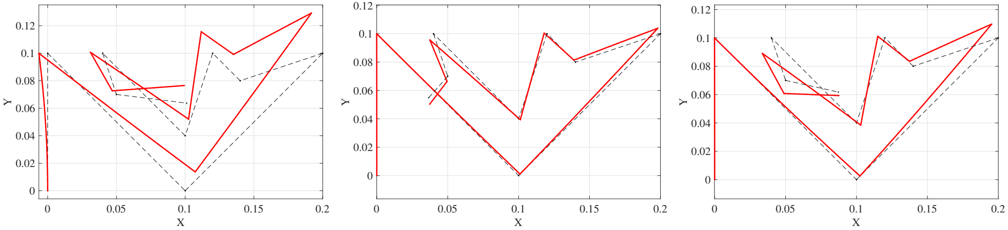
(c) Mechanism subject to pure moment



(d) Mechanism subject to pure transverse force

Center of elasticity of multiple serial beams (contd..)

What happens if the members are not uniform?



(a) Width of element 1 scaled down by factor of 10 (CoM≠CoE)

(b) Depth of element 1 scaled up by factor of 10 (CoM≠CoE)

(c) Depth of element 1 scaled up by factor of 10 (CoM*≠CoE)

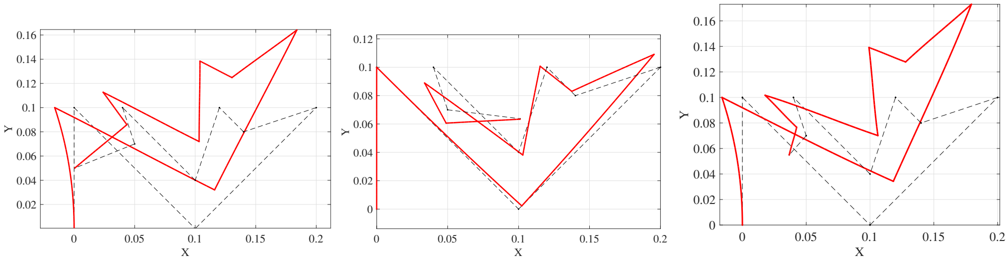
CoE for non-uniform members

The trick is that the weighing factor is $L/(EI)$, which is the rotational compliance of each element about its CoE!

*calculated by assuming uniform depth and width

Center of elasticity of multiple serial beams (contd..)

What happens if the members are not uniform?



(a) Width of element 1 scaled down by factor of 10

(b) Depth of element 1 scaled up by factor of 10

(c) Depth of element 1 scaled down by factor of 10

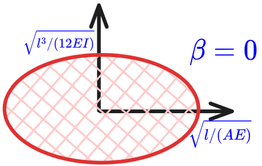
Outline

Stiffness/Compliance ellipsoids

Earlier, we had derived the stiffness matrix for a cantilever beam element about its CoE. The matrix is as follows:

$$\mathbf{K}' = \begin{bmatrix} AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 0 \\ 0 & 0 & EI/L \end{bmatrix}$$

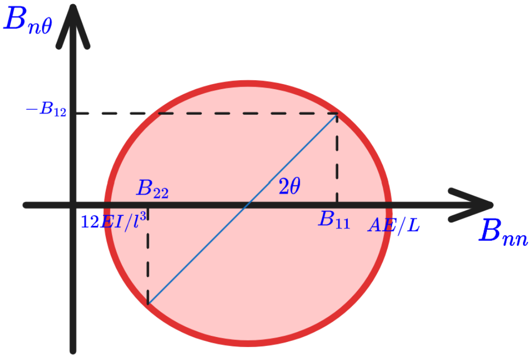
Now let us ignore the $\boldsymbol{K'}_{3,3}$ and consider only the top 2×2 sub-matrix ($\boldsymbol{B}_{2\times 2}$). Here, the rigid body rotation of the Cartesian basis is allowed for this matrix. Thus we will be able to generate an ellipse ($\boldsymbol{R'BR} = c$, $\boldsymbol{R} = [x, y]'$, let c=1) whose principal axis is oriented along the global x-axis at an angle $\beta = 0$.



Stiffness/Compliance ellipsoids (contd..)

Another useful way to look at this is through a Mohr circle transformation. β is not necessarily zero and we are usually more interested in reference to global coordinates. Let us represent the new stiffness matrix obtained from rotation of angle θ as:

$$\mathbf{B}_\theta = \begin{bmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{bmatrix}$$



Serial assembly of elements

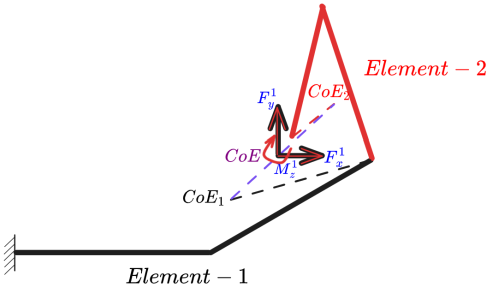
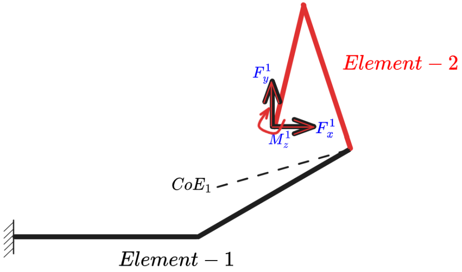
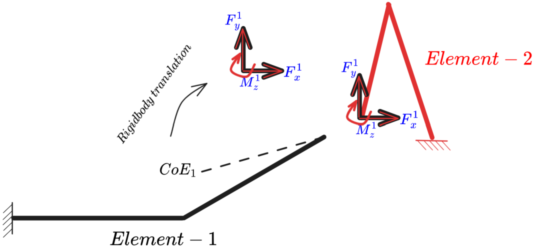


Figure: Element 2 is serially assembled with the terminal node of element 1. *CoE* of resultant structure lies in line joining CoE_1 and CoE_2

Serial assembly of elements (contd..)

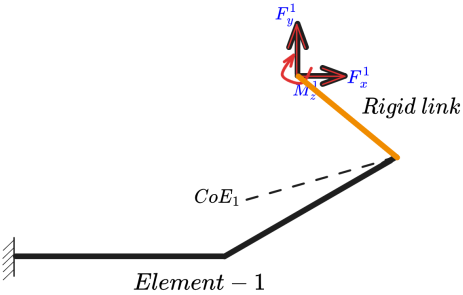


(a) The assembled element is assumed to be loaded about the terminal end of element-2

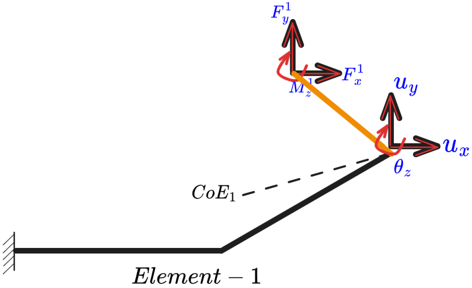


(b) The intersection joint is assumed fixed for element-2 and decoupled from element-1

Serial assembly of elements (contd..)



(a) For element-1, a rigid element is assumed to join the point to the load tip



(b) The displacements of the point due to the load can be calculated, and then rigid body transformation can be performed to find displacements at the other end of rigid link

Serial assembly of elements (contd..)

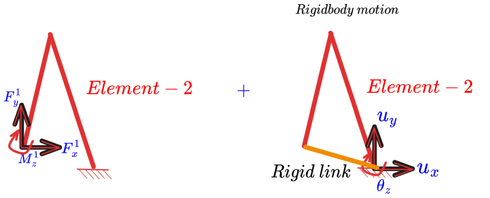


Figure: Now using the compliance matrix of element-2, the displacements of the loaded tip are found out and superimposed with the displacement at the loaded tip due to tip displacement of element-1 (constitutes rigid body motion)

Serial assembly of elements (contd..)

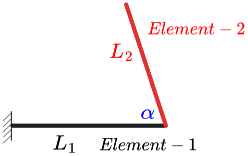
Thus, the net compliance matrix can be written as:

$$\boldsymbol{C}_{net,t} = \boldsymbol{C}_{2,t} + \boldsymbol{Q}\boldsymbol{C}_{\boldsymbol{1},\boldsymbol{CoE_1}}\boldsymbol{Q}'$$

where $\boldsymbol{C}_{net,o}$ is the compliance matrix relating tip displacement to tip load of the entire assembly, $\boldsymbol{C}_{2,t}$ is the compliance matrix relating tip displacement to tip load of the element-2 assuming the joint to be fixed, $\boldsymbol{C}_{\boldsymbol{1},\boldsymbol{CoE_1}}$ s the compliance matrix of element-1 about its CoE and Q is the transformation matrix from CoE_1 to the tip point.

The $\boldsymbol{C}_{net,t}$ can be now represented about the center of elasticity of the assembled structure by rigid body translation to its centroid (provided properties mentioned earlier are unifrom)

Example: Serial assembly of two beam elements



The coordinates of interest are:

$$
\begin{aligned}
\boldsymbol{r}_{CoE_1} &= [L_1/2, 0]' \\
\boldsymbol{r}_{CoE_2} &= [L_1 - L_2/2 \cos\alpha, L_2/2 \sin\alpha]' \\
\boldsymbol{r}_{CoE} &= (L_1 \boldsymbol{r}_{CoE_1} + L_2 \boldsymbol{r}_{CoE_2})/(L_1 + L_2) \\
\boldsymbol{r}_t &= [L_1 - L_2 \cos\alpha, L_2 \sin\alpha]'
\end{aligned}
$$

Example: Serial assembly of two beam elements (contd..)

$$\boldsymbol{C}_{2,t} = \boldsymbol{R'}_z(\pi - \alpha) \begin{bmatrix} L_2/AE & 0 & 0 \\ 0 & L_2^3/3EI & L_2^2/2EI \\ 0 & L_2^2/2EI & L_2/EI \end{bmatrix} \boldsymbol{R'}_z(\pi - \alpha)$$

$$= \begin{bmatrix} L_2(3I\cos^2\alpha + AL_2^2\sin^2\alpha)/(3AEI) & (AL_2^2 - 3I)\sin 2\alpha & -(L_2^2\sin\alpha)/2EI \\ (AL_2^2 - 3I)\sin 2\alpha & L_2(3I\sin^2\alpha + AL_2^2\cos^2\alpha)/(3AEI) & -(L_2^2\cos\alpha)/2EI \\ -(L_2^2\sin\alpha)/2EI & -(L_2^2\cos\alpha)/2EI & L_2/EI \end{bmatrix}$$

Example: Serial assembly of two beam elements (contd..)

$$\boldsymbol{Q}\boldsymbol{C}_{\boldsymbol{1},\boldsymbol{CoE_1}}\boldsymbol{Q}' = \boldsymbol{Q}\begin{bmatrix} AE/L_1 & 0 & 0 \\ 0 & 12EI/L_1^3 & 0 \\ 0 & 0 & EI/L_1 \end{bmatrix}^{-1} \boldsymbol{Q}'$$

$$= \begin{bmatrix} AE/L_1 + L_1 L_2^2 \sin^2\alpha/EI & (L_1 L_2 \sin\alpha)(L_1 - L_2\cos\alpha)/2EI & L_1 L_2 \sin\alpha/EI \\ (L_1 L_2 \sin\alpha)(L_1 - L_2\cos\alpha)/2EI & 12EI/L_1^3 + L_1(L_1 - L_2\cos\alpha)/2EI & L_1(L_1 - L_2\cos\alpha)/2EI \\ L_1 L_2 \sin\alpha/EI & L_1(L_1 - L_2\cos\alpha)/2EI & L_1/EI \end{bmatrix}$$

where $\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & L_2\sin\alpha \\ 0 & 1 & (L_1 - L_2\cos\alpha)/2 \\ 0 & 0 & 1 \end{bmatrix}^T$

Example: Serial assembly of two beam elements (contd..)

Summing them and using the fact that $\boldsymbol{B}_{2\times 2}$ is invariant under rigid body translation:

$$\boldsymbol{B}_{net,CoE} = \begin{bmatrix} L_2(3I\cos^2\alpha + AL_2^2\sin^2\alpha)/(3AEI) + AE/L_1 + L_1L_2^2\sin^2\alpha/EI & (AL_2^2 - 3I)\sin 2\alpha + (L_1L_2\sin\alpha)(L_1 - L_2\cos\alpha)/2EI \\ (AL_2^2 - 3I)\sin 2\alpha + (L_1L_2\sin\alpha)(L_1 - L_2\cos\alpha)/2E & L_2(3I\sin^2\alpha + AL_2^2\cos^2\alpha)/(3AEI) + 12EI/L_1^3 + L_1(L_1 - L_2\cos\alpha)/2EI \end{bmatrix}$$

and $\boldsymbol{C}_{net,t,3,3} = L_2/AE + L_1/AE$ (rotational compliance being summed in series). The coupling vector ($\bar{\boldsymbol{c}}$) comprising elements of ($\boldsymbol{C}_{3,1}$ and $\boldsymbol{C}_{3,2}$) are:

$$\bar{\mathbf{c}} = \begin{Bmatrix} L_2(L_1 - L_2/2)\sin\alpha/EI \\ (L_1^2 - (L_1 + L_2)L_2\cos\alpha)/2EI \end{Bmatrix}$$

Example: Serial assembly of two beam elements (contd..)

Thus, the new compliance matrix about the Center of Elasticity of resultant structure is:

$$[\boldsymbol{C}_{net,CoE}]_{3\times 3} = \begin{bmatrix} [\boldsymbol{B}_{net,CoE}]_{2\times 2} & \mathbf{0}_{2\times 1} \\ \mathbf{0}_{2\times 1}^T & \boldsymbol{C}_{net,t,3,3} + (x_{CoE} - x_t)^2 \bar{\boldsymbol{c}}_2 + +(y_{CoE} - y_t)^2 \bar{\boldsymbol{c}}_1 \end{bmatrix}$$

Using this procedure, we should be able to serially concatenate any frame element (at least in principle).

Outline

Center of elasticity of a parallel assembly of frame elements

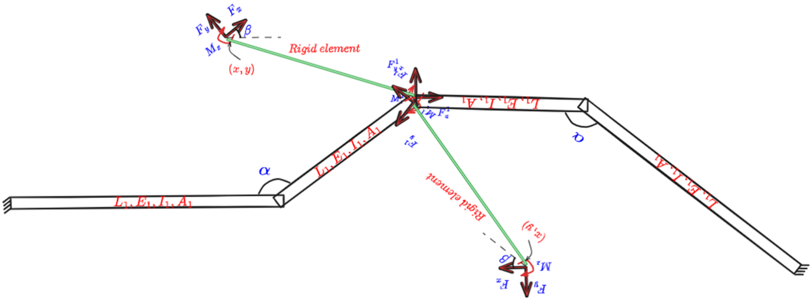
Simialir to the procedure explained above for the determination of centre of elasticity, we first add the stiffness matrices of each element in the global coordinate system and then perform rigid body transformation to diagonalise the matrix.

Equivalence between CoE and CoM?

Here, the CoM is not equivalent to CoE and no trivial transformation which maps the center of elasticity is known (here, it depends on other parameters such as stiffness and orientation).

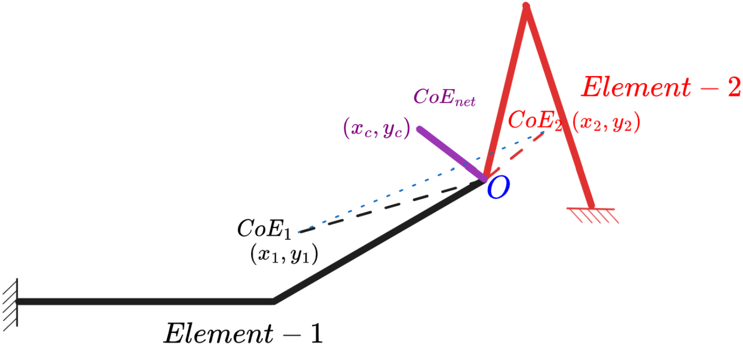
Center of elasticity of a parallel assembly of frame elements

The center of elasticity is generally not straightforward to calculate; however, if appropriate symmetry exists, then it can be calculated pretty easily.



Center of elasticity of a parallel assembly of frame elements

In the absence of symmetry the general procedure to follow is prescribed here:



Center of elasticity of a parallel assembly of frame elements

Let the stiffness matrix of element-1 about CoE_1 be:

$$\boldsymbol{K_1} = \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{12} & K_{22} & 0 \\ 0 & 0 & K_{33} \end{bmatrix}$$

and the stiffness of matrix of elemnt-2 about CoE_2 be:

$$\boldsymbol{K_2} = \begin{bmatrix} P_{11} & P_{12} & 0 \\ P_{12} & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix}$$

We know the force is being distributed between two elements, but the displacement at the point O must be the same to satisfy compatibility.

Center of elasticity of a parallel assembly of frame elements

Performing rigid body translation to relate both matrices about CoE_{net}:

$$\hat{\boldsymbol{K}}_1 = \begin{bmatrix} K_{11} & K_{12} & K_{11}(y_1 - y_c) - K_{12}(x_1 - x_c) \\ K_{12} & K_{22} & K_{12}(y_1 - y_c) - K_{22}(x_1 - x_c) \\ K_{11}(y_1 - y_c) - K_{12}(x_1 - x_c) & K_{12}(y_1 - y_c) - K_{22}(x_1 - x_c) & K_{33} + K_{11}(y_1 - y_c)^2 + K_{22}(x_1 - x_c)^2 - 2K_{12}(x_1 - x_c)(y_1 - y_c) \end{bmatrix}$$

$$\hat{\boldsymbol{P}}_1 = \begin{bmatrix} P_{11} & P_{12} & P_{11}(y_2 - y_c) - P_{12}(x_2 - x_c) \\ P_{12}) & P_{22} & P_{12}(y_2 - y_c) - P_{22}(x_2 - x_c) \\ P_{11}(y_2 - y_c) - P_{12}(x_2 - x_c) & P_{12}(y_2 - y_c) - P_{22}(x_2 - x_c) & P_{33} + P_{11}(y_2 - y_c)^2 + P_{22}(x_2 - x_c)^2 - 2P_{12}(x_2 - x_c)(y_2 - y_c) \end{bmatrix}$$

Center of elasticity of a parallel assembly of frame elements

Summing both the elements in entries (3,1) and (3,2) must be zero. Which leads to the following condition:

$$\begin{bmatrix} K_{11}+P_{11} & K_{12}+P_{12} \\ K_{12}+P_{12} & K_{22}+P_{22} \end{bmatrix} \begin{Bmatrix} y_c \\ -x_c \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \begin{Bmatrix} y_1 \\ -x_1 \end{Bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{Bmatrix} y_2 \\ -x_2 \end{Bmatrix}$$

$$(\boldsymbol{k}+\boldsymbol{p}) \begin{Bmatrix} y_c \\ -x_c \end{Bmatrix} = \boldsymbol{k} \begin{Bmatrix} y_1 \\ -x_1 \end{Bmatrix} + \boldsymbol{p} \begin{Bmatrix} y_2 \\ -x_2 \end{Bmatrix}$$

This is a matrix-weighed average of individual *CoE*. In general, matrix-weighed average will not lie in the line joining the two points (For more info on matrix-weighed average and its properties, refer[‡]).

If all the off-diagonal terms are zero,

$$y_c = (K_{11}y_1 + P_{11}y_1)/(K_{11}+P_{11})$$
$$x_c = (K_{22}x_1 + P_{22}x_1)/(K_{22}+P_{22})$$

[‡] Gary Chamberlain and Edward E. Leamer. "Matrix Weighted Averages and Posterior Bounds". In: *Journal of the Royal Statistical Society: Series B (Methodological)* 38.1 ().

Outline

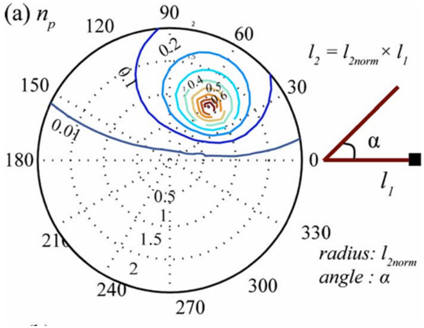
Synthesis using CoE

- The principal disadvantage of the CoE-based synthesis method is that the stiffness ellipsoids derived in this method only relate the displacement to the force at the same point. In general, the input and output can be at different locations.
- There have been attmepts such as[§], but they involve assumptions which do not scale well with large number of elements. And at best is only a minor improvement (if at all) to the Instant center method.
- The main advantage of this method is its ability to easily perform **single point synthesis** (compliance/stiffness matrix relating the displacement at a node to the forces/moment applied at the node, for the case where all the other nodes do not carry any force, essentially the 3×3 block matrix corresponding to the node in the global compliance matrix).

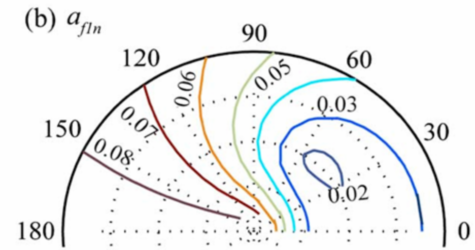
[§] *.Functional Characterization of Compliant Building Blocks Utilizing Eigentwists and Eigenwrenches*. Vol. 32nd Mechanisms and Robotics Conference, Parts A and B. 2008.

Synthesis using CoE (contd..)

We will make use of design charts[¶] for a compliant dyad building block.



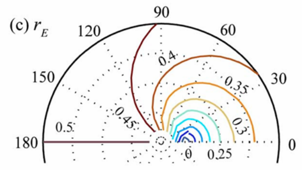
(a) n_p is the ratio of the minor axis and major axis of the ellipse subtended by compliance (only due to translation) $[n_p = a_{f_2}/a_{f_1}]$



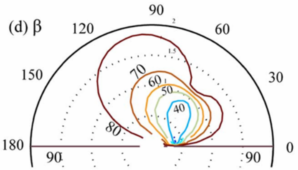
(b) Normalized linear compliance $[a_{f_{1n}} = a_{f_1} EI/(L_1 + L_2)^3]$

[¶]Girish Krishnan, Charles Kim, and Sridhar Kota. "An Intrinsic Geometric Framework for the Building Block Synthesis of Single Point Compliant Mechanisms". In: *Journal of Mechanisms and Robotics* (Feb

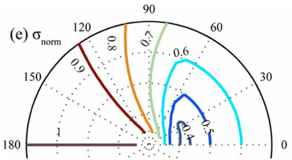
Synthesis using CoE (contd..)



(a) Normalised radius (measure from intersection of two beams) of CoE [$r_{E_n} = r_E/l_{dyad}$]



(b) Orientation of CoE (β)



(c) Stress concentration factor
$[\sigma = 6(L_1 + L_2)/(b^2 h)(\sigma_n \sqrt{f_x^2 + f_y^2} + m/(L_1 + L_2))]$

Synthesis using CoE (contd..)

Now let us say, we want to synthesize a mechanism which has the following stiffness matrix (let us ignore the units and dimensions here) about some node:

$$K = \begin{bmatrix} 20 & 10 & -30 \\ 10 & 60 & -50 \\ -30 & -50 & 300 \end{bmatrix}$$

Step-1

Draw the Mohr circle and find the principal translational stiffness, which in this case happens to be $40 \pm 20\sqrt{2}$.

Step-2

The ratio of the two translational principal compliance is 0.17$\sim$0.2 (let's say).

Synthesis using CoE (contd..)

Step-3

Solve the following equation:

$$\begin{bmatrix} 20 & 10 \\ 10 & 60 \end{bmatrix} \begin{Bmatrix} y_t - y_c \\ -x_t + x_c \end{Bmatrix} = \begin{Bmatrix} -30 \\ -50 \end{Bmatrix} \implies \begin{Bmatrix} y_t - y_c \\ x_t - x_c \end{Bmatrix} = \begin{Bmatrix} -13/11 \\ 7/11 \end{Bmatrix}$$

Obtained from relation in page no-46.

Step-4

We know $K_{33,CoE}$ is $EI/(l_{dyad})$ and from relation in page no-46, we can write:

$$EI/(L_1 + L_2) = 300 - 20(13/11)^2 - 60(7/11)^2 + 2 \times 10(13 \times 7)/11^2 \sim 263$$

Obtained from relation in page no-46.

Synthesis using CoE (contd..)

Step-5

Consider the curve $n_p = 0.2$ in fig. 12(a) and virtually superimpose it over fig. 12(b); all the points in the curve $n_p = 0.2$ are admissible solutions, and multiple solutions for L_2/L_1 and α are possible, thus allowing design flexibility. However, be wary of the stress concentration factor in fig. 13(c).

Here, let us choose $L_2/L_1 = 1.5$ and $\alpha \sim 85^\circ$. The corresponding normalised compliance is around 0.045. As the value of $EI/(L_1 + L_2)$ is already fixed, thus
$(L_1 + L_2)^{-2} = (40 - 20\sqrt{2}) \times 0.045/(263) \implies L_1 + L_2 = 22.34$
$\implies L_1 = 8.94, \; L_2 = 13.40$ and $EI \sim 5875$.

Synthesis using CoE (contd..)

Step-6

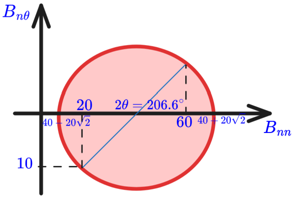
Calculate the CoE of the assembly and find the position of (x_t, y_t)

Step-7

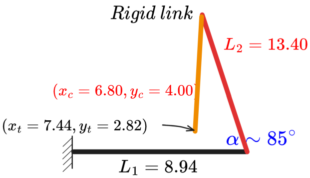
Now rotate the whole assembly by angle β or $90^\circ - \beta$ (measured from fig. 13(b)) in clockwise direction and then rotate it θ or $\theta - 90^\circ$(θ measured from Mohr circle) in counter-clockwise direction (rotation about the same point), the second case when $K_{22} > K_{11}$.

Here, $\beta \sim 75^\circ$ and $\theta = 103.3^\circ$. Here $K_{22} > K_{11}$, thus the basis has to be rotated counter-clockwise an angle $180^\circ - \beta - \theta = 1.70^\circ$.

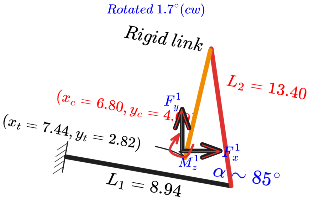
Synthesis using CoE (contd..)



(a) Step-1



(b) Step-6



(c) Final design

Synthesis using CoE (contd..)

Verification using FEA beam MATLAB code:

$$K = \begin{bmatrix} 16.70 & 13.93 & -28.87 \\ 13.93 & 55.05 & -53.36 \\ -28.87 & -53.36 & 332.11 \end{bmatrix}$$

The value is pretty close considering the approximations we made along the way (however, it is possible in principle to synthesise this stiffness matrix accurately).
To realise the rigid element, we must appropriately modify the dimensions of that element.

Synthesis using CoE (contd..)

Design using a single dyad maynot always suffice, consider a case where we require $K_{11} = 0.9K_{22}$, we notice from fig. 12(a), that such a design is highly restrictive in terms of design flexibility, thus, we split the stiffness matrix (we know stiffness matrix sums when frame elements are assembled in a parallel manner). For example:

$$K = \begin{bmatrix} 60 & 1 & -30 \\ 1 & 54 & -50 \\ -30 & -50 & 300 \end{bmatrix} = \begin{bmatrix} 8 & 0.5 & -15 \\ 0.5 & 48 & -50 \\ -30 & -50 & 150 \end{bmatrix} + \begin{bmatrix} 52 & 0.5 & -15 \\ 0.5 & 6 & -25 \\ -15 & -25 & 150 \end{bmatrix}$$

We can synthesise separate assemblies for each with good design flexibility and then assemble them in a parallel manner.

Summary

1. CoE can be computed very easily for any assembly of frame elements.
   - For serial assembly of elements, the position of CoE is just a weighted average of the CoE of individual elements (weighing factor=L/EI).
   - For parallel assembly of elements, the position of CoE is a matrix weighted average of the CoE of individual elements.
2. CoE can be used to efficiently find a block stiffness matrix of a particular node, assuming all other nodes are not loaded (In an conventional manner, we have to invert the global stiffness matrix (thus $O(n^3)$), but this way we can find it in $\sim O(n)$ steps). The drawback is that no other relation between the deformation of different nodes can be inferred.
3. This method is only suitable (practically) for single-point synthesis.

Thank you!