M6: Stability Analysis of Mechanical Systems

ME 204: Mechanics of Elastica

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Abstract

In this module, the notion of stability of a mechanical system is established, and the procedure for analysis of the response of the mechanical system is elucidated with examples (of 1-DoF systems) through both the energy criterion method and the method of small vibrations. The variation of steady-state system response with system parameters is also studied. The energy criterion method is also applied to perform stability analysis of a continuous system.

1 Notion of stability

The stability of mechanical systems is inferred from the nature of equilibrium solutions. Equilibrium solutions, in general, do not need to be an isolated point (fixed point) in the phase plane. Periodic motion, as in the case of an undamped 1-DoF pendulum, is also a valid equilibrium solution. For most structural systems with damping, periodic orbits satisfying equilibrium conditions are usually absent unless the system characteristics are highly non-linear in a certain way. Thus, we are generally interested in the static solutions which correspond to isolated fixed points where time derivatives of the generalized coordinates (unique and independent representation of each DoF) of the system are zero in the phase plane (For example, in the case of a 1-DoF pendulum, θ is the generalized coordinate and $\theta = \theta^*$ corresponding to $\dot{\theta}(\theta^*, t),\ \ddot{\theta}(\theta^*, t)\ ... = 0\ \forall t$ is the fixed point of the system). Equivalently, suppose the system is subject to initial conditions corresponding to that of the fixed point. In that case, the system is expected to remain in the same configuration for all time, assuming the system is free from any perturbation. However, such an assumption is unrealizable in practice, so it is of interest to understand how such perturbation can affect the response of the system.

1.1 Static stability

The typical schematic illustration of balls placed on a locally smooth *viscous* surface is shown in fig. 1. For small perturbations (small in this context means infinitesimal), the ball would return to its original position in case of stable equilibrium. In contrast, in unstable equilibrium, the ball would move disproportionately farther away from its original position, and in the case of neutral equilibrium, it neither moves farther nor returns, the ball would remain in the perturbed position.
The definition of stability of a fixed point is similar to that shown in the figure. When the system is initialized about a fixed point and when perturbed, the fixed point is said to be *stable* if it returns to the same fixed point and *unstable* if it does not return nor remain stationary in the perturbed position. There is also the *half-stable* fixed point (which is uncommon), in which the system returns to the initial configuration upon perturbation in some direction approaching the point and does not return nor remain stationary for perturbations in the same direction away from the point (for a defined orientation), It is analogous to the neutral equilibrium configuration. It could be understood by imagining a stable and unstable fixed point arbitrarily close. An illustrative schematic is shown in fig. 2. There also exists another type of fixed point where the system shows characteristics of a stable fixed point along perturbations along specific directions and characteristics of an unstable fixed point along other directions, but this fixed point is also deemed unstable.

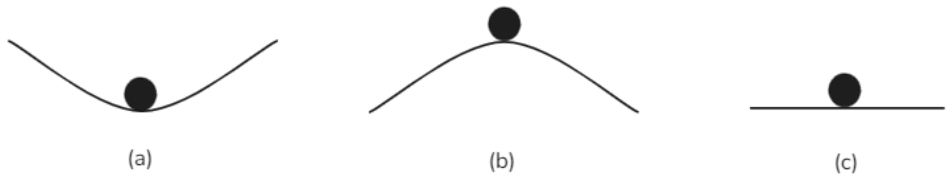


Figure 1: Stability of equilibrium configuration: (a) Stable equilibrium (b) Unstable equilibrium (c) Neutral equilibrium

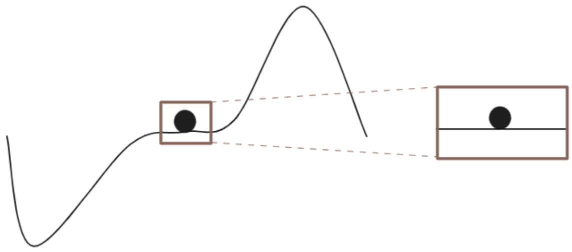


Figure 2: Illustrative schematic of an equilibrium configuration where the stable and unstable fixed points are close

2 Stability analysis of discrete systems

In this section, we will study how to perform static stability analysis of discrete mechanical systems (discrete - finite number of DoF). To perform stability analysis, the *method of small vibrations* is usually used. For which the following steps are to be performed in order:

Method of small vibrations

1. Derive the equations of motion,
2. Represent the equations in the state space form ($\dot{\underline{x}} = f(\underline{x})$[a]),
3. Find the fixed points of the system,
4. Initialize the system at the fixed point with a slight perturbation in an arbitrary direction (direction in the phase plane),
5. Linearize the non-linear terms in the expression around the fixed point using the Taylor series,
6. A system of linear first-order ordinary differential equation is obtained, and the stability can be inferred from the matrix in the expression (Jacobian of the function $f(\underline{x})$ evaluated at the fixed point).

[a]Here for simplicity, we will assume the system is autonomous, i.e. the function f does not depend on the independent variable of the DEs.

The method of small vibrations is a general procedure followed to find the stability of fixed points for any system. However, there is an equivalent *energy criterion method* for a restricted class of problems in structural mechanics that do not involve non-conservative forces.

The energy criterion method is based on the minimum potential energy principle, which states, "For conservative structural systems, of all the kinematically admissible deformations, those corresponding to the static-equilibrium state extremize (i.e., minimize or maximize) the total potential energy. The corresponding equilibrium state is stable if the extremum is a minimum and unstable if the extremum

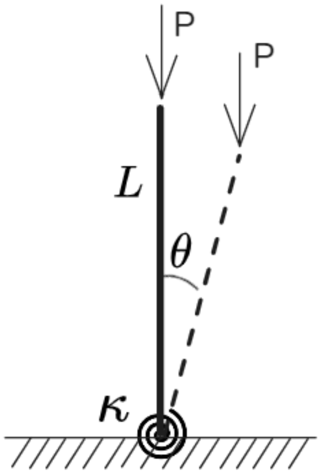


Figure 3: Compressive load is applied on a rigid bar held with a torsional spring

is not a minimum". The following steps are to be performed:

Energy criterion method

1. Write the potential energy of the system (including work potential),
2. Find fixed points which extremize the expression (through the first order derivative test in its multi-variable form),
3. Classify stability by observing whether the extremum is minimum or not (through higher-order derivative tests).

2.1 Example-1

Consider the schematic shown in fig. 3, where a rigid rod held with a torsional spring is applied a compressive load. We wish to determine the stability of the system at its fixed points. We will verify using both methods.

Method of small vibrations

1. Derive the equations of motion:

$$\Sigma M = P \times (L\sin\theta) - \kappa(\theta - \cancel{\theta_0})\text{[a]},$$
$$I_o\ddot{\theta} = P \times (L\sin\theta) - \kappa\theta.$$

2. Represent the equations in the state space form: Let $y = \dot{\theta}$

$$\begin{Bmatrix} \dot{\theta} \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} y \\ (PL\sin\theta - \kappa\theta)/I_0 \end{Bmatrix}$$

3. Find the fixed points of the system:

$$\begin{Bmatrix} y \\ (PL\sin\theta - \kappa\theta)/I_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \implies \dot{\theta}^* = 0 \text{ and } PL\sin\theta^* = \kappa\theta^*$$

$(\theta^*, 0)$ are the coordinates of the fixed points of the system.

We immediately note that $\theta^* = 0$ is one solution, and depending on the value of $1/\lambda = PL/\kappa$ we may have other solutions,

$$f(\theta) = \frac{\sin\theta}{\theta} = \lambda$$

The plot of $f(\theta)$ is shown in fig. 4, based on which we can infer that multiple fixed points are possible depending on the value of parameter λ, note that if the value of parameter λ

4. Introduce perturbation: Let us say the fixed point of interest is $(\theta^*, 0)$ and is perturbed along (e, y).

5. Linearizing the DE about this point,

$$\begin{Bmatrix} \dot{\theta} \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} y \\ (PL\sin\theta^* - \kappa\theta^* + (PL\cos\theta^* - \kappa)(\theta - \theta^*))/I_0 \end{Bmatrix} = \begin{Bmatrix} y \\ ((PL\cos\theta^* - \kappa)(\theta - \theta^*))/I_0 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{e} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ (PL\cos\theta^* - \kappa)/I_0 & 0 \end{bmatrix} \begin{Bmatrix} e = \theta - \theta^* \\ y \end{Bmatrix}$$

This system will continue to oscillate unless it is damped out, so we assume the existence of some fictional damper which brings it into a steady state.

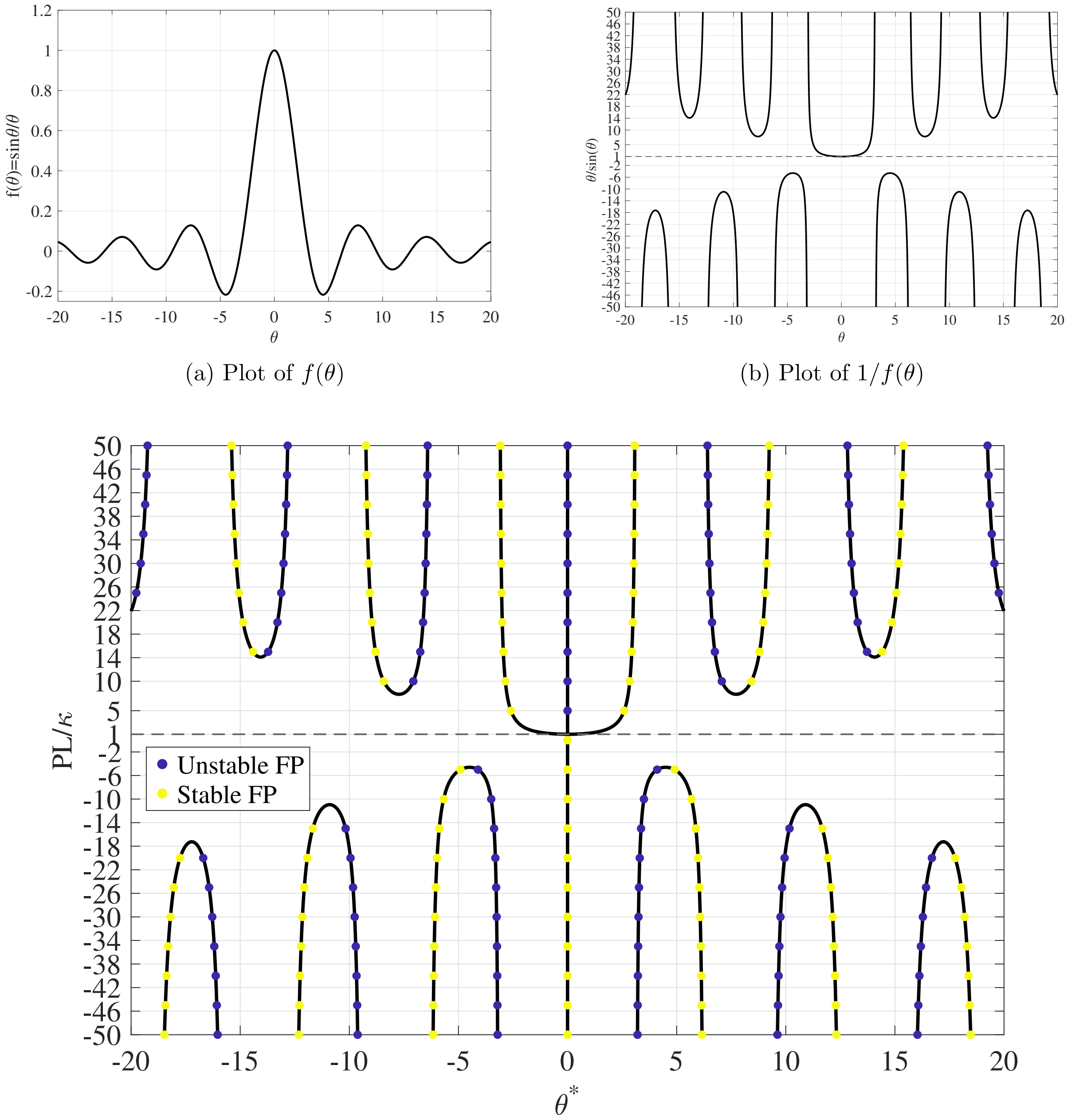
$$\begin{Bmatrix} \dot{e} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ (PL\cos\theta^* - \kappa)/I_0 & -(c > 0) \end{bmatrix} \begin{Bmatrix} e = \theta - \theta^* \\ y \end{Bmatrix}$$

6. Now, the trace of the matrix is negative, and if the determinant is positive, both eigenvalues of the matrix are negative, implying the system response will decrease exponentially in magnitude in the corresponding eigenvector directions, system response is stable. Otherwise, if the determinant is negative, the system response will grow along one eigenvector direction (corresponding to positive eigenvalue) and decrease along the other, and as a whole, the system will be unstable.

$$\begin{aligned} \kappa > PL\cos\theta^* &\implies \text{F.P. is Stable} \\ \kappa < PL\cos\theta^* &\implies \text{F.P. is Unstable} \end{aligned}$$

The variation of fixed points and its stability with parameter PL/κ is shown in fig. 4c.

[a] clockwise-positive



(a) Plot of $f(\theta)$

(b) Plot of $1/f(\theta)$

(c) Variation of fixed points with parameter PL/κ

Figure 4

We note that the trivial solution $\theta^* = 0$ remains stable only in the range of load $P \in (-\infty, \kappa/L)$, as the load is slowly increased after the critical load, the fixed point moves away to some other location, but in a continuous manner (i.e. there is no discontinuity, if PL/κ is increased from 1 to $1+\epsilon$, the fixed point would show change by a value proportional to ϵ). Based on the figure, we note that the trivial solution undergoes a bifurcation[1], this type of bifurcation is usually referred to as supercritical pitchfork bifurcation, The fixed points outside range $(-\pi, \pi)$ are solutions which will not be realized in practice unless the system is started from a configuration where the initial wound angle of torsional spring (θ_0) is present and significant.

[1]Usually used to denote a change in system behaviour with respect to parameters, which are not extrapolatable from the system response at neighbouring values of the parameter. In our case, if we were to guess the system response looking at the solution at $P/L\kappa = 1^-$, we'd guess the solution must always remain stable, which is not true.

Energy criterion method

1. Potential energy of the system (including work potential):

$$E(\theta) = \frac{1}{2}\kappa(\theta - \cancel{\theta_0})^2 - PL(1 - \cos\theta)$$ [a]

2. Fixed points which extremize the expression: The equation is dependent on one variable, thus differentiating with θ,

$$\left.\frac{dE}{d\theta}\right|_{\theta^*} = \kappa\theta^* - PL\sin\theta^* = 0$$
$$\implies sin\theta^* = (\kappa/PL)\theta^*$$

3. Classify stability by observing whether the extremum is minimum or not:

$$\left.\frac{d^2E}{d\theta^2}\right|_{\theta^*} = \kappa - PL\cos\theta^* \begin{cases} > 0 \implies \text{Stable} \\ < 0 \implies \text{Unstable} \\ = 0 \implies \text{Higher order derivatives need to be computed} \end{cases}$$

Here, we note that we end up with the same expression as in the method of small vibrations but with much less work (for a single variable problem), and the same conclusions about the fixed points and their stability are inferred here.

[a] Work potential is negative as work is performed externally from the surrounding to system

2.2 Example-2

Consider the schematic shown in fig. 5, where a rigid bar is pivoted at one end, and the free end of the bar is attached to a spring which is mounted on a vertical slider. We will follow the energy criterion method to identify the fixed points and their stability.

Energy criterion method

1. Potential energy of the system (including work potential):

$$E(\theta) = \frac{1}{2}k(\cancel{l_0} + L\sin\theta - \cancel{l_0})^2 - PL(1 - \cos\theta)$$ [a]

2. Fixed points which extremize the expression: The equation is dependent on one variable, thus differentiating with θ,

$$\left.\frac{dE}{d\theta}\right|_{\theta^*} = kL^2\sin\theta^*\cos\theta^* - PL\sin\theta^* = 0$$
$$\implies \cos\theta^* = (P/Lk) \text{ or } \sin\theta^* = 0$$

We immediately note that the latter term will have infinite solutions, and the former may or may not have a solution depending on the value of parameter $\lambda = P/kl$.

3. Classify stability by observing whether the extremum is minimum or not:

$$\left.\frac{d^2E}{d\theta^2}\right|_{\theta^*} = k(\cos^2\theta^* - \sin^2\theta^*) - PL\cos\theta^*$$
$$\begin{cases} > 0 \implies \text{Stable} \\ < 0 \implies \text{Unstable} \\ = 0 \implies \text{Higher order derivatives need to be computed} \end{cases}$$

Based on the above two equations, the variations of fixed points are classified by their stability and shown in fig. 6.

[a]Work potential is negative as work is performed externally from the surrounding to system

Based on the figure, we note that the trivial solution undergoes a bifurcation, this type of bifurcation is usually referred to as subcritical pitchfork bifurcation, in this case, the systems long term response immediately shifts from $\theta^* = 0$ to $\theta^* = \pm\pi$ if the parameter is slightly raised above 1, It reaches $+\pi$ or $-\pi$ depending on the perturbation applied and the asymmetry in the system assembly.

Note 2.1

In both figs. 4c and 6, we notice that for a given parameter value, there exists a stable FP between two consecutive unstable fixed points and vice-versa. This must be true for any system.

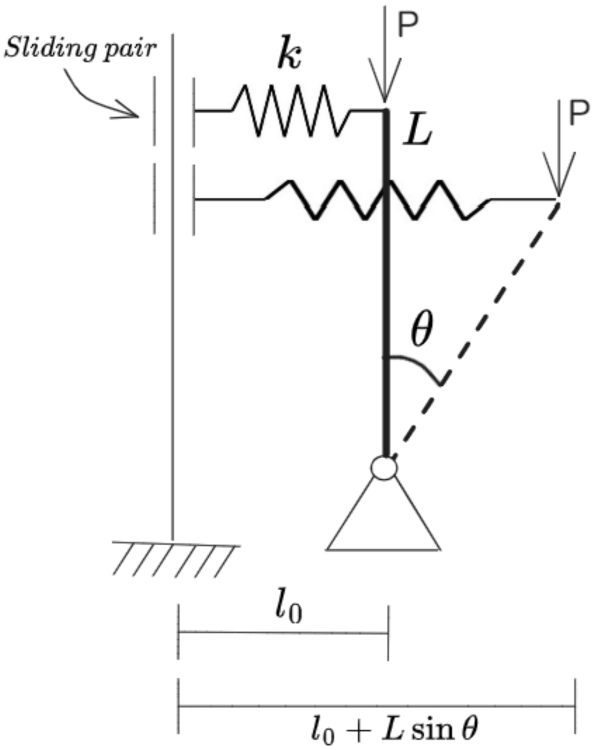


Figure 5: A rigid bar is pivoted at one end and carries a spring at the other end, the spring is attached to a vertical slider

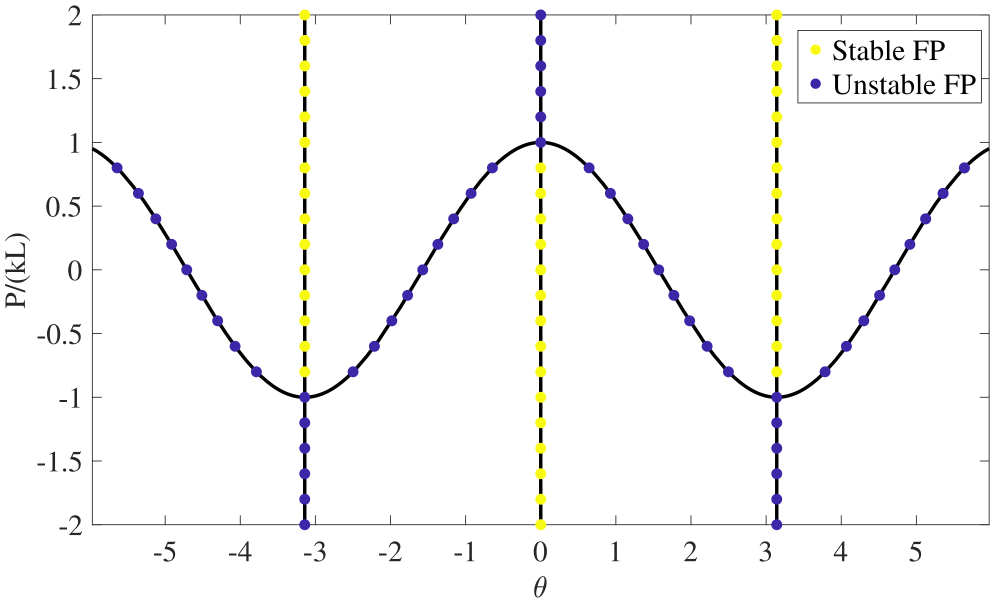


Figure 6: Variation of fixed points with parameter $P/(kL)$

3 Stability analysis of buckling problem

Here the system has infinite DoF, but the procedure we adopt will be similar to the finite DoF case. We will use the energy criterion method here. Here we expect to find functions $\theta^*(s)$ which extermize the potential energy expression while satisfying the kinematic constraints ($\theta^*(0) = 0$). The schematic of the problem is shown in fig. 7.

Energy criterion method

1. Potential energy of the system (including work potential):

$$\Psi(\theta(s)) = \frac{1}{2}\int_0^L EI\theta'^2(s)\ ds - PL + P\int_0^L \cos\theta(s)\ ds$$

2. Now let us claim that we have found a solution $\theta^*(s)$, which extremizes the above expres-

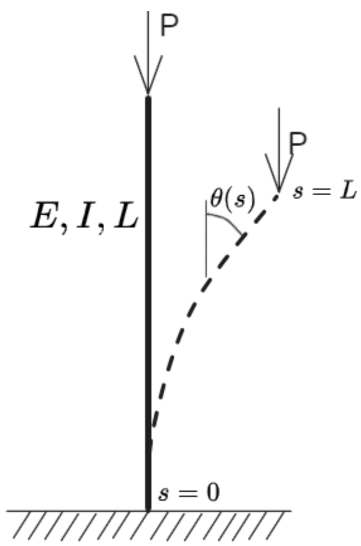


Figure 7: Buckling of cantilever beam

sion. Then, let us apply a perturbation in the direction $(v(s))$[a] scaled by an arbitrarily small parameter ϵ. It is essential that the new function also satisfies the kinematic constraint. Then the new potential energy is

$$\Psi(\theta^*(s)+\epsilon v(s)) = \frac{1}{2}\int_0^L EI(\theta'^*(s)+\epsilon v'(s))^2\ ds - PL + P\int_0^L \cos(\theta^*(s)+\epsilon v(s))\ ds$$

We note that the as we have specified both $\theta^*(s)$ and $v(s)$, the only free variable in the problem is ϵ. As we have claimed $\theta^*(s)$ extremizes the solution, then $\epsilon = 0$ must extremize the potential energy term, now we have essentially converted the problem into a single variable optimization problem, thus using the same principle as we have used for the prior examples, then the condition on θ^* so that it extremizes the functional is:

$$\left.\frac{d\Psi(\theta^*(s)+\epsilon v(s))}{d\epsilon}\right|_{\epsilon=0} = 0\ \forall v \in V^{b} \tag{1}$$

The condition must be satisfied for all v as it must extremize compared to all points in the local vicinity of the point in its phase plane.
In our problem,

$$\left.\int_0^L EI(\theta'^*(s)+\epsilon v'(s))v'(s)\ ds - P\int_0^L \sin(\theta^*(s)+\epsilon v(s))v(s)\ ds\right|_{\epsilon=0}$$
$$= \int_0^L EI\theta'^*(s)v'(s)\ ds - P\int_0^L \sin\theta^*(s)v(s)\ ds = 0\ \forall v \in V$$

Integrating by parts,

$$[EI\theta'^*v]_0^L - \int_0^L (EI\theta'^*(s) + P\sin\theta^*(s))v(s)\ ds = 0\ \forall v \in V$$
$$\implies EI\theta'^*v\big|_{s=L} = 0\ \forall v \in V \implies EI\theta^*(L) = 0$$
$$\text{and } EI\theta'^*(s) + P\sin\theta^*(s) = 0\ \forall s \in (0, L)$$

3. Classify stability by observing whether the extremum is minimum or not:

$$\left.\frac{d^2\Psi(\theta^*(s)+\epsilon v(s))}{d\epsilon^2}\right|_{\epsilon=0} = \int_0^L EI(v'(s))^2\ ds - P\int_0^L \cos(\theta^*(s))v^2(s)\ ds \tag{2}$$
$$\begin{cases} > 0\ \forall v \in V \implies \text{Stable} \\ < 0\ \exists v \in V \implies \text{Unstable} \\ = 0\ \forall v \in V \implies \text{Higher derivatives need to be computed} \end{cases}$$

Let us consider the solution $\theta^*(s) = 0\ \forall s \in [0, L]$ which satisfies the first order condition, then this solution is stable iff

$$\int_0^L EI(v'(s))^2\ ds - P\int_0^L \cos(\theta^*(s))v^2(s)\ ds > 0\ \forall v \in V$$
$$\implies \frac{\int_0^L (v'(s))^2\ ds}{\int_0^L v^2(s)\ ds} > P/EI\ \forall v \in V$$

We note that $P < 0$ is stable. Now, we wish to find the minimum of the first term of the last expression. We will state it as k, then the critical load in which the trivial solution is no longer stable is

$$P_{cr} = \frac{kEI}{L^2},$$

which represents the form of Euler buckling load for cantilever beams $(\pi^2 EI/4L^2)$.

[a] The direction could be imagined as an arbitrarily weighted sum of Dirac-delta distributions centred about different points along s or it can equivalently thought of as another function such that $v(s) = 0$.

[b] V is the set of kinematically admissible solutions

Note 3.1

Here it is to be noted that eqn-2 will be linear in v and eqn-2 is quadratic[a] in v (courtesy of chain rule of differentiation).

[a] $Q(\alpha v) = \alpha^2 Q(v)$