Solution to Q1 =>

Q1

$$\frac{f(t+dt, x) - f(t-dt, x)}{2dt} = -v\,\frac{f(t, x+dx) - f(t, x-dx)}{2dx} \quad -(1)$$

Assuming the solution to be of the form:

$$f(x,t) = \xi_q^{t} \exp(ikx) \quad (2)$$

This equation will be unstable if $|\xi_q(k)| > 1$ for some k. Thus, stability requires $|\xi_q|^2 \le 1$.

Substituting (2) in (1) $\Rightarrow$

$$\frac{\xi_q^{t+dt} \exp(ikx) - \xi_q^{t-dt} \exp(ikx)}{2dt}$$

$$= -V \frac{\xi_q^{t} \exp(ik(x+dx)) - \xi_q^{t} \exp(ik(x-dx))}{2dx}$$

Dividing $\xi_q^{t} \exp(ikx)$ from both sides $\Rightarrow$

$$\left(\xi_q^{dt} - \xi_q^{-dt}\right)/2dt =$$

$$-V\left(\exp(ikdx) - \exp(-ikdx)\right)$$

$$\left(\frac{\quad}{2dx}\right)$$

$$\left(\frac{\varepsilon_l^{2dt}-1}{\varepsilon_l^{dt}}\right)\left(\frac{1}{2dt}\right) =$$

$$-v\left(\frac{\cancel{2}i\sin(kdx)}{\cancel{2}dx}\right)$$

$$\frac{\varepsilon_l^{2dt}-1}{2\varepsilon_l^{dt}} = -\frac{v\,dt}{dx}\, i\sin(kdx)$$

$$\varepsilon_l^{2dt}-1 = 2\varepsilon_l^{dt}\left(-\frac{v\,dt}{dx}\right) i\sin(kdx)$$

Setting $dt = 1$ (the grid-spacing) $\Rightarrow$

$$\varepsilon_l^{2}-1 = -2\varepsilon_l\, i\sin(kdx)\,\frac{v\,dt}{dx}$$

This is a quadratic equation whose solution is $\Rightarrow$

$$e_t = -i \frac{v\,dt}{dx} \sin(k\,dx) \pm \sqrt{1 - \left(\frac{v\,dt}{dx} \sin(k\,dx)\right)^2}$$

$\Downarrow$

for this term to be real $\Rightarrow$

$$1 - \frac{v\,dt}{dx} \sin(k\,dx) \geq 0$$

$$\frac{v\,dt}{dx} \leq 1$$ (since $\sin(k\,dx)$ is bounded by -1 and 1)

$\Downarrow$

CFL condition

Thus, the leapfrog scheme conserves energy if the CFL condition is satisfied.