Q5 We can add an extra invertible matrix S into χ^2 to give as

$$\chi^2 = (d - Am)^T S^T S^{T^{-1}} N^{-1} S^{-1} S (d - Am)$$

$$= (Sd - SAm)^T (S N S^T)^{-1} (Sd - SAm)$$

Now we define the new rotated variables as:-

$$\tilde{d} = Sd \qquad \tilde{A} = SA \qquad \tilde{N} = SNS^T$$

Then the new χ^2 becomes as

$$\chi^2 = (\tilde{d} - \tilde{A}m)^T \tilde{N}^{-1} (\tilde{d} - \tilde{A}m)$$

This can be true only if as

$$\tilde{N}_{ij} = \langle \tilde{n}_i \tilde{n}_j \rangle \quad - (1)$$

(Since in the original definition of χ^2, $N = \langle n_i n_j \rangle$)

Let's prove (1) as

$$RHS = \langle \tilde{n}_i \tilde{n}_j \rangle$$

$\cancel{\tilde{A} = S_{ni}}$ $\tilde{n} = Sn$

Thus, $\tilde{n}_i = \sum_k S_{ik} n_k$

So, $\langle n_i n_j \rangle =$

$$\left\langle \left(\sum_k S_{ik} n_k \right) \left(\sum_l S_{jl} n_l \right) \right\rangle$$

Since the original noises are independent of each other $\Rightarrow n_i n_j = \delta_{ij} \sigma_i^2$

Thus $\langle n_i n_j \rangle =$

$$\left\langle \sum_k S_{ik} S_{jk} n_k^2 \right\rangle$$

$$= \sum_k S_{ik} S_{jk} \sigma_k^2 \quad \text{— I}$$

$$\text{LHS} = \tilde{N}_{ij} = \sum_k (SN)_{ik} S^T_{kj}$$

$$= \sum_k S_{ik} \sigma_k^2 \cdot S_{jk}$$

$$= \sum_k S_{ik} S_{jk} \sigma_k^2 \quad \text{— II}$$

$$\text{I} = \text{II} \Rightarrow \tilde{N}_{ij} = \langle \tilde{n}_i \tilde{n}_j \rangle$$