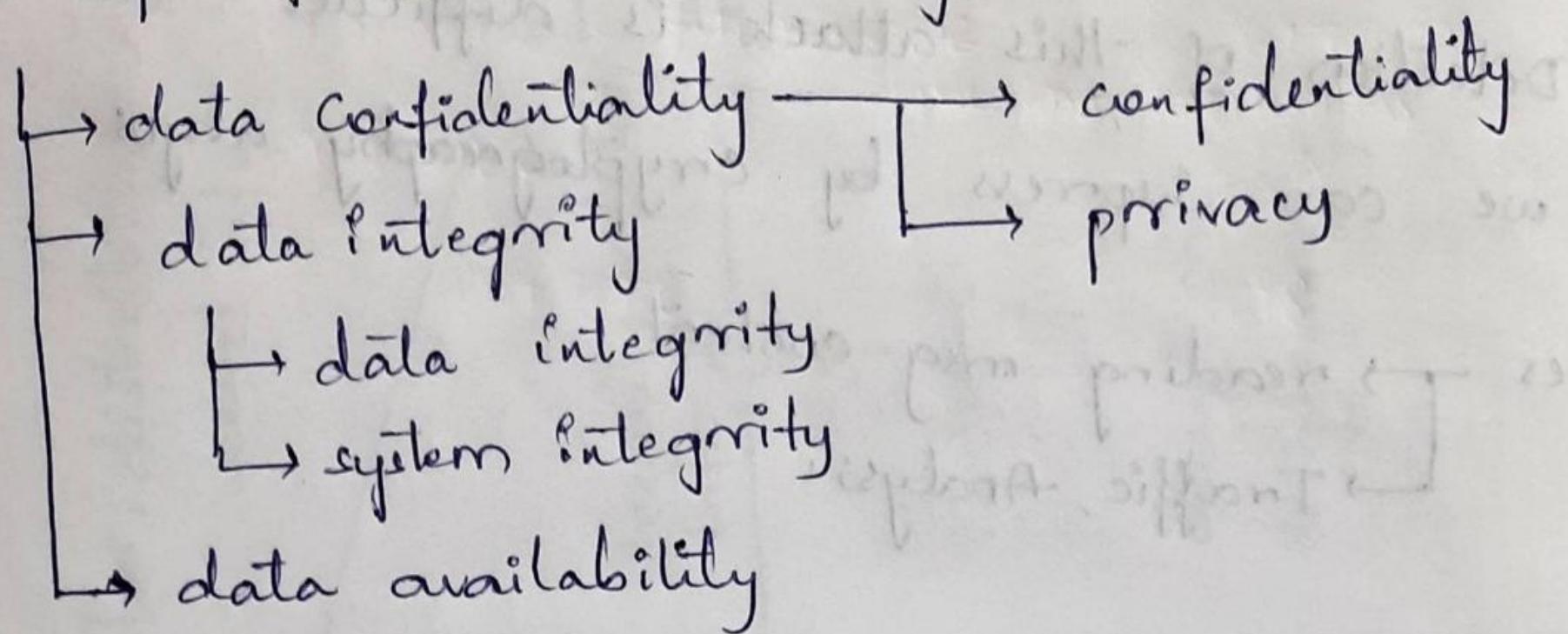


Cryptography and Network Security

Concepts of Network Security



Data Availability:

↳ Data is made to be available for authorised person

Data Authentication:

↳ Data is need to be sent by authorised person

Data Accountability

↳ maintain the track records of activities being performed

Security Attacks:

- ↳ passive attacks
- ↳ Active attacks

Passive attacks:

- unauthorised won't modify it
- Detection of this attack is difficult
- we can suppress by cryptography algos

Types → reading msg content

→ Traffic Analysis

Active Attacks:

→ unauthorised user will alter the message

→ Types → masquerade

→ replay

→ denial of service

→ modifying message content

Cryptography

Normal

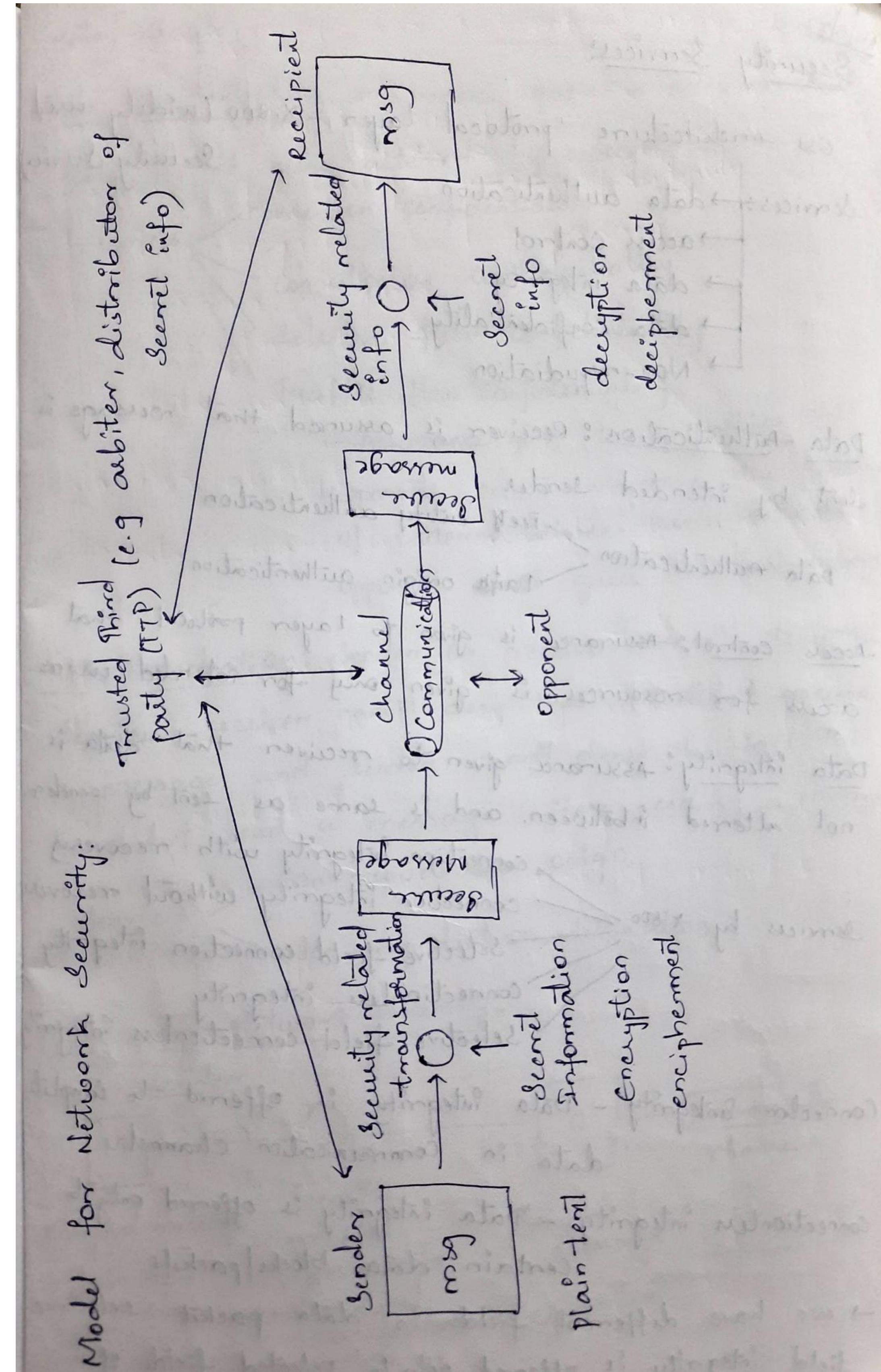
plain text → original msg

cipher text → Secure msg

Security related transformation → cryptography

encipherment → process of converting plain text to
cipher text

Decipherment → process of converting cipher text to
plain text.

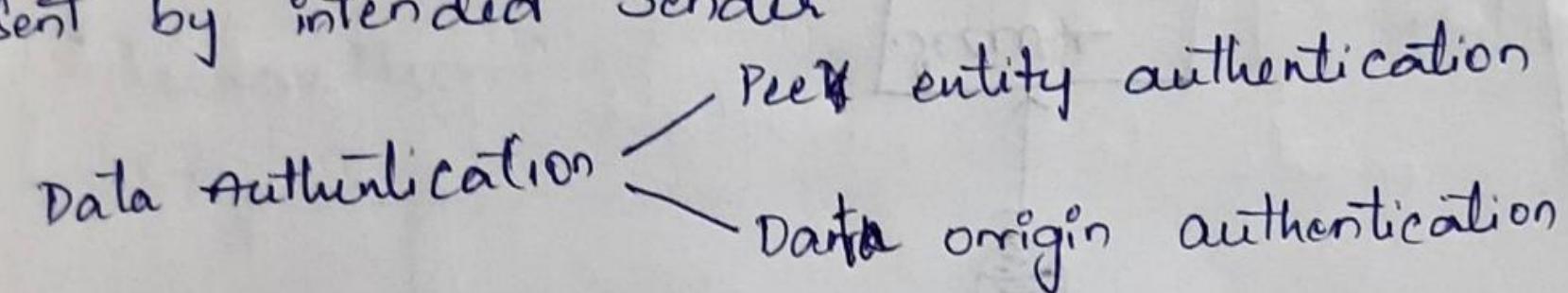


16/6/22
Security Services:

OSI architecture protocol layer - X.800 (widely used
Security Service)

- Services:
- data authentication
 - access control
 - data integrity
 - data confidentiality
 - Non-repudiation

Data Authentication: Receiver is assured that message is sent by intended sender



Access Control: Assurance is given to Layer protocol that access for resources is given only for intended users

Data Integrity: Assurance given to receiver that data is not altered in between. and is same as sent by sender.

- Services by X.800 →
- connection integrity with recovery
 - connection integrity without recovery
 - selective field connection integrity
 - connectionless integrity
 - selective field connectionless integrity

Connection Integrity - Data integrity is offered to complete data in communication channel

connectionless integrity - Data integrity is offered only to certain data blocks/packets

→ we have different fields in data packet. selective field integrity is offered only to selected fields of

data packet.

Data Confidentiality: Assurance given to receiver that data is not being altered by opponent.

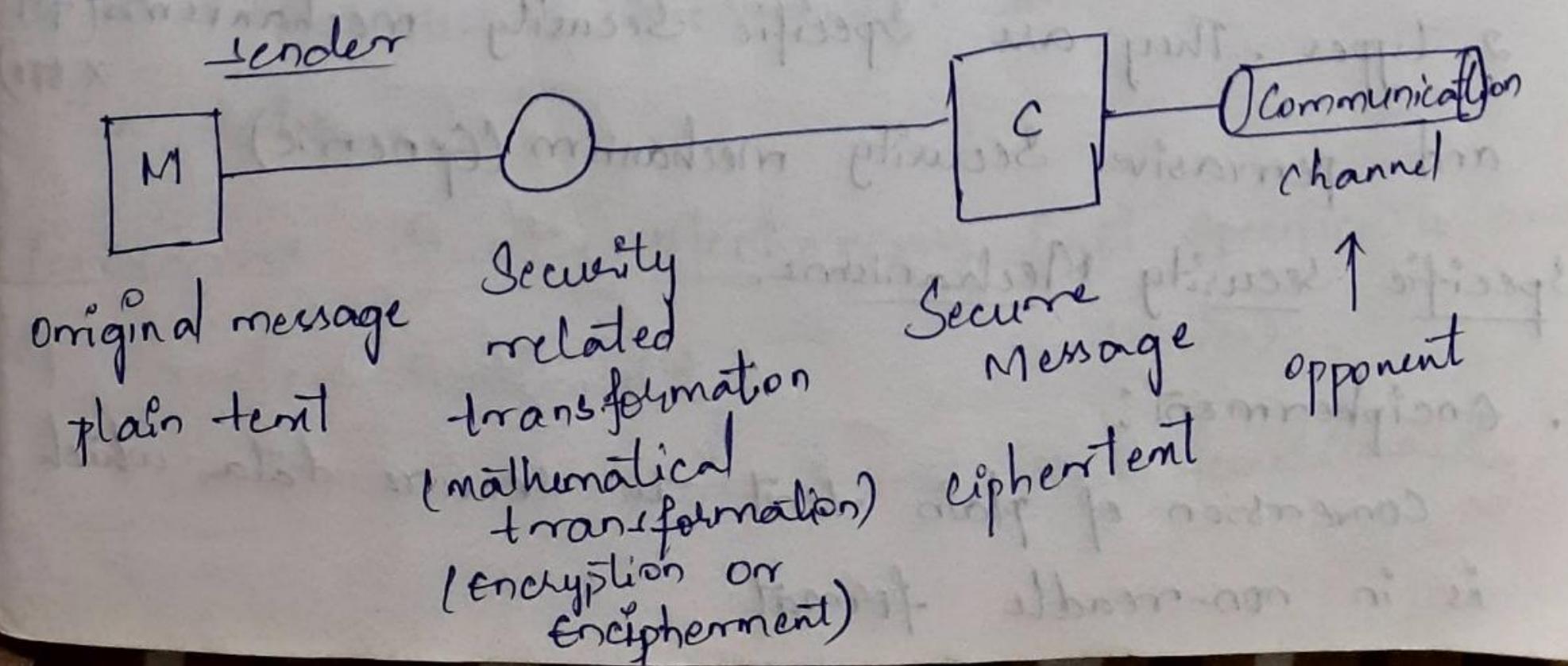
Services → connection confidentiality
connectionless - confidentiality
selective field confidentiality
Traffic flow confidentiality

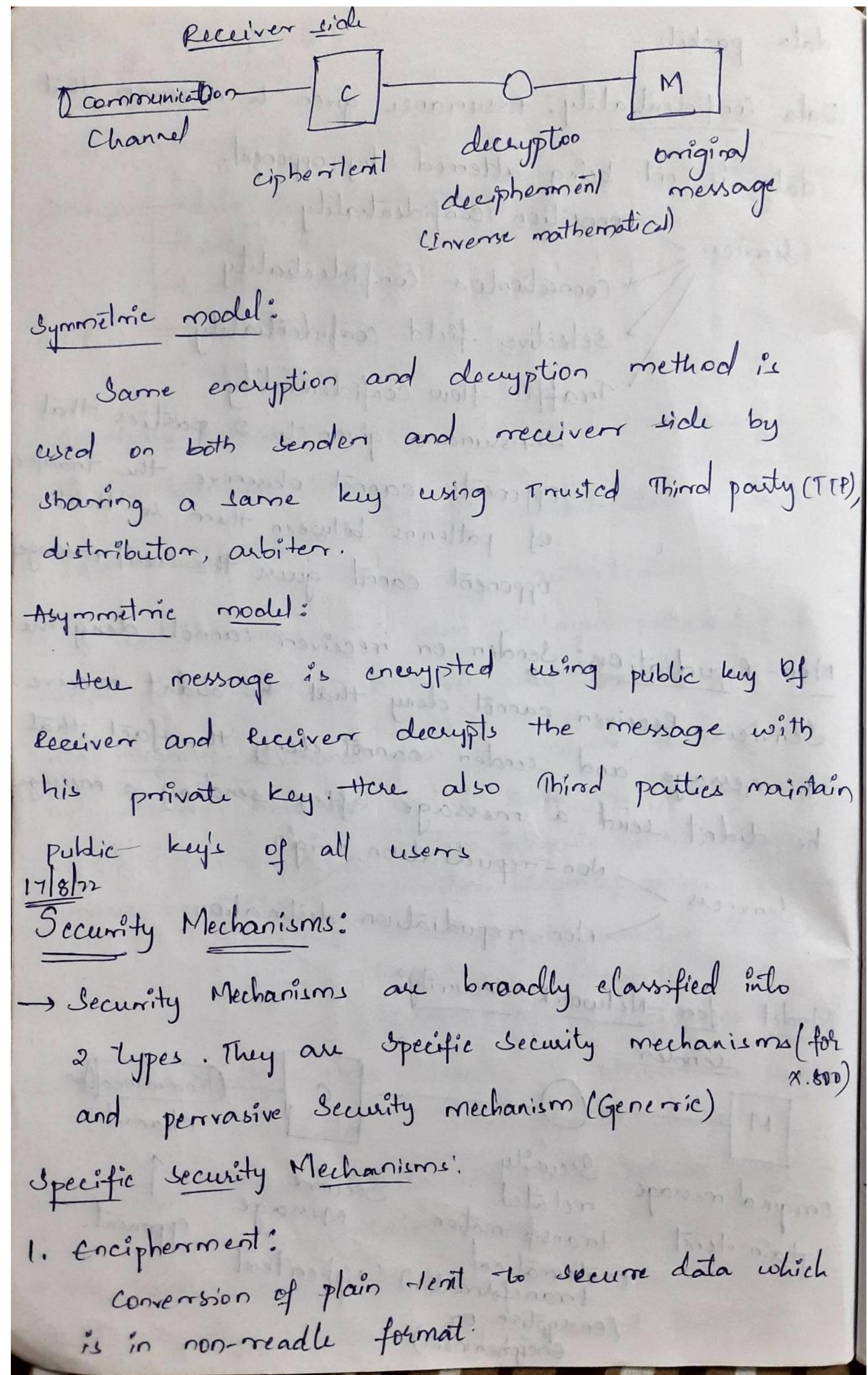
↳ Assurance given to 2 parties that opponents cannot observe the transfer of patterns between them so that opponent cannot guess the next message

Non-Repudiation: Sender or receiver cannot deny the Services. Receiver cannot deny that he didn't receive a message and sender cannot deny the fact that he didn't send a message after sending a message

Services → Non-repudiation, origin
Non-repudiation, destination

Model for Network Security:





2. Digital signature:

It is a procedure to add an additional message to original message so that it acts as signature to ensure a specific sender.

3. access control

4. Data integrity

5. Authentication Exchange

Exchanging some blocks between sender and receiver before original message to prove the authentication.

6. Traffic padding:

It is a mechanism where some bits are padded in the msg such that opponent cannot analyze the msg.

7. Routing control:

It is a mechanism where after a security breach is happened, this allows the source to select a physically secure channel to send data to receiver.

8. Notarization:

TPP (Trusted Third party) are used to transfer the data.

Pervasive security Mechanism (not specific to any service)

1. Trusted functionality

Communication peers trust this functionality.

2. using security label
3. event detection
4. security audit trail
5. security recovery (some mechanisms are available to recover after a security breach)

Number Theory Concepts:

1. Divisibility:

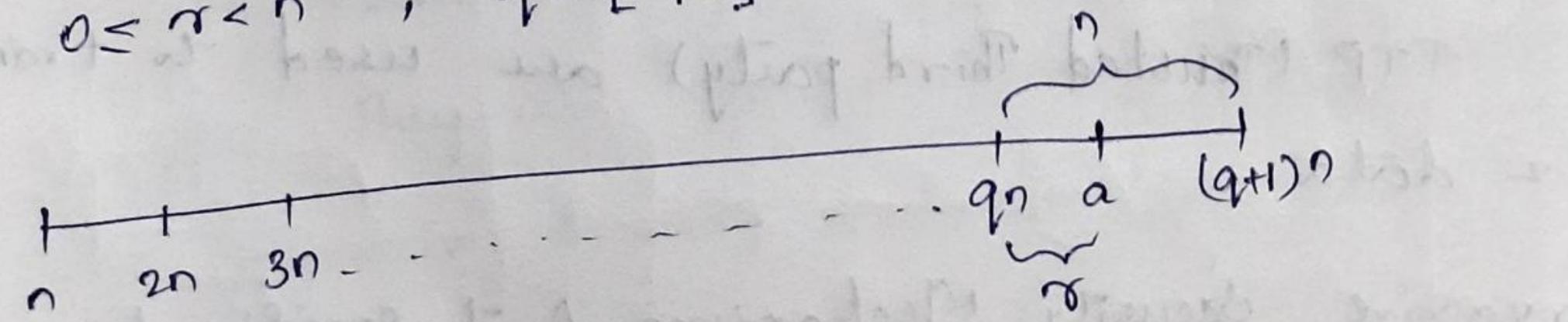
Properties:

- If $a|1$, then $a=\pm 1$
- If $a|b$ and $b|a$ then $a=\pm b$
- Any $b \neq 0$ divides 0
- If $a|b$ and $b|c$ then $a|c$
- If $b|g$ and $b|h$ then $b|(m \cdot g + n \cdot h)$

divisibility algorithm:

$$a = q \cdot n + r$$

$$0 \leq r < n, \quad q = \lfloor n/a \rfloor$$



Modular arithmetic

$$11 \bmod 7 = 4$$

$$-11 \bmod 7 \Rightarrow (-11+7) \bmod 7 \Rightarrow (-4+7) \bmod 7 \Rightarrow 3$$

2. Congruence:

if $a \bmod n = b \bmod n$ then $a \equiv b \pmod{n}$

$$\begin{aligned} 73 \bmod 23 &= 4 \\ 4 \bmod 23 &= 4 \end{aligned} \quad 73 \equiv 4 \pmod{23}$$

Properties:

- $a \equiv b \pmod{n}$ if $n \mid (a-b)$
- $a \equiv b \pmod{n}$ implies $b \equiv a \pmod{n}$
- $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ then $a \equiv c \pmod{n}$
- $(a \bmod n + b \bmod n) \bmod n = (a+b) \bmod n$
- $(a \bmod n - b \bmod n) \bmod n = (a-b) \bmod n$
- $(a \bmod n \times b \bmod n) \bmod n = (a \times b) \bmod n$

Exponentiation

$$11^7 \bmod 13 = (11^{4+2+1}) \bmod 13$$

$$11 \bmod 13 = 11$$

$$11^2 \bmod 13 = 4$$

$$11^4 \bmod 13 = (11^2 \cdot 11^2) \bmod 13$$

$$= (4 \cdot 4) \bmod 13$$

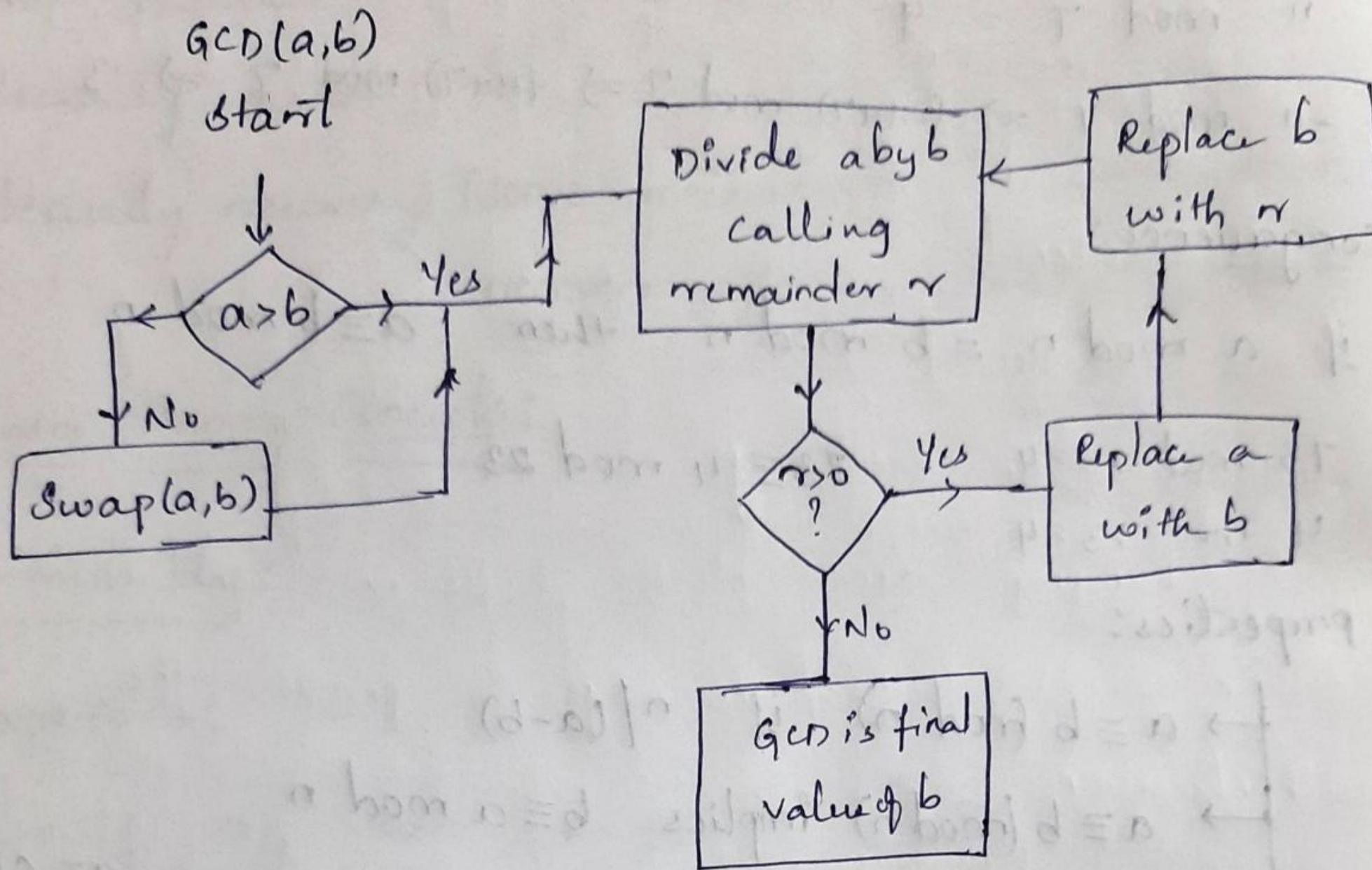
$$= 3$$

$$\text{ans: } (3 \times 4 \times 11) \bmod 13 = 2$$

$$4 \times 4 \times 4 \times 11$$

$$\begin{array}{r} 64 \\ \times 11 \\ \hline 64 \\ 704 \\ \hline 13 \end{array} \quad \begin{array}{r} 13^{704} \\ 65 \\ \hline 54 \end{array}$$

Euclidean Algorithm



GCD(20, 75)

$$\begin{aligned} & a = 20, b = 75 \quad 0 < r_1 < b \\ & \text{swap} \quad a = 75, b = 20 \quad 0 < r_2 < r_1 \\ & b = 20 \end{aligned}$$

divide $r = 15, q = 3$

$$a = 20$$

$$b = 15$$

divide $q = 1, r = 5$

$$a = 15$$

$$b = 5$$

divide $q = 3, r = 0$

5 is GCD

$$a = q_1 r_1 + r_2 \quad 0 < r_1 < b$$

$$b = q_2 r_2 + r_3 \quad 0 < r_2 < r_1$$

$$r_3 = q_3 r_3 + r_4 \quad 0 < r_3 < r_2$$

$$r_{n-2} = q_{n-1} r_{n-1} + r_n \quad 0 < r_{n-1} < r_{n-2}$$

$$r_{n-1} = q_n r_n + 0 \quad 0 < r_n < r_{n-1}$$

$$\gcd(a, b) = r_n$$

$$\text{E1 hom } (11, 11) =$$

$$c = \text{E1 hom } (ax+bx) = 2ab$$

Recursive function

```

GCD(a,b) {
    if (a < b) {
        swap(a,b)
    }
    if (b == 0)
        return a
    if (a % b == 0)
        return b
    return GCD(b, a % b)
}

```

Modular Arithmetic (Cont)

Additive Inverse

If $(x+y) \text{ mod } n = 0$

x's additive inverse is y

If $n=8$

0	8	}
1	7	
2	6	
:		
7	1	

additive
inverse

For all numbers additive inverse is there

Multiplicative Inverse

If $(x+y) \text{ mod } n = 1$

then y is multiplicative inverse of x

$$1 \times 1 \text{ mod } 8 = 1$$

$$3 \times 3 \text{ mod } 8 = 1$$

$$5 \times 5 \text{ mod } 8 = 1$$

$$2 \times 5 \text{ mod } 8 = 1$$

$$4 \times 7 \text{ mod } 8 = 1$$

1. it may have (or) may not for an element x
multiplicative inverse

ex: $n = 11$

0	x	(0,1) pair
1	1	(1,2) pair
2	6	(2,3) pair
3	4	middle
4	3	(0,1,2,3) pair
5	x	middle
6	2	(0,1,2,3,4,5) pair
7	x	middle
8	x	middle
9	x	(1,2,3,4,5,6,7,8) pair
10	x	middle

Extended Euclidean Algorithm

$$d = \text{gcd}(a, b)$$

$$d = xa + by$$

$$\text{gcd}(1547, 560)$$

$$1547 = 2 \cdot 560 + 427$$

$$560 = 1 \cdot 427 + 133$$

$$427 = 3 \cdot 133 + 28$$

$$133 = 4 \cdot 28 + 21$$

$$28 = 1 \cdot 21 + 7$$

$$21 = 3 \cdot 7 + 0$$

$$7 = 28 - 1 \cdot 21$$
$$= 28 - 1 \cdot (133 - 4 \cdot 28)$$

23/08

Extended Euclidean Algorithm

$$d = \gcd(a, b) \quad a > b$$

$$d = ax + by$$

$$\text{ex: } \gcd(1547, 560) = 7$$

to find: x, y where $d = ax + by$ & x, y are integers

$$\text{ex: } 7 = 1547x + 560y$$

$$\gcd(1547, 560)$$

$$\begin{aligned} 1547 &= 2 \cdot 560 + 427 & \rightarrow & 21(1547 - 2 \cdot 560) - 16 \cdot 560 = 21 \cdot 1547 - 58 \cdot 560 \\ 560 &= 1 \cdot 427 + 133 & \rightarrow & 5 \cdot 427 - 16(560 - 1 \cdot 427) = 21 \cdot 427 - 16 \cdot 560 \\ 427 &= 3 \cdot 133 + 28 & \rightarrow & 5(427 - 3 \cdot 133) - 133 = 5 \cdot 427 - 16 \cdot 133 \\ 133 &= 4 \cdot 28 + 21 & \rightarrow & 7 = 28 - (133 - 4 \cdot 28) = 5 \cdot 28 - 133 \\ 28 &= 1 \cdot 21 + 7 & \uparrow & \\ 21 &= 3(7) + 0 & \rightarrow & 7 = 28 - 1 \cdot 21 \\ && \downarrow & \\ && \text{GCD} & \end{aligned}$$

$$7 = 21 \times 1547 - 58 \times 560$$

Euclidean algorithm

$$q_1 = \lfloor a/b \rfloor$$

$$r_1 = a \bmod b \quad a = q_1 b + r_1$$

$$r_2 = b \bmod r_1 \quad b = q_2 r_1 + r_2$$

$$r_3 = r_1 \bmod r_2$$

⋮

$$r_{n+1} = r_{n-1} \bmod r_n$$

Extended Euclidean algorithm

calculate

$$r_{-1} = a \quad x_{-1} = 1 \quad y_{-1} = 0$$

$$r_0 = b \quad x_0 = 0 \quad y_0 = 1$$

$$r_1 = a \bmod b \quad x_1 = x_{-1} - q_1 x_0$$

$$q_1 = \lfloor a/b \rfloor \quad y_1 = y_{-1} - q_1 y_0$$

$$r_2 = b \bmod r_1 \quad x_2 = x_0 - q_2 x_1$$

$$q_2 = \lfloor b/r_1 \rfloor \quad y_2 = y_0 - q_2 y_1$$

$$r_3 = r_1 \bmod r_2 \quad x_3 = x_1 - q_3 x_2$$

$$q_3 = \lfloor r_1/r_2 \rfloor \quad y_3 = y_1 - q_3 y_2$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$r_n = r_{n-2} \bmod r_{n-1} \quad x_n = x_{n-2} - q_n x_{n-1}$$

$$q_n = \lfloor r_{n-2}/r_{n-1} \rfloor \quad y_n = y_{n-2} - q_n y_{n-1}$$

$$r_{n+1} = r_{n-1} \bmod r_n$$

\downarrow
GCD
 \downarrow
 $r_n = 0$

find GCD (1759, 550) = 1

i	r_i	q_i	x_i	y_i	
-1	1759		1	0	
0	550	3	1	1	
1	109	1	16	-3	$1 - 5(-3)$
2	5	21	106	-5	$(1-3 \cdot 0) \quad (0-3 \cdot 1)$
3	4	1	-111	16	$1 - 5(-1) \quad 1 - (-3)^5$
4	0	gcd	355		$1 - 21(-5) \quad -3 - 21(16)$
5	0	4			$-5 - 1(106) \quad 16 - 1(-339)$

$$1 = -111 \times 1759 + 355 \times 550$$

for each of the following eqns find x

$$3x \equiv 4 \pmod{5}$$

$$x = 3$$

$$9x \equiv 3 \pmod{7}$$

$$x = 5$$

$$\Rightarrow x \equiv 6 \pmod{9}$$

$$x = 6$$

4	9
15	9
24	10
78	17
42	24
52	31
240	38 + 7
1760	45

find $\text{GCD}(2740, 1760)$ using extended euclidean algorithm also find x & y .

i	r _i	q _i	x _i	y _i
-1	2740		1	0
0	1760		0	1
1	980	1	1	-1
2	780	1	-1	2
3	200	1	2	-3
4	180	3	-7	11
5	20	1	19	-14

248 Prime numbers

If P has divisors ± 1 and $\pm P$ only

Given any integer a can be factored as prime number power some integer

$$a = p_1^{a_1} \times p_2^{a_2} \times \dots \times p_n^{a_n}$$

$$14 = 2^1 \times 7^1$$

$$15 = 3^1 \times 5^1$$

$$28 = 2^2 \times 7^1$$

P is a set of prime numbers then any integer a can be expressed as $a = \prod_{p \in P} p^{a_p}$ where each $a_p \geq 0$

Multiplication of two numbers can be expressed as

$$216 = 2^3 \times 3^3$$

$$800 = 2^4 \times 3^1 \times 5^2$$

$$18 = 2^1 \times 3^2 \times 5^0$$

$$\gcd(300, 18) = 2^1 \times 3^1 \times 5^0$$

we have to select prime factor power minimum out of the two, with prime factorization we can easily find gcd of two numbers.

$$k = \gcd(a, b)$$

$$\text{then } k_p = \min(a_p, b_p) \text{ for all } p$$

Fermat's Theorem:

If p is a prime number and a is an integer not divisible by p then $a^{p-1} \equiv 1 \pmod{p}$

Alternate form

$$a^p \equiv a \pmod{p}$$

Ex: $a=7$ and $p=19$

$$7^{18} \equiv 1 \pmod{19}$$

(by)

$$7^9 \equiv 7 \pmod{19}$$

Here, the cases fail when a is divisible by p .

Euler's Totient Function

→ It is represented as $\phi(n)$

→ $\phi(n)$ is defined as the number of numbers that are less than n that are relatively prime to n .

If $n=5$ then $\phi(n)=\{1, 2, 3, 4\}$

If n is prime then $\phi(n)=n-1$

If $n=8$ then $\phi(n)=\{1, 3, 5, 7\}$, so $\phi(n)=4$.

$n=14$ $\{1, 3, 5, 9, 11, 13\}$ $\phi(n)=6$

two prime numbers $P, q\}$ different prime numbers

If a number $n=Pq$

$$\phi(n) = \phi(P) \times \phi(q)$$

$$\phi(n) = (P-1)(q-1)$$

$$P=2 \quad q=7$$

$$\phi(14) = (2-1) \times (7-1) = 1 \times 6 = 6$$

$$\phi(22) = (2-1)(11-1)$$

$$\phi(22) = 10$$

Euler's Theorem

→ for every a and n that are relatively prime to each other

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

alternative form

$$a^{\phi(n)+1} \equiv a \pmod{n}$$

$$a=3, n=10$$

$$\phi(10) = (2-1)(5-1) = 4$$

$$3^4 \pmod{10} = 1$$

$$3^5 \equiv 3 \pmod{10}$$

Properties of prime numbers

There are 2 useful properties of prime numbers that allows Robin Miller to construct the algorithm

i) n is odd then n can be expressed as

$$n-1 = 2^k q \text{ where } k \geq 0 \text{ and } q \text{ is odd}$$

$$\text{if } n=7$$

$$6 = 2^1 \times 3 \text{ then } k=1, q=3$$

$$\text{if } n=11$$

$$10 = 2^1 \times 5 \text{ then } k=1, q=5$$

$$\text{if } n=13$$

$$12 = 2^2 \times 3$$

$$k=2, q=3$$

q is odd and k cannot be zero also

for $a \neq 1$

2) If p is prime, a is a positive integer less than p

then $a^2 \mod p = 1$

If and only if $a \mod p = 1$ or $a \mod p = -1 \mod p$

Ex: $p=7, a=6$

$6^2 \mod 7 = 1$ as $6 \mod 7 = 6 \Rightarrow 6 \mod 7 = -1 \mod 7$

$p=11, a=3$

$11^2 \mod 11 \neq 1$ as $3 \mod 11 = 3 \neq -1 \mod 11$

2) If p is a prime number greater than 2

$p-1 = 2^k q$ with $k > 0$ and q is odd

Let a be an integer $1 < a < p-1$ then one of the following conditions is true

1. $a^q \equiv 1 \mod p$

2. one of the numbers $a^q, a^{2q}, a^{4q}, \dots, a^{2^{k-1}q} \equiv -1 \mod p$

formally $a^{2^{j-1}q} \equiv -1 \mod p = p-1$ $1 \leq j \leq k$

→ Rabin miller algorithm is constructed in a way

to say a number may be prime

if these 2 properties fail then we say it is a composite number.

Rabin miller Algorithm

Test(n)

Find integers k, q with $k > 0$, q odd so that

$$n-1 = 2^k q$$

Select a random integer $1 < a < n-1$

If $a^q \mod n = 1$ then return ("inconclusive")

for $j=0$ to $k-1$ do

If $a^{2^j q} \mod n = n-1$ then return ("inconclusive")

return ("composite")

Ex: $n = 2047$, let random number selected = 2

then the output will be inconclusive.

$$n-1 = 2046$$

$$2046 = 2 \times 1023$$

$$k=1 \quad q=1023$$

If $a = 2$

$$2^{1023} \mod 2047 = 1 \Rightarrow$$
 says it may be prime

But 2047 is a composite number.

Sol 8
Residue classes mod n (set of residues)

$$[0] \ [1] \ \dots \ [n-1] \text{ where}$$

$$[r] = \{a : a \text{ is an integer}; a \equiv r \pmod{n}\}$$

The residue classes mod 4 are

$$[0] = \{ \dots -16, -12, -8, -4, 0, 4, 8, 12, 16, \dots \}$$

$$[1] = \{ \dots -15, -11, -7, -3, 1, 5, 9, 13, 17, \dots \}$$

$$[2] = \{ \dots -14, -10, -6, -2, 2, 6, 10, 14, 18, \dots \}$$

$$[3] = \{ \dots -13, -9, -5, -1, 3, 7, 11, 15, 19, \dots \}$$

Chinese Remainder Theorem (CRT)

Any integer can be constructed from a pair of integers mod residue class, the pair is relatively prime and the residue class are relatively prime

$$x \equiv \frac{5}{a_1} \pmod{8}_{\text{mod}(m_1)}$$

$$x \equiv \frac{3}{a_2} \pmod{5}_{\text{mod}_2}$$

x can be reconstructed from 5, 3 and residue classes

$$a_1 = 5, a_2 = 3, m_1 = 8, m_2 = 5$$

$$M = m_1 \times m_2$$

$$= 8 \times 5$$

$$M = 40$$

$$M_1 = \frac{M}{m_1} = \frac{40}{8} = 5$$

$$M_2 = \frac{M}{m_2} = \frac{40}{5} = 8$$

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 \pmod{M}$$

$$\begin{aligned} M_1 y_1 &\equiv 1 \pmod{m_1} \\ 5 y_1 &\equiv 1 \pmod{8} \end{aligned}$$

$$M_2 y_2 \equiv 1 \pmod{m_2}$$

$$8 y_2 \equiv 1 \pmod{5}$$

$$y_1 \equiv 5 \pmod{8} \quad y_2 \equiv 2 \pmod{5}$$

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 \pmod{M}$$

$$= (5 \times 5 \times 5 + 3 \times 8 \times 2) \pmod{40}$$

$$x = (125 + 48) \pmod{40}$$

$$x = 173 \pmod{40}$$

$$x = 13 \pmod{40}$$

$$13 \pmod{40} \begin{cases} 5 \pmod{8} \\ 3 \pmod{5} \end{cases}$$

where 13 mod 40 is obtained from 13 mod 5 and 13 mod 8
where 13 mod 5 is obtained from 13 mod 40

Ex: $973 \pmod{1813}$

find the pair $973 = (-, -)$ where $m_1 > 37$

and $m_2 > 49$

$$973 \begin{cases} a_1 \pmod{37} \\ a_2 \pmod{49} \end{cases}$$

$$M_1 = \frac{M}{m_1} = \frac{37 \times 49}{37} = 49$$

$$M_2 = \frac{M}{m_2} = \frac{37 \times 49}{49} = 37$$

$$M_1^{-1} \equiv - \pmod{m_1}$$

$$M_1^{-1} \equiv \frac{34}{49 \times 34} \pmod{37}$$

$$M_2^{-1} \equiv \frac{-}{37 \times 4} \pmod{m_2}$$

$$M_2^{-1} \equiv 4 \pmod{49}$$

$$34 \pmod{37}$$

$$973 \equiv 4 \pmod{49}$$

$$\Rightarrow 973 \equiv 11 \pmod{37} > 973 \equiv (11, 42)$$

$$973 \equiv 42 \pmod{49}$$

$$\text{Let } M = \prod_{i=1}^k m_i$$

where m_i are relatively prime.

$$\text{i.e., } \gcd(m_i, m_j) = 1 \quad i \neq j$$

any integer in \mathbb{Z}_k can be represented by k-tuple.

$$A \leftrightarrow (a_1, a_2, \dots, a_k)$$

where $A \in \mathbb{Z}_M$ and $a_i = A \pmod{m_i}$ for $1 \leq i \leq k$

Ex: use CRT to find x ,

$$x \equiv 3 \pmod{7}$$

$$x \equiv 8 \pmod{13}$$

$$a_1 = 3, a_2 = 8, m_1 = 7, m_2 = 13$$

$$M = m_1 \times m_2 \\ = 7 \times 13$$

$$M_1 = \frac{M}{m_1} = 13$$

$$M_2 = \frac{M}{m_2} = 7$$

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 \pmod{M}$$

$$\begin{aligned} M_1 y_1 &\equiv 1 \pmod{m_1} & M_2 y_2 &\equiv 1 \pmod{m_2} \\ 13 y_1 &\equiv 1 \pmod{7} & 7 y_2 &\equiv 1 \pmod{13} \\ y_1 &\equiv 6 \pmod{7} & y_2 &\equiv 82 \pmod{13} \end{aligned}$$

$$\begin{aligned} x &= a_1 M_1 y_1 + a_2 M_2 y_2 \pmod{M} \\ &= (3 \times 13 \times 6 + 3 \times 7 \times 2) \pmod{91} \\ &\equiv (234 + 42) \pmod{91} \\ &\equiv 276 \pmod{91} \\ x &\equiv 3 \pmod{91} \\ 3 &\pmod{91} \quad 3 \pmod{7} \\ 3 &\pmod{91} \quad 3 \pmod{13} \end{aligned}$$

Discrete logarithms

$\mathbb{Z}_n = \{ \text{the collection of numbers that are relative prime to } n \}$

$$\mathbb{Z}_{14} = \{ 1, 3, 5, 9, 11, 13 \}$$

Given

1) $n = 2, 4, P^k, 2 \cdot P^k$ for $k \in \mathbb{N}$

and P is odd prime

2) Given $\alpha \in \mathbb{Z}_n^*$ is a generator (primitive root)

3) $\beta \in \mathbb{Z}_n^*$ (not necessarily a generator)

Compute $\log_2 B$.

i.e., find n such that $2^n = B$

Ex: Given $n=9$

$$Z_n = \{1, 2, 4, 5, 7, 8\}$$

$$\phi(n) = 6$$

$$\alpha = 2$$

$$2^0 \bmod 9 \equiv 1 \quad 0 \Rightarrow \log_2 1$$

$$2^1 \bmod 9 \equiv 2 \quad 1 \Rightarrow \log_2 2$$

$$2^2 \bmod 9 \equiv 4 \quad 2 \Rightarrow \log_2 4$$

$$2^3 \bmod 9 \equiv 8 \quad 3 \Rightarrow \log_2 8$$

$$2^4 \bmod 9 \equiv 7 \quad 4 \Rightarrow \log_2 7$$

$$2^5 \bmod 9 \equiv 5 \quad 5 \Rightarrow \log_2 5$$

from 2^0 to $2^{\phi(n)-1}$ we are able to generate all numbers

in Z_n , i.e. α is a generator

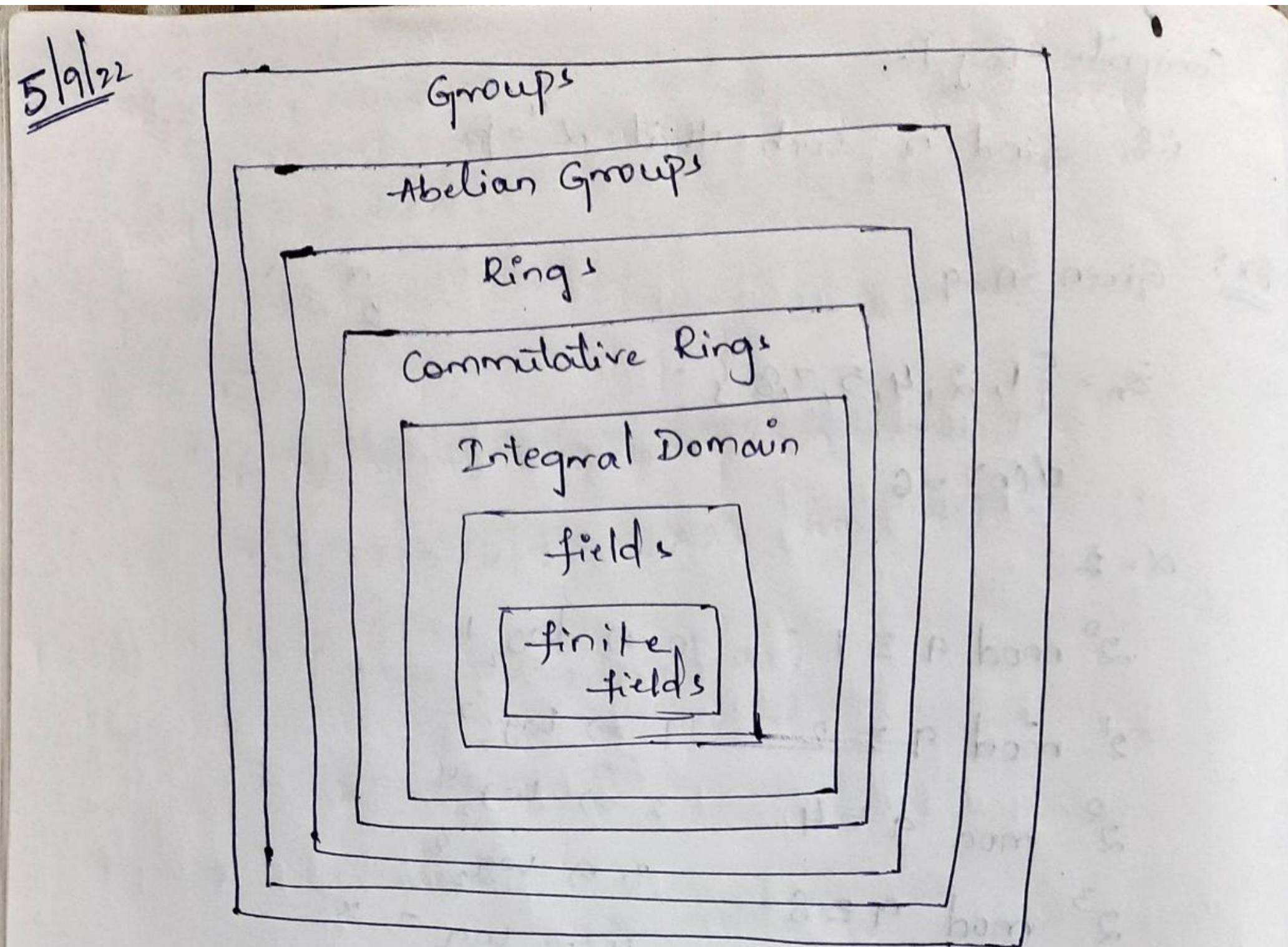
logarithm	0	1	2	3	4	5
number	1	2	4	8	7	5

1) multiplicative

2) exponentiation

3) generating 10^{24} prime number using rabin miller.

4) chinese remainder theorem



Groups

denoted by $\{G, \circ\}$ is a set of elements with binary operation \circ , binary operation limits to addition, permutation

Axioms that groups hold:

(a,b) ordered pair, a.b

A₁: If a and b $\in G$ then a.b belongs to G

A₂: a.(b.c) = (a.b).c & a,b,c in G

A₃: Identity element: there is an element e in G

such that a.e = e.a = a & a in G

A₄: Inverse element: for each a in G there is an inverse a' such that

$$a.a' = a'.a = e$$

Abelian Groups:

It satisfies the axioms of group.

A₁: commutative: $a.b = b.a \quad \forall a, b \in G$

$$A_1: \{3, 2, 1\} \cdot \{1, 3, 2\} = \{2, 3, 1\}$$

A₂: $a.(b.c) = (a.b).c$

$$\{1, 2, 3\} \cdot (\{3, 2, 1\} \cdot \{1, 3, 2\}) = \{2, 3, 1\}$$

$$(\{1, 2, 3\} \cdot \{3, 2, 1\}) \cdot \{1, 3, 2\} = \{2, 3, 1\}$$

A₃: $\{1, 2, 3, \dots, n\}$

A₄: $a.b \neq b.a$

$$\{3, 2, 1\} \cdot \{1, 3, 2\} = \{2, 3, 1\}$$

$$\{1, 3, 2\} \cdot \{3, 2, 1\} = \{3, 1, 2\}$$

\therefore The permutation is a group but not abelian group

Cyclic Groups:

A group is called cyclic if every element in G is a power of a^k , k is an integer and a is a fixed element $\in G$.

Ex: $n=5 \quad \mathbb{Z}_5 = \{1, 2, 3, 4\}$

Consider $a=2$

$2^0, 2^1, 2^2, 2^3$ should produce all elements in group

$$2^0 \bmod 5 = 1$$

$$2^1 \bmod 5 = 2$$

$$2^2 \bmod 5 = 4$$

$$2^3 \bmod 5 = 3$$

\therefore It is a cyclic group.

Rings:

- It is an abelian group with some additional conditions represented by $\{R, +, \times\}$
- A_1 to A_5 should be satisfied
- $A_1 - A_5$: R is an abelian group w.r.t addition i.e.,
 R satisfies axioms A_1 to A_5 for the additive group

M_1 : If a and $b \in R$, then $a \cdot b \in R$

M_2 : Associativity of multiplication

$$a(bc) = (ab)c$$

M_3 : Distributive class

$$a(b+c) = ab + ac \quad \forall a, b, c$$

$$(a+b)c = ac + bc \quad \forall a, b, c$$

+ - talking about distributive

\times - talking about closure and associativity

Commutative Ring:

M_4 : commutativity of multiplication

$$ab = ba \quad \forall a, b \in R$$

Example: Let \mathbb{E} be the set of even integers. \mathbb{E} is a commutative ring.

Standard Notations

- \mathbb{N} : set of positive integers $\{1, 2, 3, \dots\}$
- \mathbb{Z} : set of all integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$
- \mathbb{Q} : set of all rational numbers
- \mathbb{R} : set of real numbers
- \mathbb{C} : set of complex numbers

Integral domain

An integral domain, which is a commutative ring that obeys following axioms

M₅: Multiplicative Identity

↳ There is an element 1 in R such that

$$a \cdot 1 = 1 \cdot a = a, \forall a \in R$$

M₆: No zero divisors: If $a, b \in R$ & $a \neq 0$, then either $a = 0$ or $b = 0$

Ex: Z is an integral domain

Fields:

- Type 1
- Type 2 - Galois field, GF(2), Binary field
- Type 3 - GF(P), prime, GF(Pⁿ)

A field F, denoted by $\{F, +, \times\}$, is a set of elements with two binary operations, called addition and multiplication, such that $\forall a, b, c \in F$ the following axioms are obeyed

(M₁-M₆) i.e., F is an integral domain

M₇: Multiplicative inverse: $\forall a \in F$ (except 0), $\exists a' \in F$ such that $a \cdot a' = a' \cdot a = 1$

Ex: Q, R, C are fields

Z is not a field because every integer does not have a multiplicative inverse.

→ In cryptographic algorithms, we use fields.

Arithmetic on polynomials

Let R be an arbitrary ring. A polynomial over R is an expression of the form:

$$f = \sum_{i=0}^n a_i x^i = a_0 + a_1 x + \dots + a_n x^n$$

where n is a non-negative integers, the coefficients a_i ($0 \leq i \leq n$) are elements of R and x is symbol not belonging to R , called an indeterminate over R .

→ Type 1 - coefficient can be any number

→ Type 2 - coefficients are 0 or 1, coefficient mod 2.

Ex: Let $f(x) = x^3 + x^2 + x + 2$, $g(x) = x^2 - x + 1$

Type 1
S is set of integers

$$f(x) + g(x) = x^3 + 2x^2 - x + 3$$

$$f(x) - g(x) = x^3 + x + 1$$

$$f(x) * g(x) = x^5 + 3x^4 - 2x^3 + 2$$

$$f(x) / g(x) \Rightarrow a = x+2$$

Type 2
Definition: $f = \sum_{i=0}^n a_i x^i$, $g = \sum_{i=0}^m b_i x^i$

$$f+g = \sum_{i=0}^{\max(n,m)} (a_i + b_i) x^i$$

Type 2 - GF(2) or GF(P) where P is only 2

$$f(x) = x^3 + 2x + 1$$

$$g(x) = x^2 - x + 2$$

$$\begin{aligned} f(x) + g(x) &= x^3 + x^2 + x + 3 \\ &\quad \text{mod } 2 \\ &= x^3 + x^2 + x + 1 \end{aligned}$$

→ polynomial arithmetic in which the arithmetic on the coefficient is performed modulo p , that is the coefficients are in $\text{GF}(p)$ - Type 2

→ polynomial arithmetic in which the coefficients are in $\text{GF}(p)$ & the polynomial are defined modulo a polynomial $m(x)$ whose highest power

Eg: over $\text{GF}(2)$

$$f(x) = x^7 + x^5 + x^4 + x^3 + x + 1$$

$$g(x) = x^3 + x + 1$$

$$f(x) + g(x) = x^7 + x^5 + x^4$$

$$f(x) - g(x) = x^7 + x^5 + x^4$$

$$f(x) * g(x) = x^{10} + x^4 + x^3 + 1$$

$$f(x) \mid g(x) = x^4 + 1$$

GCD and inverse on Type 2 polynomials

Divisibility

Let F denote a field from now on, we consider polynomials over fields. we say that the polynomial $g \in F[x]$ divides $f \in F[x]$ if

let $g \neq 0$ be polynomial in $F[x]$. Then for any $f \in F[x]$ there exist polynomial $q, r \in F[x]$ such

that $f = qg + r$, where $\deg(r) < \deg(g)$
using the division algorithm, we can show that every ideal of $F[x]$ is principal.

GF(2) Addition (XOR)			multiplication (AND)		
+ mod 2	0	1	x	0	1
0	0	1	0	0	0
1	1	0	1	0	1

GF(2) - mod 2

GF(2ⁿ) - mod m(x) - represents irreducible polynomial

GF(2³) : n=3 m(x) = x³ + x + 1

f(x) = x² + x + 1 111 = 7

g(x) = x² + 1 101 = 5

f(x) + g(x) = x² + x + 1 + x² + 1
= 2x² + x + 2
= x
= 2

In GF(2³) addition of 7 and 5 is 2, 101 + 101 = 1000 = x³.

f(x). g(x) = (x² + x + 1) . (x² + 1)
= x⁴ + x³ + x² + x² + x + 1
= x⁴ + x³ + 2x² + x + 1
= x⁴ + x³ + x + 1 mod x³ + x + 1
= -x² - x
= x² + x
= b

mod 8 (\mathbb{Z}_8)								GF(2 ³)									
x	0	1	2	3	4	5	6	7	x	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	0	2	0	2	4	6	3	1	7	5
3	0	3	6	1	4	7	2	5	3	0	3	6	5	7	4	1	2
4	0	4	0	4	0	4	0	4	4	0	4	3	7	6	2	5	9
5	0	5	2	7	4	1	6	3	5	0	5	1	4	2	7	3	6
6	0	6	4	2	0	6	4	2	6	0	6	7	1	5	3	2	4
7	0	7	0	5	4	3	2	1	7	0	7	5	2	1	6	4	3

when we move to galoian fields we can find
multiplicative inverse of all elements.

→ In mod(2^k) frequency of elements is not uniformly distributed. (there is a chance of guessing)

integer	0	1	2	3	4	5	6	7
freq in \mathbb{Z}_8	22	4	8	4	12	4	8	4
freq in GF(2 ³)	7	7	7	7	7	7	7	7

→ GF(2³) makes cryptograph more powerful so that
there will be a less chance of guessing the message.

$$\text{Ex: } x \equiv 1 \pmod{5}, x \equiv 1 \pmod{7}, x \equiv 3 \pmod{11}$$

$$a_1 = 1, a_2 = 1, a_3 = 3$$

$$m_1 = 5, m_2 = 7, m_3 = 11$$

$$M = m_1 \times m_2 \times m_3 = 5 \times 7 \times 11 = 385$$

$$M_1 = \frac{M}{m_1} = \frac{5 \times 7 \times 11}{5} = 77$$

$$M_2 = \frac{M}{m_2} = \frac{5 \times 7 \times 11}{7} = 55$$

$$M_3 = \frac{M}{m_3} = \frac{5 \times 7 \times 11}{11} = 35$$

$$M_1 y_1 \equiv 1 \pmod{5} \quad M_2 y_2 \equiv 1 \pmod{7} \quad M_3 y_3 \equiv 1 \pmod{11}$$

$$77 y_1 \equiv 1 \pmod{5} \quad 55 y_2 \equiv 1 \pmod{7} \quad 35 y_3 \equiv 1 \pmod{11}$$

$$y_1 \equiv 3 \pmod{5} \quad 6 y_2 \equiv 1 \pmod{7} \quad 2 y_3 \equiv 1 \pmod{11}$$

$$y_2 \equiv 6 \pmod{7} \quad y_3 \equiv 6 \pmod{11}$$

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + a_3 M_3 y_3$$

$$= 1 \times 77 \times 3 + 1 \times 55 \times 6 + 3 \times 35 \times 6$$

$$= 1191 \pmod{385}$$

$$= 36 \pmod{385}$$

$$36 \pmod{385} \begin{cases} 1 \pmod{5} \\ 1 \pmod{7} \\ 3 \pmod{11} \end{cases}$$

$$\text{Ex: } x \equiv 3 \pmod{7}, x \equiv 3 \pmod{13}, x \equiv 0 \pmod{12}$$

$$a_1 = 3, a_2 = 3, a_3 = 0$$

$$m_1 = 7, m_2 = 13, m_3 = 12$$

$$M = 7 \times 13 \times 12 = 1092$$

$$M_1 = 13 \times 12 = 156$$

$$M_2 = 7 \times 12 = 84$$

$$M_3 = 7 \times 13 = 91$$

$$\begin{aligned}
 M_1y_1 &\equiv 1 \pmod{7} \quad \frac{104}{24} \quad M_2y_2 \equiv 1 \pmod{13} \quad M_3y_3 \equiv 1 \pmod{12} \\
 156y_1 &\equiv 1 \pmod{7} \quad 84y_2 \equiv 1 \pmod{13} \\
 2y_1 &\equiv 1 \pmod{7} \quad 6y_2 \equiv 1 \pmod{13} \\
 y_1 &\equiv 4 \pmod{7} \quad y_2 \equiv 9 \pmod{13} \\
 x &= a_1M_1y_1 + a_2M_2y_2 + a_3M_3y_3 \\
 x &= 3 \times 156 \times 4 + 3 \times 84 \times 11 + 0 \\
 x &= 4644 \pmod{1092} \\
 x &= 276 \pmod{1092}
 \end{aligned}$$

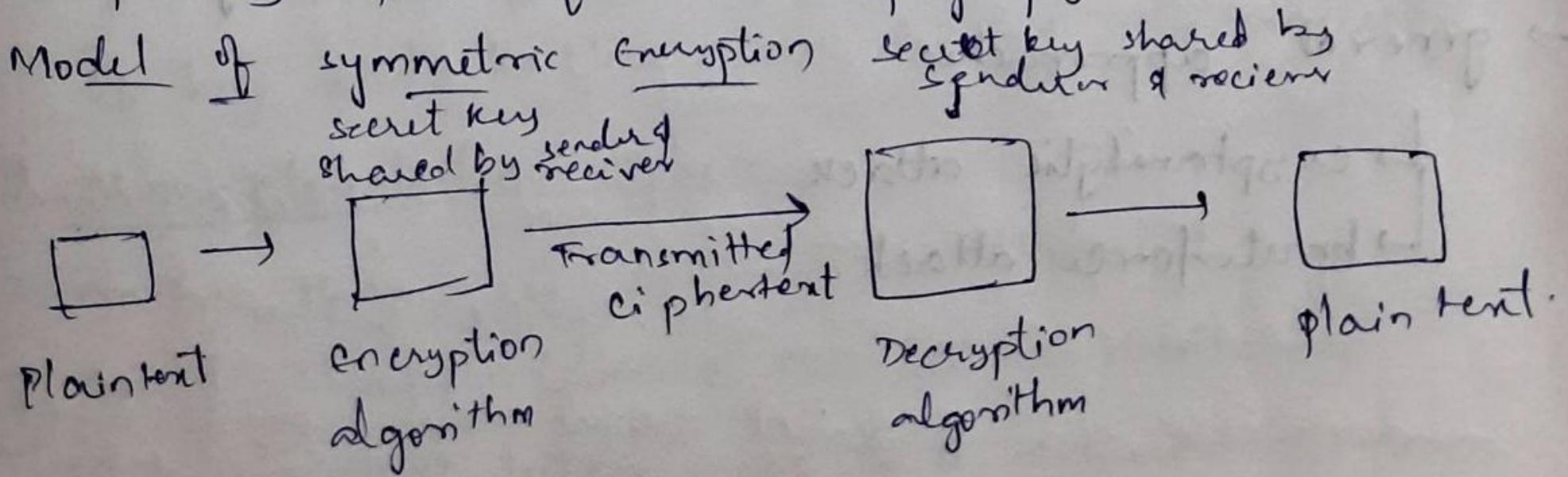
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Symmetric encryption

- or conventional / private-key / single-key
- sender and recipient share a common key
- all classical encryption algorithms are private-key
- was only type used prior to 70's
- key - info used in cipher known only to sender/receiver
- cryptanalysis - study of encryption and decryption

Cryptanalysis (code breaking) - study of methods of deciphering ciphertext without knowing key

Cryptography - field of both cryptography and cryptanalysis



→ Two requirement for secure use of symmetric encryption

↳ a strong encryption algorithm

↳ a secret key known only to sender/receiver.

Encryption algot operations

→ substitution - each letter is replaced with other letter in message

→ Transposition - re-arranging letters in original message

Mathematically we write

$y = E_k(x)$ x - original msg y - ciphered msg

$x = D_k(y)$ k - same key.

→ characterize cryptographic system by

↳ type of encryption operations used

↳ substitution/transposition/product

↳ number of keys used

↳ single-key or private/public key

↳ way in which plaintext is processed

↳ block/stream.

Cryptanalysis

→ objective - to recover key not just messages

→ general approaches

↳ cryptanalytic attack

↳ brute force attack.

classical substitution ciphers

↳ where letters of plaintext are replaced by other letters or numbers

1. Caesar cipher

→ earliest known substitution cipher

→ by Julius Caesar

→ replaces each letter by 3rd letter

Ex: meet me after the toga party

PHHW PH DIWHU WKH WRJD SDUWB

$$C = (P + K) \bmod 26$$

$$P = (C - K) \bmod 26$$

→ can define transformation as
a b c d e f g h i j k l m n o p q r s t u v w x y z
D E F G H I J K L M N O P A R S T U V W X Y Z A B C

→ then have Caesar cipher as

$$C = E(P) = (P + K) \bmod 26$$

$$P = D(C) = (C - K) \bmod 26$$

→ only 26 possible ciphers

→ couch simply try each in turn

→ a brute force search

→ given ciphertext, we can check the messages with
26 key and hacker can dectrypt the message.

2. Monoalphabetic cipher

→ rather than just shifting the alphabet

→ could shuffle (jumble) the letters arbitrarily.

→ each plaintext letter maps to a different random
cipher text letter

- hence key is 26 letters long
- Plain abcdefghijklmnopqrstuvwxyz
- Cipher: DKVQFIBJW
- because of characteristics of english language, we can still break the code.
- in English e is by far the most common letter followed by T, R,
- Playfair key Matrix
- a 5x5 matrix of letters based on a keyword
- fill in letters of keyword
- fill rest of matrix with other letters
- Eg:- using the keyword MONARCHY

M	O	N	A	R
C	H	Y	B	P
E	F	G	I	K
L	P	Q	S	T
U	V	W	X	Z

- plaintext is encrypted two letters at a time
 - if a pair is a repeated letter, insert filler like X
 - if both letters fall in same row, replace each letter with letter right to it.
 - if both letters are in same column then replace each letter with letter below it
 - if both are different letters rows and columns replace with a element in same row but column is of other element.

NITWARANGAL*

AGA ZRMRAINSU

C2:
Play-fair matrix:

J	I	K	C	D	E	F
U	N	P	A	S		
Z	V	W	X	Y		
R	A	L	G	B		
B	I	T	H	M		

Plaintext: I only regret that I have but one life to
give for my country.

Ciphertext: M A P A Z O Q H G J H W I G H T I G X C J Z M L Q C
A T J F M L A H X C S M O B V F R S P I O Z

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Polyalphabetic ciphers

- polyalphabetic substitution ciphers
- improve security using multiple cipher alphabets
- make cryptanalysis harder with more alphabets to guess and flatten frequency distribution
- use a key to select which alphabet is used for each letter of the message
- use each alphabet in turn

Vigenere cipher

- simplest polyalphabetic substitution cipher
- effectively
- Encryption key is given → sample

key: Sample samples

Plain: I am in warangal

Cipher: $(S+I) \bmod 26$ (plain + keyletter) mod 26

Decryption

Plain: (cipher - keyletter) mod 26

frequency cannot be found easily.

Autocorrelation cipher

- plain text characters is used as key for rest of the message after the key is completed.
- keyword is prefixed to msg as a key.

One-Time Pad

- some key is arbitrarily selected and for one message one key is selected.
- it makes frequency analysis very tough for the cryptanalysis.

- sharing the key everytime is a problem here, It depends on TTP, every time a message has to TTP has to

generate a key, and share between Sender & Receiver.

Hill Cipher

- key is taken as a matrix based on the block size of message.
- encryption algorithm takes m successive plain text and substitute for them m cipher text letters.
- each character is assigned a numeric value ($a=0, \dots, z=25$)

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \bmod 26$$

$$C = KP \bmod 26$$

$$P = K^{-1}C \bmod 26 \Rightarrow K K^{-1}P = P$$

key K

$$K^{-1} = (\det K)^{-1} (-1)^{i+j} D_{ji}$$

where D_{ji} is subdeterminant - formed by deleting i^{th} row & j^{th} column of K .

$\det(K^{-1}) \Rightarrow$ multiplicative inverse of $(\det K) \bmod 26$

for decrypting:

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

example:

$$P = E G \quad K_2 = \begin{pmatrix} 3 & 2 \\ 3 & 5 \end{pmatrix} \quad K^{-1} = \begin{pmatrix} 15 & 20 \\ 17 & 9 \end{pmatrix}$$

$C_2 = K_2 P \pmod{26}$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \end{pmatrix} \pmod{26} = \begin{pmatrix} 24 \\ 16 \end{pmatrix}$$

$$\det(K) = 15 - 6 = 9$$

$$(9)^{-1} = 9x \pmod{26}$$

$$= 3$$

$$K_2^{-1} = 3 \times \begin{pmatrix} 5 & -2 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} 15 & -6 \\ -9 & 9 \end{pmatrix} = \begin{pmatrix} 15 & 20 \\ 17 & 9 \end{pmatrix}$$

$$P = K_2^{-1} C \pmod{26}$$

$$= \begin{pmatrix} 15 & 20 \\ 17 & 9 \end{pmatrix} \begin{pmatrix} 24 \\ 16 \end{pmatrix} \pmod{26}$$

$$= \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$P = E G$$

Transposition Ciphers

- The original plain text characters are just rearranged
- also called as transposition ciphers.
- They are easy to break as frequency of letters in plain text and cipher text are same.

Rail Fence Cipher

→ write message letters out diagonally over a number of rows

→ then read off cipher row by row

→ ~~mete~~^m
et e f e t h e o g a a t
c a r + t p y

cipher: m m t h g r e t e f e o a a t e a n t p y

Row Transposition Ciphers

→ a more complex transposition

→ write letters of message out in rows over a specified number of columns

→ then reorder the columns according to some key before reading off the rows

key: 3 4 2 1 5 6 7

Plaintext: a t t a c k p
o s t p o n e
d u n t i l t
w o a m x y z

Ciphertext: t t n a a p t m t s u o a o d w c o i x k n l y

Petz.

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Block cipher Modes of Operation:

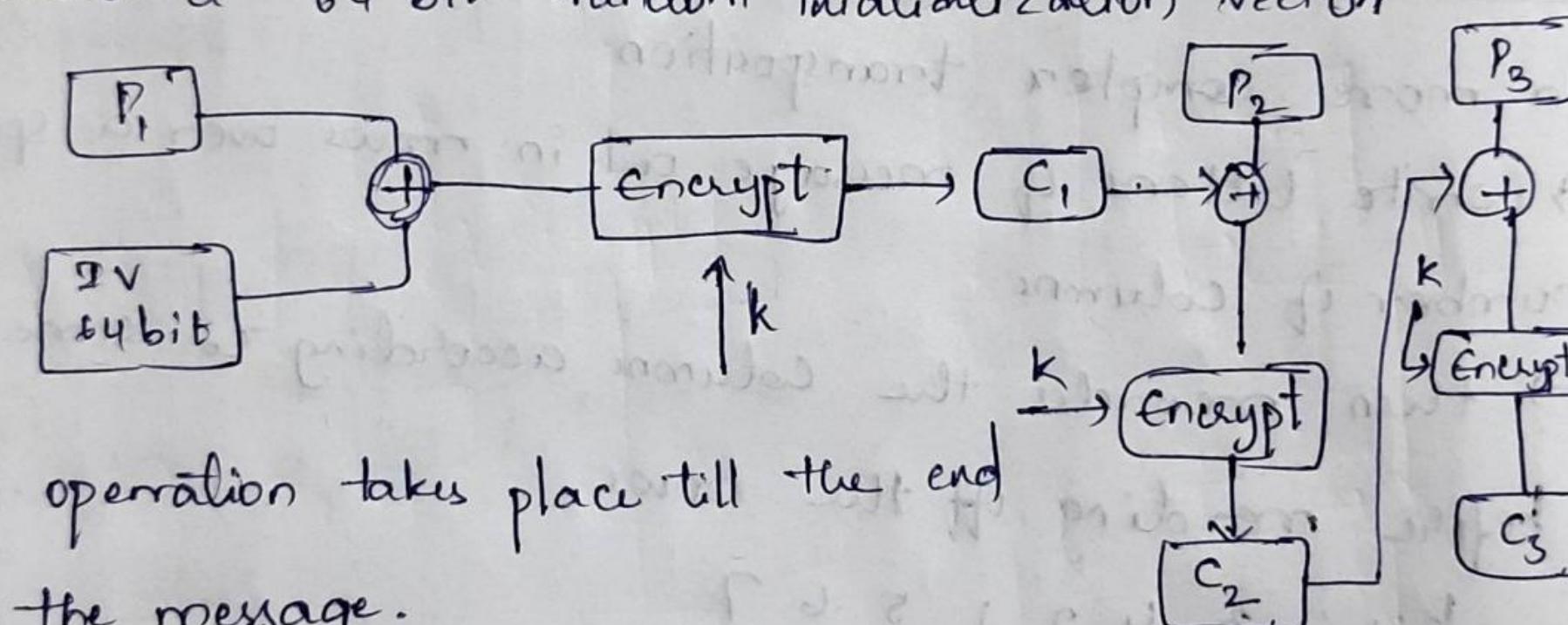
1. ECB mode

→ when you have less amount of data
msg is divided into blocks of bits, same key is used for
all blocks and the plain text is encrypted.

2. Cipher block chaining mode (CBC mode):

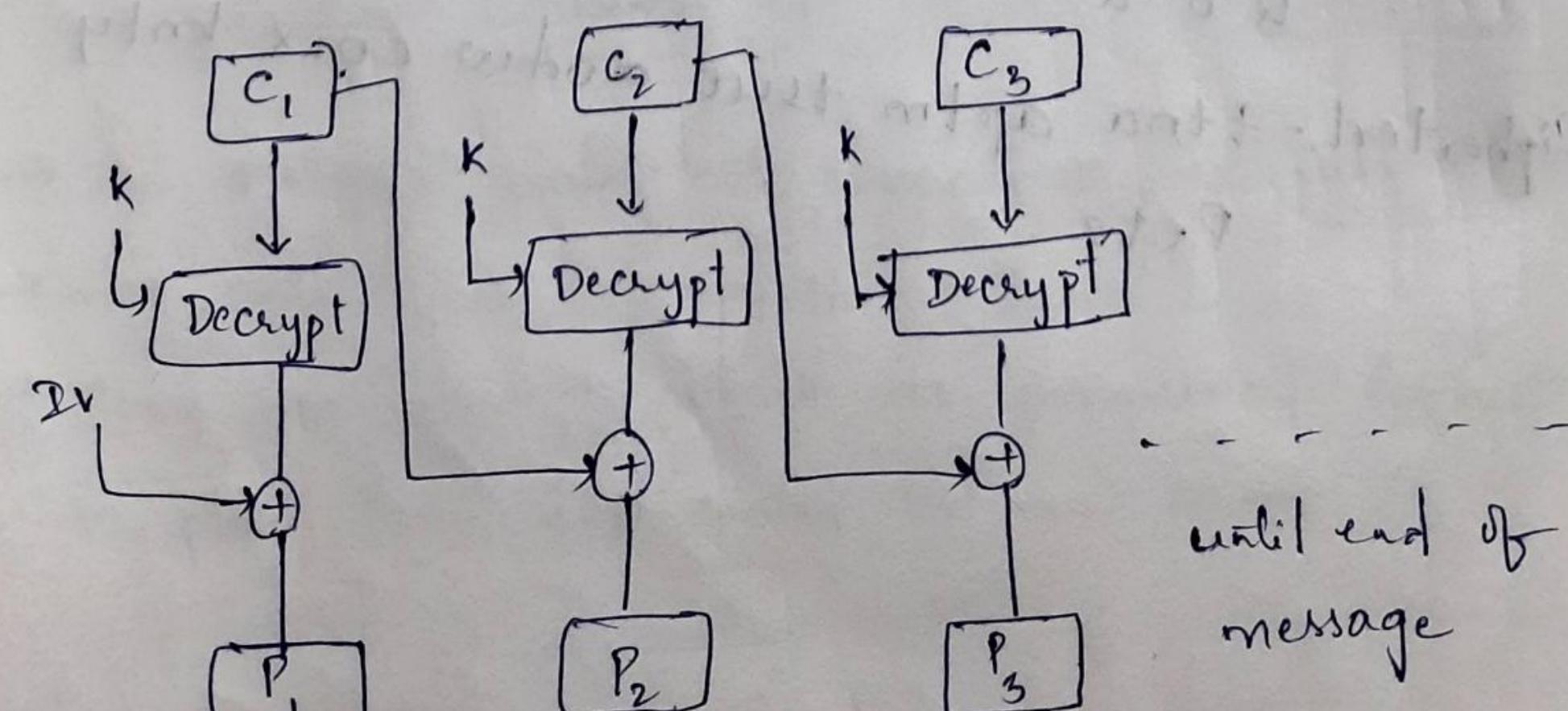
P_i - 64 bit

generate a 64 bit random initialization vector



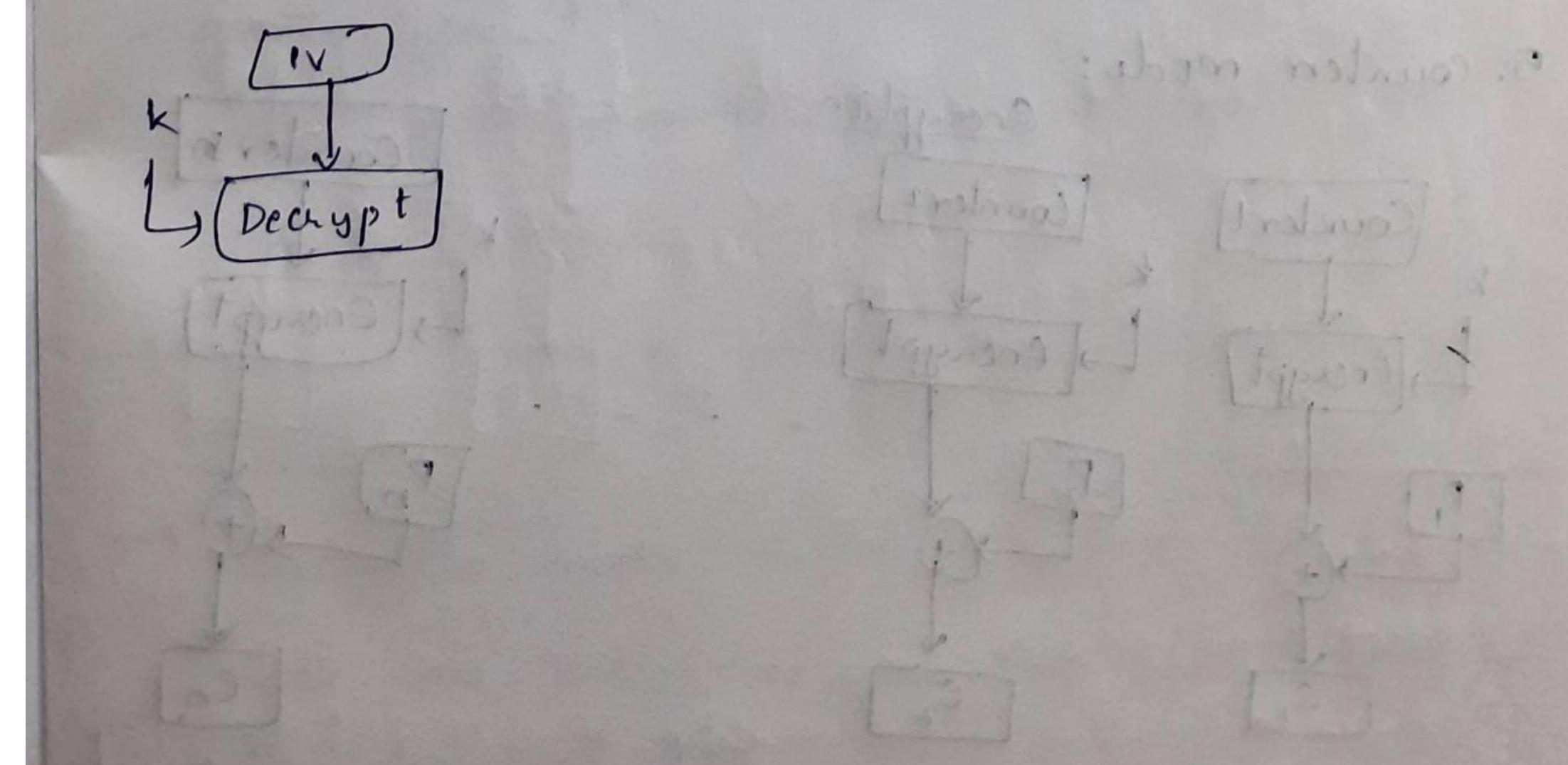
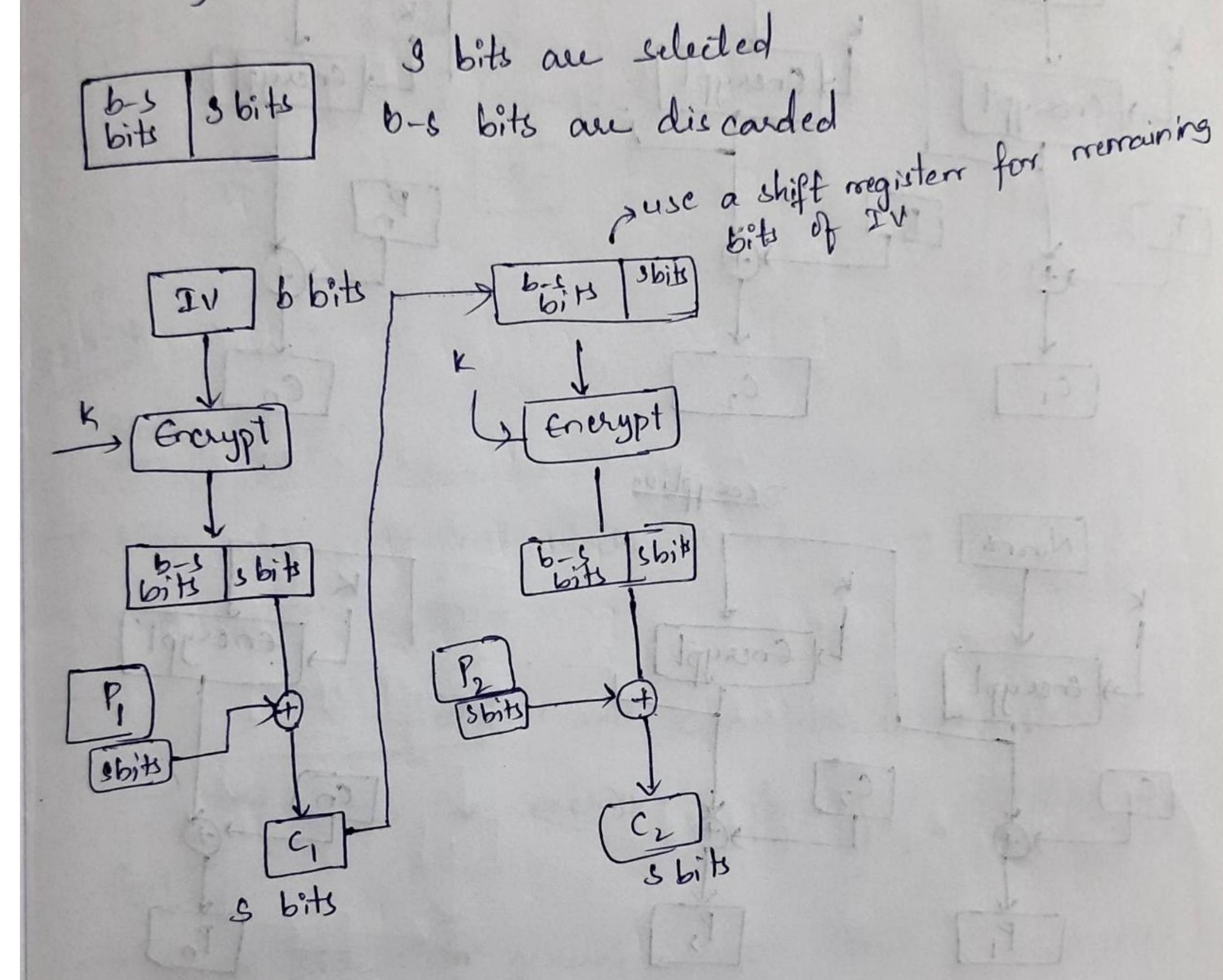
this operation takes place till the end
of the message.

key and IV has to be shared by requested third party
among two parties

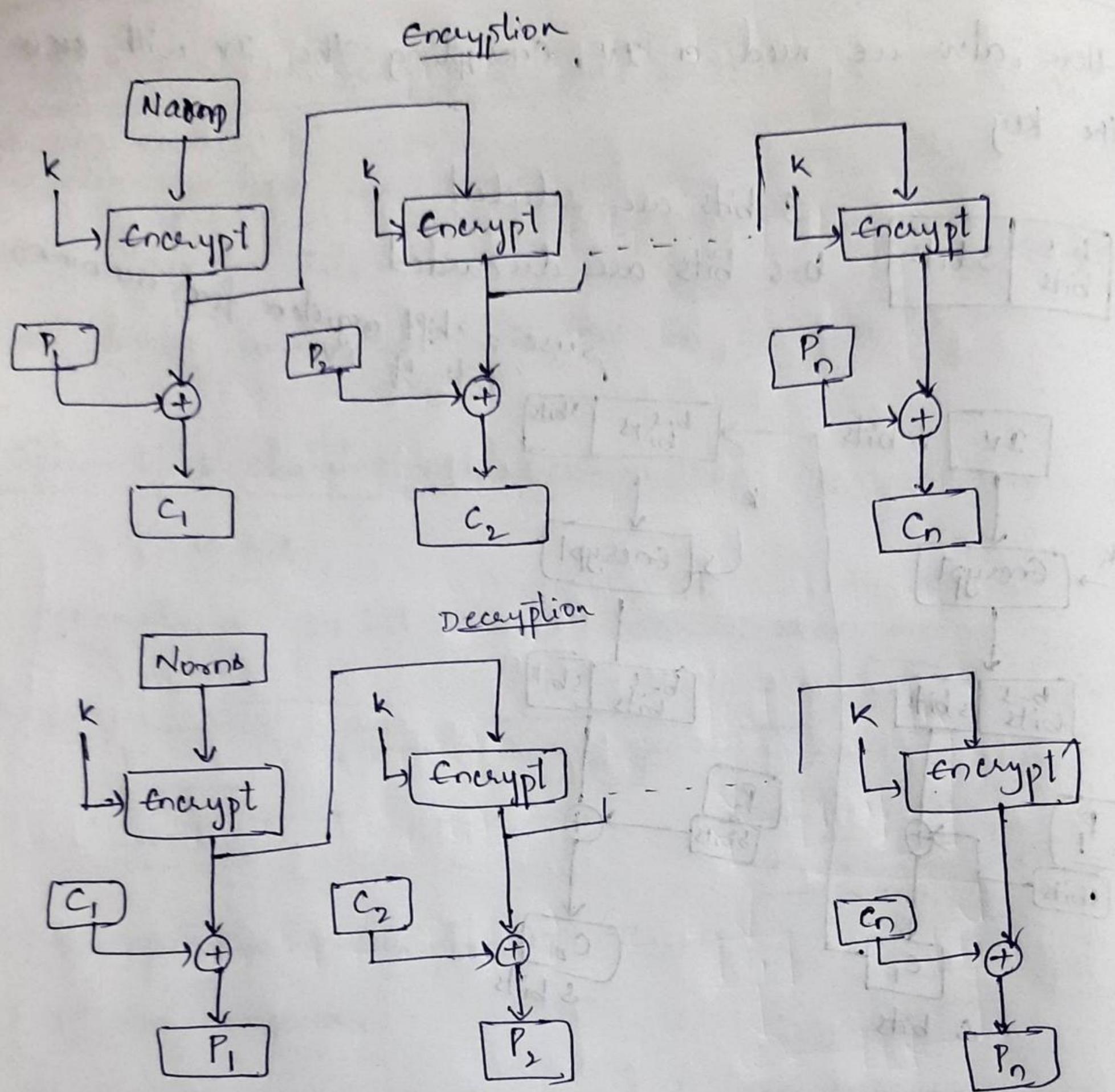


3. Cipher Feedback mode (CFB mode):

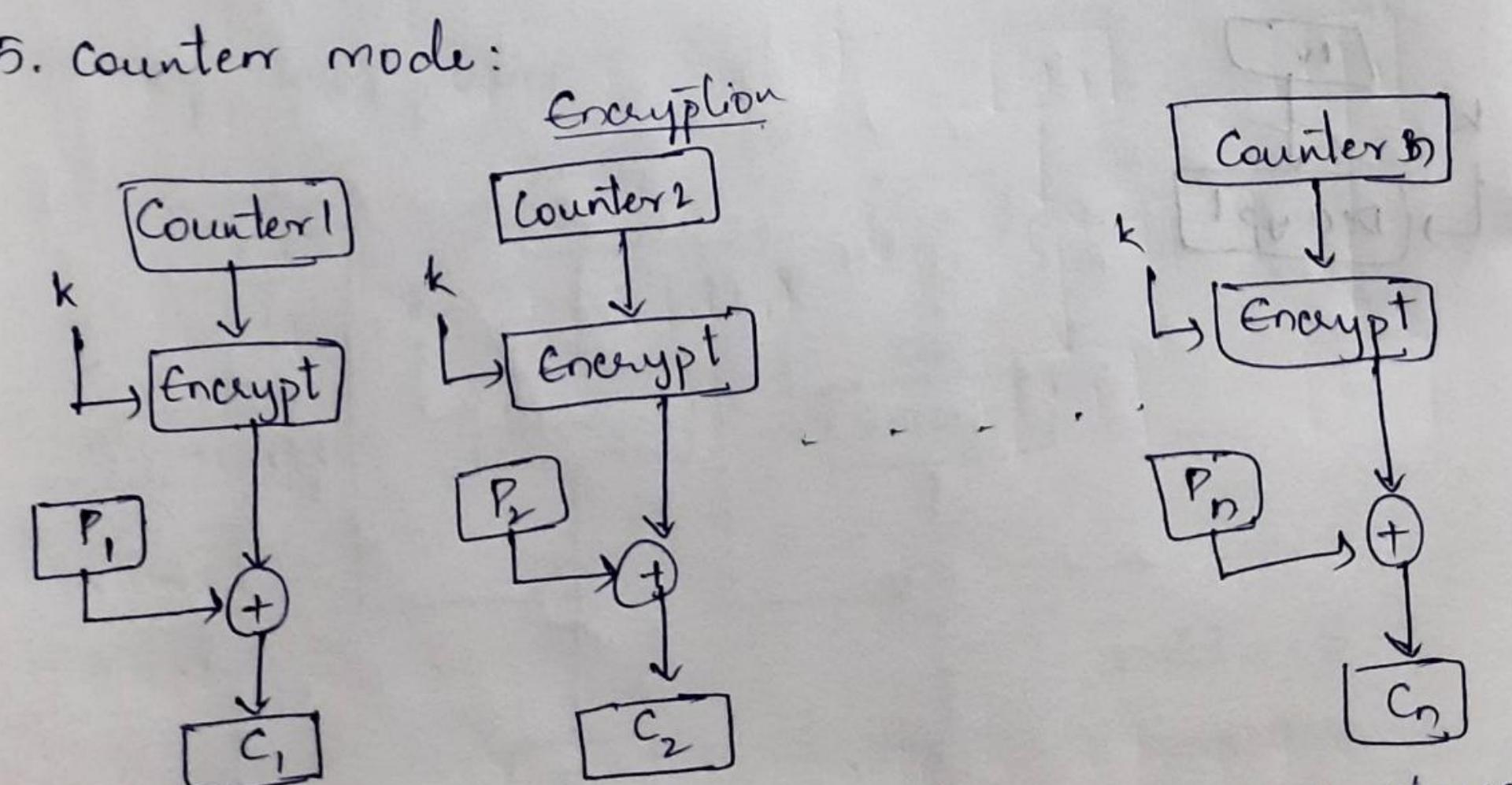
Here, also we need a IV. Encrypting the IV with ~~key~~ the key



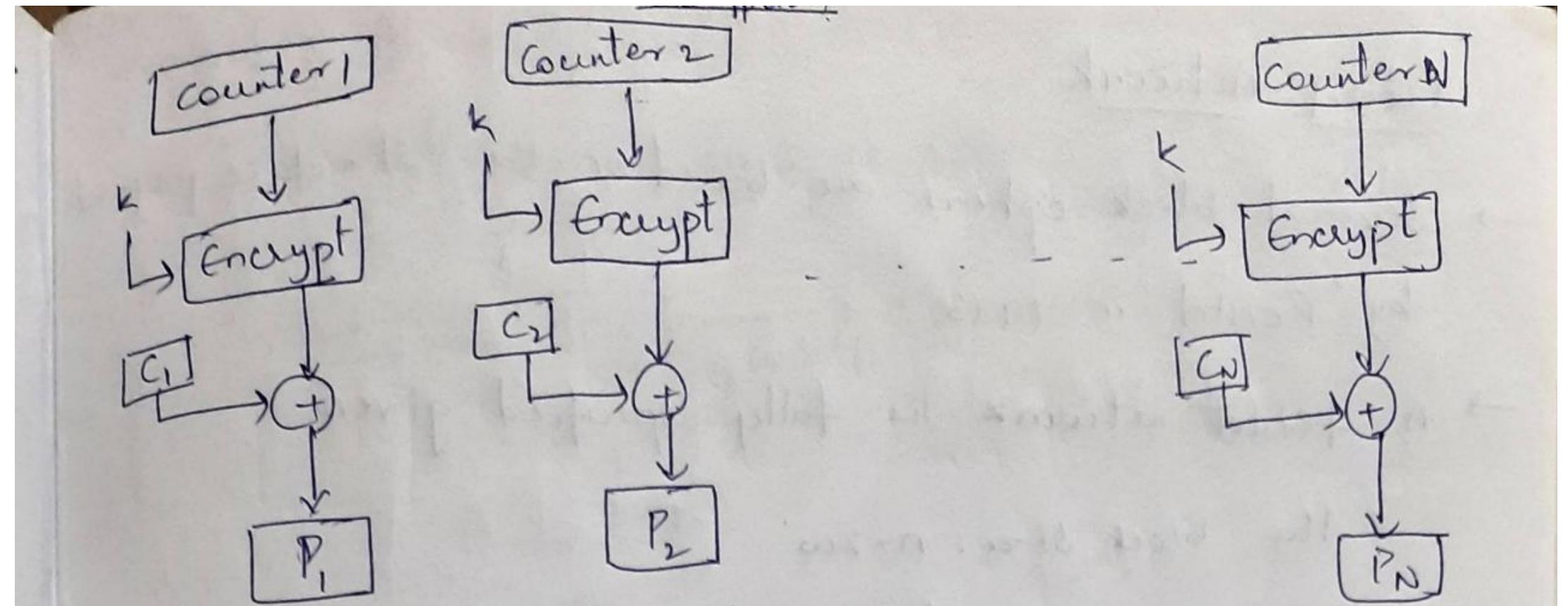
4. Output Feedback mode (OFB mode):



5. Counter mode:



Apart from key, counter initialization has to be shared between two parties.



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Block ciphers

- Map n-bit plaintext blocks to n-bit ciphertext blocks
(n = block length)
- For n-bit plaintext and ciphertext block and a fixed key, the encryption function is bijection.

features:

- Block size: In general longer block sizes mean greater security
- Key size: Larger key size means greater security (larger key space)
- Number of rounds: Multiple rounds after increasing security
- Encryption modes: Define how msg's larger than block are encrypted, very imp for security of the encrypted msg.

Feistel Network

- Several block ciphers are based on the structure proposed by Feistel in 1973
- A feistel network is fully specified given
 - The block size: $n = 2w$
 - no. of rounds = d
 - d round function, $f_1, \dots, f_d : (0,1)^w \rightarrow (0,1)^w$
- used in DES, IDEA, RC5 not used in AES.

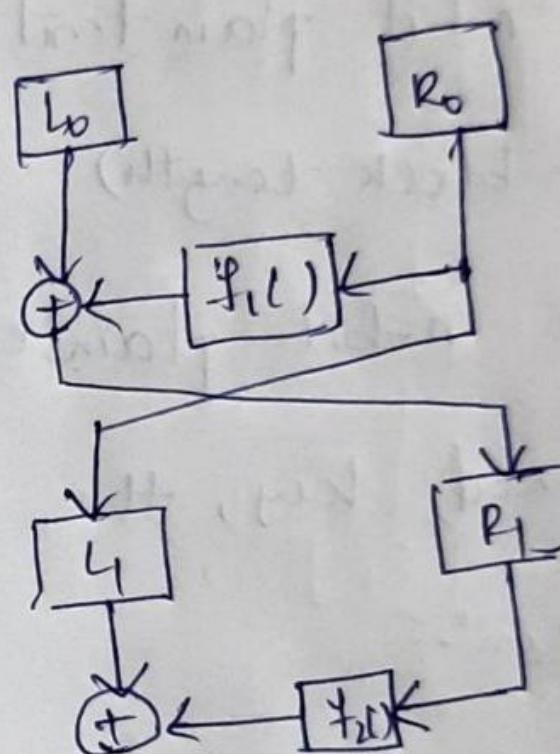
• Encryption

$$L_0 = R_0 \quad R_0 \rightarrow L_0 \oplus f_1(R_0)$$

$$L_1 = R_1 \quad R_1 \rightarrow L_1 \oplus f_2(R_1)$$

$$\vdots$$

$$L_d = R_{d-1} \quad R_d = L_{d-1} \oplus f_d(R_{d-1})$$

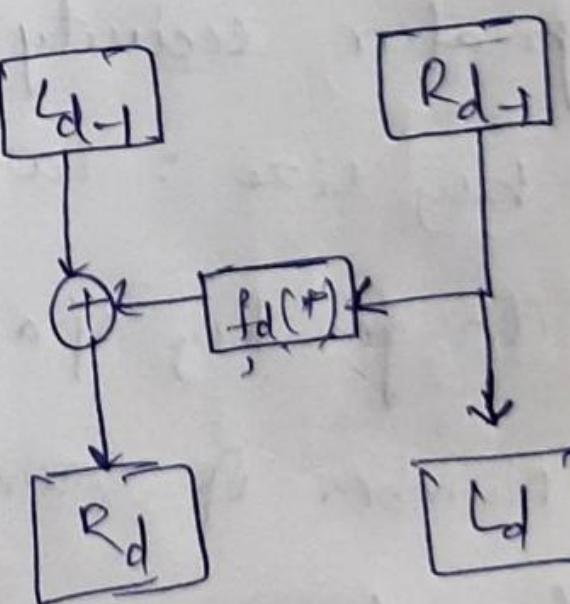


• Decryption

$$R_{d-1} = L_d \quad L_{d-1} = R_d \oplus f_d(L_d)$$

$$\vdots$$

$$R_0 = L_1; \quad L_0 = R_1 \oplus f_1(L_1)$$



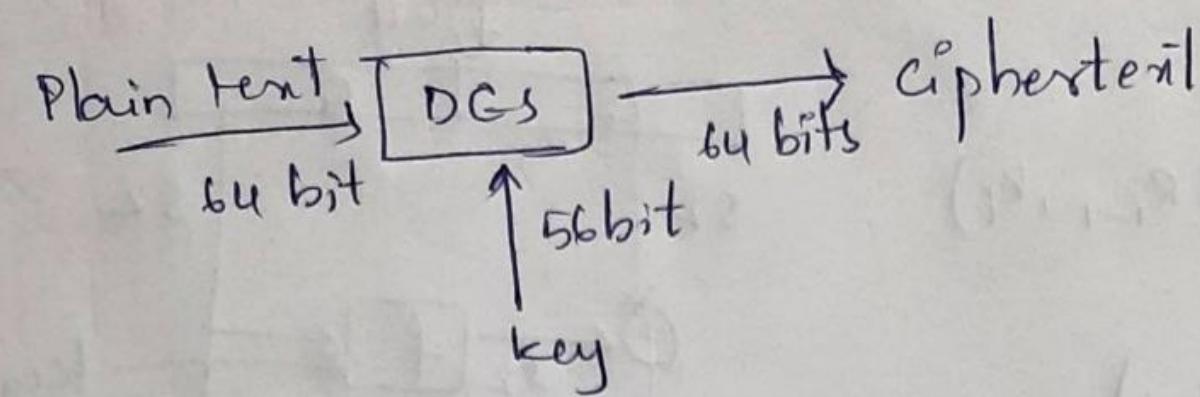
DES Features

• Features

- Block size = 64 bits

- key size = 56 bits (in reality, 64 bits, but 8 are used as parity-check bits for error control)

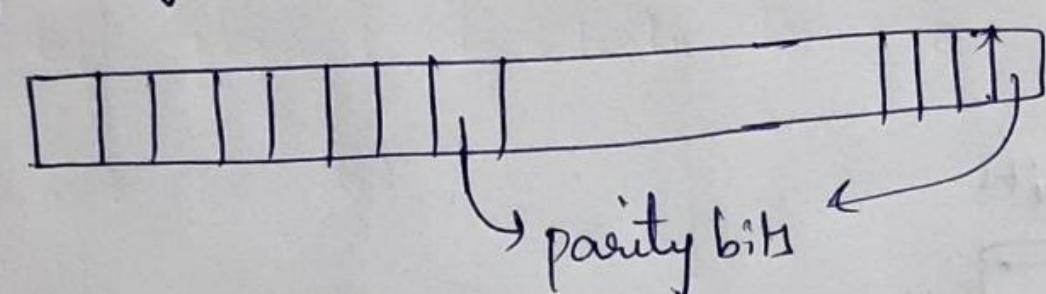
- number of rounds = 16
- 16 intermediary keys, each 48 bits



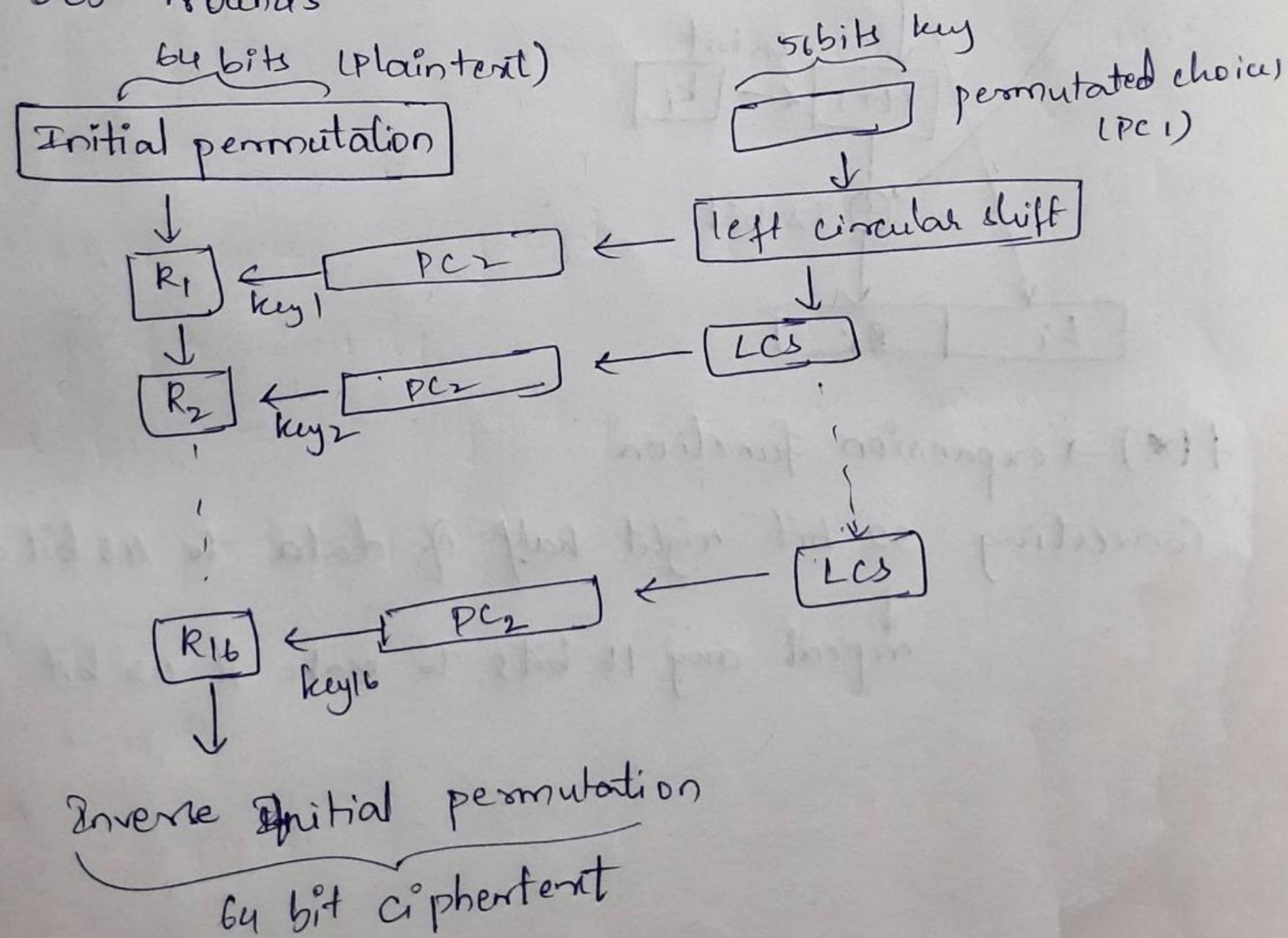
In key, every 8th bit is parity bit of 7 bits

$$\text{So key size} = 64 - 8 = 56 \text{ bits}$$

→ each parity check bit is XOR of previous 7 bits



→ DES rounds



Details:

$$IP(x) = L_0 R_0$$

$$L_i = R_{i-1}$$

$$R_i = L_{i-1} \oplus f(R_{i-1}, k_i)$$

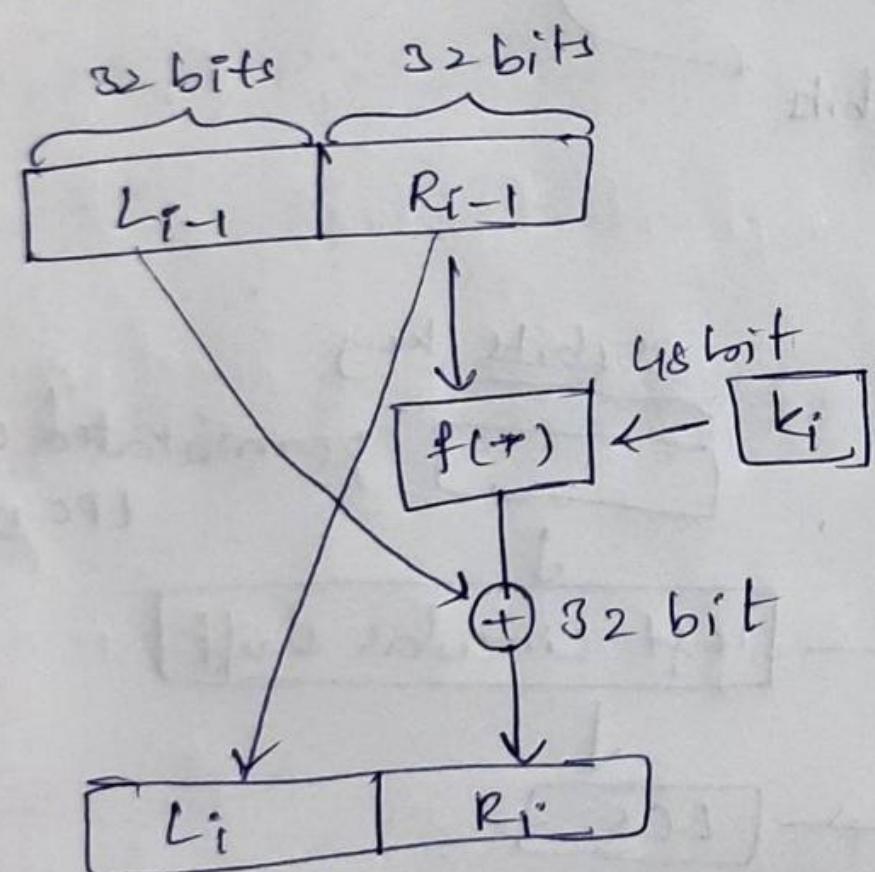
$$Y = IP^{-1}(R_{16} L_0)$$

Initial permutation

58 - 1st bit

50 - 2nd bit

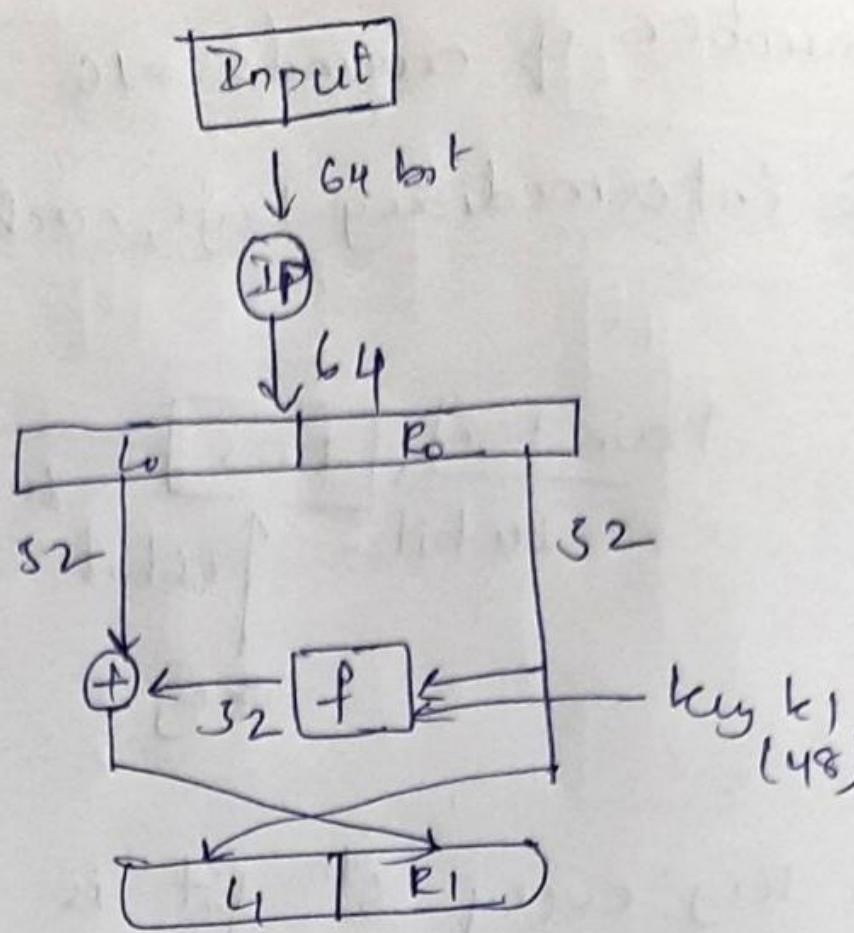
DES Round i



$f(*) \rightarrow$ expansion function

Converting 32 bit right half of data to 48 bit data

↓ repeat any 16 bits to make it 48 bit



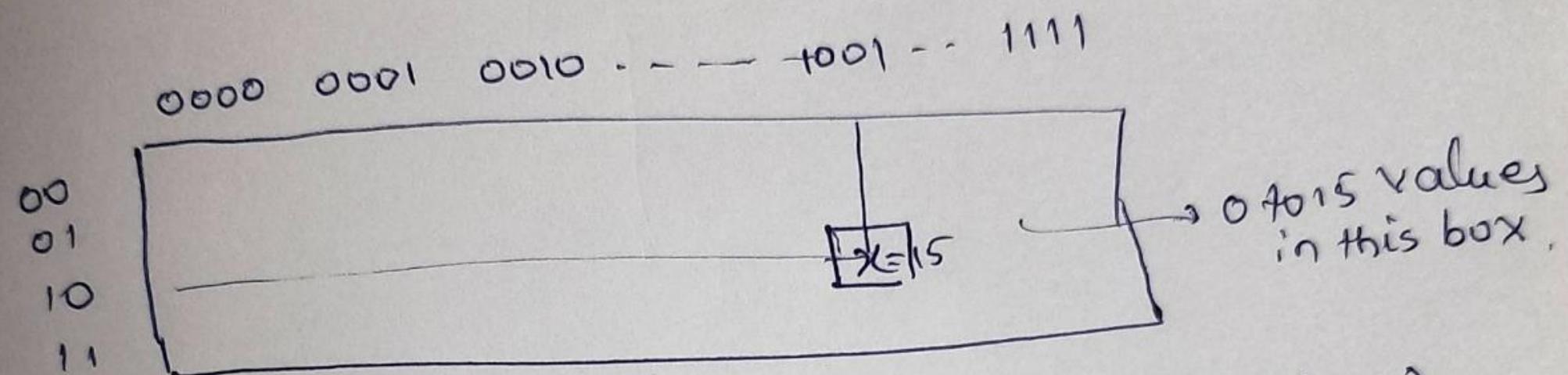
$\boxed{6\text{ bits}}$

uses 8 substitution boxes $= 8 \times 6 = 48$ bits
 ↓
 each 6 bits.

Eg: $\boxed{110010}$

\downarrow row

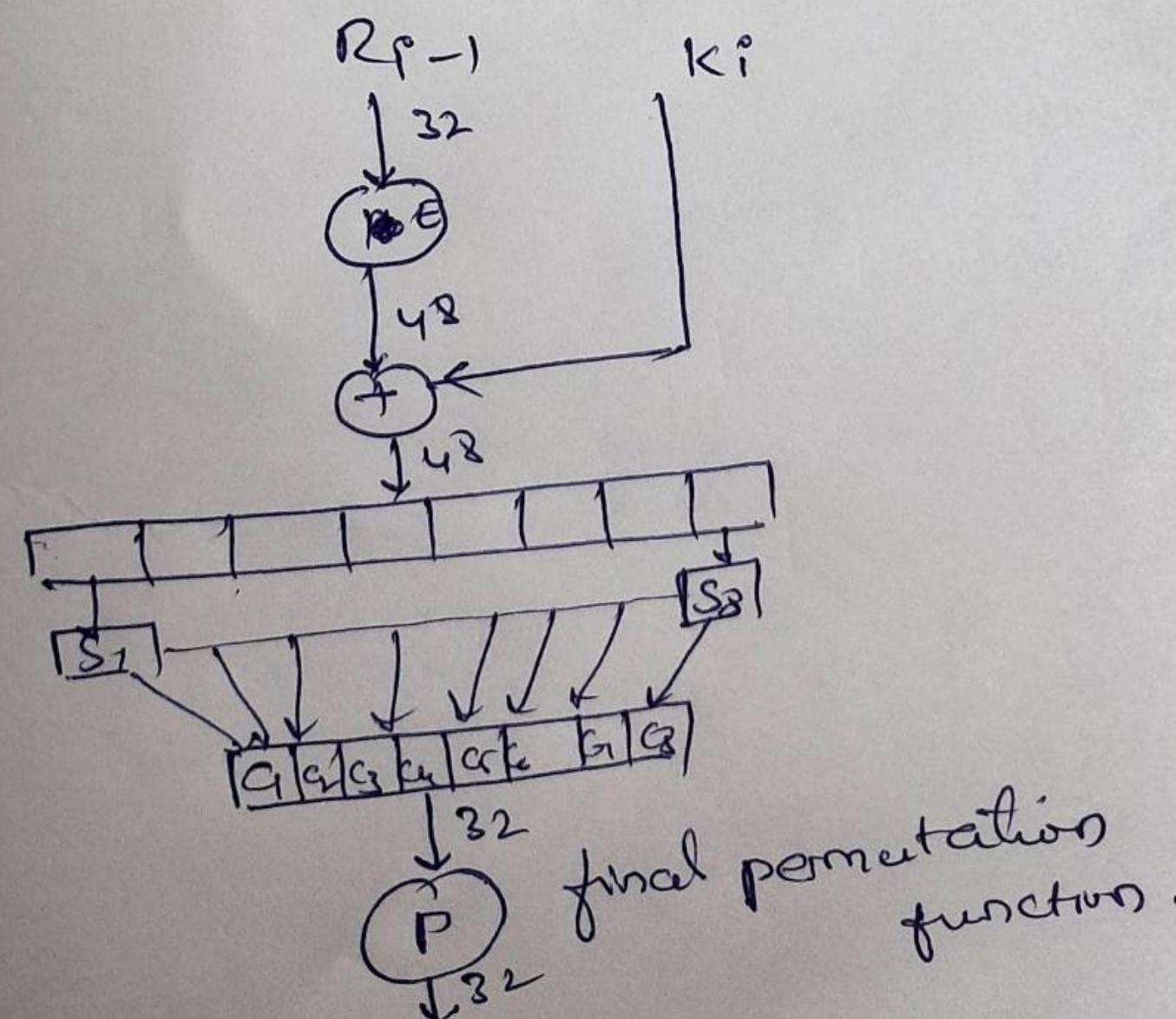
column } for substitution
 box.



Replace 6 bits (110010) by $\boxed{15(1111)}$
 [4 bits]

output : Each box $\times 4$
 so, final output $8 \times 4 = 32$ bits.

$\boxed{32}$
 $\boxed{4\text{ bits}}$



final permutation
 function.

DES Weak keys

0101010101010101

FEFEFEFE FEFEFEFE

EEEOEOEO AF, AF, AF

IF, IF, IF, IF, IF, IF

need to be avoided

XOR

XOR

Copy error
XOR error

exit

(111) or (000)
(exit) did a bridge

→ Xor does not change
exit is px8 times long as

exit

