Advanced Applications of Probability & Statistics

Linear Regression

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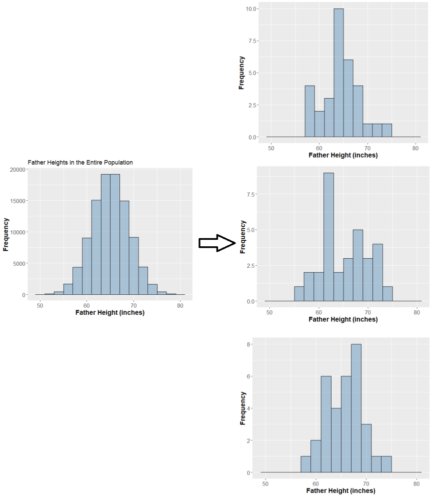
- Note that sample statistic (or just statistic) is a *random variable*.

Population & Sample - Example with Sample Size = 32



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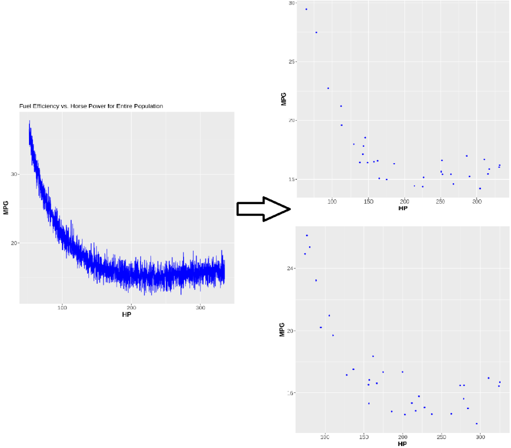
Population & Sample - Another Example with Sample Size = 32



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The Geometric Idea Behind Simple Linear Regression Model (SLRM)

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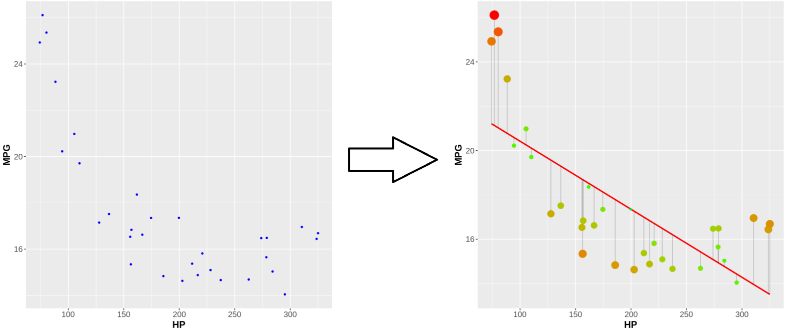
Given a dataset (*random samples from the population*), find a straight line that *fits* the data (*response variable and a single predictor*) well in an *average sense*:



Population & Sample - Revisited in the Context of SLRM

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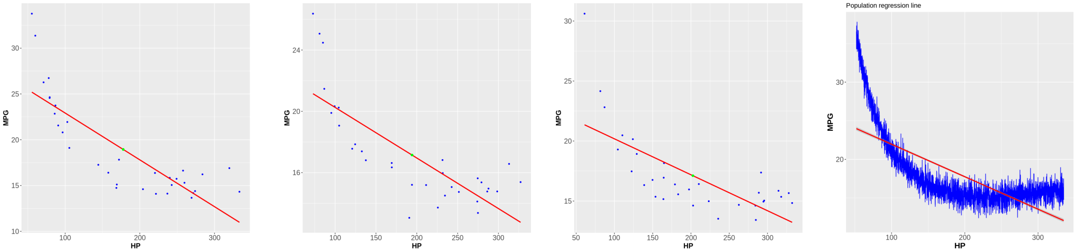
Note that the straight line of best fit will depend on the dataset but there is only one unique straight line of best fit for the entire population data:



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- Note that in the SLRM above, we use the same symbol ε for the random error term which now additionally includes the effect of missing out a possibly nonlinear relationship between Y and X_1.

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- Solving this results in the estimates

$$\begin{aligned} \hat{\beta}_0 &= \bar{y_n} - \hat{\beta}_1 \bar{x_n}, \\ \hat{\beta}_1 &= \frac{s_{xy}}{s_{xx}}, \end{aligned}$$

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where

$$s_{xy} = \underbrace{\sum_{i=1}^{n} \left(x_1^{(i)} - \bar{x}_n\right)\left(y^{(i)} - \bar{y}_n\right)}_{\text{sample covariance-like measure}}$$

$$s_{xx} = \underbrace{\sum_{i=1}^{n} \left(x_1^{(i)} - \bar{x}_n\right)^2}_{\text{sample variance-like measure in the predictor}},$$

$$\bar{x}_n = \underbrace{\frac{1}{n}\sum_{i=1}^{n} x_1^{(i)}}_{\text{sample mean of predictors}} \quad \text{and } \bar{y}_n = \underbrace{\frac{1}{n}\sum_{i=1}^{n} y^{(i)}}_{\text{sample mean of responses}}.$$

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- Later, for the purpose of constructing hypotheses tests and confidence intervals for the least squares estimates, we will also assume that $\varepsilon^{(i)}$ is normally distributed.

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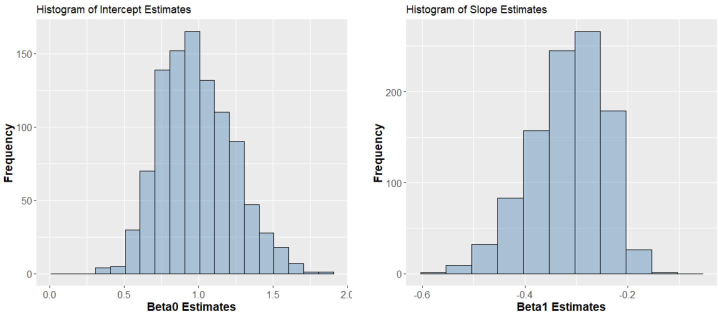

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- A 95% CI for β_0 is $\left[\hat{\beta}_0 - 1.96 \times SE(\hat{\beta}_0),\ \hat{\beta}_0 + 1.96 \times SE(\hat{\beta}_0)\right]$.
- How to interpret this? Each dataset gives us one such CI; approximately 95% of those intervals will contain the true population parameter β_0.
- In practice, however, we have one dataset and one CI that comes out of it.
- For calculating CI as above, additional assumption on the random error term $\varepsilon^{(i)}$ that it is *normally distributed* is needed.

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- If the p-value is greater than the threshold, we fail to reject the null hypothesis; this means that there possibly is no relationship between mpg and hp.

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- The RSE is the amount by which the response will deviate from the true regression line on an average.

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- The R^2 statistic varies between 0 & 1 is and is a measure of the variability in the response Y that the SLRM (built using the predictor X_1) is able to explain.

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- This is helpful typically in the multiple linear regression setup where different scales may be present in the data.

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- This means, $\hat{\beta}_1$ is the proportionate change in the response value for a unit increase in the predictor value.

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	heatinghot air	heatinghot water/steam
electric	0	0
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hot water/steam	0	1

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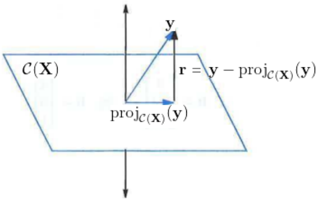
Minimizing $\|\mathbf{r}\|^2 = \|\mathbf{y} - \mathbf{X}\beta\|^2 = \left\|\mathbf{y} - \left(\underbrace{\beta_0\mathbf{x}_1 + \beta_1\mathbf{x}_2 + \cdots + \beta_p\mathbf{x}_{p+1}}_{\text{linear combination of columns of } \mathbf{X}}\right)\right\|^2$ is equivalent to solving the equation $\mathbf{X}\hat{\beta} = \text{proj}_{\mathcal{C}(\mathbf{X})}(\mathbf{y})$ which represents the *orthogonal projection* of $\mathbf{y}$ on to the column space of the design matrix $\mathcal{C}(\mathbf{X})$ (set of all possible linear combinations of the columns of $\mathbf{X}$):



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- Using the fact that the design matrix $\mathbf{X}$ has full rank (that is, its columns are linearly independent), we arrive at the unique OLS solution $\hat{\beta} = \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{y}$.

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- This is a reiteration of the fact that linear regression works best on an average.

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- If the p-value is smaller than a threshold, typically 0.05, then we reject the null hypothesis, and conclude that the jth predictor contributes to the linear model.
- A better hypothesis test, when the number of predictors is large, is the F test, in which the null hypothesis is that all population coefficients are zeros except the intercept.

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- The intercept estimate can now be interpreted as the (approximate) average value of $price$ around the average value of $livingArea$ and average value of age.

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- The response variable may be logarithmically transformed if it cannot be predicted to be negative (for example, height, weight etc.).

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- Both ridge and lasso approaches for regularization shrink the coefficient estimates towards 0 but lasso typically yields a much smaller subset of nonzero coefficient estimates.

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- Correlation matrix for some continuous predictors from the `saratogaHouses` dataset:

	livingArea	lotSize	age	landValue	bedrooms	rooms
livingArea	1.00	0.16	-0.17	0.42	0.66	0.73
lotSize	0.16	1.00	-0.02	0.06	0.11	0.14
age	-0.17	-0.02	1.00	-0.02	0.03	-0.08
landValue	0.42	0.06	-0.02	1.00	0.20	0.30
bedrooms	0.66	0.11	0.03	0.20	1.00	0.67
rooms	0.73	0.14	-0.08	0.30	0.67	1.00

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- If all the predictors are perfectly collinear, then the R^2 metric when one predictor is regressed upon the others will be exactly 1.
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- A small value of the *tolerance* (< 0.1, for example) indicates that the predictor under consideration is highly correlated with the other predictors.
- The variance inflation factor (VIF) of the predictor under consideration is $1/tolerance = 1/(1 - R^2)$.
- A large vale of VIF (> 10, for example) indicates additional study about the correlation between predictors.

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Residual plots: Continued

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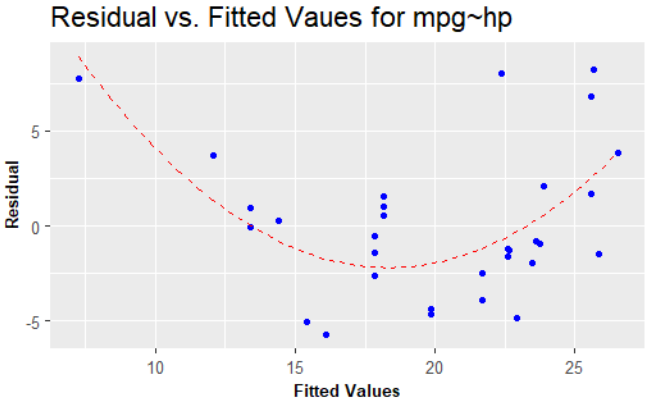
A residual plot shows the relationship between the residuals and the fitted values.



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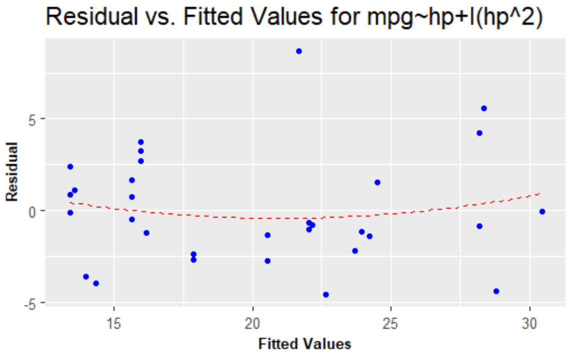

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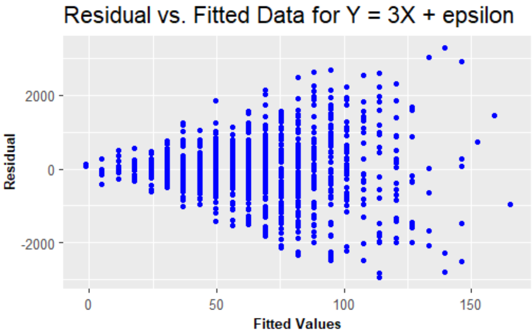
A funnel-shaped residual plot is an indication of heteroskedasticity, which means that the random error term <u>does not</u> have a constant variance impacting the *standard error*, *confidence interval*, and hypothesis test calculations.



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$$\underbrace{[\hat{\beta}_0 + \hat{\beta}_1 \times (livingArea + 1) + \hat{\beta}_2 \times rooms + \hat{\beta}_3 \times (livingArea + 1) \times rooms]}_{\widehat{price}_{\text{new}}}$$

$$-\underbrace{[\hat{\beta}_0 + \hat{\beta}_1 \times livingArea + \hat{\beta}_2 \times rooms + \hat{\beta}_3 \times livingArea \times rooms]}_{\widehat{price}_{\text{old}}}$$

$$= \hat{\beta}_1 + \hat{\beta}_3 \times rooms.$$

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- Suppose we have a house that is hot air heated and not centrally air-conditioned; the predicted price of the house is $\hat{\beta}_0 + \hat{\beta}_1$.
- Now consider another house that is hot air heated and centrally air-conditioned; the predicted price of this house is $\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_3 + \hat{\beta}_4$.
- Subtracting the two results above, we see that $\hat{\beta}_3 + \hat{\beta}_4 =$ difference between average prices of centrally air-conditioned and not centrally air-conditioned houses among hot air-heated houses.

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- Note the differences in both intercept and slope for new houses.

Interaction Between Continuous & Categorical Predictors: Continued

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The scatter plot indicates that a higher slope is needed for new houses: