Deep Learning Principles & Applications

Chapter 2 – Linear Classifiers

Sudarsan N.S. Acharya (sudarsan.acharya@manipal.edu)

Classification in Practice

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is this sample a $\underbrace{\text{lion/tiger/leopard}}_{\text{known 3 classes}}$?

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Recall that this color image is internally represented as a $337 \times 600 \times 3$-tensor of integer values ranging from 0 to 255.

Classification in Practice – continued

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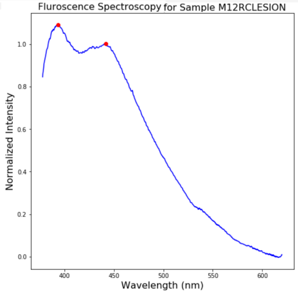
Medical Signal
Processing

Classification in Practice – continued

Medical Signal Processing

	0	1	...	1017	1018
M12RCLESION	0.845890	0.861290	...	0.000037	0.001354



Classification in Practice – continued

MANIPAL ACADEMY of HIGHER EDUCATION
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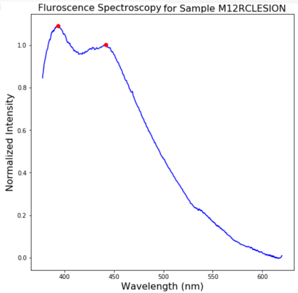

What kind of an oral tumor does this patient have: benign / premalignant / malignant?

known 3 classes

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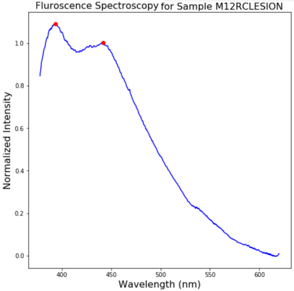

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Language Application

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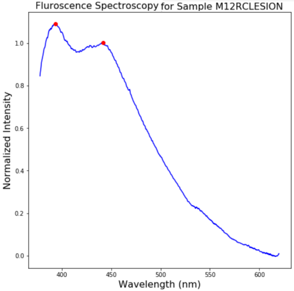

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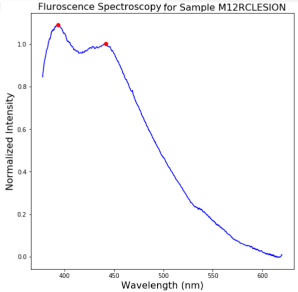

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Language Application

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Is this movie review positive/negative?

known 2 classes

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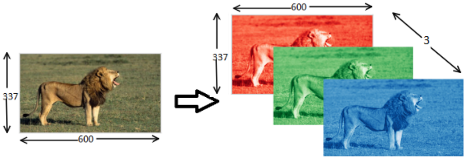

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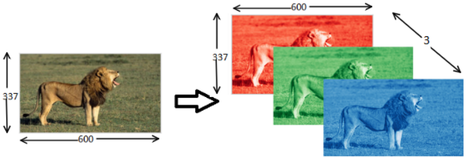

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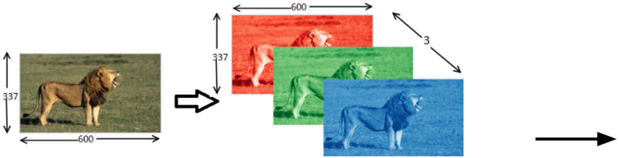

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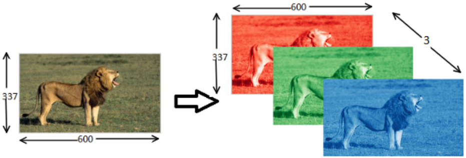

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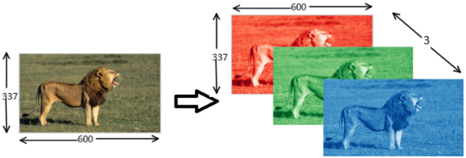

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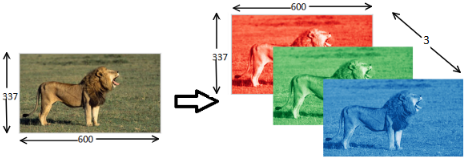

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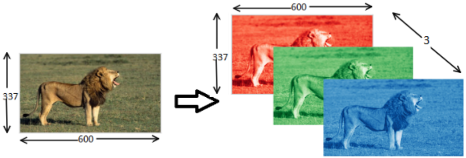

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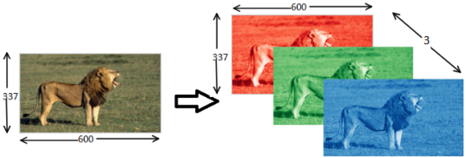

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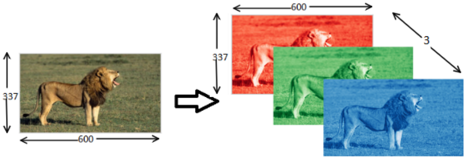

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What are $\mathbf{W}$ and $\mathbf{b}$ (the parameters), and how do we know what they are?

Weights, Bias, & Raw Scores

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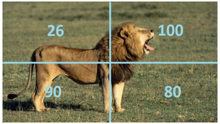

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Loss for Sample-1

$$L_1 = \begin{cases} \max(0, 6.4 - 5.6) \\ \quad + \\ \max(0, -4.6 - 5.6) \end{cases}$$

$$= 0.8$$

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Loss for Sample-2

$$L_2 = \begin{cases} \max(0, -1.8 - 10.2) \\ + \\ \max(0, 3.5 - 10.2) \end{cases}$$

$$= 0$$

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Loss for Sample-3

$$L_3 = \begin{cases} \max(0, 2.0 - (-8.6)) \\ + \\ \max(0, 5.4 - (-8.6)) \end{cases}$$
$$= 24.6$$

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Average training loss

$$\frac{0.8 + 0 + 24.6}{3} = 8.5$$

Hinge Loss Function

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- Suppose there are n training samples $\left(\mathbf{x}^{(i)}, y^{(i)}\right)$.

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Hinge Loss Function

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correct class/label

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Hinge Loss Function

MANIPAL ACADEMY of HIGHER EDUCATION
(Institution of Eminence Deemed to be University)

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Visualizing Loss Functions

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Visualizing different loss functions considering contribution from one incorrect class:

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Perceptron Loss

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Perceptron Loss	Hinge Loss	Squared Hinge Loss
$\max\left(0, z_j^{(i)} - z_{y^{(i)}}^{(i)}\right)$	$\max\left(0, z_j^{(i)} - z_{y^{(i)}}^{(i)} + 1\right)$	$\max\left(0, z_j^{(i)} - z_{y^{(i)}}^{(i)} + 1\right)^2$

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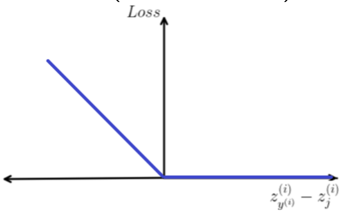

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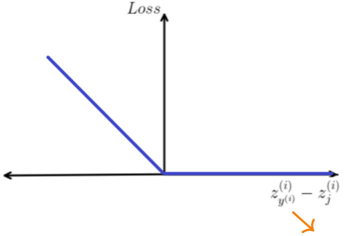

difference between correct and incorrect class raw scores

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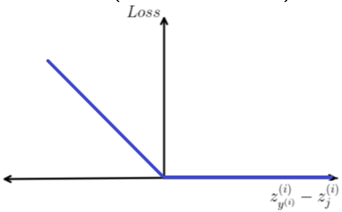

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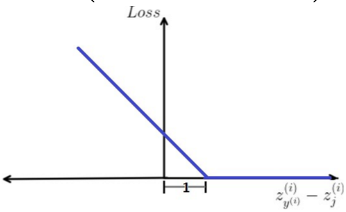

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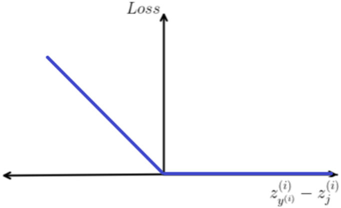

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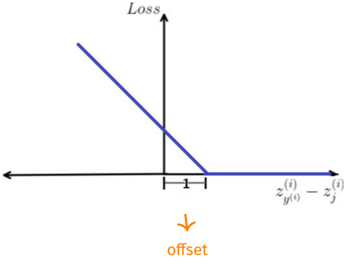

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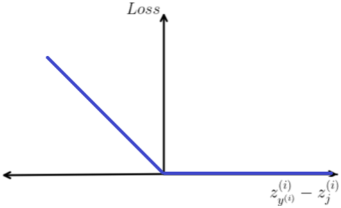

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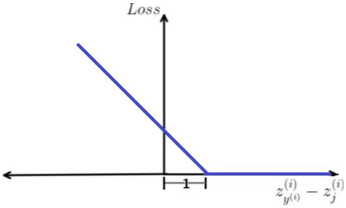

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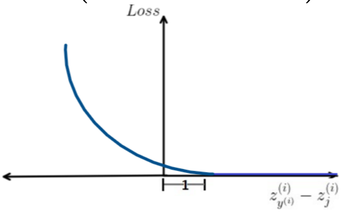

Softmax Function

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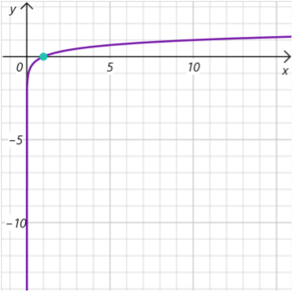

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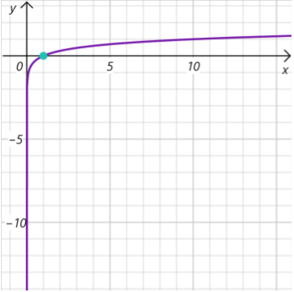

- Note that $\begin{cases} \log(1) = 0, \\ \log(x) \to -\infty \text{ as } x \to 0. \end{cases}$

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- In plain English, it is the negative of the logarithm of the correct class's probability.
- Note that $\begin{cases} [\text{softmax}(\mathbf{z})]_y = 1 & \Rightarrow ☺ \Rightarrow \text{loss} = -\log(1) = 0, \\ [\text{softmax}(\mathbf{z})]_y = 0 & \Rightarrow ☹ \Rightarrow \text{loss} = -\log(0) \to \infty. \end{cases}$

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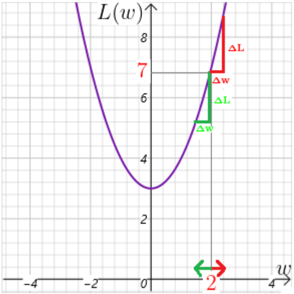

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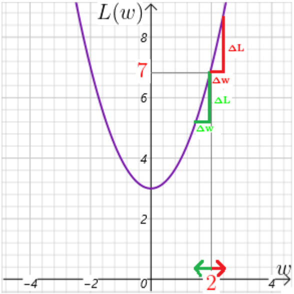

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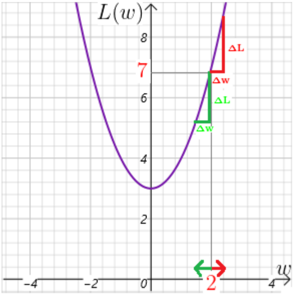

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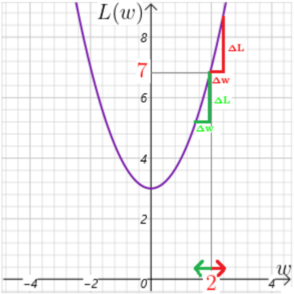
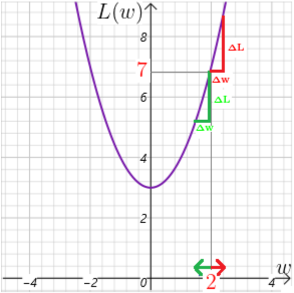


- How can we tweak the input w from it's current value of 2 so that the output L decreases from its current value of 7?
- w can be increased (move right) or decreased (move left) from the current value 2.
- Can we quantify the sensitivity of the output L w.r.t. small changes in the input w?

Sensitivity

Sensitivity



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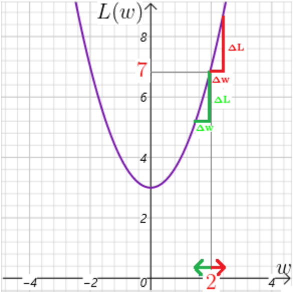

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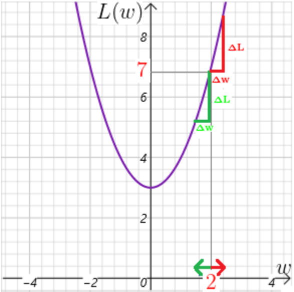

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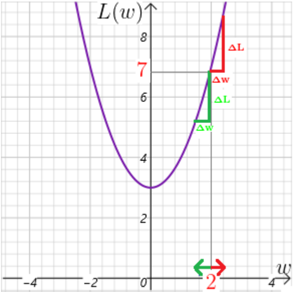

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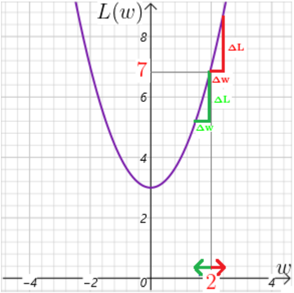

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$$\frac{\text{change in output}}{\text{change in input}} : \begin{cases} \text{moving right} & = \\ & \\ & \end{cases}$$

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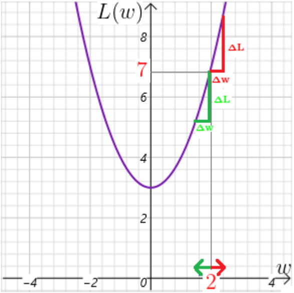
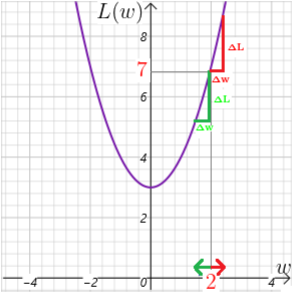


$$\frac{\text{change in output}}{\text{change in input}} : \begin{cases} \textcolor{red}{\text{moving right}} & = \dfrac{\underbrace{\textcolor{red}{\Delta L}}_{+ve}}{\underbrace{\textcolor{red}{\Delta w}}_{+ve}} \\ & \end{cases}$$

Sensitivity



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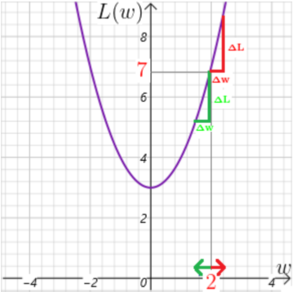

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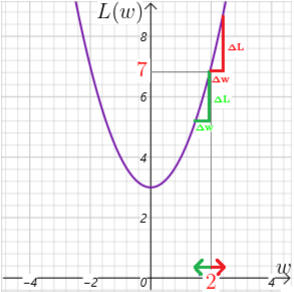

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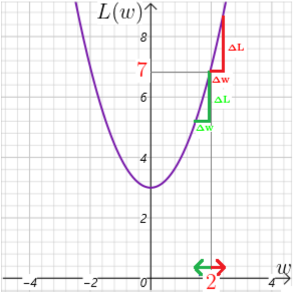

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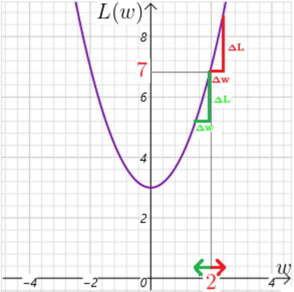

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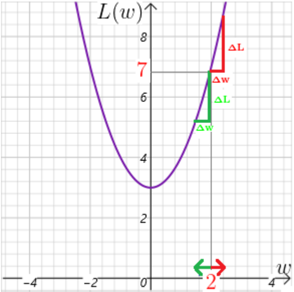

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$\boxed{+ve \text{ sensitivity}}$ w increases $\Rightarrow$ L increases & w decreases $\Rightarrow$ L decreases

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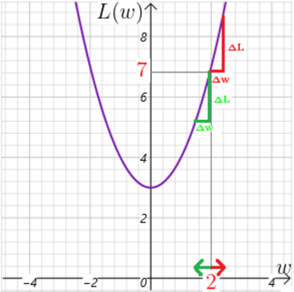

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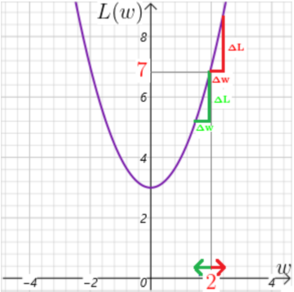

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In this case, we move left (decrease) w to decrease L.

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$$
\begin{tikzcd}
\boxed{L} \arrow[d] \\
\boxed{z} \arrow[d] \\
\boxed{w}
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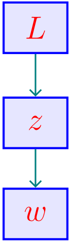

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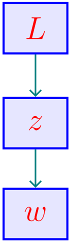

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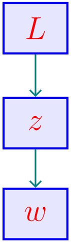

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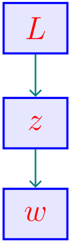

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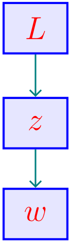

Gradient calculation using chain rule:

$$\begin{aligned} \nabla_w(L) &= \nabla_w(z) \times \nabla_z(L) \\ &= \nabla_w\left(1+e^{-w}\right) \times \nabla_z\left(\frac{1}{z}\right) \\ &= \qquad\qquad -\frac{1}{z^2} \end{aligned}$$

Chain Rule

The chain rule is used to calculate gradients in a hierarchical way by using intermediate variable(s).

E.g. $L(w) = 1/\left(1 + e^{-w}\right) \xrightarrow{\text{use } z=1+e^{-w}} \begin{cases} L(z) & = 1/z, \\ z(w) & = 1 + e^{-w}. \end{cases}$

Computation graph:

$$\begin{tikzcd} \boxed{L} \arrow[d] \\ \boxed{z} \arrow[d] \\ \boxed{w} \end{tikzcd}$$

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\nabla_w(L) &= \nabla_w(z) \times \nabla_z(L) \\
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&= \left[\nabla_w(1) + \nabla_w\left(e^{-w}\right)\right] \times -\frac{1}{z^2} \\
&= \left[0 + \left(-e^{-w}\right)\right] \times -\frac{1}{z^2}
\end{aligned}$$

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$$\begin{tikzcd} \boxed{L} \arrow[d] \\ \boxed{z} \arrow[d] \\ \boxed{w} \end{tikzcd}$$

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Gradient in Higher Dimensions

Gradient in Higher Dimensions

Gradient $\nabla_{input}(output)$ has shape $input\ shape \times output\ shape$:

Gradient in Higher Dimensions

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Function	I/O Shapes	Grad. Shape	Gradient

Gradient in Higher Dimensions

Gradient $\nabla_{input}(output)$ has shape $input\ shape \times output\ shape$:

Function	I/O Shapes	Grad. Shape	Gradient
$L(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{w}$			

Gradient in Higher Dimensions

Gradient $\nabla_{input}(output)$ has shape $input\ shape \times output\ shape$:

Function	I/O Shapes	Grad. Shape	Gradient
$L(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{w}$	$input : n$ $output : 1$		

Gradient in Higher Dimensions

Gradient $\nabla_{input}(output)$ has shape $input\ shape \times output\ shape$:

Function	I/O Shapes	Grad. Shape	Gradient
$L(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{w}$	$input : n$ $output : 1$	$n \times 1$	

Gradient in Higher Dimensions

Gradient $\nabla_{input}(output)$ has shape $input\ shape \times output\ shape$:

Function	I/O Shapes	Grad. Shape	Gradient
$L(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{w}$	$input : n$ $output : 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = 2\mathbf{w}$

Gradient in Higher Dimensions

Gradient $\nabla_{input}(output)$ has shape $input\ shape \times output\ shape$:

Function	I/O Shapes	Grad. Shape	Gradient
$L(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{w}$	$input: n$ $output: 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = 2\mathbf{w}$
$L(\mathbf{w}) = \mathbf{x}^{\mathrm{T}}\mathbf{w}$ $\mathbf{x}$ known n-vector			

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$L(\mathbf{w}) = \mathbf{x}^{\mathrm{T}}\mathbf{w}$ $\mathbf{x}$ known n-vector	$input : n$ $output : 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = \mathbf{x}$
$L(\mathbf{w}) = (w_1 - 2)^2 + (w_2 + 3)^2$			

Gradient in Higher Dimensions

Gradient $\nabla_{input}(output)$ has shape $input\ shape \times output\ shape$:

Function	I/O Shapes	Grad. Shape	Gradient
$L(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{w}$	$input : n$ $output : 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = 2\mathbf{w}$
$L(\mathbf{w}) = \mathbf{x}^{\mathrm{T}}\mathbf{w}$ $\mathbf{x}$ known n-vector	$input : n$ $output : 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = \mathbf{x}$
$L(\mathbf{w}) = (w_1 - 2)^2 + (w_2 + 3)^2$	$input : 2$ $output : 1$		

Gradient in Higher Dimensions

Gradient $\nabla_{input}(output)$ has shape $input\ shape \times output\ shape$:

Function	I/O Shapes	Grad. Shape	Gradient
$L(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{w}$	$input : n$ $output : 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = 2\mathbf{w}$
$L(\mathbf{w}) = \mathbf{x}^{\mathrm{T}}\mathbf{w}$ $\mathbf{x}$ known n-vector	$input : n$ $output : 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = \mathbf{x}$
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$\mathbf{l}(\mathbf{z}) = \mathbf{W}\mathbf{z}$ W known $p \times k$-matrix			

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$L(\mathbf{w}) = (w_1 - 2)^2 + (w_2 + 3)^2$	$input : 2$ $output : 1$	2×1	$\nabla_{\mathbf{w}}(L) = \begin{bmatrix} \nabla_{w_1}(L) \\ \nabla_{w_2}(L) \end{bmatrix} = \begin{bmatrix} 2(w_1 - 2) \\ 2(w_2 + 3) \end{bmatrix}$
$\mathbf{l}(\mathbf{z}) = \mathbf{W}\mathbf{z}$ W known $p \times k$-matrix	$input : k \times 1$ $output : p \times 1$		

Gradient in Higher Dimensions

Gradient $\nabla_{input}(output)$ has shape $input\ shape \times output\ shape$:

Function	I/O Shapes	Grad. Shape	Gradient
$L(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{w}$	$input : n$ $output : 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = 2\mathbf{w}$
$L(\mathbf{w}) = \mathbf{x}^{\mathrm{T}}\mathbf{w}$ $\mathbf{x}$ known n-vector	$input : n$ $output : 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = \mathbf{x}$
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$L(\mathbf{w}) = \mathbf{x}^{\mathrm{T}}\mathbf{w}$ $\mathbf{x}$ known n-vector	$input : n$ $output : 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = \mathbf{x}$
$L(\mathbf{w}) = (w_1 - 2)^2 + (w_2 + 3)^2$	$input : 2$ $output : 1$	2×1	$\nabla_{\mathbf{w}}(L) = \begin{bmatrix} \nabla_{w_1}(L) \\ \nabla_{w_2}(L) \end{bmatrix} = \begin{bmatrix} 2(w_1 - 2) \\ 2(w_2 + 3) \end{bmatrix}$
$\mathbf{l}(\mathbf{z}) = \mathbf{W}\mathbf{z}$ W known $p \times k$-matrix	$input : k \times 1$ $output : p \times 1$	$k \times p$	$\nabla_{\mathbf{z}}(\mathbf{l}) = \mathbf{W}^{\mathrm{T}}$

Gradient in Higher Dimensions

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Gradient $\nabla_{input}(output)$ has shape $input\ shape \times output\ shape$:

Function	I/O Shapes	Grad. Shape	Gradient
$L(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{w}$	$input : n$ $output : 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = 2\mathbf{w}$
$L(\mathbf{w}) = \mathbf{x}^{\mathrm{T}}\mathbf{w}$ $\mathbf{x}$ known n-vector	$input : n$ $output : 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = \mathbf{x}$
$L(\mathbf{w}) = (w_1 - 2)^2 + (w_2 + 3)^2$	$input : 2$ $output : 1$	2×1	$\nabla_{\mathbf{w}}(L) = \begin{bmatrix} \nabla_{w_1}(L) \\ \nabla_{w_2}(L) \end{bmatrix} = \begin{bmatrix} 2(w_1 - 2) \\ 2(w_2 + 3) \end{bmatrix}$
$\mathbf{l}(\mathbf{z}) = \mathbf{W}\mathbf{z}$ W known $p \times k$-matrix	$input : k \times 1$ $output : p \times 1$	$k \times p$	$\nabla_{\mathbf{z}}(\mathbf{l}) = \mathbf{W}^{\mathrm{T}}$ note the transpose

Chain Rule Example in Higher Dimensions

Chain Rule Example in Higher Dimensions

Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^{\mathrm{T}}\mathbf{x}}\right)$:

Chain Rule Example in Higher Dimensions

Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^\mathrm{T}\mathbf{x}}\right)$: $\begin{cases} & \\ z_2(\mathbf{w}) & = -\mathbf{w}^\mathrm{T}\mathbf{x}. \end{cases}$

Chain Rule Example in Higher Dimensions

Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^\mathrm{T}\mathbf{x}}\right)$: $\begin{cases} z_1(z_2) & = 1 + e^{z_2}, \\ z_2(\mathbf{w}) & = -\mathbf{w}^\mathrm{T}\mathbf{x}. \end{cases}$

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Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^{\mathrm{T}}\mathbf{x}}\right)$: $\begin{cases} L(z_1) & = 1/z_1, \\ z_1(z_2) & = 1 + e^{z_2}, \\ z_2(\mathbf{w}) & = -\mathbf{w}^{\mathrm{T}}\mathbf{x}. \end{cases}$

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Computation graph:

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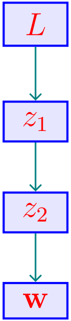
Computation graph:

$$
\begin{tikzcd}
\boxed{L} \arrow[d] \\
\boxed{z_1} \arrow[d] \\
\boxed{z_2} \arrow[d] \\
\boxed{\mathbf{w}}
\end{tikzcd}
$$

Chain Rule Example in Higher Dimensions

Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^{\mathrm{T}}\mathbf{x}}\right)$: $\begin{cases} L(z_1) &= 1/z_1, \\ z_1(z_2) &= 1 + e^{z_2}, \\ z_2(\mathbf{w}) &= -\mathbf{w}^{\mathrm{T}}\mathbf{x}. \end{cases}$

Computation graph:



Gradient calculation using chain rule:

Chain Rule Example in Higher Dimensions

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Computation graph:

$$\begin{array}{c} \boxed{L} \\ \downarrow \\ \boxed{z_1} \\ \downarrow \\ \boxed{z_2} \\ \downarrow \\ \boxed{\mathbf{w}} \end{array}$$

Gradient calculation using chain rule:

$$\nabla_{\mathbf{w}}(L) =$$

Chain Rule Example in Higher Dimensions

Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^{\mathrm{T}}\mathbf{x}}\right)$: $\begin{cases} L(z_1) &= 1/z_1, \\ z_1(z_2) &= 1 + e^{z_2}, \\ z_2(\mathbf{w}) &= -\mathbf{w}^{\mathrm{T}}\mathbf{x}. \end{cases}$

Computation graph:

$$
\begin{tikzcd}
\boxed{L} \arrow[d] \\
\boxed{z_1} \arrow[d] \\
\boxed{z_2} \arrow[d] \\
\boxed{\mathbf{w}}
\end{tikzcd}
$$

Gradient calculation using chain rule:

$$\nabla_{\mathbf{w}}(L) = \qquad\qquad \nabla_{z_1}(L)$$

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Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^\mathrm{T}\mathbf{x}}\right)$: $\begin{cases} L(z_1) &= 1/z_1, \\ z_1(z_2) &= 1 + e^{z_2}, \\ z_2(\mathbf{w}) &= -\mathbf{w}^\mathrm{T}\mathbf{x}. \end{cases}$

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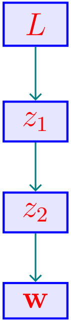

Gradient calculation using chain rule:

$$\nabla_{\mathbf{w}}(L) = \qquad \nabla_{z_2}(z_1) \times \nabla_{z_1}(L)$$

Chain Rule Example in Higher Dimensions

Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^{\mathrm{T}}\mathbf{x}}\right)$: $\begin{cases} L(z_1) & = 1/z_1, \\ z_1(z_2) & = 1 + e^{z_2}, \\ z_2(\mathbf{w}) & = -\mathbf{w}^{\mathrm{T}}\mathbf{x}. \end{cases}$

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$$\begin{tikzcd} \boxed{L} \arrow[d] \\ \boxed{z_1} \arrow[d] \\ \boxed{z_2} \arrow[d] \\ \boxed{\mathbf{w}} \end{tikzcd}$$

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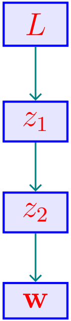
Computation graph:



Gradient calculation using chain rule:

$$\nabla_{\mathbf{w}}(L) = \nabla_{\mathbf{w}}(z_2) \times \nabla_{z_2}(z_1) \times \nabla_{z_1}(L)$$

$$\nabla_{z_1}\left(\frac{1}{z_1}\right)$$

Chain Rule Example in Higher Dimensions

Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^{\mathrm{T}}\mathbf{x}}\right)$: $\begin{cases} L(z_1) &= 1/z_1, \\ z_1(z_2) &= 1 + e^{z_2}, \\ z_2(\mathbf{w}) &= -\mathbf{w}^{\mathrm{T}}\mathbf{x}. \end{cases}$

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$$\nabla_{\mathbf{w}}(L) = \nabla_{\mathbf{w}}(z_2) \times \nabla_{z_2}(z_1) \times \nabla_{z_1}(L)$$

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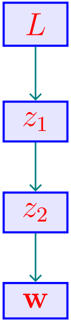
Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^{\mathrm{T}}\mathbf{x}}\right)$: $\begin{cases} L(z_1) & = 1/z_1, \\ z_1(z_2) & = 1 + e^{z_2}, \\ z_2(\mathbf{w}) & = -\mathbf{w}^{\mathrm{T}}\mathbf{x}. \end{cases}$

Computation graph:



Gradient calculation using chain rule:

$$\begin{aligned} \nabla_{\mathbf{w}}(L) &= \nabla_{\mathbf{w}}(z_2) \times \nabla_{z_2}(z_1) \times \nabla_{z_1}(L) \\ &= \nabla_{\mathbf{w}}\left(-\mathbf{w}^{\mathrm{T}}\mathbf{x}\right) \times \nabla_{z_1}\left(1 + e^{z_2}\right) \times \nabla_{z_1}\left(\frac{1}{z_1}\right) \end{aligned}$$

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<u>Computation graph:</u>

$$
\begin{tikzcd}
L \arrow[d] \\
z_1 \arrow[d] \\
z_2 \arrow[d] \\
\mathbf{w}
\end{tikzcd}
$$

<u>Gradient calculation using chain rule:</u>

$$
\begin{aligned}
\nabla_\mathbf{w}(L) &= \nabla_\mathbf{w}(z_2) \times \nabla_{z_2}(z_1) \times \nabla_{z_1}(L) \\
&= \nabla_\mathbf{w}\left(-\mathbf{w}^\mathrm{T}\mathbf{x}\right) \times \nabla_{z_1}\left(1 + e^{z_2}\right) \times \nabla_{z_1}\left(\frac{1}{z_1}\right) \\
& \qquad\qquad\qquad\qquad - 1/z_1^2
\end{aligned}
$$

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Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^{\mathrm{T}}\mathbf{x}}\right)$: $\begin{cases} L(z_1) &= 1/z_1, \\ z_1(z_2) &= 1 + e^{z_2}, \\ z_2(\mathbf{w}) &= -\mathbf{w}^{\mathrm{T}}\mathbf{x}. \end{cases}$

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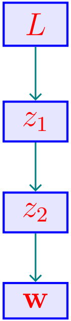

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$$\nabla_{\mathbf{w}}(L) = \nabla_{\mathbf{w}}(z_2) \times \nabla_{z_2}(z_1) \times \nabla_{z_1}(L)$$

$$= \nabla_{\mathbf{w}}\left(-\mathbf{w}^{\mathrm{T}}\mathbf{x}\right) \times \nabla_{z_1}\left(1 + e^{z_2}\right) \times \nabla_{z_1}\left(\frac{1}{z_1}\right)$$

$$e^{z_2} \times -1/z_1^2$$

Chain Rule Example in Higher Dimensions

Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^{\mathrm{T}}\mathbf{x}}\right)$: $\begin{cases} L(z_1) &= 1/z_1, \\ z_1(z_2) &= 1 + e^{z_2}, \\ z_2(\mathbf{w}) &= -\mathbf{w}^{\mathrm{T}}\mathbf{x}. \end{cases}$

Computation graph:

$$\begin{tikzcd} \boxed{L} \arrow[d] \\ \boxed{z_1} \arrow[d] \\ \boxed{z_2} \arrow[d] \\ \boxed{\mathbf{w}} \end{tikzcd}$$

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&= \frac{e^{-\mathbf{w}^T\mathbf{x}}}{\left(1 + e^{-\mathbf{w}^T\mathbf{x}}\right)^2}\mathbf{x}.
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Gradient Descent Method

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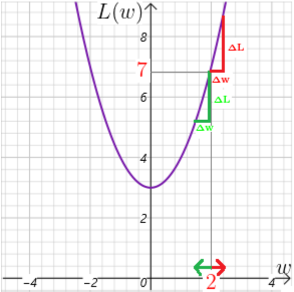

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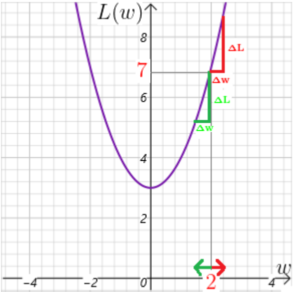

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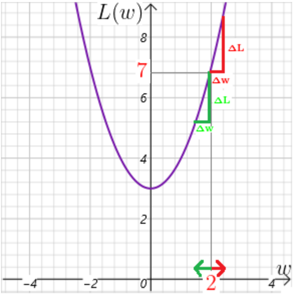

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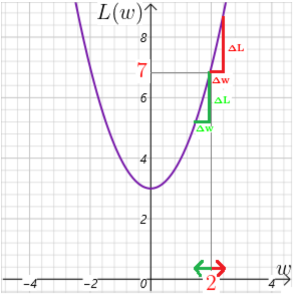

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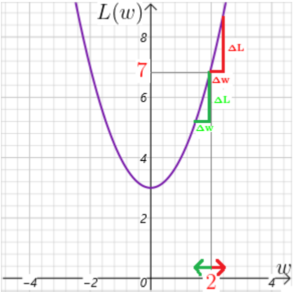

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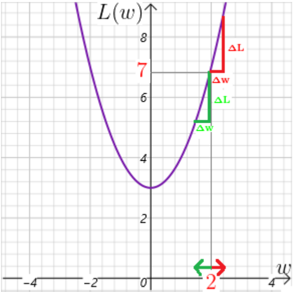

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- How much should we move? We move by a specific small amount known as the learning rate α.

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gradient at old w value

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$$\Rightarrow \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} - \alpha \left. \begin{bmatrix} \nabla_{w_1}\left((w_1 - 2)^2 + (w_2 + 3)^2\right) \\ \nabla_{w_2}\left((w_1 - 2)^2 + (w_2 + 3)^2\right) \end{bmatrix} \right|_{\mathbf{w}}$$

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&= \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} - \alpha \left.\begin{bmatrix} 2(w_1 - 2) \\ 3(w_2 + 3) \end{bmatrix}\right|_{\mathbf{w}}.
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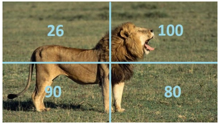

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- This helps in simplifying gradient calculations for computing optimal weights and bias values without having to account for the bias separately.

Softmax Classifier Setup

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- Suppose we have n samples each with p features: $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$, with labels $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$ belonging to k classes.

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- The ith sample's softmax loss is $-\log\left(\left[\text{softmax}\left(\mathbf{z}^{(i)}\right)\right]_{y^{(i)}}\right)$.

Softmax Classifier Gradient

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- The average training softmax loss

$$L(\mathbf{W}) = \frac{1}{n}\sum_{i=1}^{n} L_i = \frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{j=1}^{k} e^{\mathbf{w}_j^{\mathrm{T}}\mathbf{x}^{(i)}}}\right).$$

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- The gradient $\nabla_{\mathbf{W}}(L)$ has shape $\underbrace{k \times (p+1)}_{\text{input shape}} \times \underbrace{1}_{\text{output shape}} = k \times (p+1)$.

Softmax Classifier Gradient

- The average training softmax loss

$$L(\mathbf{W}) = \frac{1}{n}\sum_{i=1}^{n} L_i = \frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{j=1}^{k} e^{\mathbf{w}_j^{\mathrm{T}}\mathbf{x}^{(i)}}}\right).$$

- The gradient $\nabla_{\mathbf{W}}(L)$ has shape $\underbrace{k \times (p+1)}_{\text{input shape}} \times \underbrace{1}_{\text{output shape}} = k \times (p+1)$.

- Weights matrix

Softmax Classifier Gradient

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- The average training softmax loss

$$L(\mathbf{W}) = \frac{1}{n}\sum_{i=1}^{n} L_i = \frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{j=1}^{k} e^{\mathbf{w}_j^{\mathrm{T}}\mathbf{x}^{(i)}}}\right).$$

- The gradient $\nabla_{\mathbf{W}}(L)$ has shape $\underbrace{k \times (p+1)}_{\text{input shape}} \times \underbrace{1}_{\text{output shape}} = k \times (p+1)$.

- Weights matrix $\mathbf{W} = \begin{bmatrix} \underbrace{\mathbf{w}_1}_{1\times(p+1)}{}^{\mathrm{T}} \\ \underbrace{\mathbf{w}_2}_{1\times(p+1)}{}^{\mathrm{T}} \\ \vdots \\ \underbrace{\mathbf{w}_k}_{1\times(p+1)}{}^{\mathrm{T}} \end{bmatrix}$,

Softmax Classifier Gradient

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- The gradient $\nabla_{\mathbf{W}}(L)$ has shape $\underbrace{k \times (p+1)}_{\text{input shape}} \times \underbrace{1}_{\text{output shape}} = k \times (p+1)$.

- Weights matrix $\mathbf{W} = \begin{bmatrix} \underbrace{\mathbf{w}_1{}^{\mathrm{T}}}_{1\times(p+1)} \\ \underbrace{\mathbf{w}_2{}^{\mathrm{T}}}_{1\times(p+1)} \\ \vdots \\ \underbrace{\mathbf{w}_k{}^{\mathrm{T}}}_{1\times(p+1)} \end{bmatrix}$, gradient $\nabla_{\mathbf{W}}(L) = \begin{bmatrix} \underbrace{\left(\nabla_{\mathbf{w}_1}(L)\right)^{\mathrm{T}}}_{1\times(p+1)} \\ \underbrace{\left(\nabla_{\mathbf{w}_2}(L)\right)^{\mathrm{T}}}_{1\times(p+1)} \\ \vdots \\ \underbrace{\left(\nabla_{\mathbf{w}_k}(L)\right)^{\mathrm{T}}}_{1\times(p+1)} \end{bmatrix}$.

Softmax Classifier Gradient

- The average training softmax loss

$$L(\mathbf{W}) = \frac{1}{n}\sum_{i=1}^{n} L_i = \frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{j=1}^{k} e^{\mathbf{w}_j^{\mathrm{T}}\mathbf{x}^{(i)}}}\right).$$

- The gradient $\nabla_{\mathbf{W}}(L)$ has shape $\underbrace{k \times (p+1)}_{\text{input shape}} \times \underbrace{1}_{\text{output shape}} = k \times (p+1)$.

- Weights matrix $\mathbf{W} = \begin{bmatrix} \underbrace{\mathbf{w}_1^{\mathrm{T}}}_{1\times(p+1)} \\ \underbrace{\mathbf{w}_2^{\mathrm{T}}}_{1\times(p+1)} \\ \vdots \\ \underbrace{\mathbf{w}_k^{\mathrm{T}}}_{1\times(p+1)} \end{bmatrix}$, gradient $\nabla_{\mathbf{W}}(L) = \begin{bmatrix} \underbrace{\left(\nabla_{\mathbf{w}_1}(L)\right)^{\mathrm{T}}}_{1\times(p+1)} \\ \underbrace{\left(\nabla_{\mathbf{w}_2}(L)\right)^{\mathrm{T}}}_{1\times(p+1)} \\ \vdots \\ \underbrace{\left(\nabla_{\mathbf{w}_k}(L)\right)^{\mathrm{T}}}_{1\times(p+1)} \end{bmatrix}$.

focus on term like this

Softmax Classifier Gradient – continued

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L)$$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right)$$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

$$\log(a/b) = \log(a) - \log(b)$$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

$$= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right)$$

Softmax Classifier Gradient – continued

$$
\begin{aligned}
\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) &= \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right) \\
&= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right) \\
&= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]
\end{aligned}
$$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

$$= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$\log(e^{a}) = a$$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

$$= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

Softmax Classifier Gradient – continued

$$
\begin{aligned}
\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) &= \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right) \\
&= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right) \\
&= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right] \\
&= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\underset{\uparrow}{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]
\end{aligned}
$$

$$
\begin{cases} = \mathbf{x}^{(i)} & \text{if } j = y^{(i)}, \\ = 0 & \text{if } j \neq y^{(i)}. \end{cases}
$$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

$$= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$I\left(y^{(i)} = j\right)\mathbf{x}^{(i)}$$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

$$= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

use chain rule: $\begin{cases} L &= \log(z), \\ z &= \sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}. \end{cases}$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

$$= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}\right) - \underbrace{\nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)}_{\boxed{\frac{e^{\mathbf{w}_j^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}}\mathbf{x}^{(i)}}\right]$$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

$$= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[-I\left(y^{(i)} = j\right)\mathbf{x}^{(i)} + \hat{p}_{ji}\mathbf{x}^{(i)}\right]$$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

$$= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[-I\left(y^{(i)} = j\right)\mathbf{x}^{(i)} + \underbrace{\hat{p}_{ji}}_{\text{predicted probability that sample-}i\text{ belongs to class }j}\mathbf{x}^{(i)}\right]$$

Softmax Classifier Gradient – continued

Softmax Classifier Gradient – continued

$$\Rightarrow \nabla_{\mathbf{W}}(L) = \begin{bmatrix} \left(\nabla_{\mathbf{w}_1}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_j}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_k}(L)\right)^{\mathrm{T}} \end{bmatrix}$$

Softmax Classifier Gradient – continued

$$\Rightarrow \nabla_{\mathbf{W}}(L) = \begin{bmatrix} \left(\nabla_{\mathbf{w}_1}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_j}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_k}(L)\right)^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{1i} - I\left(y^{(i)} = 1\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \\ \vdots \\ \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{ji} - I\left(y^{(i)} = j\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \\ \vdots \\ \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{ki} - I\left(y^{(i)} = k\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \end{bmatrix}$$

Softmax Classifier Gradient – continued

$$\Rightarrow \nabla_{\mathbf{W}}(L) = \begin{bmatrix} \left(\nabla_{\mathbf{w}_1}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_j}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_k}(L)\right)^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{1i} - I\left(y^{(i)} = 1\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \\ \vdots \\ \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{ji} - I\left(y^{(i)} = j\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \\ \vdots \\ \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{ki} - I\left(y^{(i)} = k\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \end{bmatrix} = \frac{1}{n}\sum_{i=1}^{n}\left(\begin{bmatrix} \left[\hat{p}_{1i} - I\left(y^{(i)} = 1\right)\right]\mathbf{x}^{(i)\mathrm{T}} \\ \vdots \\ \left[\hat{p}_{ji} - I\left(y^{(i)} = j\right)\right]\mathbf{x}^{(i)\mathrm{T}} \\ \vdots \\ \left[\hat{p}_{ki} - I\left(y^{(i)} = k\right)\right]\mathbf{x}^{(i)\mathrm{T}} \end{bmatrix}\right)$$

Softmax Classifier Gradient – continued

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$$\Rightarrow \nabla_{\mathbf{W}}(L) = \begin{bmatrix} \left(\nabla_{\mathbf{w}_1}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_j}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_k}(L)\right)^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{1i} - I\left(y^{(i)}=1\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \\ \vdots \\ \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{ji} - I\left(y^{(i)}=j\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \\ \vdots \\ \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{ki} - I\left(y^{(i)}=k\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \end{bmatrix} = \frac{1}{n}\sum_{i=1}^{n}\left(\begin{bmatrix} \left[\hat{p}_{1i} - I\left(y^{(i)}=1\right)\right]\mathbf{x}^{(i)\,\mathrm{T}} \\ \vdots \\ \left[\hat{p}_{ji} - I\left(y^{(i)}=j\right)\right]\mathbf{x}^{(i)\,\mathrm{T}} \\ \vdots \\ \left[\hat{p}_{ki} - I\left(y^{(i)}=k\right)\right]\mathbf{x}^{(i)\,\mathrm{T}} \end{bmatrix}\right)$$

$$= \frac{1}{n}\left(\begin{bmatrix} \hat{p}_{11} - I\left(y^{(1)}=1\right) \\ \vdots \\ \hat{p}_{j1} - I\left(y^{(1)}=j\right) \\ \vdots \\ \hat{p}_{k1} - I\left(y^{(1)}=k\right) \end{bmatrix}\mathbf{x}^{(1)\,\mathrm{T}} + \begin{bmatrix} \hat{p}_{12} - I\left(y^{(2)}=1\right) \\ \vdots \\ \hat{p}_{j2} - I\left(y^{(2)}=j\right) \\ \vdots \\ \hat{p}_{k2} - I\left(y^{(2)}=k\right) \end{bmatrix}\mathbf{x}^{(2)\,\mathrm{T}} + \cdots + \begin{bmatrix} \hat{p}_{1n} - I\left(y^{(n)}=1\right) \\ \vdots \\ \hat{p}_{jn} - I\left(y^{(n)}=j\right) \\ \vdots \\ \hat{p}_{kn} - I\left(y^{(n)}=k\right) \end{bmatrix}\mathbf{x}^{(n)\,\mathrm{T}}\right)$$

Softmax Classifier Gradient – continued

Softmax Classifier Gradient – continued

$$\Rightarrow \nabla_{\mathbf{w}}(L) = \frac{1}{n} \begin{bmatrix} \hat{p}_{11} - I\left(y^{(1)} = 1\right) & \hat{p}_{12} - I\left(y^{(2)} = 1\right) & \ldots & \hat{p}_{1n} - I\left(y^{(n)} = 1\right) \\ \vdots & \vdots & \vdots & \\ \hat{p}_{j1} - I\left(y^{(1)} = j\right) & \hat{p}_{j2} - I\left(y^{(2)} = j\right) & \ldots & \hat{p}_{jn} - I\left(y^{(n)} = j\right) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{p}_{k1} - I\left(y^{(1)} = k\right) & \hat{p}_{k2} - I\left(y^{(2)} = k\right) & \ldots & \hat{p}_{kn} - I\left(y^{(n)} = k\right) \end{bmatrix} \begin{bmatrix} \mathbf{x}^{(1)^{\mathrm{T}}} \\ \mathbf{x}^{(2)^{\mathrm{T}}} \\ \vdots \\ \vdots \\ \mathbf{x}^{(n)^{\mathrm{T}}} \end{bmatrix}$$

Softmax Classifier Gradient – continued

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for each sample, correct class predicted probability minus one; incorrect class predicted probabilities untouched

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check shape : $(k \times n) \times (n \times (p+1)) = (k \times (p+1))$-matrix

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- But how do we get the predicted probability that a sample belongs to its correct class?

Logistic Regression Classifier Setup – continued

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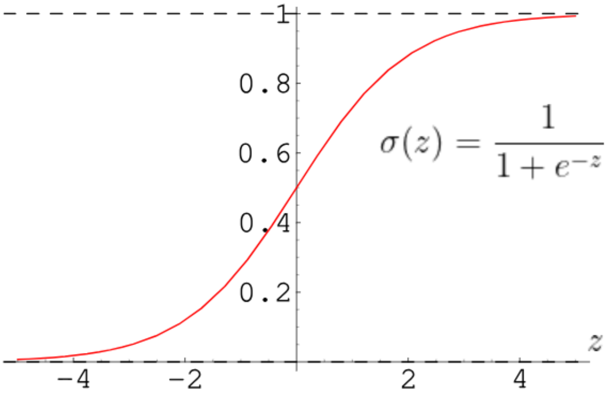

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- Note that we can write this compactly as

$$\hat{y}^{(i)} = \left(\sigma\left(\mathbf{w}^{\mathrm{T}}\mathbf{x}^{(i)}\right)\right)^{y^{(i)}} \left(1 - \sigma\left(\mathbf{w}^{\mathrm{T}}\mathbf{x}^{(i)}\right)\right)^{1-y^{(i)}}.$$

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sigmoid broadcasted to all elements of vector

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vector of correct classes

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Following intermediate variables and computation graph can be used to derive the ith samples logistic regression loss function's gradient $\nabla_{\mathbf{w}}(L_i)$:

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$$
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$$\begin{array}{c} \boxed{L_i} \\ \downarrow \\ \boxed{a^{(i)}} \\ \downarrow \\ \boxed{\mathbf{z}^{(i)}} \\ \downarrow \\ \boxed{\mathbf{w}} \end{array}$$

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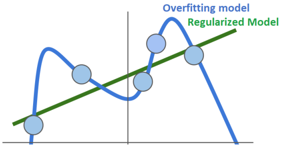
Potentially overfit feature

Oil	Density	Crispy	Fracture	Hardness	Taste
16.5	2955	10	23	97	fair
17.7	2660	14	9	139	excellent
16.2	2870	12	17	143	poor
16.7	2920	10	31	95	good
16.3	2975	11	26	143	fair
19.1	2790	13	16	189	good
18.4	2750	13	17	114	poor



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MANIPAL ACADEMY of HIGHER EDUCATION
(Institution of Eminence Deemed to be University)

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Regularization is achieved by
adding to the average training data loss constraints on the weights (not to the bias values):

$$\underbrace{\frac{1}{n}\sum_{i=1}^{n} L_i(\mathbf{W})}_{\text{average training data loss}} + \begin{cases} \lambda \sum_{r,s} |w_{rs}|^2 \\ (\text{ridge or } L_2) \\ \quad \text{or} \\ \lambda \sum_{r,s} |w_{rs}| \\ (\text{lasso or } L_1) \\ \quad \text{or} \\ \lambda \left[\alpha \sum_{r,s} |w_{rs}| + (1-\alpha) \sum_{r,s} |w_{rs}|^2\right] \\ (\text{elastic-net}), \end{cases}$$

where $\lambda > 0$ is the strength of regularization.

- λ is a hyperparameter that has to be tuned.
- All regularization approaches tend to drive the weight values close to zero.
- L_1-regularization typically results in a smaller subset of nonzero weights than L_2-regularization.

Regularization Loss & Gradient

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