Deep Learning Principles & Applications

Chapter 2 – Linear Classifiers

Sudarsan N.S. Acharya (sudarsan.acharya@manipal.edu)

Classification in Practice

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is this sample a $\underbrace{\text{lion/tiger/leopard}}_{\text{known 3 classes}}$?

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Recall that this color image is internally represented as a $337 \times 600 \times 3$-tensor of integer values ranging from 0 to 255.

Classification in Practice – continued

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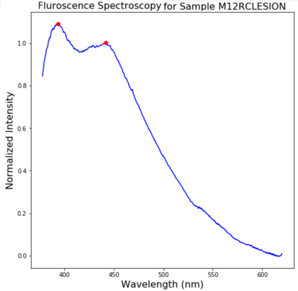
Medical Signal Processing

Classification in Practice – continued

Medical Signal Processing

	0	1	...	1017	1018
M12RCLESION	0.845890	0.861290	...	0.000037	0.001354



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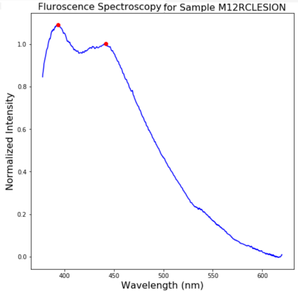

What kind of an oral tumor does this patient have:

benign / premalignant / malignant?

known 3 classes

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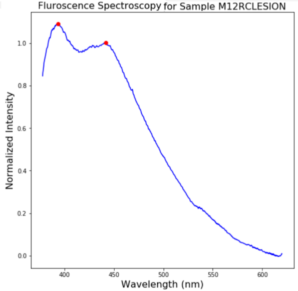

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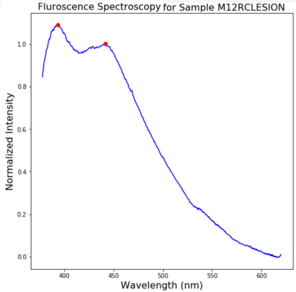

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Language Application

The movie was goat

Classification in Practice – continued

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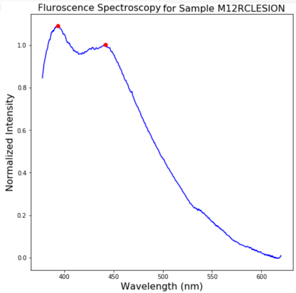

What kind of an oral tumor does this patient have: benign/premalignant/malignant?

known 3 classes

Language Application

The movie was goat

Is this movie review positive/negative?

known 2 classes

Linear Classifier: Basic Idea

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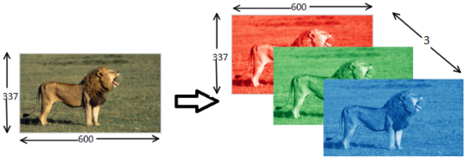

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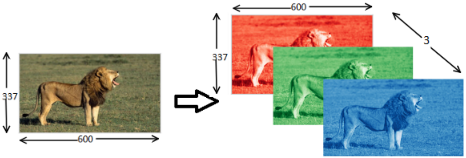

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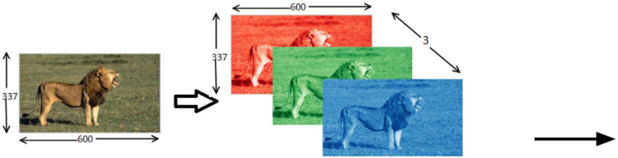

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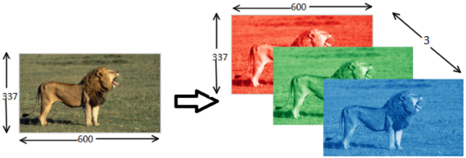

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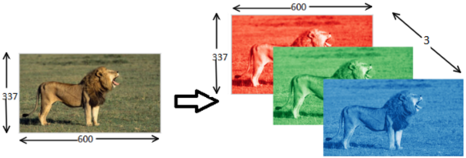

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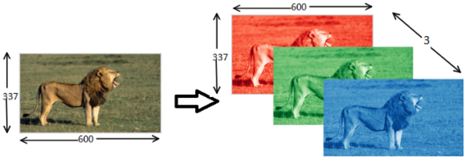

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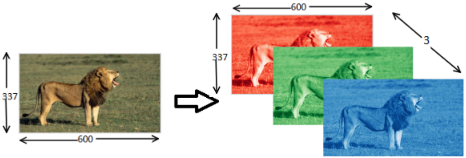

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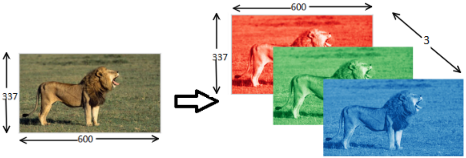

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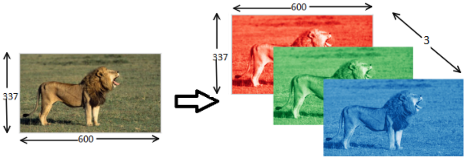

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What are $\mathbf{W}$ and $\mathbf{b}$ (the parameters), and how do we know what they are?

Weights, Bias, & Raw Scores

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MANIPAL ACADEMY of HIGHER EDUCATION (Institution of Eminence Deemed to be University)

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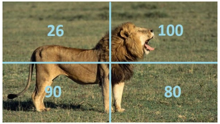

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Loss Function – Intuition

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Loss for Sample-1

$$L_1 = \begin{cases} \max(0, 6.4 - 5.6) \\ + \\ \max(0, -4.6 - 5.6) \end{cases}$$

$$= 0.8$$

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Loss for Sample-2

$$L_2 = \begin{cases} \max(0, -1.8 - 10.2) \\ + \\ \max(0, 3.5 - 10.2) \end{cases}$$

$$= 0$$

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Loss for Sample-3

$$L_3 = \begin{cases} \max(0, 2.0 - (-8.6)) \\ + \\ \max(0, 5.4 - (-8.6)) \end{cases}$$

$$= 24.6$$

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Average training loss

$$\frac{0.8 + 0 + 24.6}{3} = 8.5$$

Hinge Loss Function

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$$\underset{\text{sample vector}}{\mathbf{x}^{(i)}}$$

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correct class/label

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- The average training data loss is $\frac{1}{n}\sum_{i=1}^{n} L_i$, which is a function of the weights and bias values.

Visualizing Loss Functions

Visualizing Loss Functions

Visualizing different loss functions considering contribution from one incorrect class:

Visualizing Loss Functions

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Perceptron Loss

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Visualizing different loss functions considering contribution from one incorrect class:

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Squared Hinge Loss

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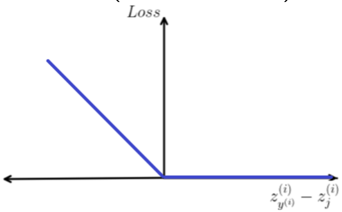

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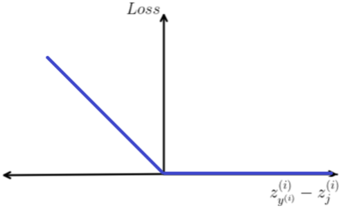

difference between correct and incorrect class raw scores

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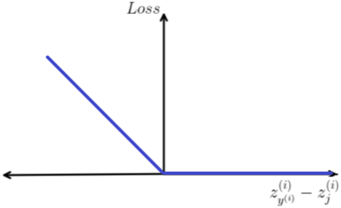

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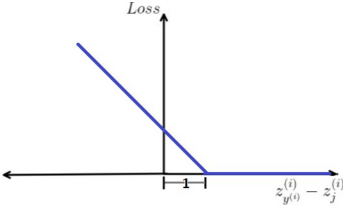

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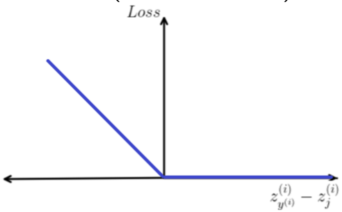

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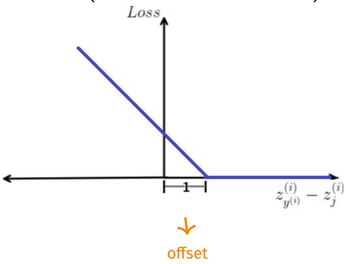

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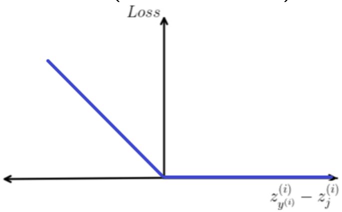

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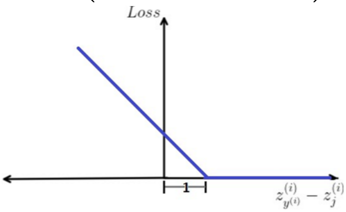

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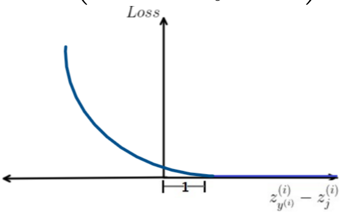

Softmax Function

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It is possible to turn the raw scores vector into a a vector of probabilities:

Softmax Function

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Raw score	
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Raise to power of e $\longrightarrow$

Softmax Function

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Raw score			**Exponentiated raw score**
Lion score	5.6	Raise to power of e $\rightarrow$	$e^{5.6}$
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Raise to power of e →

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normalize →

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Lion score	5.6	Raise to power of e	$e^{5.6}$	normalize	$\frac{e^{5.6}}{e^{5.6}+e^{6.4}+e^{-4.6}} \approx 0.31$
Tiger score	6.4		$e^{6.4}$		$\frac{e^{6.4}}{e^{5.6}+e^{6.4}+e^{-4.6}} \approx 0.69$
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Raise to power of e →

normalize →

Formally, the softmax function takes a vector as input, and outputs a vector (of the same size) of probabilities through exponentiation and normalization. The lion probability is not 1.0 rather $0.39 \Rightarrow$ 😕

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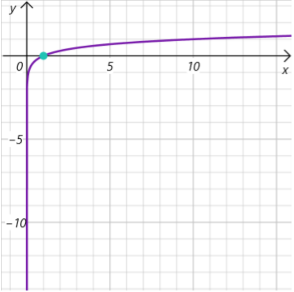

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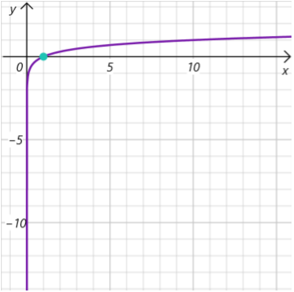

- Note that $\begin{cases} \log(1) = 0, \\ \log(x) \to -\infty \text{ as } x \to 0. \end{cases}$

Softmax Loss Function – continued

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raw score

probability

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- In plain English, it is the negative of the logarithm of the correct class's probability.
- Note that $\begin{cases} [\text{softmax}(\mathbf{z})]_y = 1 & \Rightarrow ☺ \Rightarrow \text{loss} = -\log(1) = 0, \\ [\text{softmax}(\mathbf{z})]_y = 0 & \Rightarrow ☹ \Rightarrow \text{loss} = -\log(0) \to \infty. \end{cases}$

Optimization Setup

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Given training samples, the goal is to find optimal values for the weights and biases that minimize the average training loss.

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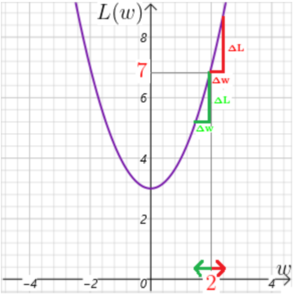

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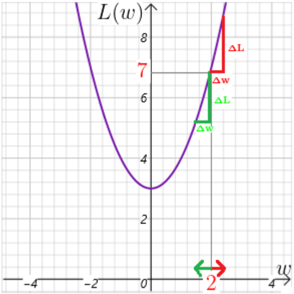

- How can we tweak the input w from it's current value of 2 so that the output L <u>decreases</u> from its current value of 7?

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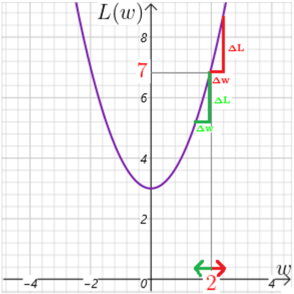

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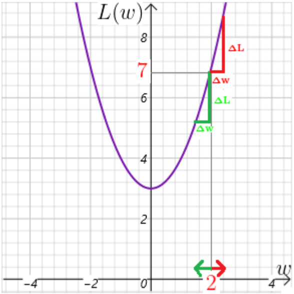
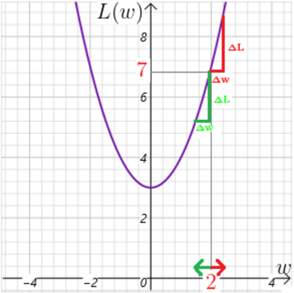


- How can we tweak the input w from it's current value of 2 so that the output L decreases from its current value of 7?
- w can be increased (move right) or decreased (move left) from the current value 2.
- Can we quantify the sensitivity of the output L w.r.t. small changes in the input w?

Sensitivity

Sensitivity



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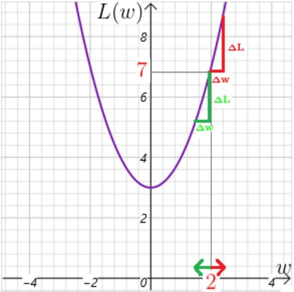

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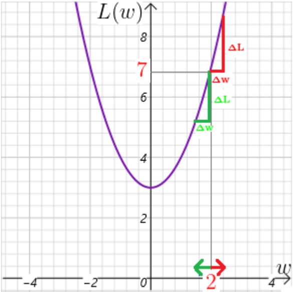

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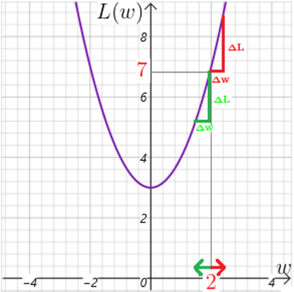

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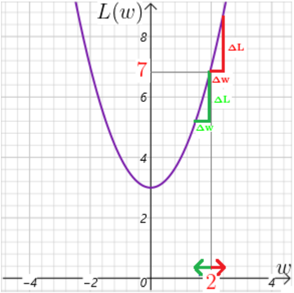

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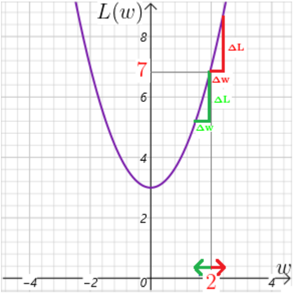
$$\frac{\text{change in output}}{\text{change in input}} : \begin{cases} \text{moving right} & = \\ & \end{cases}$$

Sensitivity



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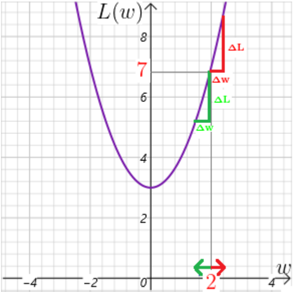

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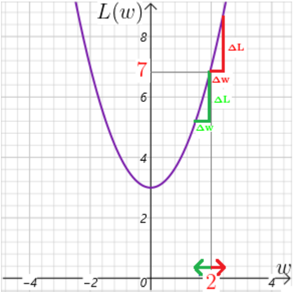

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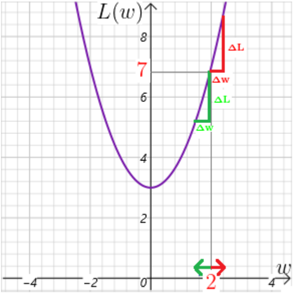

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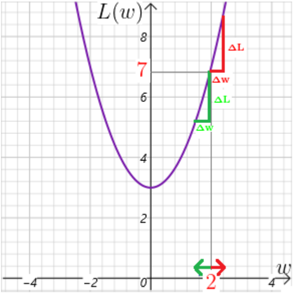

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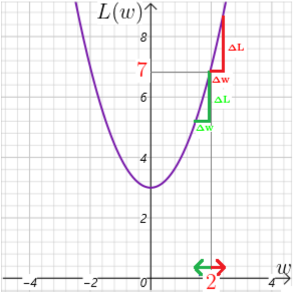

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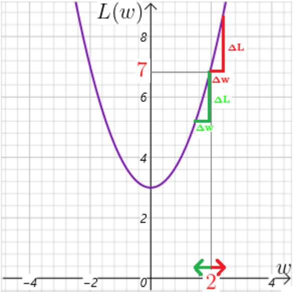

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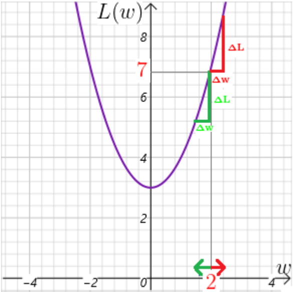

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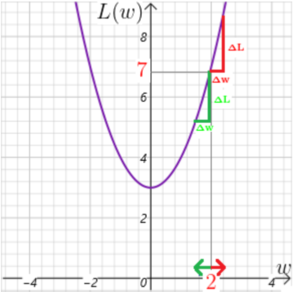

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In this case, we move left (decrease) w to decrease L.

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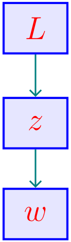

Gradient calculation using chain rule:

$$\begin{aligned}
\nabla_w(L) &= \nabla_w(z) \times \nabla_z(L) \\
&= \nabla_w\left(1 + e^{-w}\right) \times \nabla_z\left(\frac{1}{z}\right) \\
&= \left[\nabla_w(1) + \nabla_w\left(e^{-w}\right)\right] \times -\frac{1}{z^2}
\end{aligned}$$

Chain Rule

The chain rule is used to calculate gradients in a hierarchical way by using intermediate variable(s).

E.g. $L(w) = 1/\left(1+e^{-w}\right) \xrightarrow{\text{use } z=1+e^{-w}} \begin{cases} L(z) & = 1/z, \\ z(w) & = 1+e^{-w}. \end{cases}$

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Gradient in Higher Dimensions

Gradient in Higher Dimensions

Gradient $\nabla_{input}(output)$ has shape $input\ shape \times output\ shape$:

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Function	I/O Shapes	Grad. Shape	Gradient

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Function	I/O Shapes	Grad. Shape	Gradient
$L(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{w}$			

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Function	I/O Shapes	Grad. Shape	Gradient
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$L(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{w}$	$input : n$ $output : 1$	$n \times 1$	

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Function	I/O Shapes	Grad. Shape	Gradient
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$L(\mathbf{w}) = (w_1 - 2)^2 + (w_2 + 3)^2$			

Gradient in Higher Dimensions

Gradient $\nabla_{input}(output)$ has shape $input\ shape \times output\ shape$:

Function	I/O Shapes	Grad. Shape	Gradient
$L(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{w}$	$input : n$ $output : 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = 2\mathbf{w}$
$L(\mathbf{w}) = \mathbf{x}^{\mathrm{T}}\mathbf{w}$ $\mathbf{x}$ known n-vector	$input : n$ $output : 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = \mathbf{x}$
$L(\mathbf{w}) = (w_1 - 2)^2$ $+(w_2 + 3)^2$	$input : 2$ $output : 1$		

Gradient in Higher Dimensions

Gradient $\nabla_{input}(output)$ has shape $input\ shape \times output\ shape$:

Function	I/O Shapes	Grad. Shape	Gradient
$L(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{w}$	$input : n$ $output : 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = 2\mathbf{w}$
$L(\mathbf{w}) = \mathbf{x}^{\mathrm{T}}\mathbf{w}$ $\mathbf{x}$ known n-vector	$input : n$ $output : 1$	$n \times 1$	$\nabla_{\mathbf{w}}(L) = \mathbf{x}$
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$\mathbf{l}(\mathbf{z}) = \mathbf{W}\mathbf{z}$ W known $p \times k$-matrix			

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Chain Rule Example in Higher Dimensions

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Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^{\mathrm{T}}\mathbf{x}}\right)$:

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Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^{\mathrm{T}}\mathbf{x}}\right)$: $\begin{cases} \\ z_2(\mathbf{w}) & = -\mathbf{w}^{\mathrm{T}}\mathbf{x}. \end{cases}$

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Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^\mathrm{T}\mathbf{x}}\right)$: $\begin{cases} L(z_1) &= 1/z_1, \\ z_1(z_2) &= 1 + e^{z_2}, \\ z_2(\mathbf{w}) &= -\mathbf{w}^\mathrm{T}\mathbf{x}. \end{cases}$

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Computation graph:

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Computation graph:

$$\begin{tikzcd} \boxed{L} \arrow[d] \\ \boxed{z_1} \arrow[d] \\ \boxed{z_2} \arrow[d] \\ \boxed{\mathbf{w}} \end{tikzcd}$$

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Gradient calculation using chain rule:

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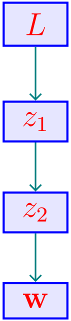

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$$\nabla_{\mathbf{w}}(L) =$$

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Gradient calculation using chain rule:

$$\nabla_{\mathbf{w}}(L) = \qquad \nabla_{z_1}(L)$$

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Gradient calculation using chain rule:

$$\nabla_{\mathbf{w}}(L) = \qquad \nabla_{z_2}(z_1) \times \nabla_{z_1}(L)$$

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Computation graph:

$$
\begin{tikzcd}
\boxed{L} \arrow[d] \\
\boxed{z_1} \arrow[d] \\
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\end{tikzcd}
$$

Gradient calculation using chain rule:

$$\nabla_{\mathbf{w}}(L) = \nabla_{\mathbf{w}}(z_2) \times \nabla_{z_2}(z_1) \times \nabla_{z_1}(L)$$

$$\nabla_{z_1}\left(\frac{1}{z_1}\right)$$

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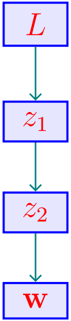

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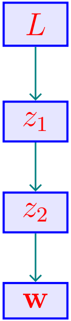

Gradient calculation using chain rule:

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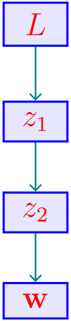

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$$\nabla_{\mathbf{w}}(L) = \nabla_{\mathbf{w}}(z_2) \times \nabla_{z_2}(z_1) \times \nabla_{z_1}(L)$$

$$= \nabla_{\mathbf{w}}\left(-\mathbf{w}^{\mathrm{T}}\mathbf{x}\right) \times \nabla_{z_1}\left(1 + e^{z_2}\right) \times \nabla_{z_1}\left(\frac{1}{z_1}\right)$$

$$e^{z_2} \times -1/z_1^2$$

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$$\begin{tikzcd} L \arrow[d] \\ z_1 \arrow[d] \\ z_2 \arrow[d] \\ \mathbf{w} \end{tikzcd}$$

Gradient calculation using chain rule:

$$\begin{aligned} \nabla_{\mathbf{w}}(L) &= \nabla_{\mathbf{w}}(z_2) \times \nabla_{z_2}(z_1) \times \nabla_{z_1}(L) \\ &= \nabla_{\mathbf{w}}\left(-\mathbf{w}^{\mathrm{T}}\mathbf{x}\right) \times \nabla_{z_1}\left(1 + e^{z_2}\right) \times \nabla_{z_1}\left(\frac{1}{z_1}\right) \\ &= -\mathbf{x} \times e^{z_2} \times -1/z_1^2 \end{aligned}$$

Chain Rule Example in Higher Dimensions

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Calculate the gradient of $L(\mathbf{w}) = 1/\left(1 + e^{-\mathbf{w}^{\mathrm{T}}\mathbf{x}}\right)$: $\begin{cases} L(z_1) & = 1/z_1, \\ z_1(z_2) & = 1 + e^{z_2}, \\ z_2(\mathbf{w}) & = -\mathbf{w}^{\mathrm{T}}\mathbf{x}. \end{cases}$

<u>Computation graph:</u>

$$
\begin{tikzcd}
\boxed{L} \arrow[d] \\
\boxed{z_1} \arrow[d] \\
\boxed{z_2} \arrow[d] \\
\boxed{\mathbf{w}}
\end{tikzcd}
$$

<u>Gradient calculation using chain rule:</u>

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&= -\mathbf{x} \times e^{z_2} \times -1/z_1^2 \\
&= \frac{e^{-\mathbf{w}^{\mathrm{T}}\mathbf{x}}}{\left(1 + e^{-\mathbf{w}^{\mathrm{T}}\mathbf{x}}\right)^2}\mathbf{x}.
\end{aligned}
$$

Gradient Descent Method

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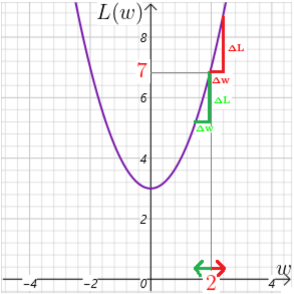

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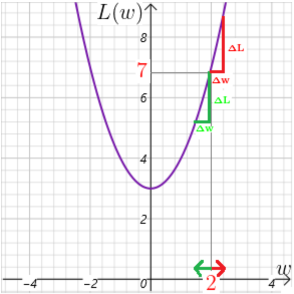

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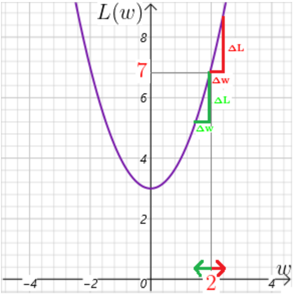

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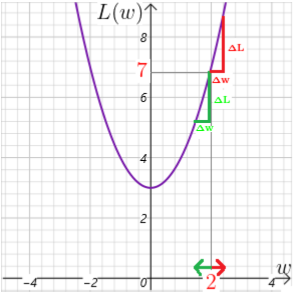

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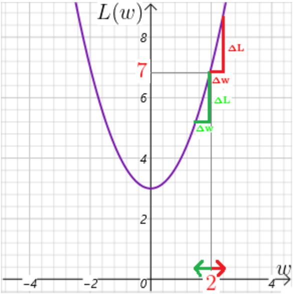

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- How much should we move?

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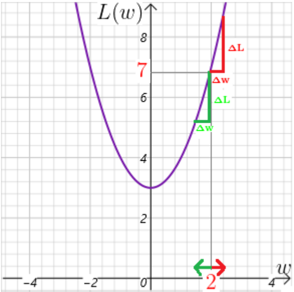

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- How much should we move? We move by a specific small amount known as the learning rate α.

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gradient at old w value

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$$\Rightarrow \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} - \alpha \left.\begin{bmatrix} \nabla_{w_1}\left((w_1 - 2)^2 + (w_2 + 3)^2\right) \\ \nabla_{w_2}\left((w_1 - 2)^2 + (w_2 + 3)^2\right) \end{bmatrix}\right|_{\mathbf{w}}$$

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&= \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} - \alpha \left.\begin{bmatrix} 2(w_1 - 2) \\ 3(w_2 + 3) \end{bmatrix}\right|_{\mathbf{w}}.
\end{aligned}
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Bias Trick

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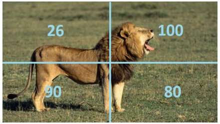

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- Last column of the weights matrix hold the bias values.
- The bias feature with value 1 gets *appended* to the sample vector.
- This helps in simplifying gradient calculations for computing optimal weights and bias values without having to account for the bias separately.

Softmax Classifier Setup

Softmax Classifier Setup

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Samples

- Suppose we have n samples each with p features: $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$, with labels $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$ belonging to k classes.

label

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sample

raw score

Raw score for the 1st sample

They are independent of each other calculation.

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- The ith sample's softmax loss is $-\log\left(\left[\text{softmax}\left(\mathbf{z}^{(i)}\right)\right]_{y^{(i)}}\right)$.

Softmax Classifier Gradient

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- The average training softmax loss

$$L(\mathbf{W}) = \frac{1}{n}\sum_{i=1}^{n} L_i = \frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{j=1}^{k} e^{\mathbf{w}_j^{\mathrm{T}}\mathbf{x}^{(i)}}}\right).$$

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- The gradient $\nabla_{\mathbf{W}}(L)$ has shape $\underbrace{k \times (p+1)}_{\text{input shape}} \times \underbrace{1}_{\text{output shape}} = k \times (p+1)$.
- Weights matrix

Softmax Classifier Gradient

- The average training softmax loss

$$L(\mathbf{W}) = \frac{1}{n}\sum_{i=1}^{n} L_i = \frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{j=1}^{k} e^{\mathbf{w}_j^{\mathrm{T}}\mathbf{x}^{(i)}}}\right).$$

- The gradient $\nabla_{\mathbf{W}}(L)$ has shape $\underbrace{k \times (p+1)}_{\text{input shape}} \times \underbrace{1}_{\text{output shape}} = k \times (p+1)$.

- Weights matrix $\mathbf{W} = \begin{bmatrix} \underbrace{\mathbf{w}_1}_{1\times(p+1)}{}^{\mathrm{T}} \\ \underbrace{\mathbf{w}_2}_{1\times(p+1)}{}^{\mathrm{T}} \\ \vdots \\ \underbrace{\mathbf{w}_k}_{1\times(p+1)}{}^{\mathrm{T}} \end{bmatrix},$

Softmax Classifier Gradient

- The average training softmax loss
$$L(\mathbf{W}) = \frac{1}{n}\sum_{i=1}^{n} L_i = \frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{j=1}^{k} e^{\mathbf{w}_j^{\mathrm{T}}\mathbf{x}^{(i)}}}\right).$$
- The gradient $\nabla_{\mathbf{W}}(L)$ has shape $\underbrace{k \times (p+1)}_{\text{input shape}} \times \underbrace{1}_{\text{output shape}} = k \times (p+1)$.
- Weights matrix $\mathbf{W} = \begin{bmatrix} \underbrace{\mathbf{w}_1^{\mathrm{T}}}_{1\times(p+1)} \\ \underbrace{\mathbf{w}_2^{\mathrm{T}}}_{1\times(p+1)} \\ \vdots \\ \underbrace{\mathbf{w}_k^{\mathrm{T}}}_{1\times(p+1)} \end{bmatrix}$, gradient $\nabla_{\mathbf{W}}(L) = \begin{bmatrix} \underbrace{\left(\nabla_{\mathbf{w}_1}(L)\right)^{\mathrm{T}}}_{1\times(p+1)} \\ \underbrace{\left(\nabla_{\mathbf{w}_2}(L)\right)^{\mathrm{T}}}_{1\times(p+1)} \\ \vdots \\ \underbrace{\left(\nabla_{\mathbf{w}_k}(L)\right)^{\mathrm{T}}}_{1\times(p+1)} \end{bmatrix}$.

Softmax Classifier Gradient

- The average training softmax loss
$$L(\mathbf{W}) = \frac{1}{n}\sum_{i=1}^{n} L_i = \frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{j=1}^{k} e^{\mathbf{w}_j^{\mathrm{T}}\mathbf{x}^{(i)}}}\right).$$
- The gradient $\nabla_{\mathbf{W}}(L)$ has shape $\underbrace{k \times (p+1)}_{\text{input shape}} \times \underbrace{1}_{\text{output shape}} = k \times (p+1)$.
- Weights matrix $\mathbf{W} = \begin{bmatrix} \underbrace{\mathbf{w}_1{}^{\mathrm{T}}}_{1\times(p+1)} \\ \underbrace{\mathbf{w}_2{}^{\mathrm{T}}}_{1\times(p+1)} \\ \vdots \\ \underbrace{\mathbf{w}_k{}^{\mathrm{T}}}_{1\times(p+1)} \end{bmatrix}$, gradient $\nabla_{\mathbf{W}}(L) = \begin{bmatrix} \underbrace{\left(\nabla_{\mathbf{w}_1}(L)\right)^{\mathrm{T}}}_{1\times(p+1)} \\ \underbrace{\left(\nabla_{\mathbf{w}_2}(L)\right)^{\mathrm{T}}}_{1\times(p+1)} \\ \vdots \\ \underbrace{\left(\nabla_{\mathbf{w}_k}(L)\right)^{\mathrm{T}}}_{1\times(p+1)} \end{bmatrix}$.

focus on term like this

Softmax Classifier Gradient – continued

Softmax Classifier Gradient – continued

$\nabla_{\mathbf{w}_j}(L)$

↳ how sensitivity of weights change in j

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right)$$

Softmax Classifier Gradient – continued

MANIPAL ACADEMY of HIGHER EDUCATION (Institution of Eminence Deemed to be University)

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

$$\log(a/b) = \log(a) - \log(b)$$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

$$= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right)$$

Softmax Classifier Gradient – continued

MANIPAL ACADEMY of HIGHER EDUCATION
(Institution of Eminence Deemed to be University)

$$
\begin{aligned}
\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) &= \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right) \\
&= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right) \\
&= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]
\end{aligned}
$$

Softmax Classifier Gradient – continued

$$
\begin{aligned}
\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) &= \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right) \\
&= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right) \\
&= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]
\end{aligned}
$$

$\log(e^a) = a$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

$$= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

$$= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$\begin{cases} = \mathbf{x}^{(i)} & \text{if } j = y^{(i)}, \\ = 0 & \text{if } j \neq y^{(i)}. \end{cases}$$

vector

Softmax Classifier Gradient – continued

$$
\begin{aligned}
\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) &= \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right) \\
&= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right) \\
&= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right] \\
&= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\underset{\uparrow}{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]
\end{aligned}
$$

$$I\left(y^{(i)} = j\right)\mathbf{x}^{(i)}$$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

$$= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

use chain rule: $\begin{cases} L &= \log(z), \\ z &= \sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}. \end{cases}$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

$$= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$\boxed{\frac{e^{\mathbf{w}_j^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}}\mathbf{x}^{(i)}$$

Softmax Classifier Gradient – continued

$$
\begin{aligned}
\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) &= \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right) \\
&= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right) \\
&= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right] \\
&= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[-I\left(y^{(i)} = j\right)\mathbf{x}^{(i)} + \hat{p}_{ji}\mathbf{x}^{(i)}\right]
\end{aligned}
$$

Softmax Classifier Gradient – continued

$$\nabla_{\mathbf{w}_j}(L) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) = \nabla_{\mathbf{w}_j}\left(\frac{1}{n}\sum_{i=1}^{n} -\log\left(\frac{e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}}{\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}}\right)\right)$$

For one feature

$$= -\frac{1}{n}\nabla_{\mathbf{w}_j}\left(\sum_{i=1}^{n}\left[\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right) - \log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right]\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\log\left(e^{\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\nabla_{\mathbf{w}_j}\left(\mathbf{w}_{y^{(i)}}^{\mathrm{T}}\mathbf{x}^{(i)}\right) - \nabla_{\mathbf{w}_j}\left(\log\left(\sum_{r=1}^{k} e^{\mathbf{w}_r^{\mathrm{T}}\mathbf{x}^{(i)}}\right)\right)\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[-I\left(y^{(i)} = j\right)\mathbf{x}^{(i)} + \underbrace{\hat{p}_{ji}}_{\text{predicted probability that sample-}i\text{ belongs to class }j}\mathbf{x}^{(i)}\right]$$

Softmax Classifier Gradient – continued

Softmax Classifier Gradient – continued

$$\Rightarrow \nabla_{\mathbf{W}}(L) = \begin{bmatrix} \left(\nabla_{\mathbf{w}_1}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_j}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_k}(L)\right)^{\mathrm{T}} \end{bmatrix}$$

Softmax Classifier Gradient – continued

$$\Rightarrow \nabla_{\mathbf{W}}(L) = \begin{bmatrix} \left(\nabla_{\mathbf{w}_1}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_j}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_k}(L)\right)^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{1i} - I\left(y^{(i)} = 1\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \\ \vdots \\ \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{ji} - I\left(y^{(i)} = j\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \\ \vdots \\ \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{ki} - I\left(y^{(i)} = k\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \end{bmatrix}$$

Softmax Classifier Gradient – continued

$$\Rightarrow \nabla_{\mathbf{W}}(L) = \begin{bmatrix} \left(\nabla_{\mathbf{w}_1}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_j}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_k}(L)\right)^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{1i} - I\left(y^{(i)} = 1\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \\ \vdots \\ \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{ji} - I\left(y^{(i)} = j\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \\ \vdots \\ \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{ki} - I\left(y^{(i)} = k\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \end{bmatrix} = \frac{1}{n}\sum_{i=1}^{n}\left(\begin{bmatrix} \left[\hat{p}_{1i} - I\left(y^{(i)} = 1\right)\right]\mathbf{x}^{(i)\mathrm{T}} \\ \vdots \\ \left[\hat{p}_{ji} - I\left(y^{(i)} = j\right)\right]\mathbf{x}^{(i)\mathrm{T}} \\ \vdots \\ \left[\hat{p}_{ki} - I\left(y^{(i)} = k\right)\right]\mathbf{x}^{(i)\mathrm{T}} \end{bmatrix}\right)$$

Softmax Classifier Gradient – continued

$$\Rightarrow \nabla_{\mathbf{W}}(L) = \begin{bmatrix} \left(\nabla_{\mathbf{w}_1}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_j}(L)\right)^{\mathrm{T}} \\ \vdots \\ \left(\nabla_{\mathbf{w}_k}(L)\right)^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{1i} - I\left(y^{(i)} = 1\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \\ \vdots \\ \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{ji} - I\left(y^{(i)} = j\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \\ \vdots \\ \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{p}_{ki} - I\left(y^{(i)} = k\right)\right]\mathbf{x}^{(i)}\right)^{\mathrm{T}} \end{bmatrix} = \frac{1}{n}\sum_{i=1}^{n}\left(\begin{bmatrix} \left[\hat{p}_{1i} - I\left(y^{(i)} = 1\right)\right]\mathbf{x}^{(i)\mathrm{T}} \\ \vdots \\ \left[\hat{p}_{ji} - I\left(y^{(i)} = j\right)\right]\mathbf{x}^{(i)\mathrm{T}} \\ \vdots \\ \left[\hat{p}_{ki} - I\left(y^{(i)} = k\right)\right]\mathbf{x}^{(i)\mathrm{T}} \end{bmatrix}\right)$$

$$= \frac{1}{n}\left(\begin{bmatrix} \hat{p}_{11} - I\left(y^{(1)} = 1\right) \\ \vdots \\ \hat{p}_{j1} - I\left(y^{(1)} = j\right) \\ \vdots \\ \hat{p}_{k1} - I\left(y^{(1)} = k\right) \end{bmatrix}\mathbf{x}^{(1)\mathrm{T}} + \begin{bmatrix} \hat{p}_{12} - I\left(y^{(2)} = 1\right) \\ \vdots \\ \hat{p}_{j2} - I\left(y^{(2)} = j\right) \\ \vdots \\ \hat{p}_{k2} - I\left(y^{(2)} = k\right) \end{bmatrix}\mathbf{x}^{(2)\mathrm{T}} + \cdots + \begin{bmatrix} \hat{p}_{1n} - I\left(y^{(n)} = 1\right) \\ \vdots \\ \hat{p}_{jn} - I\left(y^{(n)} = j\right) \\ \vdots \\ \hat{p}_{kn} - I\left(y^{(n)} = k\right) \end{bmatrix}\mathbf{x}^{(n)\mathrm{T}}\right)$$

Softmax Classifier Gradient – continued

Softmax Classifier Gradient – continued

$$\Rightarrow \nabla_{\mathbf{w}}(L) = \frac{1}{n} \begin{bmatrix} \hat{p}_{11} - I\left(y^{(1)} = 1\right) & \hat{p}_{12} - I\left(y^{(2)} = 1\right) & \dots & \hat{p}_{1n} - I\left(y^{(n)} = 1\right) \\ \vdots & \vdots & \vdots & \\ \hat{p}_{j1} - I\left(y^{(1)} = j\right) & \hat{p}_{j2} - I\left(y^{(2)} = j\right) & \dots & \hat{p}_{jn} - I\left(y^{(n)} = j\right) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{p}_{k1} - I\left(y^{(1)} = k\right) & \hat{p}_{k2} - I\left(y^{(2)} = k\right) & \dots & \hat{p}_{kn} - I\left(y^{(n)} = k\right) \end{bmatrix} \begin{bmatrix} {\mathbf{x}^{(1)}}^{\mathrm{T}} \\ {\mathbf{x}^{(2)}}^{\mathrm{T}} \\ \vdots \\ \vdots \\ {\mathbf{x}^{(n)}}^{\mathrm{T}} \end{bmatrix}$$

Softmax Classifier Gradient – continued

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for each sample, correct class predicted probability minus one; incorrect class predicted probabilities untouched

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check shape : $(k \times n) \times (n \times (p+1)) = (k \times (p+1))$-matrix

Softmax Classifier Gradient – continued

$$\Rightarrow \nabla_{\mathbf{w}}(L) = \frac{1}{n}\begin{bmatrix} \hat{p}_{11} - I\left(y^{(1)}=1\right) & \hat{p}_{12} - I\left(y^{(2)}=1\right) & \ldots & \hat{p}_{1n} - I\left(y^{(n)}=1\right) \\ \vdots & \vdots & \vdots & \\ \hat{p}_{j1} - I\left(y^{(1)}=j\right) & \hat{p}_{j2} - I\left(y^{(2)}=j\right) & \ldots & \hat{p}_{jn} - I\left(y^{(n)}=j\right) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{p}_{k1} - I\left(y^{(1)}=k\right) & \hat{p}_{k2} - I\left(y^{(2)}=k\right) & \ldots & \hat{p}_{kn} - I\left(y^{(n)}=k\right) \end{bmatrix} \begin{bmatrix} \mathbf{x}^{(1)^{\mathrm{T}}} \\ \mathbf{x}^{(2)^{\mathrm{T}}} \\ \vdots \\ \vdots \\ \mathbf{x}^{(n)^{\mathrm{T}}} \end{bmatrix}$$

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Gradient descent iteration for softmax:

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- But how do we get the predicted probability that a sample belongs to its correct class?

Logistic Regression Classifier Setup – continued

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The sigmoid function $\sigma(z)$:

Logistic Regression Classifier Setup – continued

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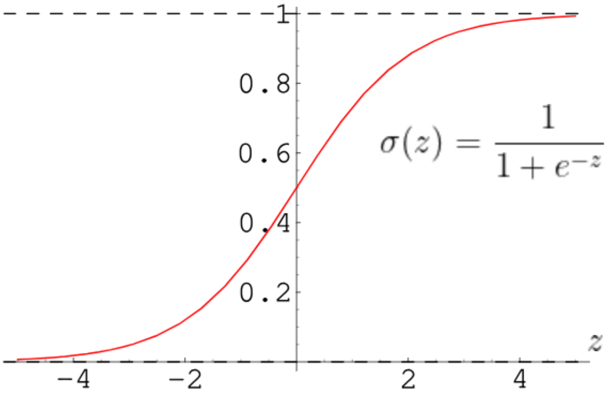

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- Note that we can write this compactly as

$$\hat{y}^{(i)} = \left(\sigma\left(\mathbf{w}^{\mathrm{T}}\mathbf{x}^{(i)}\right)\right)^{y^{(i)}} \left(1 - \sigma\left(\mathbf{w}^{\mathrm{T}}\mathbf{x}^{(i)}\right)\right)^{1-y^{(i)}}.$$

Logistic Regression Classifier Loss Function and Gradient

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sigmoid broadcasted to all elements of vector

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vector of correct classes

Logistic Regression Classifier Loss Function and Gradient – continued

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Following intermediate variables and computation graph can be used to derive the ith samples logistic regression loss function's gradient $\nabla_{\mathbf{w}}(L_i)$:

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$$L_i = -\log\left(\left(a^{(i)}\right)^{y^{(i)}} \times \left(1 - a^{(i)}\right)^{1-y^{(i)}}\right),$$

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$$
\begin{aligned}
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$$\begin{tikzcd} \boxed{L_i} \arrow[d] \\ \boxed{a^{(i)}} \arrow[d] \\ \boxed{\mathbf{z}^{(i)}} \arrow[d] \\ \boxed{\mathbf{w}} \end{tikzcd}$$

Why Regularizarization?

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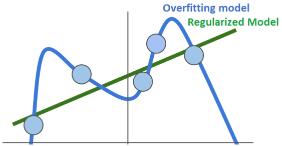
Potentially overfit feature

Oil	Density	Crispy	Fracture	Hardness	Taste
16.5	2955	10	23	97	fair
17.7	2660	14	9	139	excellent
16.2	2870	12	17	143	poor
16.7	2920	10	31	95	good
16.3	2975	11	26	143	fair
19.1	2790	13	16	189	good
18.4	2750	13	17	114	poor



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where $\lambda > 0$ is the strength of regularization.

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- λ is a hyperparameter that has to be tuned.

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