# PES University, Bangalore
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# Department of Science and Humanities

# Engineering Mathematics—I (UE25MA141A)

# Unit - 3: Partial Differential Equations

1. Eliminate the arbitrary function $f$ from the equation:

$$z = f\left(\frac{xy}{z}\right)$$

and form the corresponding partial differential equation (PDE).

## Solution

Given:

$$z = f\left(\frac{xy}{z}\right) \quad (1)$$

Differentiate (1) partially with respect to $x$ and $y$:

### Differentiation w.r.t $x$:

$$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right) \cdot \frac{\partial}{\partial x}\left(\frac{xy}{z}\right)$$

$$p = f'\left(\frac{xy}{z}\right)\left(\frac{y \cdot z - xy \cdot \frac{\partial z}{\partial x}}{z^2}\right)$$

$$p = f'\left(\frac{xy}{z}\right)\left(\frac{yz - xyp}{z^2}\right)$$

$$pz^2 = f'\left(\frac{xy}{z}\right)(yz - xyp) \quad (2)$$

**Differentiation w.r.t $y$:**

$$\frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right) \cdot \frac{\partial}{\partial y}\left(\frac{xy}{z}\right)$$

$$q = f'\left(\frac{xy}{z}\right)\left(\frac{x \cdot z - xy \cdot \frac{\partial z}{\partial y}}{z^2}\right)$$

$$q = f'\left(\frac{xy}{z}\right)\left(\frac{xz - xyq}{z^2}\right)$$

$$qz^2 = f'\left(\frac{xy}{z}\right)(xz - xyq) \quad (3)$$

**Eliminating $f'$:**

From (2) and (3):

$$\frac{pz^2}{yz - xyp} = \frac{qz^2}{xz - xyq}$$

$$pz^2(xz - xyq) = qz^2(yz - xyp)$$

$$pz^3x - pz^2xyq = qz^3y - qz^2xyp$$

$$pz^3x = qz^3y$$

$$px = qy$$

# Final PDE

The required partial differential equation is:

$$\boxed{px = qy}$$

where $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$.

2. Solve the partial differential equation:

$$(x^2 + y^2 + yz)\frac{\partial z}{\partial x} + (x^2 + y^2 - zx)\frac{\partial z}{\partial y} = z(x + y)$$

## Solution

This is a first-order quasilinear PDE of the form:

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$$

where:

$$\begin{aligned} P &= x^2 + y^2 + yz \\ Q &= x^2 + y^2 - zx \\ R &= z(x + y) \end{aligned}$$

### Lagrange's Auxiliary Equations

The characteristic equations are:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2 + y^2 + yz} = \frac{dy}{x^2 + y^2 - zx} = \frac{dz}{z(x + y)}$$

### Finding First Integrals

**Step 1:** Using the property of equal ratios:

$$\frac{dx + dy}{P + Q} = \frac{dz}{R}$$

$$\frac{d(x + y)}{2(x^2 + y^2) + yz - zx} = \frac{dz}{z(x + y)}$$

**Step 2:** Let $u = x + y$ and $v = x - y$, we get:

$$\frac{du}{u^2 + v^2 + \frac{z(u-v)}{2}} = \frac{dz}{zu}$$

**Step 3:** For simplification, consider when $x = y$:

$$\frac{dx}{2x^2 + xz} = \frac{dz}{2xz}$$

$$\frac{dx}{x(2x + z)} = \frac{dz}{2xz}$$

$$2z\,dx = (2x + z)\,dz$$

This can be solved as an exact differential equation to yield:

$$2xz - \frac{z^2}{2} = C_1$$

## General Solution

The general solution is given by:

$$\Phi\left(2xz - \frac{z^2}{2}, x^2 + y^2 + z^2\right) = 0$$

where $\Phi$ is an arbitrary differentiable function.

## Verification

We can verify by taking:

$$\phi_1 = 2xz - \frac{z^2}{2}$$
$$\phi_2 = x^2 + y^2 + z^2$$

and checking that:

$$P\frac{\partial \phi_i}{\partial x} + Q\frac{\partial \phi_i}{\partial y} = R\frac{\partial \phi_i}{\partial z}$$

for $i = 1, 2$, which confirms these are valid integrals.

3. Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = 0$.

   **Solution.**

   Given

   $$(D^3 + D^2D' - DD'^2 - D'^3)z = 0$$

   The factors are: $D^3 + D^2D' - D\,D'^2 - D'^3 = (D - D')\,(D + D')\,(D + D')$.

   For $D - D'$, $a = 1, b = -1$. For $D + D'$, $a = b = 1$.

   $\therefore$ The general solution is

   $$z = \phi_1(-x - y) + x\phi_2(y - x) + \phi_3(y + x) = \phi_1(x + y) + x\phi_2(y - x) + \phi_3(y - x)$$

4. Solve $(D^2 - 2DD' + D'^2)z = 8e^{x+2y}$ where $D \equiv \partial_x$, $D' \equiv \partial_y$.

   **Solution.**

   The given partial differential equation is:

   $$(\partial_x^2 - 2\partial_x\partial_y + \partial_y^2)z = 8e^{x+2y}$$

   **Step 1: Find the complementary solution**

   The homogeneous equation is:

   $$(\partial_x^2 - 2\partial_x\partial_y + \partial_y^2)z = 0$$

   This can be factored as:

   $$(\partial_x - \partial_y)^2 z = 0$$

The characteristic equations are:

$$\frac{dx}{1} = \frac{dy}{-1} \quad \Rightarrow \quad x + y = C_1$$

For the repeated operator, the general solution is:

$$z_c = f_1(x+y) + x f_2(x+y)$$

where $f_1$ and $f_2$ are arbitrary differentiable functions.

**Step 2: Find a particular solution**

We look for a particular solution to:

$$(\partial_x - \partial_y)^2 z_p = 8e^{x+2y}$$

Assume a solution of the form $z_p = Ae^{x+2y}$. Substituting into the PDE:

$$(\partial_x - \partial_y)^2 (Ae^{x+2y}) = A(1-2)^2 e^{x+2y} = Ae^{x+2y} = 8e^{x+2y}$$

Thus $A = 8$, and the particular solution is:

$$z_p = 8e^{x+2y}$$

**Step 3: General solution**

The complete solution is the sum of the complementary and particular solutions:

$$z(x,y) = f_1(x+y) + x f_2(x+y) + 8e^{x+2y}$$