ENGINEERING MATHEMATICS - I

Unit - 3: Partial Differential Equations

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Contents

1 Problems on special case

Find the solution of the differential equation
$[2D^2 + 5DD' + 3(D')^2]z = ye^x$.
Solution We write

$$[2D^2 + 5DD' + 3(D')^2]z = (2D + 3D')(D + D')z = ye^x.$$

The complementary function as

$$z = \phi_1(3x - 2y) + \phi_2(x - y).$$

The particular integral is given by

$$z = (2D + 3D')^{-1}(D + D')^{-1}(ye^x).$$

We first obtain $(D + D')^{-1}(ye^x)$ as in case 4. For the sake of completeness, we repeat the procedure used in this case. Denote

$$u = (D + D')^{-1}(ye^x) \quad \text{or} \quad (D + D')u = ye^x.$$

Problem (contd.)

The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{ye^x}.$$

The first two terms give $y = x + c$. Using the first and third terms, we get

$$\frac{dx}{1} = \frac{du}{(x+c)e^x}$$

and

$$u = \int (x+c)e^x dx = (x+c-1)e^x = (y-1)e^x.$$

Now, denote

$$z = (2D + 3D')^{-1}u = (2D + 3D')^{-1}(y-1)e^x$$

or

$$(2D + 3D')z = (y-1)e^x.$$

Problem (contd.)

The auxiliary equations are

$$\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{(y-1)e^x}.$$

The first two terms give $2y = 3x + c_1$. The first and third terms give

$$\frac{dx}{2} = \frac{dz}{[(3x+c_1)/2 - 1]e^x}$$

and

$$z = \frac{1}{4}\int (3x + c_1 - 2)e^x dx = \frac{1}{4}(3x + c_1 - 5)e^x = \frac{1}{4}(2y-5)e^x$$

which is the required particular integral. The general solution of the differential equation is

$$z = \phi_1(3x - 2y) + \phi_2(x - y) + \frac{1}{4}(2y-5)e^x.$$

Problem

Find the solution of the differential equation

$$\left[D^2 + D\,D' - 2(D')^2\right] z \;=\; 8\,\ln\!\left(x + 5y\right)$$

Solution:

We write $\left[D^2 + D\,D' - 2(D')^2\right] z = (D + 2D')(D - D')\,z = 8\,\ln\!\left(x + 5y\right)$.

The complementary function as

$$z = \phi_1(2x - y) \;+\; \phi_2(x + y)$$

Problem (contd.)

The particular integral is given by

$$z = (D + 2D')^{-1}\,(D - D')^{-1}\Big(8\,\ln(x + 5y)\Big)$$

Denote

$$u \;=\; (D - D')^{-1}\big(8\,\ln(x + 5y)\big), \quad \text{or} \quad (D - D')\,u = 8\,\ln(x + 5y).$$

The auxiliary equations are

$$\frac{dx}{1} \;=\; \frac{dy}{-1} \;=\; \frac{du}{8\,\ln\big(x + 5y\big)}$$

The first and second terms give $x + y = c$.

The first and third terms give $\dfrac{dx}{1} = \dfrac{du}{8\,\ln\big(5c - 4x\big)}$

Thus,

$$u = 8\int \ln\big(5c - 4x\big)\,dx = 8\left[x\,\ln\big(5c - 4x\big) + \int \frac{4x}{5c - 4x}\,dx\right]$$

$$\Rightarrow u \;=\; 8\int \ln\big(5c-4x\big)\,dx \;=\; 8\left[x\,\ln\big(5c-4x\big)\;-\;\int \frac{4x}{4x-5c}\,dx\right]$$

$$\Rightarrow u \;=\; 8\int \ln\big(5c-4x\big)\,dx \;=\; 8\left[x\,\ln\big(5c-4x\big)\;-\;\int \left[1+\frac{5c}{4x-5c}\,dx\right]\right]$$

$$\Rightarrow u = 8\left[x\,\ln\big(5c-4x\big)\;-\;x\;-\;\int \frac{5c}{4x-5c}\,dx\right]$$

$$\Rightarrow u = 8\left[x\,\ln\big(5c-4x\big)\;-\;x\;+\;\int \frac{5c}{5c-4x}\,dx\right]$$

Problem (contd.)

$$\Rightarrow u = 8\left[x\,\ln\bigl(5c - 4x\bigr)\ -\ x\ -\frac{5c}{4}\ \ln(5c - 4x)\right]$$

$$\Rightarrow u = 8\Bigl[\bigl(x - \tfrac{5c}{4}\bigr)\ln\bigl(5c - 4x\bigr)\ -\ x\Bigr]$$

$$\Rightarrow u = -2(x + 5y)\ \ln(x + 5y)\ -\ 8x$$

Problem (contd.)

$$\text{Now, denote } z = (D + 2D')^{-1} u$$

$$\text{or} \quad (D + 2D')\, z = u = -\big[\, 8x + 2(x + 5y)\, \ln(x + 5y) \big]$$

$$\text{The auxiliary equations are} \quad \frac{dx}{1} = \frac{dy}{2} = \frac{dz}{-\big[\, 8x + 2(x + 5y) \ln(x + 5y) \big]}$$

The first two terms give $2x - y = c_1$

Problem (contd.)

The I and III terms give $\dfrac{dx}{1} = \dfrac{dz}{-\left[\,8x + 2(11x - 5c_1)\,\ln(11x - 5c_1)\right]}$

$$z = -\int \left[\,8x + 2(11x - 5c_1)\,\ln(11x - 5c_1)\right] dx$$

Integrate w.r.t. x by taking $11x - 5c_1 = t$. Then

$$z = -\left[\,4x^2 + \frac{2}{11}\left(\tfrac{1}{2}\,(11x - 5c_1)^2\,\ln(11x - 5c_1) - \tfrac{1}{4}\,(11x - 5c_1)^2\right)\right]$$

$$\Rightarrow z = -\left[\,4x^2 + \frac{1}{11}\,(11x - 5c_1)^2\,\ln(11x - 5c_1) - \tfrac{1}{22}\,(11x - 5c_1)^2\right]$$

$$\Rightarrow z = -\left[4x^2 + \frac{1}{22}(11x - 5c_1)^2 \{2 \ln(11x - 5c_1) - 1\} \right]$$

Since $2x - y = c_1$, $5c_1 = 10x - 5y$. Then $11x - 5c_1$ becomes $x + 5y$. Hence,

$$z = -\left[4x^2 + \frac{1}{22}(x + 5y)^2 \{2 \ln(x + 5y) - 1\} \right]$$

This is the required particular integral.

The general solution of the differential equation is

$$z = \phi_1(2x - y) + \phi_2(x + y) - \left[4x^2 + \frac{1}{22}(x + 5y)^2 \left(2 \ln(x + 5y) - 1\right) \right]$$