PES University, Bangalore

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Department of Science and Humanities

Engineering Mathematics - I (UE25MA141A)

Unit - 2: Higher Order Differential Equations

1. Solve the differential equation:

$$y''' - 6y'' + 11y' - 6y = 0$$

with initial conditions:

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0.$$

Solution:

(a) Find the characteristic equation:

$$r^3 - 6r^2 + 11r - 6 = 0$$

(b) Factor the equation:

$$(r-1)(r-2)(r-3) = 0 \implies r = 1, 2, 3$$

(c) General solution:

$$y(x) = C_1 e^x + C_2 e^{2x} + C_3 e^{3x}$$

(d) Apply initial conditions:

$$\begin{aligned} y(0) &= C_1 + C_2 + C_3 = 1 \\ y'(0) &= C_1 + 2C_2 + 3C_3 = 0 \\ y''(0) &= C_1 + 4C_2 + 9C_3 = 0 \end{aligned}$$

(e) Solve the system:

$$C_1 = \frac{5}{2}, \quad C_2 = -2, \quad C_3 = \frac{1}{2}$$

(f) Final solution:

$$y(x) = \frac{5}{2}e^x - 2e^{2x} + \frac{1}{2}e^{3x}$$

2. Solve the differential equation:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$$

Solution:

The given equation is:

$$(D^2 + 4D + 5)y = -2\cosh x = -\left(e^x + e^{-x}\right)$$

1. Complementary Function (C.F)

To find the complementary function, solve the homogeneous equation:

$$(D^2 + 4D + 5)y = 0$$

The auxiliary equation is:

$$m^2 + 4m + 5 = 0$$

Solving for m:

$$m = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$$

The roots are complex numbers with $\alpha = -2$ and $\beta = 1$.

Thus, the complementary function is:

$$\text{C.F} = e^{-2x}\left[C_1 \cos x + C_2 \sin x\right]$$

2. Particular Integral (P.I)

We need to find the particular integral for the non-homogeneous term $-2\cosh x = -e^x - e^{-x}$.

The particular integral can be written as:

$$\text{P.I} = \frac{1}{D^2 + 4D + 5}\left(-e^x - e^{-x}\right) = \text{P.I}_1 + \text{P.I}_2$$

where:

$$\text{P.I}_1 = \frac{1}{D^2 + 4D + 5}(-e^x), \quad \text{P.I}_2 = \frac{1}{D^2 + 4D + 5}(-e^{-x})$$

Calculating P.I_1

Replace D with 1:

$$\text{P.I}_1 = \frac{-1}{1^2 + 4(1) + 5}e^x = \frac{-1}{1 + 4 + 5}e^x = -\frac{e^x}{10}$$

Calculating P.I$_2$

Replace D with -1:

$$\text{P.I}_2 = \frac{-1}{(-1)^2 + 4(-1) + 5} e^{-x} = \frac{-1}{1 - 4 + 5} e^{-x} = -\frac{e^{-x}}{2}$$

Thus, the total particular integral is:

$$\text{P.I} = \text{P.I}_1 + \text{P.I}_2 = -\frac{e^x}{10} - \frac{e^{-x}}{2}$$

3. General Solution

The general solution is the sum of the complementary function and the particular integral:

$$y = \text{C.F} + \text{P.I} = e^{-2x} \left[C_1 \cos x + C_2 \sin x\right] - \frac{e^x}{10} - \frac{e^{-x}}{2}$$

3. Solve the differential equation:

$$(D^2 - 2D + 2)y = e^x x^2 + 5 + e^{-2x}$$

Solution:

1. Complementary Function (C.F)

The homogeneous equation is:

$$(D^2 - 2D + 2)y = 0$$

The auxiliary equation:

$$m^2 - 2m + 2 = 0 \implies m = 1 \pm i$$

Thus, the complementary solution is:

$$\text{C.F} = e^x (C_1 \cos x + C_2 \sin x)$$

2. Particular Integral (P.I)

We find the P.I for each non-homogeneous term.

(a) For $e^x x^2$ (Resonance Case)

Using the exponential shift theorem:

$$\begin{aligned}
\text{P.I}_1 &= \frac{1}{D^2 - 2D + 2} e^x x^2 \\
&= e^x \frac{1}{(D+1)^2 - 2(D+1) + 2} x^2 \\
&= e^x \frac{1}{D^2 + 1} x^2 \\
&= e^x (1 - D^2) x^2 \quad (\text{since } D^3 x^2 = 0) \\
&= e^x (x^2 - 2)
\end{aligned}$$

(b) For the constant term 5

$$\text{P.I}_2 = \frac{1}{D^2 - 2D + 2} \cdot 5 = \frac{5}{2}$$

(c) For e^{-2x}

$$\text{P.I}_3 = \frac{1}{D^2 - 2D + 2} e^{-2x} = \frac{e^{-2x}}{10}$$

3. General Solution

Combining all components:

$$y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2 + \text{P.I}_3$$

$$\boxed{y = e^x (C_1 \cos x + C_2 \sin x) + e^x (x^2 - 2) + \frac{5}{2} + \frac{e^{-2x}}{10}}$$

4. Solve

$$(x+2)^2 \frac{d^2 y}{dx^2} - (x+2) \frac{dy}{dx} + y = 3x + 4.$$

Solution.

The given equation is

$$\left[(x+2)^2 D^2 - (x+2) D + 1\right] y = 3x + 4 \quad (1)$$

which is Legendre's equation.

Let $x + 2 = e^z$ (i.e., $z = \log(x+2)$) and $u = \frac{d}{dz}$. Then:

$$(x+2) D = u, \quad (x+2)^2 D^2 = u(u-1).$$

Substituting into (1):

$$[u(u-1) - u + 1]\, y = 3e^z - 2.$$

Simplifying:

$$(u^2 - 2u + 1)y = 3e^z - 2.$$

Complementary Function (C.F.): The auxiliary equation is:

$$m^2 - 2m + 1 = 0 \implies m = 1 \text{ (double root)}.$$

Thus:

$$y_{\text{CF}} = (C_1 + C_2 z)e^z.$$

Particular Integral (P.I.): For the nonhomogeneous term $3e^z - 2$:

$$y_{\text{PI}} = \frac{1}{(u-1)^2}(3e^z) - \frac{2}{(u-1)^2}(1).$$

- For $3e^z$:

$$\frac{1}{(u-1)^2}e^z = \frac{z^2}{2}e^z \implies \frac{3z^2}{2}e^z.$$

- For -2:

$$\frac{1}{(u-1)^2}(1) = 1 \implies -2.$$

Thus:

$$y_{\text{PI}} = \frac{3z^2}{2}e^z - 2.$$

General Solution:

$$y = y_{\text{CF}} + y_{\text{PI}} = (C_1 + C_2 z)e^z + \frac{3z^2}{2}e^z - 2.$$

Substituting back $z = \log(x+2)$:

$$y = C_1(x+2) + C_2(x+2)\log(x+2) + \frac{3}{2}(x+2)\log^2(x+2) - 2.$$

5. An uncharged condenser of capacity C is charged by applying an e.m.f $E\sin\left(\frac{t}{\sqrt{LC}}\right)$ through leads of self inductance L and negligible resistance. Prove that, at time t, the charge q on one of the plates is:

$$q = \frac{1}{2}\left[\sin\left(\frac{t}{\sqrt{LC}}\right) - \left(\frac{t}{\sqrt{LC}}\right)\cos\left(\frac{t}{\sqrt{LC}}\right)\right]$$

Note: The differential equation governing the circuit is:

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = \frac{E}{L}\sin\left(\frac{t}{\sqrt{LC}}\right).$$

Take $\omega^2 = \frac{1}{LC}$.

Solution to the Differential Equation

The governing equation is:

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = \frac{E}{L}\sin\left(\frac{t}{\sqrt{LC}}\right)$$

Step 1: Simplification

Let $\omega = \frac{1}{\sqrt{LC}}$, then the equation becomes:

$$\frac{d^2q}{dt^2} + \omega^2 q = \frac{E}{L}\sin(\omega t)$$

Step 2: Complementary Function

Solve the homogeneous equation:

$$(D^2 + \omega^2)q = 0 \quad \text{where} \quad D = \frac{d}{dt}$$

The auxiliary equation is:

$$m^2 + \omega^2 = 0 \Rightarrow m = \pm i\omega$$

Thus, the complementary solution is:

$$q_{CF} = A\cos(\omega t) + B\sin(\omega t)$$

Step 3: Particular Integral

For the non-homogeneous term:

$$q_{PI} = \frac{1}{D^2 + \omega^2}\left(\frac{E}{L}\sin(\omega t)\right)$$

Using the method of undetermined coefficients:

$$q_{PI} = \frac{E}{L} \cdot \frac{1}{D^2 + \omega^2}\sin(\omega t) = -\frac{Et}{2\omega L}\cos(\omega t)$$

Step 4: General Solution

Combine CF and PI:

$$q = A\cos(\omega t) + B\sin(\omega t) - \frac{Et}{2\omega L}\cos(\omega t)$$

Step 5: Apply Initial Conditions

At $t = 0$, $q = 0$:

$$0 = A \Rightarrow A = 0$$

Differentiating and applying $\frac{dq}{dt}\big|_{t=0} = 0$:

$$\frac{dq}{dt} = B\omega\cos(\omega t) - \frac{E}{2\omega L}\cos(\omega t) + \frac{Et\omega}{2\omega L}\sin(\omega t)$$

At $t = 0$:

$$0 = B\omega - \frac{E}{2\omega L} \Rightarrow B = \frac{E}{2\omega^2 L}$$

Step 6: Final Solution

Substituting back:

$$q = \frac{E}{2\omega^2 L}\sin(\omega t) - \frac{Et}{2\omega L}\cos(\omega t)$$

Substitute $\omega = \frac{1}{\sqrt{LC}}$:

$$q = \frac{EC}{2}\sin\left(\frac{t}{\sqrt{LC}}\right) - \frac{Et\sqrt{LC}}{2L}\cos\left(\frac{t}{\sqrt{LC}}\right)$$

Simplify:

$$q = \frac{1}{2}\left[\sin\left(\frac{t}{\sqrt{LC}}\right) - \left(\frac{t}{\sqrt{LC}}\right)\cos\left(\frac{t}{\sqrt{LC}}\right)\right]$$

Final Answer

$$\boxed{q = \frac{1}{2}\left[\sin\left(\frac{t}{\sqrt{LC}}\right) - \frac{t}{\sqrt{LC}}\cos\left(\frac{t}{\sqrt{LC}}\right)\right]}$$