Unit I Superposition of states – a soft introduction to QUBITs

The wave function stores information about the physical observables associated with state of the system. The eigen values of the observables can be extracted using the corresponding operators.

A system can exist in any one of the available states $\boldsymbol{\psi_1}, \boldsymbol{\psi_2}, \boldsymbol{\psi_3}, \ldots . \boldsymbol{\psi_n}$, **and** the same will be revealed in a measurement of the state. Until such a measurement is done, any quantum mechanical system can be in a superposition of all states with certain probabilities, such that the total probability of finding the system is 1. When the measurement is performed the systems gives us the classical result of being in any one of the states which have a maximum probability.

In the case of classical computing which uses the bits – either a 0 or a 1 – as the basic elements for representing information as a combination of the two bits. The measurement in this case reveals that the CMOS transistor connected with the bit is in the ON state or the OFF state. This leads to the proposition that to represent N bits of information one would require N^2 computational operation which grows exponentially when one uses large numbers in computation

Quantum computation relies on three fundamental principles namely, superposition, entanglement and decoherence. The quantum bits (qubits) on the other hand can use the principle of superposition and create qubits which can generally be in the state of superposition between two states represented by the state vectors called as the ket vectors $|\mathbf{0}\rangle$ and $|\mathbf{1}\rangle$ using the Dirac notation for the states of system. These ket vectors are column vectors given by

$|\mathbf{0}\rangle = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}$ **and** $|\mathbf{1}\rangle = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$.

The transpose of these vectors are row vectors referred to as bra vectors and written as $\langle \mathbf{0}| \ \mathbf{and} \ \langle \mathbf{1}|$.

$\langle \mathbf{0}| = [\mathbf{1} \quad \mathbf{0}]$ and $\langle \mathbf{1}| = [\mathbf{0} \quad \mathbf{1}]$.

Inner product of the state vectors

The inner product of the two vectors defined as the product of the transpose of one of the vectors with the other vector.

Obviously, the inner products $\langle \mathbf{0}|\mathbf{0}\rangle$ **or** $\langle \mathbf{1}|\mathbf{1}\rangle$ will result in a 1 indicating that they can be in the state $|\mathbf{0}\rangle$ or $|\mathbf{1}\rangle$ respectively with a probability of 1

The inner product of the two vectors defined by $\langle \mathbf{0}|\mathbf{1}\rangle$ **or** $\langle \mathbf{1}|\mathbf{0}\rangle$ can be shown by matrix multiplication that two vectors are orthogonal, meaning that the system cannot be observed to be in the two states simultaneously. This concept is identical to concept of the collapse of the wave function.

The superposition state can be written as $|\boldsymbol{\psi}\rangle = \boldsymbol{\alpha}|\mathbf{0}\rangle + \boldsymbol{\beta}|\mathbf{1}\rangle$ which is equivalent to stating that the state $|\mathbf{0}\rangle$ has a complex probability amplitude of $\boldsymbol{\alpha}$ and the state $|\mathbf{1}\rangle$ has a complex probability amplitude of $\boldsymbol{\beta}$.

If the state vector is a normalised vector, the inner product $\langle\boldsymbol{\psi}|\boldsymbol{\psi}\rangle$ results in $\boldsymbol{\alpha}^2 + \boldsymbol{\beta}^2 = \mathbf{1}.$

This concept is the same as the normalisation concept for wave functions in the quantum wave mechanics.

The condition $\boldsymbol{\alpha}^2 + \boldsymbol{\beta}^2 = \mathbf{1}$ is stating that the total probability of finding the system in the $|\mathbf{0}\rangle$ or $|\mathbf{1}\rangle$ is 1 which is termed as normalisation.

When we take the inner product of $|\mathbf{0}\rangle$ and $|\mathbf{1}\rangle$ we find that $\langle\mathbf{1}|\mathbf{0}\rangle = \mathbf{0} = \langle\mathbf{0}|\mathbf{1}\rangle$**.** This implies that the basis state $|\mathbf{0}\rangle$ **has no component in** $|\mathbf{1}\rangle$ and vice versa. These vectors are then said to be orthonormal.

The superposition of states using multiple qubits can then hold information exponentially higher than classical information storage because the superposition states are infinitely large. Using 300 qubits one can create a superposition of $\mathbf{2^{300}}$ states which equates to $\approx \mathbf{10^{90}}$ possible superpositions of the individual states. This implies that $\mathbf{10^{90}}$ numbers of complex numbers are required as coefficients to describe the superposition correctly. To give an idea of the magnitude of this number is understood by realising that there are only $\mathbf{10^{28}}$ atoms in the known universe!

<u>Outer product of state vectors</u>

The outer product of the basis vectors can be written as

$$|\mathbf{0}\rangle\langle\mathbf{0}| = \begin{bmatrix}\mathbf{1}\\\mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{1} & \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathbf{1} & \mathbf{0}\\\mathbf{0} & \mathbf{0}\end{bmatrix}$$

$$|\mathbf{1}\rangle\langle\mathbf{1}| = \begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\begin{bmatrix}\mathbf{0} & \mathbf{1}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\\mathbf{0} & \mathbf{1}\end{bmatrix}$$

These are the tensor product of the single qubit states which result in a matrix (operator) of order $\mathbf{2} \times \mathbf{2}$. These operators are also referred to as the projection operators.

Consider the vector $|\boldsymbol{\psi}\rangle = \boldsymbol{\alpha}|\mathbf{0}\rangle + \boldsymbol{\beta}|\mathbf{1}\rangle = \begin{bmatrix}\boldsymbol{\alpha}\\\boldsymbol{\beta}\end{bmatrix}$ which is a superposition of the states $|\mathbf{0}\rangle\ and|\mathbf{1}\rangle$ with probability amplitudes $\boldsymbol{\alpha}\ and\ \boldsymbol{\beta}$**.**

The operation $|\mathbf{0}\rangle\langle\mathbf{0}|\boldsymbol{\psi}\rangle$ gives the projection of $|\boldsymbol{\psi}\rangle$ on to the $|\mathbf{0}\rangle$ basis

$$|\mathbf{0}\rangle\langle\mathbf{0}|\boldsymbol{\psi}\rangle = |\mathbf{0}\rangle\langle\mathbf{0}|(\boldsymbol{\alpha}|\mathbf{0}\rangle + \boldsymbol{\beta}|\mathbf{1}\rangle) = |\mathbf{0}\rangle\langle\mathbf{0}|\boldsymbol{\alpha}(|\mathbf{0}\rangle) + |\mathbf{0}\rangle\langle\mathbf{0}|(\boldsymbol{\beta}|\mathbf{1}\rangle) = \boldsymbol{\alpha}|\mathbf{0}\rangle$$

And the operation

$$|\mathbf{1}\rangle\langle\mathbf{1}|\boldsymbol{\psi}\rangle = |\mathbf{1}\rangle\langle\mathbf{1}|(\boldsymbol{\alpha}|\mathbf{0}\rangle + \boldsymbol{\beta}|\mathbf{1}\rangle) = |\mathbf{1}\rangle\langle\mathbf{1}|\boldsymbol{\alpha}(|\mathbf{0}\rangle) + |\mathbf{1}\rangle\langle\mathbf{1}|(\boldsymbol{\beta}|\mathbf{1}\rangle) = \boldsymbol{\beta}|\mathbf{1}\rangle$$

These operators are also referred to as the measurement operators. In this example quoted above the operator $|\mathbf{0}\rangle\langle\mathbf{0}|$ operates on $|\boldsymbol{\psi}\rangle$ and gives out the component of $|\boldsymbol{\psi}\rangle$ along the $|\mathbf{0}\rangle$ basis and the operator $|\mathbf{1}\rangle\langle\mathbf{1}|$ operates on $|\boldsymbol{\psi}\rangle$ and gives out the component of $|\boldsymbol{\psi}\rangle$ along the $|\mathbf{1}\rangle$ basis.

For any square matrix $\boldsymbol{\pi} = |\boldsymbol{v}\rangle\langle\boldsymbol{v}|$ where $|\boldsymbol{v}\rangle$ is an orthonormal vector to be a projection operator if it satisfies the following two conditions:

1. The matrix is equal to its conjugate transpose $\boldsymbol{\pi} = (\boldsymbol{\pi})^{\dagger}$
2. The matrix squared is equal to itself $\boldsymbol{\pi}^{2} = \boldsymbol{\pi}$

Quantum Technologies

Quantum information processing requires a reliable quantum hardware generally referred to as circuit Quantum Electrodynamics Devices (QED).

Such an approach has been implemented for a wide range of quantum technologies -

- Analog quantum computers
- Digital or gate based computers
- Single photon sources
- Quantum memories
- Components of quantum communication systems.

These devices rely on the quantum dynamics of the electromagnetic fields in superconducting circuits to generate and process quantum information.

Circuit QED devices are designed by embedding a Josephson junction into complex systems of planar microwave circuitry. The JJ can behave as a nonlinear inductor in a simple LC circuit and the resulting nonlinear spectra of the circuit can be optimized to form the basis states of individually addressable quantum states. Furthermore, the associated microwave circuitry can be made with the same superconductors to ensure non dissipative circuit elements. This allows the quantum states to survive for long enough time scales for interaction of the EM radiations with matter which is necessary for creation and processing of quantum information systems.

Qubits are sensitive to measurements. When a system of n qubits is in the superposition state, it is essential to measure the state of the system to ascertain the probabilities of the system being found in the different basis states. This can only give an information about any changes to the information bought about by transformations. However, as in the wave mechanics, we find that on measurement the system collapses to one of the possible states with a probability 1 and all other states the probability goes to zero. But this is not the information one is interested in and need to get the output of the superposed states with meaningful probability information of the individual states without the state collapsing into one of the possible states. This is one of the challenges of quantum computing technologies.

- **Superconducting Qubits:**

  A widely attempted Qubit system is the Transmon Qubit: A charge-insensitive qubit design based on a Josephson junction shunted by a large capacitor. The circuit element is equivalent to an anharmonic oscillator circuit and is implementable in macro scales – on appropriate substrates using conventional IC manufacturing techniques. Other superconducting designs that are being researched are flux qubits, phase qubits, Xmon)

- **Trapped Ion Qubits:** Individual ions (charged atoms) are confined using electromagnetic fields, and their electronic states are manipulated with lasers. Known for long coherence times and high gate fidelities.
- **Photonic Qubits:** Utilize individual photons (particles of light) as qubits, encoding information in their polarization or other degrees of freedom. Promising for quantum communication due to their ability to travel long distances.
- **Quantum Dot Qubits (Spin Qubits):** Confine single electrons in tiny semiconductor structures (quantum dots). The spin of the electron (up or down) serves as the qubit state. Potentially compatible with existing semiconductor manufacturing.
- **Neutral Atom Qubits:** Use lasers to trap and manipulate neutral atoms. Their electronic states form the qubit. Offers potential for large, dense arrays of qubits.
- **Diamond NV (Nitrogen-Vacancy) Center Qubits:** Utilize defects in a diamond's crystal lattice where a nitrogen atom and an adjacent vacancy create a quantum system whose spin can be used as a qubit. Can operate at room temperature for some applications.
- **Topological Qubits:** Based on exotic quasiparticles (like Majorana fermions) whose quantum states are inherently protected by their topological properties, potentially leading to more robust and error-resistant qubits. Still largely in the experimental and theoretical stages.