ENGINEERING MATHEMATICS I

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Class content

Problems on recurrence relations

Recurrence relations

1. $$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$$

2. $$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$$

3. $$\frac{d}{dx} J_n(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$$

4. $$\frac{d}{dx} J_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

5. $$\frac{d}{dx} J_n(x) = \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right)$$

6. $$2n J_n(x) = x\left(J_{n-1}(x) + J_{n+1}(x)\right)$$

Problems on recurrence relation

1. Using the values of $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$ find $J_{\frac{3}{2}}(x)$ and $J_{-\frac{3}{2}}(x)$

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Problems on recurrence relations

Solution: We have, $2nJ_n(x) = x\left(J_{n-1}(x) + J_{n+1}(x)\right)$

put $n = \frac{1}{2}$ we get,

$$J_{\frac{1}{2}}(x) = x\left(J_{-\frac{1}{2}}(x) + J_{\frac{3}{2}}(x)\right)$$

$$\Rightarrow J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

Thus, $\boxed{J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)}$

Problems on recurrence relations

Put $n = -\frac{1}{2}$ in $2nJ_n(x) = x\left(J_{n-1}(x) + J_{n+1}(x)\right)$

Then we get,

$$-J_{-\frac{1}{2}}(x) = x\left(J_{-\frac{3}{2}}(x) + J_{\frac{1}{2}}(x)\right)$$

$$\Rightarrow J_{-\frac{3}{2}}(x) = -\frac{1}{x}J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$= -\frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= -\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} + \cos x\right)$$

Thus, $\boxed{J_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} + \cos x\right)}$

Problems on recurrence relations

2. Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$.

Problems on recurrence relations

Solution: We have, $2nJ_n(x) = x\left(J_{n-1}(x) + J_{n+1}(x)\right)$

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x) \quad ----\rightarrow (1)$$

Put n=1 in (1), we get, $J_2(x) = \frac{2}{x}J_1(x) - J_0(x) \quad ----\rightarrow (2)$

Put n=2 in (1), we get, $J_3(x) = \frac{4}{x}J_2(x) - J_1(x) \quad ----\rightarrow (3)$

Put n=3 in (1), we get, $J_4(x) = \frac{6}{x}J_3(x) - J_2(x) \quad ----\rightarrow (4)$

Put n=4 in (1), we get, $J_5(x) = \frac{8}{x}J_4(x) - J_3(x) \quad ----\rightarrow (5)$

Problems on recurrence relations

Using equation (4) in (5), we get,

$$J_5(x) = \frac{8}{x}\left(\frac{6}{x}J_3(x) - J_2(x)\right) - J_3(x)$$

$$= \left(\frac{48}{x^2} - 1\right)J_3(x) - \frac{8}{x}J_2(x)$$

Using equation (3) , we get,

$$J_5(x) = \left(\frac{48}{x^2} - 1\right)\left(\frac{4}{x}J_2(x) - J_1(x)\right) - \frac{8}{x}J_2(x)$$

$$= \left(\frac{192}{x^3} - \frac{12}{x}\right)J_2(x) - \left(\frac{48}{x^2} - 1\right)J_1(x)$$

Problems on recurrence relations

Using equation (2) , we get,

$$J_5(x)=\left(\frac{192}{x^3}-\frac{12}{x}\right)\left(\frac{2}{x}J_1(x)-J_0(x)\right)-\left(\frac{48}{x^2}-1\right)J_1(x)$$

$$=\left(\frac{192}{x^3}-\frac{12}{x}\right)\frac{2}{x}J_1(x)-\left(\frac{192}{x^3}-\frac{12}{x}\right)J_0(x)-\left(\frac{48}{x^2}-1\right)J_1(x)$$

$$=\left(\frac{384}{x^4}-\frac{24}{x^2}\right)J_1(x)-\left(\frac{48}{x^2}-1\right)J_1(x)-\left(\frac{192}{x^3}-\frac{12}{x}\right)J_0(x)$$

$$=\left(\frac{384}{x^4}-\frac{72}{x^2}+1\right)J_1(x)-\left(\frac{192}{x^3}-\frac{12}{x}\right)J_0(x)$$

Problems on recurrence relation

3. Prove that, $J_1''(x) = -J_1(x) + \frac{1}{x} J_2(x)$

Problems on recurrence relation

Solution: We know that, $J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$ -----> (1)

$$\therefore J_1'(x) = J_0(x) - \frac{1}{x} J_1(x) \quad \text{-----> (2)}$$

Differentiating (2) with respect to x,

$$J_1''(x) = J_0'(x) - \frac{1}{x} J_1'(x) - J_1(x)\left(-\frac{1}{x^2}\right) \quad \text{-----> (3)}$$

$$= J_0'(x) - \frac{1}{x} J_1'(x) + \frac{1}{x^2} J_1(x)$$

Put $n = 0$ in (1) then,

$$J_0'(x) = J_{-1}(x) - 0$$

$$J_0'(x) = -J_1(x) \quad \text{-----> (4)} \quad \left(\because J_{-n}(x) = (-1)^n J_n(x)\right)$$

Problems on recurrence relation

Substituting (4) and (2) in (3), we get,

$$J_1''(x) = -J_1(x) - \frac{1}{x}\left[J_0(x) - \frac{1}{x}J_1(x)\right] + \frac{1}{x^2}J_1(x) \text{ ----- > (5)}$$

By recurrence relation (6), we have,

$$2n\,J_n(x) = x\left(J_{n-1}(x) + J_{n+1}(x)\right)$$

$$i.e.,\ J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

Put $n = 1$, then,

$$J_2(x) = \frac{2}{x}J_1(x) - J_0(x) \text{ or } J_0(x) = \frac{2}{x}J_1(x) - J_2(x) \text{ ----- > (6)}$$

Problems on recurrence relation

Substituting (6) in (5), we get,

$$J_1''(x) = -J_1(x) - \frac{1}{x}\left[\frac{2}{x}J_1(x) - J_2(x) \; - \frac{1}{x}J_1(x)\right] + \frac{1}{x^2}J_1(x)$$

$$= -J_1(x) - \frac{2}{x^2}J_1(x) + \frac{1}{x}J_2(x) \; + \frac{1}{x^2}J_1(x) + \frac{1}{x^2}J_1(x)$$

$$= -J_1(x) + \frac{1}{x}J_2(x)$$

Problems on recurrence relation

4. Prove that, $\int J_3(x)dx = c - J_2(x) - \frac{2}{x} J_1(x)$

Problems on recurrence relation

Solution: We know that, $\frac{d}{dx}\left(x^{-n}J_n(x)\right)=-x^{-n}J_{n+1}(x)$

$$\therefore \int x^{-n}J_{n+1}(x)dx=-x^{-n}J_n(x) \quad ----->(1)$$

Now $\int J_3(x)dx=\int x^2x^{-2}J_3(x)dx+c$

$$=x^2\int x^{-2}J_3(x)dx-\int 2x\left[\int x^{-2}J_3(x)dx\right]dx+c$$

$$=x^2\left(-x^{-2}J_2(x)\right)-\int 2x\left[-x^{-2}J_2(x)\right]dx+c \qquad \text{(using (1))}$$

$$=c-J_2(x)+\int \frac{2}{x}J_2(x)dx$$

$$=c-J_2(x)-\frac{2}{x}J_1(x)$$

Problems on recurrence relations

5. Prove that, $\int xJ_0^{2}(x)\,dx=\frac{1}{2}x^2\left(J_0^{2}(x)+J_1^{2}(x)\right)$

Problems on recurrence relations

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Solution:

$$\int xJ_0^{\,2}(x)dx = J_0^{\,2}(x)\frac{1}{2}x^2 - \int 2J_0(x)J_0'(x)\frac{1}{2}x^2dx$$

$$= \frac{1}{2}x^2J_0^{\,2}(x) - \int x^2J_0(x)J_0'(x)dx$$

$$= \frac{1}{2}x^2J_0^{\,2}(x) + \int x^2J_0(x)J_1(x)dx$$

$$= \frac{1}{2}x^2J_0^{\,2}(x) + \int xJ_0(x)xJ_1(x)dx$$

$$= \frac{1}{2}x^2J_0^{\,2}(x) + \int \frac{d}{dx}\left(xJ_1(x)\right)xJ_1(x)dx$$

$$= \frac{1}{2}x^2J_0^{\,2}(x) + \frac{1}{2}\left(xJ_1(x)\right)^2$$

$$= \frac{1}{2}x^2\left(J_0^{\,2}(x) + J_1^{\,2}(x)\right)$$

THANK YOU

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