ENGINEERING MATHEMATICS - I

Unit - 3: Partial Differential Equations

Department of Science and Humanities

PES UNIVERSITY

Contents

Method of Separation of Variables

- The method of separation of variables is a powerful and widely used technique to solve certain types of Partial Differential Equations (PDEs), especially when the PDE and boundary conditions are defined over a rectangular or regular domain
- It is based on the idea that a solution can be written as a product of functions, each depending on only one independent variable
- Separation of variables provides closed-form solutions for many fundamental partial differential equations—the one-dimensional heat equation, the wave equation, and Laplace's equation—by reducing each PDE to a set of ordinary differential equations

Working Rule

1. For a P.D.E. in the function u of two independent variables x and y, assume that the required solution is separable, i.e.,
$$u(x, y) = X(x)Y(y), \tag{1}$$
where $X(x)$ is a function of x alone and $Y(y)$ is a function of y alone
2. Substitution of u from (1) and its derivatives reduces the P.D.E. to the form
$$f(X, X', X'', \ldots) = g(Y, Y', Y'', \ldots) \tag{2}$$
which is separable in X and Y
3. Since the L.H.S. of (2) is a function of x alone and the R.H.S. of (2) is a function of y alone, then (2) must be equal to a common constant, say k

Working Rule (contd.)

1. Thus (2) reduces to

$$f(X, X', X'', \dots) = k \tag{3}$$

$$g(Y, Y', Y'', \dots) = k \tag{4}$$

2. The determination of a solution to P.D.E. reduces to the determination of solutions to two O.D.E.s (with appropriate conditions).

Example

Use the separation of variables technique to solve

$$3u_x + 2u_y = 0 \quad \text{with} \quad u(x,0) = 4e^{-x}.$$

Solution: Assume $u(x,y) = X(x)Y(y)$.
Then the P.D.E. becomes $3X'Y + 2XY' = 0$ or

$$\frac{X'}{X} = -\frac{2}{3}\frac{Y'}{Y} = k \quad \text{(constant)}$$

Solving $X' - kX = 0$, we get $X(x) = c_1 e^{kx}$.
Similarly, $Y' + \frac{3}{2}kY = 0$, so $Y(y) = c_2 e^{-\frac{3}{2}ky}$.
So $u(x,y) = X(x)Y(y) = c_1 e^{kx} \cdot c_2 e^{-\frac{3}{2}ky} = ce^{kx}e^{-\frac{3}{2}ky} = ce^{k(x-\frac{3}{2}y)}$.
Given that $4e^{-x} = u(x,0) = X(x)Y(0) = ce^{kx}$,
Thus $c = 4$, $k = -1$.
Hence, the required solution is

$$u(x,y) = 4e^{-\frac{1}{2}(2x-3y)}$$

Example

Use separation of variables to solve

$$5\,u_x \;-\; 4\,u_y \;=\; 0, \qquad u(x,0) = 3e^{2x}.$$

Solution. Assume $u(x,y) = X(x)\,Y(y)$. Then,
$u_x = X'(x)Y(y), \quad u_y = X(x)Y'(y)$, so $5X'Y \;-\; 4X\,Y' = 0$
$\implies \frac{X'}{X} = \frac{4}{5}\,\frac{Y'}{Y} = k.$
Solve ODEs.

$$X'/X = k \;\Rightarrow\; X(x) = A\,e^{kx},$$
$$\frac{Y'}{Y} = \frac{5}{4}k \;\Rightarrow\; Y(y) = B\,e^{\frac{5}{4}k\,y}.$$

Combine.

$$u(x,y) = AB\,e^{kx}\,e^{\frac{5}{4}k\,y} = C\,e^{k\,(x+\frac{5}{4}y)}.$$

Initial condition.

$$u(x,0) = C\,e^{kx} = 3e^{2x} \implies C = 3,\; k = 2.$$

Final answer. $u(x,y) = 3\,e^{2x+\frac{5}{2}y}.$