UE25MA141A: ENGINEERING MATHEMATICS - I

Unit - 2: Higher Order Differential Equations

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When X is any other function of x

Rules for Finding the Particular Integral
Consider the differential equation,
$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1} D + a_n)y = X$
It can be written as:

$$f(D)y = X$$

$$\therefore \quad \text{P.I.} = \frac{1}{f(D)} X$$

Resolve $f(D)$ into linear factors.

Let

$$f(D) = (D - m_1)(D - m_2)\ldots(D - m_n)$$

Then,

$$\text{P.I.} = \frac{1}{f(D)} X = \frac{1}{(D - m_1)(D - m_2)\ldots(D - m_n)} X$$

Use partial fractions:

$$P.I = \left(\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}\right) X$$

$$P.I = A_1 \frac{1}{D - m_1} X + A_2 \frac{1}{D - m_2} X + \ldots + A_n \frac{1}{D - m_n} X$$

$$P.I = A_1 e^{m_1 x} \int X e^{-m_1 x} dx + A_2 e^{m_2 x} \int X e^{-m_2 x} dx + \ldots$$

$$+ \mathrm{A}_n e^{m_n x} \int X e^{-m_n x} dx$$

Problem 1

Solve $(D^2 + a^2)y = \tan(ax)$.

Solution:

The given equation in symbolic form is:

$$(D^2 + a^2)y = \tan(ax)$$

Auxiliary equation is:

$$m^2 + a^2 = 0 \Rightarrow m = \pm ia$$

Therefore, the Complementary Function (C.F.) is:

$$y_c = c_1 \cos(ax) + c_2 \sin(ax)$$

Particular Integral:

$$\text{P.I.} = \frac{1}{D^2 + a^2} \tan(ax) = \frac{1}{(D + ia)(D - ia)} \tan(ax)$$

Problem 1 (Contd...)

Now,

$$\text{P.I.} = \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]\tan(ax)$$

$$\frac{1}{D-ia}\tan ax = e^{iax}\int \tan ax \cdot e^{-iax}\,dx$$

$$= e^{iax}\int \tan ax\,(\cos ax - i\sin ax)\,dx = e^{iax}\int\left(\sin ax - i\frac{\sin^2 ax}{\cos ax}\right)dx$$

$$= -\frac{1}{a}e^{iax}\left[(\cos ax - i\sin ax) + i\log(\sec ax + \tan ax)\right]$$

$$= -\frac{1}{a}\left[1 + ie^{iax}\log(\sec ax + \tan ax)\right]$$

Problem 1 (Contd...)

Changing i to -i,

$$\frac{1}{D+ia}\tan ax = -\frac{1}{a}\left[1 - ie^{-iax}\log(\sec ax + \tan ax)\right]$$

$$\text{P.I.} = \frac{1}{2ia}\Bigg[-\frac{1}{a}\left\{1 + ie^{iax}\log(\sec ax + \tan ax)\right\}$$

$$+\frac{1}{a}\left\{1 - ie^{-iax}\log(\sec ax + \tan ax)\right\}\Bigg]$$

$$= -\frac{1}{a^2}\log(\sec ax + \tan ax)\cdot\cos ax$$

$$y = c_1\cos ax + c_2\sin ax - \frac{1}{a^2}\cos ax\ \log(\sec ax + \tan ax).$$

Problem 2

Solve $\dfrac{d^2y}{dx^2} = \ln x$.

<u>Solution:</u>

The Auxiliary equation is

$$m^2 = 0 \Rightarrow\Rightarrow m = 0, 0$$

$$\Rightarrow \text{C.F.} = c_1 + xc_2$$

$$\text{P.I.} = \frac{1}{D^2} \ln x$$

$$= \iint \ln x \, dx$$

Problem 2 (Contd...)

$$= \int (x \ln x - x) dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx - \int x \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2} - \frac{x^2}{2}$$

$$\text{P.I.} = \frac{x^2 \ln x}{2} - \frac{3x^2}{4}$$

$$\therefore y = c_1 + x \; c_2 + \frac{x^2 \ln x}{2} - \frac{3x^2}{4}$$

Problem 3

Find the particular integral for the differential equation: $(D^2 + 3D + 2)y = e^{e^x}$.

Solution:

$$(D^2 + 3D + 2)y = e^{e^x}$$
$$\Rightarrow \frac{1}{(D+2)(D+1)} e^{e^x}$$

Use partial fractions:

$$\frac{1}{(D+2)(D+1)} = \frac{A}{D+2} + \frac{B}{D+1}$$
$$1 = A(D+1) + B(D+2) \Rightarrow A = -1, \quad B = 1$$

So,

$$P.I = \left(\frac{1}{D+1} - \frac{1}{D+2} \right) e^{e^x}$$

Problem 3 (Contd...)

$$= e^{-x}\int e^{x}e^{e^{x}}dx - e^{-2x}\int e^{2x}e^{e^{x}}dx$$

Substitute: $e^{x} = t \Rightarrow e^{x}dx = dt$

$$= e^{-x}\int e^{t}dt - e^{-2x}\int te^{t}dt$$

$$= e^{-x}e^{t} - e^{-2x}\left[te^{t} - \int e^{t}dt\right]$$

$$= e^{-x}e^{e^{x}} - e^{-2x}\left[e^{x}e^{e^{x}} - e^{e^{x}}\right]$$

$$= e^{e^{x}}\left[e^{-x} + e^{-2x} - e^{-x}\right]$$

$$\therefore P.I = e^{-2x}\cdot e^{e^{x}}$$