



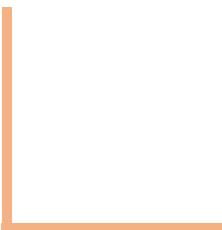
ENGINEERING MATHEMATICS I

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Class content

- ❖ Relation between Beta and Gamma functions
- ❖ Duplication formula



Relationship between beta and gamma functions

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Relationship between beta and gamma functions

Proof: By definition, $\Gamma(m) = 2 \int_0^{\infty} e^{-x^2} x^{2m-1} dx$

and $\Gamma(n) = 2 \int_0^{\infty} e^{-y^2} y^{2n-1} dy$

$$\text{Therefore, } \Gamma(m)\Gamma(n) = 2 \int_0^{\infty} e^{-x^2} x^{2m-1} dx \times 2 \int_0^{\infty} e^{-y^2} y^{2n-1} dy$$

$$= 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx dy$$

Relationship between beta and gamma functions

Put $x = r \cos \theta$ and $y = r \sin \theta$

then $dxdy = rdrd\theta$

$$\begin{aligned}\therefore \Gamma(m)\Gamma(n) &= 4 \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} (r \cos \theta)^{2m-1} (r \sin \theta)^{2n-1} r dr d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} \cos^{2m-1} \theta \sin^{2n-1} \theta d\theta \times 2 \int_0^{\infty} e^{-r^2} r^{2m-1} r^{2n-1} r dr\end{aligned}$$

Put $r^2 = t$, $rdr = \frac{dt}{2}$ in second integral, we get,

Relationship between beta and gamma functions

$$\Gamma(m)\Gamma(n) = \beta(m, n).2.\frac{1}{2} \int_0^{\infty} e^{-t} t^{m-\frac{1}{2}+n-\frac{1}{2}} dt$$

$$= \beta(m, n) \int_0^{\infty} e^{-t} t^{m+n-1} dt$$

$$= \beta(m, n) \Gamma(m+n)$$

Thus,

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Legendre duplication formula for gamma function

$$\sqrt{\pi} \Gamma(2m) = 2^{2m-1} \Gamma(m) \Gamma\left(m + \frac{1}{2}\right)$$

Proof: We know that, $\beta(m, n) = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$

$$\text{Therefore, } \beta(m, m) = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2m-1} \theta d\theta$$

$$= 2 \int_0^{\frac{\pi}{2}} \frac{\sin^{2m-1} 2\theta}{2^{2p-1}} d\theta$$

$$= \frac{2}{2^{2p-1}} \int_0^{\frac{\pi}{2}} \sin^{2m-1} 2\theta d\theta$$

Legendre duplication formula for gamma function

Let $2\theta = t \Rightarrow 2d\theta = dt$

$$\begin{aligned}\text{Then } \beta(m, m) &= \frac{2}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1} t \frac{dt}{2} \\ &= \frac{1}{2^{2m-1}} \int_0^{\frac{\pi}{2}} \sin^{2m-1} t dt \\ &= \frac{1}{2^{2m-1}} \beta\left(\frac{1}{2}, m\right)\end{aligned}$$

Legendre duplication formula for gamma function

$$\text{i.e. } \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \frac{\Gamma(\frac{1}{2})\Gamma(m)}{\Gamma\left(m + \frac{1}{2}\right)}$$

$$\Rightarrow \frac{\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \frac{\sqrt{\pi}}{\Gamma\left(m + \frac{1}{2}\right)}$$

$$\Rightarrow \sqrt{\pi} \Gamma(2m) = 2^{2m-1} \Gamma(m) \Gamma\left(m + \frac{1}{2}\right)$$

ENGINEERING MATHEMATICS I

Note:

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}, 0 < n < 1$$

Proof: We know that, $\Gamma(n)\Gamma(m) = \beta(n, m)\Gamma(m+n)$

Put $m = 1 - n$

$$\Rightarrow \Gamma(n)\Gamma(1-n) = \beta(n, 1-n)\Gamma(1)$$

$$\Rightarrow \Gamma(n)\Gamma(1-n) = \beta(n, 1-n)$$

Also we know that, $\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

Therefore

$$\beta(n, 1-n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx = \frac{\pi}{\sin n\pi}$$



THANK YOU

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