ENGINEERING MATHEMATICS I

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Class content

- Bessel Integral Formula
- Orthogonal property of Bessel Function

Bessel Integral Formula

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos\left(n\theta - x\sin\theta\right) d\theta$$

Bessel Integral Formula

Proof: Consider the Jacobi series

$$\cos(x\sin\theta) = J_0 + 2\left(J_2\cos 2\theta + J_4\cos 4\theta + \ldots\ldots\ldots\right) \quad ----(1)$$

$$\sin(x\sin\theta) = 2\left(J_1\sin\theta + J_3\sin 3\theta + .J_5\sin 5\theta\ldots\ldots\right) \quad ----(2)$$

$$(1)\times\cos n\theta \Rightarrow$$

$$\cos(x\sin\theta)\cos n\theta = J_0\cos n\theta + 2\left(J_2\cos 2\theta\cos n\theta + J_4\cos 4\theta\cos n\theta + \ldots\ldots\ldots\right)$$

where n is an even integer

Bessel Integral Formula

Integrating between the limits 0 to π, we get,

$$\int_0^{\pi} \cos(x\sin\theta)\cos n\theta\, d\theta = \int_0^{\pi} J_0 \cos n\theta\, d\theta + 2\int_0^{\pi} \left(J_2 \cos 2\theta \cos n\theta + J_4 \cos 4\theta \cos n\theta + \ldots\ldots\ldots\right) d\theta$$

$$= 0 + 2J_n \frac{\pi}{2} \qquad \because \int_0^{\pi} \cos n\theta \cos m\theta\, d\theta = \begin{pmatrix} \frac{\pi}{2} \text{ when } n = m, \\ \text{otherwise} = 0 \end{pmatrix}$$

$$\Rightarrow \int_0^{\pi} \cos(x\sin\theta)\cos n\theta\, d\theta = J_n \pi \quad ----(3)$$

Bessel Integral Formula

$(2) \times \sin n\theta \Rightarrow$

$$\sin(x\sin\theta)\sin n\theta = 2\left(J_1 \sin\theta \sin n\theta + J_3 \sin 3\theta \sin n\theta + J_5 \sin 5\theta \sin n\theta \ldots\ldots..\right)$$

where n is an even integer

Integrating between the limits 0 to π, we get,

$$\int_0^{\pi} \sin(x\sin\theta)\sin n\theta \, d\theta = 2\int_0^{\pi} \left(J_1 \sin\theta \sin n\theta + J_3 \sin 3\theta \sin n\theta + .J_5 \sin 5\theta \sin n\theta \ldots\ldots..\right) d\theta$$

$$\Rightarrow \int_0^{\pi} \sin(x\sin\theta)\sin n\theta \, d\theta = 0 \quad ----(4)$$

Bessel Integral Formula

Adding (3) and (4), we get,

$$\int_0^{\pi}\left(\cos(x\sin\theta)\cos n\theta+\sin(x\sin\theta)\sin n\theta\right)d\theta = J_n\pi$$

i.e., $\displaystyle\int_0^{\pi}\cos(x\sin\theta-n\theta)d\theta=J_n\pi$

Therefore,

$$\boxed{J_n=\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta-n\theta)d\theta}$$

Orthogonal property of Bessel function

If α and β are the roots of the equation $J_n(ax)$=0, then

$$\int_0^a J_n(\alpha x) J_n(\beta x)\, dx = \begin{cases} 0 \quad when\ \alpha \neq \beta \\ \dfrac{a^2}{2} J_{n+1}^2(a\alpha) \quad when\ \alpha = \beta \end{cases}$$

Orthogonal property of Bessel function

Proof: Consider the Bessel differential equations

$$x^2 u'' + x u' + (\alpha^2 x^2 - n^2)u = 0 \quad ----(1)$$

$$x^2 v'' + x v' + (\beta^2 x^2 - n^2)v = 0 \quad ----(2)$$

Solutions of (2) and (3) are

$u = J_n(\alpha x)$ and $v = J_n(\beta x)$

$$(1) \times \frac{v}{x} \Rightarrow x u'' v + u' v + (\alpha^2 x^2 - n^2)\frac{uv}{x} = 0 \quad ----(4)$$

$$(2) \times \frac{u}{x} \Rightarrow x v'' u + u v' + (\beta^2 x^2 - n^2)\frac{uv}{x} = 0 \quad ----(5)$$

Orthogonal property of Bessel function

$$(4)-(5) \;\Rightarrow\; x\left(u''v-uv''\right)+(u'v-uv')+(\alpha^2-\beta^2)uvx=0$$

$$\Rightarrow \frac{d}{dx}\left(x(u'v-uv')\right)=(\beta^2-\alpha^2)uvx$$

$$\Rightarrow uvx=\frac{1}{(\beta^2-\alpha^2)}\frac{d}{dx}\left(x(u'v-uv')\right) \quad ----(5)$$

Orthogonal property of Bessel function

Integrating wrt x between 0 to a, we get,

$$\int_0^a xJ_n(\alpha x)\ J_n(\beta x)\,dx = \int_0^a \frac{1}{(\beta^2-\alpha^2)}\frac{d}{dx}\left(x(u'v-uv')\right)dx$$

$$= a\frac{(J'(\alpha a).\alpha.J_n(\beta a) - J_n'(\beta a).J_n(\alpha a).\beta)}{(\beta^2-\alpha^2)}$$

$$= 0 \qquad (when\ \alpha \neq \beta)$$

$$(because\ J_n(\alpha a) = 0\ and\ \ J_n(\beta a) = 0\ )$$

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Orthogonal property of Bessel function

when $\alpha = \beta$

$$\lim_{\beta \to \alpha} \int_0^a x J_n(\alpha x)\, J_n(\beta x)\, dx = \lim_{\beta \to \alpha} \frac{a(J_n'(\alpha a).\alpha.J_n(\beta a) - J_n'(\beta a) J_n(\alpha a).\alpha)}{(\beta^2 - \alpha^2)}$$

Applying L'Hospital's rule on RHS, we get,

$$\int_0^a x J_n^2(\alpha x)\, dx = \lim_{\beta \to \alpha} \frac{a(J_n'(a\alpha).\alpha.J_n{}'(\beta a)a}{2\beta}$$

$$= \frac{a^2 (J_n'(a\alpha))^2}{2} \qquad ----(6)$$

Orthogonal property of Bessel function

But, $\frac{d}{dx} J_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$

$$\Rightarrow J_n'(a\alpha) = \frac{n}{a\alpha} J_n(a\alpha) - J_{n+1}(a\alpha)$$

$$= 0 - J_{n+1}(a\alpha)$$

$$= - J_{n+1}(a\alpha)$$

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Orthogonal property of Bessel function

Substituting in (6), we get,

$$\int_0^a x J_n^2(\alpha x)\ dx = \frac{a^2\left(-J_{n+1}(a\alpha)\ \right)^2}{2}$$

$$= \frac{a^2 J_{n+1}{}^2(a\alpha)}{2}$$

Combining both the results, we get

$$\int_0^a x J_n(\alpha x) J_n(\beta x) dx = \begin{cases} 0 & when\ \alpha \neq \beta \\ \dfrac{a^2}{2} J_{n+1}^2(a\alpha) & when\ \alpha = \beta \end{cases}$$

THANK YOU

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