PES University, Bangalore

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Department of Science and Humanities

Engineering Mathematics - I (UE25MA141A)

$\boxed{\text{Notes}}$

Unit - 1: Partial Differentiation

Function of several variables: We studied the calculus of functions of a single real variable defined by $y = f(x)$. We shall extend the concepts of functions of one variable to functions of two or more variables.

If to each point (x, y) of a certain part of the x-y plane, $x \in \mathbb{R}, y \in \mathbb{R}$ or $(x, y) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, there corresponds a real value z according to some rule $f(x, y)$, then $f(x, y)$ is called a *real valued function of two variables* x and y and is written as

$$z = f(x, y), \quad x \in \mathbb{R}, \ y \in \mathbb{R}, \quad \text{or} \quad (x, y) \in \mathbb{R}^2, \ z \in \mathbb{R}.$$

We call x, y as the independent variables and z as the dependent variable.

In general, we define a real valued function of n variables as

$$z = f(x_1, x_2, \ldots, x_n), \quad (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \ z \in \mathbb{R} \tag{1}$$

where $x_1, x_2, \ldots, x_n$ are the n independent variables and z is the dependent variable. The point $(x_1, x_2, \ldots, x_n)$ is called an n-tuple and lies in an n-dimensional space. In this case, the function f maps $\mathbb{R}^n$ into $\mathbb{R}$.

The function as defined by Eq. 1 is called an *explicit function*, whereas a function defined by $\phi(z, x_1, x_2, \ldots, x_n) = 0$ is called an *implicit function*.

We shall discuss the calculus of the functions of two variables in detail and then generalize to the case of several variables.

Functions of Two Variables

Consider the function of two variables

$$z = f(x, y).$$

The set of points (x, y) in the x-y plane for which $f(x, y)$ is defined is called the *domain* of definition of the function and is denoted by D. This domain may be the entire x-y plane or a part of the x-y plane. The collection of the corresponding values of z is called the *range* of the function. The following are some examples.

$$z = \sqrt{1 - x^2 - y^2} :$$

z is real. Therefore, we have $1 - x^2 - y^2 \geq 0$, or $x^2 + y^2 \leq 1$, that is, the domain is the region $x^2 + y^2 \leq 1$. The range is the set of all real, positive numbers.

$$z = \frac{1}{(x^2 - y^2)} :$$

The domain is the set of all points (x, y) such that $x^2 - y^2 \neq 0$, that is $y \neq \pm x$. The range is $\mathbb{R}$.

$$z = \log(x + y):$$

The domain is the set of all points (x, y) such that $x + y > 0$. The range is $\mathbb{R}$.

0.0.1 Limits

Let $z = f(x, y)$ be a function of two variables defined in a domain D. Let $P(x_0, y_0)$ be a point in D. If for a given real number $\varepsilon > 0$, however small, we can find a real number $\delta > 0$ such that for every point (x, y) in the δ-neighborhood of $P(x_0, y_0)$

$$|f(x, y) - L| < \varepsilon, \quad \text{whenever} \quad \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

then the real, finite number L is called the limit of the function $f(x, y)$ as $(x, y) \to (x_0, y_0)$. Symbolically, we write it as

$$\lim_{(x,y) \to (x_0,y_0)} f(x, y) = L.$$

Note that for the limit to exist, the function $f(x, y)$ may or may not be defined at (x_0, y_0). If $f(x, y)$ is not defined at $P(x_0, y_0)$, then we write

$$|f(x, y) - L| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

This definition is called the δ-ε approach to study the existence of limits.

<p align="center">Remarks</p>

1. $\lim_{(x,y) \to (x_0,y_0)} f(x, y)$, if it exists is unique.
2. Since $(x, y) \to (x_0, y_0)$ in the two-dimensional plane, there are infinite number of paths joining (x, y) to (x_0, y_0). Since the limit is unique, the limit is same along all the paths, that is the limit is independent of the path. Thus, the limit of a function cannot be obtained by approaching the point P along a particular path and finding the limit of $f(x, y)$. If the limit is dependent on a path, then the limit does not exist.

Let $u = f(x, y)$ and $v = g(x, y)$ be two real valued functions defined in a domain D. Let

$$\lim_{(x,y) \to (x_0,y_0)} f(x, y) = L_1 \quad \text{and} \quad \lim_{(x,y) \to (x_0,y_0)} g(x, y) = L_2.$$

Then, the following results can be easily established.

(i)
$$\lim_{(x,y) \to (x_0,y_0)} [k f(x, y)] = k L_1 \quad \text{for any real constant } k.$$

(ii)
$$\lim_{(x,y) \to (x_0,y_0)} [f(x, y) + g(x, y)] = L_1 + L_2.$$

(iii)
$$\lim_{(x,y) \to (x_0,y_0)} [f(x, y) g(x, y)] = L_1 L_2.$$

(iv)
$$\lim_{(x,y)\to(x_0,y_0)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{L_1}{L_2}, \quad L_2 \neq 0.$$

Example:

Show that the following limits do not exist:

$$\text{(i)} \quad \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

$$\text{(ii)} \quad \lim_{(x,y)\to(0,0)} \frac{x+\sqrt{y}}{x^2+y^2}$$

$$\text{(iii)} \quad \lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$$

Solution.

A limit does not exist if it is not finite or if it depends on the path of approach.

(i) *Consider the path $y = mx$. Then as $(x,y) \to (0,0)$, $x \to 0$:*

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x(mx)}{x^2+(mx)^2}$$
$$= \lim_{x\to 0} \frac{mx^2}{x^2(1+m^2)}$$
$$= \frac{m}{1+m^2}$$

This value depends on m, so the limit is path-dependent and does not exist.

(ii) *Consider the path $y = mx^2$. Then as $(x,y) \to (0,0)$, $x \to 0$:*

$$\lim_{(x,y)\to(0,0)} \frac{x+\sqrt{y}}{x^2+y^2} = \lim_{x\to 0} \frac{x+\sqrt{mx^2}}{x^2+(mx^2)^2}$$
$$= \lim_{x\to 0^+} \frac{x+|x|\sqrt{m}}{x^2(1+m^2x^2)}$$
$$= \lim_{x\to 0^+} \frac{x(1+\sqrt{m})}{x^2(1+m^2x^2)}$$
$$= \lim_{x\to 0^+} \frac{1+\sqrt{m}}{x(1+m^2x^2)} = \infty$$

The limit is not finite, so it does not exist.

(iii) *Consider the path $y = mx^3$. Then as $(x,y) \to (0,0)$, $x \to 0$:*

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2} = \lim_{x\to 0} \frac{x^3(mx^3)}{x^6+(mx^3)^2}$$
$$= \lim_{x\to 0} \frac{mx^6}{x^6(1+m^2)}$$
$$= \frac{m}{1+m^2}$$

This depends on m, so the limit is path-dependent and does not exist.

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Definition of Partial Differentiation:

A partial derivative of a function of several variables is the derivative with respect to one of those variables, with the others held constant.

- Let $z = f(x, y)$ be a function of two variables x & y. Then the partial derivative of z with respect to x treating y as constant, denoted by $\frac{\partial z}{\partial x}$ or z_x or $\frac{\partial f}{\partial x}$ or f_x, is defined by

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

- Let $z = f(x, y)$ be a function of two variables x & y. Then the partial derivative of z with respect to y treating x as constant, denoted by $\frac{\partial z}{\partial y}$ or z_y or $\frac{\partial f}{\partial y}$ or f_y, is defined by

$$\frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Second Order Partial Derivatives

The second order partial derivatives are obtained by partially differentiating the first order partial derivatives.

$$z_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

With respect to y:

$$z_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

Mixed Second Order Partial Derivatives:

Mixed partial derivatives of $z = f(x, y)$:

$$z_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{xy}$$

$$z_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{yx}$$

Note 1: The crossed or mixed partial derivatives are in general equal.

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

That is, the order of differentiation is immaterial if the second order derivatives involved are continuous.

Note 2: In the subscript notation, the subscripts are written in the same order in which the differentiation is carried out, while in the ∂ notation the order is opposite. For example,

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{xy}$$

Higher Order Partial Derivatives:

The partial derivative $\frac{\partial f}{\partial x}$ obtained by differentiating once is known as a first order partial derivative, while

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}$$

which are obtained by differentiating twice are known as second-order partial derivatives. Third, fourth and nth order derivatives involve 3, 4 and n times differentiation, respectively.

Partial derivative of the function $f(x, y)$ at the point (x_0, y_0) can be visualized as the slope of the tangent line along the direction of the independent variable with respect to which we would differentiate the function.

Note:

1. If $f_{xx} > 0$ then $f(x, y)$ is concave up in the x direction.

2. If $f_{yy} > 0$ then $f(x, y)$ is concave up in the y direction.

3. Mixed partials tell us how a partial in one variable is changing in the direction of the other.

4. f_{xy} tells us how the rate of change of $f(x, y)$ in the x direction is changing as we move in the y direction.

Problem: Find all the second order partial derivatives of the function

$$f(x, y) = \ln(x^2 + y^2) + \tan^{-1}(y/x), \quad (x, y) \neq (0, 0).$$

Solution

We have

$$f_x(x, y) = \frac{2x}{x^2 + y^2} + \frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{2x - y}{x^2 + y^2}$$

$$f_y(x, y) = \frac{2y}{x^2 + y^2} + \frac{1}{1 + (y/x)^2}\left(\frac{1}{x}\right) = \frac{2y + x}{x^2 + y^2}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{2x - y}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(-1) - (2x - y)(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2 - 4xy}{(x^2 + y^2)^2}$$

$$f_{yx}(x, y) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\frac{2y + x}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(1) - (2y + x)(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2 - 4xy}{(x^2 + y^2)^2}$$

$$f_{xx}(x, y) = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{2x - y}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(2) - (2x - y)(2x)}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2 + 2yx}{(x^2 + y^2)^2}$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(\frac{2y + x}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(2) - (2y + x)(2y)}{(x^2 + y^2)^2} = \frac{2x^2 - 2y^2 - 2xy}{(x^2 + y^2)^2}$$

We note that $f_{xy}(x, y) = f_{yx}(x, y)$.

Problem: Find the first and second partial derivatives of $z = x^3 + y^3 - 3axy$.

Solution. We have

$$z = x^3 + y^3 - 3axy.$$

$$\frac{\partial z}{\partial x} = 3x^2 + 0 - 3ay(1) = 3x^2 - 3ay, \quad \text{and} \quad \frac{\partial z}{\partial y} = 0 + 3y^2 - 3ax(1) = 3y^2 - 3ax$$

Also

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3ay) = 6x, \qquad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2 - 3ay) = -3a$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 - 3ax) = 6y, \qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(3y^2 - 3ax) = -3a$$

We observe that

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}.$$

Total Derivative

If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$, then we can express u as a function of t alone by substituting the values of x and y in $f(x, y)$. Thus we can find the ordinary derivative du/dt which is called the *total derivative* of u to distinguish it from the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$.

Now to find du/dt without actually substituting the values of x and y in $f(x, y)$, we establish the following

Chain rule:

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \tag{2}$$

Obs. If $u = f(x, y, z)$, where x, y, z are all functions of a variable t, then **Chain rule** is

$$\frac{du}{dt} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

If $t = x$, then Eq. (2) becomes

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \tag{3}$$

Differentiation of implicit functions

If $f(x, y) = c$ be an implicit relation between x and y which defines y as a differentiable function of x, then Eq. (3) becomes

$$0 = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

This gives the *important formula*

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}; \quad \left[\frac{\partial f}{\partial y} \neq 0\right]$$

for the first differential coefficient of an implicit function.

Example: *Given* $u = \sin(x/y)$, $x = e^t$ *and* $y = t^2$, *find* du/dt *as a function of* t. *Verify your result by direct substitution.*

Solution. We have

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} = \left(\cos\left(\frac{x}{y}\right) \cdot \frac{1}{y} \right) e^t + \left(\cos\left(\frac{x}{y}\right) \cdot \left(-\frac{x}{y^2}\right) \right) 2t$$

$$= \cos\left(e^t/t^2\right) \cdot e^t/t^2 - 2\cos\left(e^t/t^2\right) \cdot e^t/t^3 = [(t-2)/t^3] e^t \cos\left(e^t/t^2\right)$$

Also $u = \sin(x/y) = \sin(e^t/t^2)$

$$\therefore \quad \frac{du}{dt} = \cos\left(\frac{e^t}{t^2}\right) \cdot \frac{t^2 e^t - e^t \cdot 2t}{t^4} = \frac{t-2}{t^3} e^t \cos\left(\frac{e^t}{t^2}\right) \quad \text{as before.}$$

Example: *If* $u = x\log xy$ *where* $x^3 + y^3 + 3xy = 1$, *find* du/dx.

Solution. From $f(x,y) = x^3 + y^3 + 3xy - 1$, we have

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{3x^2 + 3y}{3y^2 + 3x} = -\frac{x^2 + y}{y^2 + x} \quad \cdots (i)$$

Also

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = (1 \cdot \log xy + x \cdot \frac{1}{x}) + \left(\frac{x}{y}\right) \cdot \frac{dy}{dx}$$

Hence

$$\frac{du}{dx} = 1 + \log xy - x\frac{(x^2 + y)}{y(y^2 + x)} \quad \text{[By (i)]}$$

Partial Derivatives of composite functions:

If $u = f(x, y)$ where both x and y are functions of two independent variables r, s, then 'u' is said to be a composite function of the two variables r and s.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

If $u = f(x, y, z)$ where both x, y & z are functions of two (or more) independent variables say r, s, then 'u' is said to be a composite function of the two variables r and s.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

Example: If $u = F(x - y, y - z, z - x)$, prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Solution. Put $x - y = r$, $y - z = s$ and $z - x = t$, so that $u = f(r, s, t)$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r} \cdot (1) + \frac{\partial u}{\partial s} \cdot (0) + \frac{\partial u}{\partial t} \cdot (-1) = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r} \cdot (-1) + \frac{\partial u}{\partial s} \cdot (1) + \frac{\partial u}{\partial t} \cdot (0) = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

and

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r} \cdot (0) + \frac{\partial u}{\partial s} \cdot (-1) + \frac{\partial u}{\partial t} \cdot (1) = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}$$

Therefore,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Example: If $w = G(u, v)$ where $u = x^2 + y^2$ and $v = xy$, find $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$.

Solution: Using the chain rule:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

Now, compute the partial derivatives:

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial v}{\partial x} = y, \quad \frac{\partial v}{\partial y} = x$$

Add the two derivatives:

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}(2x + 2y) + \frac{\partial w}{\partial v}(y + x)$$

Or, simplified:

$$\boxed{\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \cdot 2(x + y) + \frac{\partial w}{\partial v} \cdot (x + y)}$$

Example:

If $z = Q(u, v)$ where $u = e^{x+y}$ and $v = x - y$, find $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$.

Answer:

By the chain rule,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

Now,

$$\frac{\partial u}{\partial x} = e^{x+y}, \quad \frac{\partial u}{\partial y} = e^{x+y}$$

$$\frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1$$

So,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}e^{x+y} + \frac{\partial z}{\partial v} \cdot 1$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}e^{x+y} + \frac{\partial z}{\partial v} \cdot (-1)$$

Therefore,

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \left(\frac{\partial z}{\partial u}e^{x+y} + \frac{\partial z}{\partial v}\right) - \left(\frac{\partial z}{\partial u}e^{x+y} - \frac{\partial z}{\partial v}\right)$$

$$= 2\frac{\partial z}{\partial v}$$

$$\boxed{\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 2\frac{\partial z}{\partial v}}$$

Homogeneous function and Euler's theorem

An expression of the form $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n$ in which every term is of the nth degree, is called a **homogeneous function of degree** n. This can be rewritten as

$$x^n[a_0 + a_1(y/x) + a_2(y/x)^2 + \ldots + a_n(y/x)^n].$$

Thus *any function $f(x, y)$ which can be expressed in the form $x^n \phi(y/x)$, is called a homogeneous function of degree n in x and y.*

For instance, $x^3 \cos(y/x)$ is a homogeneous function of degree 3, in x and y.

In general, a function $f(x, y, z, t, \ldots)$ is said to be a homogeneous function of degree n in $x, y, z, t, \ldots$, if it can be expressed in the form $x^n\phi(y/x, z/x, t/x, \ldots)$.

Euler's theorem on homogeneous functions*. *If u be a homogeneous function of degree n in x and y, then*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

In general, if u be a homogeneous function of degree n in $x, y, z, t, \ldots$, then,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + t\frac{\partial u}{\partial t} + \ldots = nu.$$

Example: *Show that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u \quad \text{where} \quad \log u = \frac{x^3 + y^3}{3x + 4y}.$$

Solution. Since

$$z = \log u = \frac{x^3 + y^3}{3x + 4y} = x^2 \cdot \frac{1 + (y/x)^3}{3 + 4(y/x)},$$

$$\therefore \quad z \text{ is a homogeneous function of degree 2 in } x \text{ and } y.$$

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \tag{1}$$

But

$$\frac{\partial z}{\partial x} = \frac{1}{u}\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{1}{u}\frac{\partial u}{\partial y}$$

Hence (1) becomes

$$x \cdot \frac{1}{u}\frac{\partial u}{\partial x} + y \cdot \frac{1}{u}\frac{\partial u}{\partial y} = 2\log u$$

or

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u.$$

Example: *If* $u = \sin^{-1}\left(\frac{x+2y+3z}{x^8+y^8+z^8}\right)$, *find the value of*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}.$$

Solution. Here u is not a homogeneous function. We therefore, write

$$\omega = \sin u = \frac{x + 2y + 3z}{x^8 + y^8 + z^8} = x^{-7} \cdot \frac{1 + 2(y/x) + 3(z/x)}{1 + (y/x)^8 + (z/x)^8}$$

Thus ω is a homogeneous function of degree -7 in x, y, z. Hence by Euler's theorem

$$x\frac{\partial \omega}{\partial x} + y\frac{\partial \omega}{\partial y} + z\frac{\partial \omega}{\partial z} = (-7)\omega \tag{1}$$

But

$$\frac{\partial \omega}{\partial x} = \cos u\frac{\partial u}{\partial x}, \quad \frac{\partial \omega}{\partial y} = \cos u\frac{\partial u}{\partial y}, \quad \frac{\partial \omega}{\partial z} = \cos u\frac{\partial u}{\partial z}$$

$$\therefore (1) \text{ becomes} \quad x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = -7\sin u$$

or

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -7\tan u.$$

Taylor's and Maclaurin series for a function of two variables

The Taylor's series expansion of a single variable function $f(x)$ in a neighbourhood of a point a is

$$f(a + h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \cdots$$

which is an infinite power series in h.

Maclaurin's series is $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$

These ideas are extended to a function $f(x, y)$ of two independent variables x, y. We state the theorem.

Taylor's theorem

Let $f(x, y)$ be a function of two independent variables x, y defined in a region R of the xy-plane and let (a, b) be a point in R. Suppose $f(x, y)$ has all its partial derivatives in a neighbourhood of (a, b), then

$$f(a + h, b + k) = f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right) f(a, b)$$

$$+ \frac{1}{2!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^2 f(a, b)$$

$$+ \frac{1}{3!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^3 f(a, b) + \cdots$$

i.e.,

$$f(a + h, b + k) = f(a, b) + [hf_x(a, b) + kf_y(a, b)]$$

$$+ \frac{1}{2!} \left[h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b) \right]$$

$$+ \frac{1}{3!} \left[h^3 f_{xxx}(a, b) + 3h^2 k f_{xxy}(a, b) + 3hk^2 f_{xyy}(a, b) + k^3 f_{yyy}(a, b) \right] + \cdots$$

Modified forms

1. Put $x = a + h, y = b + k$, then $h = x - a, k = y - b$

$\therefore$ the Taylor's series can be written as

$$f(x, y) = f(a, b) + \{(x - a)f_x(a, b) + (y - b)f_y(a, b)\}$$

$$+ \frac{1}{2!}\{(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)\} + \cdots$$

This series is known as the Taylor's series expansion of $f(x, y)$ in the neighbourhood of (a, b) or about the point (a, b).

2. Putting $a = 0, b = 0$, we get the expansion of $f(x, y)$ in the neighbourhood of $(0, 0)$

$$f(x, y) = f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)]$$

$$+ \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \cdots$$

This is called Maclaurin's series for $f(x, y)$ in powers of x and y.

Expand $\tan^{-1} \frac{y}{x}$ **about** $(1,1)$ **upto the second degree terms.**

Solution.

We know the expansion of $f(x,y)$ about the point (a,b) as Taylor's series is

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)]$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

Here $(a,b) = (1,1)$

$$\therefore \ f(x,y) = f(1,1) + (x-1)f_x(1,1) + (y-1)f_y(1,1)$$
$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right] + \cdots$$

$$f(x,y) = \frac{\pi}{4} + \left[(x-1)\left(-\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right]$$
$$+ \frac{1}{2}\left[(x-1)^2 \cdot \frac{1}{2} + (x-1)(y-1)\cdot 0 + (y-1)^2\left(-\frac{1}{2}\right)\right] + \cdots$$

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2$$

Expand $e^x \cos y$ **near the point** $\left(1, \frac{\pi}{4}\right)$ **by Taylor's series as far as quadratic terms.**

Solution. We know Taylor's series about the point (a,b) is

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)]$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

Here $(a,b) = \left(1, \frac{\pi}{4}\right)$

$$\therefore \ f(x,y) = f\left(1,\frac{\pi}{4}\right) + (x-1)f_x\left(1,\frac{\pi}{4}\right) + \left(y-\frac{\pi}{4}\right)f_y\left(1,\frac{\pi}{4}\right)$$
$$+ \frac{1}{2}\left[(x-1)^2 f_{xx}\left(1,\frac{\pi}{4}\right) + 2(x-1)\left(y-\frac{\pi}{4}\right)f_{xy}\left(1,\frac{\pi}{4}\right) + \left(y-\frac{\pi}{4}\right)^2 f_{yy}\left(1,\frac{\pi}{4}\right)\right] + \cdots$$

Given $f(x,y) = e^x \cos y$,

$$f_x(x,y) = e^x \cos y,$$
$$f_y(x,y) = -e^x \sin y,$$
$$f_{xx}(x,y) = e^x \cos y,$$
$$f_{xy}(x,y) = -e^x \sin y,$$
$$f_{yy}(x,y) = -e^x \cos y.$$

At $\left(1, \frac{\pi}{4}\right)$:

$$f\left(1, \frac{\pi}{4}\right) = e\cos\frac{\pi}{4} = \frac{e}{\sqrt{2}},$$

$$f_x\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}},$$

$$f_y\left(1, \frac{\pi}{4}\right) = -e\sin\frac{\pi}{4} = -\frac{e}{\sqrt{2}},$$

$$f_{xx}\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}},$$

$$f_{xy}\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}},$$

$$f_{yy}\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}.$$

$$\therefore \; e^x\cos y = \frac{e}{\sqrt{2}} + (x-1)\frac{e}{\sqrt{2}} + \left(y - \frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right)$$

$$+ \frac{1}{2}\left[(x-1)^2\frac{e}{\sqrt{2}} + 2(x-1)\left(y - \frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right) + \left(y - \frac{\pi}{4}\right)^2\left(-\frac{e}{\sqrt{2}}\right)\right] + \cdots$$

$$= \frac{e}{\sqrt{2}}\left[1 + (x-1) - \left(y - \frac{\pi}{4}\right) + \frac{1}{2}(x-1)^2 - (x-1)\left(y - \frac{\pi}{4}\right) - \frac{1}{2}\left(y - \frac{\pi}{4}\right)^2\right]$$

Expand $e^x\log_e(1+y)$ in powers of x and y up to terms of third degree.
Solution.
Required the expansion in powers of x and y and so Maclaurin's series is to be used.
We know

$$f(x, y) = f(0,0) + [xf_x(0,0) + yf_y(0,0)]$$

$$+ \frac{1}{2}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)]$$

$$+ \frac{1}{6}[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \cdots$$

Here $(a, b) = (0, 0)$
Given

$$f(x, y) = e^x\log(1+y), \qquad f(0,0) = e^0\log(1+0) = 0,$$

$$f_x = e^x \log(1 + y), \qquad f_x(0,0) = 0$$

$$f_y = \frac{e^x}{1 + y}, \qquad f_y(0,0) = 1$$

$$f_{xx} = e^x \log(1 + y), \qquad f_{xx}(0,0) = 0$$

$$f_{xy} = \frac{e^x}{1 + y}, \qquad f_{xy}(0,0) = 1$$

$$f_{yy} = -\frac{e^x}{(1 + y)^2}, \qquad f_{yy}(0,0) = -1$$

$$f_{xxx} = e^x \log(1 + y), \qquad f_{xxx}(0,0) = 0$$

$$f_{xxy} = \frac{e^x}{1 + y}, \qquad f_{xxy}(0,0) = 1$$

$$f_{xyy} = -\frac{e^x}{(1 + y)^2}, \qquad f_{xyy}(0,0) = -1$$

$$f_{yyy} = \frac{2e^x}{(1 + y)^3}, \qquad f_{yyy}(0,0) = 2$$

$$f(x, y) = 0 + x \cdot 0 + y \cdot 1$$
$$+ \frac{1}{2}[x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot (-1)]$$
$$+ \frac{1}{6}[x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2 \cdot (-1) + y^3 \cdot 2]$$

$$\Rightarrow e^x \log(1 + y) = y + xy - \frac{1}{2}y^2 + xy^2 + \frac{1}{2}x^2 y - \frac{1}{2}y^3 + \frac{1}{3}y^3$$

Maxima and minima of a function of two variables

Definition: A function $f(x, y)$ is said to have a **maximum** or **minimum** at $x = a, y = b$, according as

$$f(a + h, b + k) < \text{ or } > f(a, b),$$

for all positive or negative small values of h and k.

Note. *A maximum or minimum value of a function is called its **extreme value**.*
If we think of $z = f(x, y)$ as a surface:

- The highest point, like the top of a dome, is called a maximum because the surface goes down in every direction from there.

- The lowest point, like the bottom of a bowl, is called a minimum because the surface goes up in every direction from there.

- Sometimes, there can be a ridge, where the surface goes up in one direction and down in another, except along the ridge itself.

There is also a special kind of point called a **saddle point**. At a saddle point, the surface is flat (the tangent plane is horizontal), but if you move in some directions, the surface goes up, and in other directions, it goes down. It looks like the leather seat of a horse back (Figure 1 (c)).
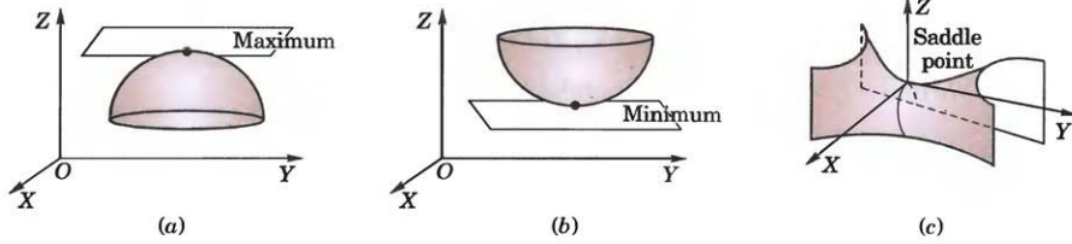
Figure 1: (a) Maximum point (dome),   (b) Minimum point (bowl),   (c) Saddle point (leather seat on horse's back)

Working rule to find the maximum and minimum values of $f(x, y)$

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and *equate each to zero.* Solve these as simultaneous equations in x and y. Let $(a, b), (c, d), \ldots$ be the pairs of values.

2. Calculate the value of $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ for each pair of values.

3.   (i) If $rt - s^2 > 0$ and $r < 0$ at (a, b), $f(a, b)$ is a max. value.

   (ii) If $rt - s^2 > 0$ and $r > 0$ at (a, b), $f(a, b)$ is a min. value.

   (iii) If $rt - s^2 < 0$ at (a, b), $f(a, b)$ is not an extreme value, i.e., (a, b) is a saddle point.

   (iv) If $rt - s^2 = 0$ at (a, b), *the case is doubtful and needs further investigation.*

   Similarly examine the other pairs of values one by one.

Critical Point: A point (a, b) is a critical point of $f(x, y)$ if $f_x = 0$ and $f_y = 0$ at (a, b) or f_x and f_y do not exist at (a, b).

Example: *Examine the following function for extreme values:*

$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2.$$

Solution. We have

$$f_x = 4x^3 - 4x + 4y, \quad f_y = 4y^3 + 4x - 4y$$

and

$$r = f_{xx} = 12x^2 - 4, \quad s = f_{xy} = 4, \quad t = f_{yy} = 12y^2 - 4$$

Now

$$f_x = 0, \ f_y = 0 \text{ give } x^3 - x + y = 0, \tag{i}$$

$$y^3 + x - y = 0 \tag{ii}$$

Adding these, we get $4(x^3 + y^3) = 0$ or $y = -x$.
Putting $y = -x$ in (i), we obtain $x^3 - 2x = 0$, i.e. $x = \sqrt{2}, \ -\sqrt{2}, \ 0$.

$$\therefore \text{ Corresponding values of } y \text{ are } -\sqrt{2}, \ \sqrt{2}, \ 0.$$

At $(\sqrt{2}, -\sqrt{2})$, $rt - s^2 = 20 \times 20 - 4^2 = +ve$ and r is also $+ve$. Hence $f(\sqrt{2}, -\sqrt{2})$ is a minimum value.
At $(-\sqrt{2}, \sqrt{2})$ also both $rt - s^2$ and r are $+ve$.

Hence $f(-\sqrt{2}, \sqrt{2})$ is also a minimum value.

At $(0,0)$, $rt - s^2 = 0$ and, therefore, further investigation is needed.

Example: *Discuss the maxima and minima of* $f(x,y) = x^3 y^2 (1 - x - y)$.

Solution. We have

$$f_x = 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3, \quad f_y = 2x^3 y - 2x^4 y - 3x^3 y^2$$

and

$$r = f_{xx} = 6xy^2 - 12x^2 y^2 - 6xy^3, \quad s = f_{xy} = 6x^2 y - 8x^3 y - 9x^2 y^2, \quad t = f_{yy} = 2x^3 - 2x^4 - 6x^3 y$$

When $f_x = 0, f_y = 0$, we have $x^2 y^2 (3 - 4x - 3y) = 0$, $x^3 y (2 - 2x - 3y) = 0$.

Solving these, the stationary points are $(1/2, 1/3), (0,0)$.

Now

$$rt - s^2 = x^4 y^2 [12(1 - 2x - y)(1 - x - 3y) - (6 - 8x - 9y)^2]$$

At $(1/2, 1/3)$,

$$rt - s^2 = \frac{1}{16} \cdot \frac{1}{9} \left[12 \left(1 - \frac{1}{2} - \frac{1}{3} \right) \left(1 - \frac{1}{2} - 1 \right) - (6 - 4 - 3)^2 \right] = \frac{1}{14} > 0$$

Also

$$r = 6 \left(\frac{1}{2} \cdot \frac{1}{9} - \frac{2}{4} \cdot \frac{1}{9} - \frac{1}{2} \cdot \frac{1}{27} \right) = -\frac{1}{9} < 0$$

Hence $f(x,y)$ has a maximum at $(1/2, 1/3)$ and maximum value

$$= \frac{1}{8} \cdot \frac{1}{9} \left(1 - \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{432}$$

At $(0,0)$, $rt - s^2 = 0$ and therefore further investigation is needed.

Example: *In a plane triangle, find the maximum value of* $\cos A \cos B \cos C$.

Solution. We have

$$A + B + C = \pi \quad \text{so that} \quad C = \pi - (A + B).$$

$$\cos A \cos B \cos C = \cos A \cos B \cos[\pi - (A+B)] = -\cos A \cos B \cos(A+B) = f(A, B), \text{ say.}$$

We get

$$\frac{\partial f}{\partial A} = \cos B [\sin A \cos(A + B) + \cos A \sin(A + B)] = \cos B \sin(2A + B)$$

and

$$\frac{\partial f}{\partial B} = \cos A \sin(A + 2B)$$

$$\frac{\partial f}{\partial A} = 0, \quad \frac{\partial f}{\partial B} = 0 \quad \text{only when} \quad A = B = \frac{\pi}{3}.$$

Also

$$r = \frac{\partial^2 f}{\partial A^2} = 2 \cos B \cos(2A + B), \quad t = \frac{\partial^2 f}{\partial B^2} = 2 \cos A \cos(A + 2B)$$

$$s = \frac{\partial^2 f}{\partial A \partial B} = -\sin B \sin(2A + B) + \cos B \cos(2A + B) = \cos(2A + 2B)$$

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When $A = B = \pi/3$, $r = -1$, $s = -1/2$, $t = -1$ so that $rt - s^2 = 3/4$.

These show that $f(A, B)$ is maximum for $A = B = \pi/3$.

Then $C = \pi - (A + B) = \pi/3$.

Hence $\cos A \cos B \cos C$ is maximum when each of the angles is $\pi/3$, i.e., triangle is equilateral and its maximum value is $1/8$.

Lagrange's Method of (undetermined) Multiplier

Let $f(x, y, z)$ be the function whose extreme values are to be found subject to the restriction

$$\phi(x, y, z) = 0 \tag{1}$$

Between the variables x, y, z construct the auxiliary function

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$$

where λ is an undetermined parameter independent of x, y, z.

λ is called Lagrange's multiplier.

Any relative extremum of $f(x, y, z)$ subject to (1) must occur at a stationary point of $F(x, y, z)$.

The stationary points of F are given by

$$\frac{\partial F}{\partial x} = 0, \qquad \frac{\partial F}{\partial y} = 0, \qquad \frac{\partial F}{\partial z} = 0, \qquad \frac{\partial F}{\partial \lambda} = 0$$

$$\Rightarrow \quad f_x + \lambda\phi_x = 0, \quad f_y + \lambda\phi_y = 0, \quad f_z + \lambda\phi_z = 0, \quad \text{and} \quad \phi(x, y, z) = 0$$

$$\Rightarrow \quad \frac{f_x}{\phi_x} = \frac{f_y}{\phi_y} = \frac{f_z}{\phi_z} = -\lambda \quad \text{and} \quad \phi(x, y, z) = 0$$

Solving these equations, we find the values of x, y, z, which are the stationary points of F, giving the maximum and minimum values of $f(x, y, z)$.

Note This method does not specify the extreme value obtained is a maximum or minimum. It is usually decided from the physical and geometrical considerations of the problem.

A method on the basis of quadratic form is given below to decide maxima or minima at the stationary point for constrained maxima and minima.

Example: *Given $x + y + z = a$, find the maximum value of $x^m y^n z^p$.*

Solution. Let

$$f(x, y, z) = x^m y^n z^p \quad \text{and} \quad \phi(x, y, z) = x + y + z - a.$$

Then

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z) = x^m y^n z^p + \lambda(x + y + z - a).$$

For stationary values of F,

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0$$

$$\Rightarrow \quad mx^{m-1}y^n z^p + \lambda = 0, \quad nx^m y^{n-1} z^p + \lambda = 0, \quad px^m y^n z^{p-1} + \lambda = 0$$

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or
$$-\lambda = mx^{m-1}y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}$$

i.e.
$$\frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a} \qquad [\because x+y+z = a]$$

$$\therefore \text{ The maximum value of } f \text{ occurs when}$$
$$x = am/(m+n+p), \quad y = an/(m+n+p), \quad z = ap/(m+n+p)$$

Hence the maximum value of $f(x,y,z)$ is
$$\frac{a^{m+n+p} \cdot m^m \cdot n^n \cdot p^p}{(m+n+p)^{m+n+p}}$$

Example: *Find the maximum and minimum distances of the point* $(3,4,12)$ *from the sphere* $x^2 + y^2 + z^2 = 4$.

Solution. Let $P(x,y,z)$ be any point on the sphere and $A(3,4,12)$ the given point so that
$$AP^2 = (x-3)^2 + (y-4)^2 + (z-12)^2 = f(x,y,z), \quad \text{say} \qquad (i)$$

We have to find the maximum and minimum values of $f(x,y,z)$ subject to the condition
$$\phi(x,y,z) = x^2 + y^2 + z^2 - 4 = 0 \qquad (ii)$$

Let
$$F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z) = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 4)$$

Then
$$\frac{\partial F}{\partial x} = 2(x-3) + 2\lambda x, \quad \frac{\partial F}{\partial y} = 2(y-4) + 2\lambda y, \quad \frac{\partial F}{\partial z} = 2(z-12) + 2\lambda z$$

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0$$

give
$$x - 3 + \lambda x = 0, \quad y - 4 + \lambda y = 0, \quad z - 12 + \lambda z = 0$$

which give
$$\lambda = \frac{x-3}{-x} = \frac{y-4}{-y} = \frac{z-12}{-z}$$

$$\lambda = \frac{\sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}}{\sqrt{x^2+y^2+z^2}} = \frac{\sqrt{F}}{2}$$

Substituting for λ in (iii), we get
$$x = \frac{3}{1+\sqrt{F}/2}, \quad y = \frac{4}{1+\sqrt{F}/2}, \quad z = \frac{12}{1+\sqrt{F}/2}$$

$$x^2 + y^2 + z^2 = \frac{9 + 16 + 144}{(1+\sqrt{F}/2)^2} = \frac{169}{(1+\sqrt{F}/2)^2}$$

Using (ii),
$$\frac{169}{(1+\sqrt{F}/2)^2} = 4 \implies 1 + \sqrt{F}/2 = \pm\frac{13}{2}, \quad \sqrt{F} = 12, 14$$

[We have left out the negative values of $\sqrt{F}$, because $\sqrt{F} = AP$ is $+ve$ by (i).]
Hence maximum $AP = 14$ and minimum $AP = 12$.