ENGINEERING MATHEMATICS I

T R Geetha

Department of Science and Humanities

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HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

T R Geetha

Department of Science and Humanities

CLASS CONTENT:

- Applications of LDE to Simple Harmonic Oscillation

APPLICATIONS OF LINEAR DIFFERENTIAL EQUATIONS

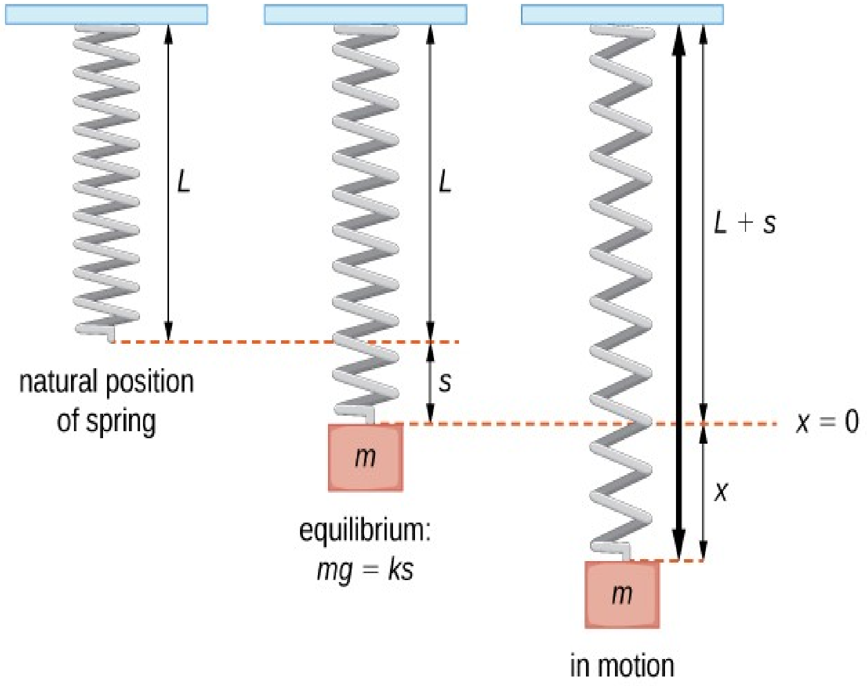
- Second order linear differential equations are used to model many situations in physics and engineering.

- Situations like an object with a mass attached to a vertical string and an electrical circuit containing a resistor , an inductor and a capacitor in series can be represented by a linear differential equation.

- Differential equations are useful for modelling and simulating phenomenon and understanding how they operate.

SIMPLE HARMONIC OSCILLATION

Consider a mass suspended from a spring which is attached to a support. Gravity is pulling the mass downward and the restoring force of the spring is pulling the spring upwards. When both the forces are equal, we say that the spring is in equilibrium position. If the mass is displaced from equilibrium , the mass oscillates up and down. This motion can be modelled by a second order LDE.



Let x(t) be the displacement of the mass from equilibrium.
There are two forces acting on the mass.

(i) Force due to gravity (mg)
(ii) The restoring force

From Hook's Law, the restoring force is given by $-k(s+x)$
(the restoring force is proportional to displacement and acts in the opposite direction of displacement)

By Newton's Second Law,
the forces acting on the system = m.a
i.e, $mg + (-k(s+x)) = m.a$ ------- (1)

SIMPLE HARMONIC OSCILLATION

But at equilibrium, mg = ks

Therefore, ma = -kx

We know that acceleration is given by, $a = \frac{d^2x}{dt^2}$

Using in eqn (1), we get

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

This equation can also be written as,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$\text{where } \omega = \sqrt{\frac{k}{m}} \text{ called angular frequency}$$

The general solution of the differential equation is given by,

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t \quad \text{-----}(2)$$

x(t) gives the position of the mass at any instant. The motion of the mass is Simple Harmonic Motion.

Introducing $c_1 = A\cos\varphi$, $c_2 = -A\sin\varphi$,equation (2) can be

rewritten as, $x(t) = A\cos(\omega t + \varphi)$

where $A = \sqrt{c_1^2 + c_2^2}$, $\tan\varphi = -\dfrac{c_2}{c_1}$

'A 'is called amplitude of the motion and gives the maximum displacement of the mass from its equilibrium position.

The period of the motion 'T' is given by $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

The natural frequency of the simple harmonic oscillator is

given by $\frac{1}{T} = \frac{\omega}{2\pi}$

EXAMPLE : A spring is such that it would be stretched to 6 inches by 12 pound weight. If the weight is pulled down 4 inches below the equilibrium point and given an upward velocity of 2 feet/sec, determine the motion of the weight assuming no damping. Find also the amplitude and period.

SOLUTION:

The differential equation of the free motion is $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$

Firstly, we need to find ‘k’ and ‘ m’.

We know that, the weight of the mass = m.g where ‘g’ is the acceleration due to gravity = 32ft/sec

Therefore, $12 = m.32$

$$\Rightarrow \quad m = \frac{12}{32} = \frac{3}{8}$$

<u>To find ‘k’</u>

At equilibrium , k.s = m . g

Here, s = 6 inches = 0.5 feet

Therefore,

$$k = \frac{m.g}{s} = \frac{\frac{3}{8}.32}{0.5} = 24$$

The differential equation now becomes,

$$\frac{d^2x}{dt^2} + \frac{24}{\frac{3}{8}}x = 0$$

$$\frac{d^2x}{dt^2} + 64x = 0$$

The general solution of the differential equation is ,

$$x(t) = c_1 \cos 8t + c_2 \sin 8t$$

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To find c_1 and c_2 :

$$x(t) = c_1 \cos 8t + c_2 \sin 8t$$

It is given that x = 4 inches = feet at t=0

$$\frac{1}{3} = c_1 \cos 0 + c_2 \sin 0 \Rightarrow c_1 = \frac{1}{3}$$

$$x(t) = \frac{1}{3} \cos 8t + c_2 \sin 8t$$

$$\frac{dx}{dt} = -\frac{8}{3} \sin 8t + c_2 . 8 \cos 8t$$

$$\frac{dx}{dt} = -\frac{8}{3}\sin 8t + c_2 .8\cos 8t$$

Using the condition,

$$\frac{dx}{dt} = -2\text{ft/sec} \quad \text{when } t = 0$$

We get, $c_2 = -\frac{1}{4}$

$$x(t) = \frac{1}{3}\cos 8t - \frac{1}{4}\sin 8t$$

$$\text{Amplitude}, A = \sqrt{c_1^2 + c_2^2}$$

$$A = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{-1}{4}\right)^2} = \frac{5}{12}$$

$$\text{The period of motion} = \frac{2\pi}{\omega}$$

$$\text{But } \omega = \sqrt{\frac{k}{m}} = 8$$

$$\text{Therefore, the period of motion} = \frac{2\pi}{8} = \frac{\pi}{4}\text{ sec}$$

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THANK YOU

T R Geetha

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