ENGINEERING MATHEMATICS - I

Unit - 3: Partial Differential Equations

Department of Science and Humanities

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Contents

Problem

Solve the PDE

$$4\,\frac{\partial^3 z}{\partial x^3} \;-\; 3\,\frac{\partial^3 z}{\partial x\,\partial y^2} \;+\; \frac{\partial^3 z}{\partial y^3} \;=\; 6\,x^2y^2$$

To find CF: The factors are
$4D^3 - 3D(D')^2 + (D')^3 = \;(D - 2D')^2\,(D + D')$.
For $D + D'$, $a = b = 1$. For $D - 2D'$, $a = 1, b = -2$.
Hence, the complementary function is
$z = \phi_1(x - y) + \;\phi_2(2x + y)\; + \;x\,\phi_3(2x + y)$

Problem (contd.)

We write $$4D^3-3D(D')^2+(D')^3 = 4D^3\left[1-\frac{3}{4}D^{-2}(D')^2+\frac{1}{4}D^{-3}(D')^3\right]$$

The PI is given by $$z = \frac{1}{4}D^{-3}\left[1-\left\{\frac{3}{4}D^{-2}(D')^2-\frac{1}{4}D^{-3}(D')^3\right\}\right]^{-1}(6x^2y^2)$$

$$= \frac{3}{2}D^{-3}\left[1+\frac{3}{4}D^{-2}(D')^2-\frac{1}{4}D^{-3}(D')^3+\cdots\right](x^2y^2)$$

$$= \frac{3}{2}D^{-3}\left[x^2y^2+\frac{3}{2}D^{-2}(x^2)\right] = \frac{3}{2}D^{-3}\left[x^2y^2+\frac{1}{8}x^4\right] = \frac{1}{80}x^5y^2+\frac{1}{1120}x^7$$

Finally, the general solution is $z = C.F + P.I$

Problem

Solve the PDE:

$$2\,\frac{\partial^2 z}{\partial x^2} \;+\; 3\,\frac{\partial^2 z}{\partial x\,\partial y} \;+\; \frac{\partial^2 z}{\partial y^2} \;+\; \frac{\partial z}{\partial x} \;+\; \frac{\partial z}{\partial y} \;=\; x-y$$

To find CF: The factors are

$$\begin{aligned}
2D^2+3DD'+D'^2+D+D' &= (2D^2+3DD'+D'^2)+(D+D') \\
&= (D+D')(2D+D')+(D+D') \\
&= (D+D')\left[(2D+D')+1\right] \\
&= (D+D')(2D+D'+1)
\end{aligned}$$

For $(D+D')$, $a=b=1$. For $2D+D'+1$, $a=2, b=1, c=1$. Hence, the complementary function is

$$z=\phi_1(x-y)+e^{\frac{-x}{2}}\phi_2(x-2y)$$

We write

$$\left[2D^2 + 3D\,D' + (D')^2 + D + D'\right]$$
$$= D\Big[\,1 + D^{-1}D' + 3D' + 2D + D^{-1}(D')^2\Big].$$

The particular integral is given by

$$z = D^{-1}\Big[\,1 + \{D^{-1}D' + 3D' + 2D + D^{-1}(D')^2\}\Big]^{-1}(x-y)$$
$$= D^{-1}\big[\,1 - \{D^{-1}D' + 3D' + 2D + D^{-1}(D')^2\} + \cdots\big](x-y)$$
$$= D^{-1}\big[x - y - \{\,D^{-1}(-1) + 3(-1) + 2(1)\}\big]$$
$$= \; D^{-1}[x - y + x + 1] \; = \; x^2 + (1-y)\,x$$

Finally, the general solution is $z = C.F + P.I$

Problem

Solve the PDE

$$\left[D^2 - DD' - 2(D')^2\right] z = 16x\, e^{2y}$$

To find CF: The factors are

$$D^2 \ - \ D\, D' \ - \ 2\,(D')^2 \ = \ \left(D - 2\,D'\right)\left(D + D'\right)$$

For $D - 2\,D'$, $a = 1, b = -2$. For $D + D'$, $a = b = 1$.
Hence, the complementary function is

$$z = \phi_1(-2x - y) + \phi_2(x - y) = \phi_1(2x + y) + \phi_2(x - y)$$

Problem (contd.)

The Particular integral is given by

$$z = \left(D - 2\,D'\right)\left(D + D'\right)16xe^{2y}$$

We first obtain $(D + D')^{-1}(16xe^{2y})$.
Denote $u = (D + D')^{-1}(16xe^{2y})$, or $(D + D')\,u = 16xe^{2y}$

The auxiliary equations are $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{du}{16xe^{2y}}$.

The first two terms give $y = x + c$

Using the first and third terms, we get

$$\frac{dx}{1} = \frac{du}{16x(e^{2c}e^{2x})}$$

$$\text{Hence} \quad u = 16\, e^{2c} \int x\, e^{2x}\, dx$$

Now, $\int x\, e^{2x}\, dx = \frac{x}{2}\, e^{2x} - \frac{1}{4}\, e^{2x} = \left(\frac{x}{2} - \frac{1}{4}\right) e^{2x}$

Hence, $u = 8\left(x - \frac{1}{2}\right) e^{2(x+c)} = (8x - 4)e^{2y}$

Problem (contd.)

$$\text{Now, denote } z = (D - D')^{-1} u = (D - D')^{-1} \big((8x-4)e^{2y}\big)$$

$$\text{or} \quad (D - D')\, z = (8x-4)e^{2y}$$

$$\text{The auxiliary equations are} \quad \frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{(8x-4)e^{2y}}$$

The first two terms give $x + y = C$

Next, the first and third equations give

$$\frac{dz}{(8x-4)\,e^{2y}} = \frac{dx}{1} \implies dz = (8x-4)\,e^{2C}e^{-2x}\,dx,$$

since $y = C - x$

$$\therefore z = e^{2C}\int (8x-4)e^{-2x}\,dx$$

But,

$$\int (8x-4)e^{-2x}\,dx = \left(-4x\,e^{-2x} - 2\,e^{-2x}\right) + 2\,e^{-2x} = -4x\,e^{-2x}.$$

$$\Rightarrow z = e^{2C}\left[-4x\,e^{-2x}\right] = -4x\,e^{2C-2x} = -4x\,e^{2(C-x)} = -4x\,e^{y},$$

which is the particular integral.