ENGINEERING MATHEMATICS - I

Unit - 3: Partial Differential Equations

Department of Science and Humanities

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Contents

Problem

Solve the partial differential equation:

$$\left[D^2 + DD' - 2(D')^2\right] z = 5e^{x+2y}$$

To find CF: The factors are:

$$D^2 + DD' - 2(D')^2 = (D + 2D')(D - D')$$

For

$$(D + 2D'), a = 1, b = 2$$

$$(D - D') = 0, a = 1, b = -1$$

Therefore, $CF = \phi_1(2x - y) + \phi_2(-x - y) = \phi_1(2x - y) + \phi_2(x + y)$

Problem (contd.)

We now solve:

$$\text{P.I.} = \frac{1}{D^2 + DD' - 2(D')^2} e^{x+2y}$$

$$\Rightarrow P.I. = 5\frac{1}{1^2 + (1)(2) - 2(2)^2} e^{x+2y}$$

Hence, the particular integral is:

$$\text{PI} = -e^{x+2y}$$

General Solution is: $z = \phi_1(2x - y) + \phi_2(x + y) - e^{x+2y}$

Problem

Solve the partial differential equation

$$(3D^2 - 2DD' - D'^2 + 4D' - 3D)\,z \;=\; 5\,e^{(2x+y)/3}$$

To find CF: The factors are:

$$3D^2 - 2DD' - D'^2 + 4D' - 3D = (3D + D' - 3)\,(D - D')$$

For $D - D'$, $a = 1, b = -1, c = 0$. For $3D + D' - 3$, $a = 3, b = 1, c = -1$
Thus, CF is $z = \phi_1(-x - y) + e^{-\frac{x}{3}}\phi_2(x - 3y)$
$\Rightarrow z = \phi_1(x + y) + e^{-\frac{x}{3}}\phi_2(x - 3y)$

Problem (contd.)

We set

$$F(D, D') \;=\; 3D^2 \;-\; 2\,DD' \;-\; D'^2 \;+\; 4D' \;-\; 3D.$$

Since the RHS is $5e^{(2x+y)/3}$, we evaluate

$$F\left(\tfrac{2}{3},\, \tfrac{1}{3}\right) = 3\left(\tfrac{2}{3}\right)^2 - 2\left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right) - \left(\tfrac{1}{3}\right)^2 + 4\left(\tfrac{1}{3}\right) - 3\left(\tfrac{2}{3}\right) = \frac{1}{9}$$

Problem (contd.)

$$\text{PI} = \frac{1}{F\left(\frac{2}{3}, \frac{1}{3}\right)} \cdot 5e^{(2x+y)/3} = \frac{1}{\frac{1}{9}} \cdot 5e^{(2x+y)/3} = 45e^{(2x+y)/3}$$

Solve the PDE

$$(4D^2 + 3DD' - D'^2 - D - D')z = 3e^{(x+2y)/2}$$

Solution: To find DF: The factors are:

$$4D^2 + 3DD' - D'^2 - D - D' \;=\; \big(D + D'\big)\big(4D - D' - 1\big)$$

For $D + D'$, $a = b = 1$. For $4D - D' - 1$, $a = 4, b = -1, c = -1$. Hence, CF is

$$z = \phi_1(x - y) + e^{\frac{x}{4}}\,\phi_2(-x - 4y) = \phi_1(x - y) + e^{\frac{x}{4}}\,\phi_2(x + 4y)$$

We have

$$F(D, D') = 4D^2 + 3DD' - D'^2 - D - D'$$

and

$$F\left(\frac{1}{2}, 1\right) = 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{2}\right)(1) - 1 - \frac{1}{2} - 1 = 0.$$

We write the particular integral as $z = \phi(x, y)e^{(x+2y)/2}$, where $\phi(x, y)$ is a function to be determined.

The given equation is: $F(D, D')[z] = 3e^{(x+2y)/2}$

$\Rightarrow F\left(D + \frac{1}{2}, D' + 1\right) \cdot \phi(x, y)e^{(x+2y)/2} = 3e^{(x+2y)/2}$

$\Rightarrow F\left(D + \frac{1}{2}, D' + 1\right) \cdot \phi(x, y) = 3$

Therefore,

$$\phi(x, y) = \left[F\left(D + \frac{1}{2}, D' + 1\right)\right]^{-1}(3)$$

Problem (contd.)

$$F\left(D+\frac{1}{2},D'+1\right)=4\left(D+\frac{1}{2}\right)^2+3\left(D+\frac{1}{2}\right)(D'+1)-(D'+1)^2-$$
$$(D+\frac{1}{2})-(D'+1)$$

$$=6D-\frac{3}{2}D'+3DD'+4D^2-(D')^2$$
$$=6D\left[1-\frac{1}{4}D'D^{-1}+\frac{1}{6}\{3D'+4D-D'(D')^2\}\right]$$

Problem (contd.)

Therefore,

$$\phi(x,y) = \left[F\left(D+\frac{1}{2}, D'+1\right)\right]^{-1}(3)$$

becomes

$$\phi(x,y) = \frac{1}{2}D^{-1}\left[1-\frac{1}{4}D'D^{-1}+\frac{1}{6}\{3D'+4D-D'(D')^2\}\right]^{-1}(1)$$

$$= \frac{1}{2}D^{-1}(1) = \frac{x}{2}.$$

Hence, the particular integral is given by

$$z = \frac{x}{2}e^{(x+2y)/2}$$

Problem

Solve the PDE

$$\left(2D^2 + 5DD' - 3D'^2\right) z \;=\; \sin\big(2x - y\big).$$

To find CF: The factors are
$2D^2 + 5DD' - 3D'^2 \;=\; \big(2D - D'\big)\,\big(D + 3D'\big).$
For $2D - D'$, $a = 2, b = -1$. For $D + 3D'$, $a = 1, b = 3$.
Therefore, CF is
$z = \phi_1(-x - 2y) + \phi_2(3x - y) = \phi_1(x + 2y) + \phi_2(3x - y)$

We have the right hand side as $\sin(ax+by)$,
where a = 2, b = -1
Hence,

$$F(D^2, DD', (D')^2)\sin(2x-y) = [2(-4)+5(2)-3(-1)]\sin(2x-y)$$
$$= 5\sin(2x-y)$$

The particular integral is given by

$$z = \frac{\sin(2x-y)}{F(-a^2, -ab, -b^2)} = \frac{1}{5}\sin(2x-y)$$