PES University, Bangalore

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Department of Science and Humanities

Engineering Mathematics - I (UE25MA141A)

Notes

Unit - 4: Special Functions

Beta and Gamma Functions

Beta and Gamma functions are improper integrals that are commonly encountered in many scientific and engineering applications. These functions are used in evaluating many definite integrals.

Gamma function: The Gamma function is defined by $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx, \quad \alpha > 0.$

Some identities of Gamma function:

1. 1. $\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1.$

2. $\Gamma(\alpha+1) = \alpha\Gamma(\alpha).$

   Integrating $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$ by parts, we get

   $\Gamma(\alpha+1) = \int_0^\infty x^{\alpha}e^{-x}dx = -\left[x^\alpha e^{-x}\right]_0^\infty + \alpha\int_0^\infty x^{\alpha-1}e^{-x}dx = \alpha\Gamma(\alpha).$

   If α is negative and not an integer, then we write $\Gamma(\alpha) = \frac{1}{\alpha}\Gamma(\alpha+1).$

3. $\Gamma(m+1) = m!,$ for any positive integer m.

   We have $\Gamma(m+1) = m\Gamma(m) = m(m-1)\Gamma(m-1) = \ldots = m(m-1)\ldots 1\Gamma(1) = m!.$

4. $\Gamma(1/2) = \sqrt{\pi}.$

   We have $\Gamma(1/2) = \int_0^\infty x^{-1/2}e^{-x}dx = 2\int_0^\infty e^{-u^2}du.$ (set $x = u^2$)

   We write

   $\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \left[2\int_0^\infty e^{-u^2}du\right]\left[2\int_0^\infty e^{-v^2}dv\right] = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}dudv.$

   Changing to polar coordinates $u = r\cos\theta, v = r\sin\theta,$ we obtain $dudv = rdrd\theta$ and

   $\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\pi/2}\int_0^\infty re^{-r^2}drd\theta = 2\pi\int_0^\infty re^{-r^2}dr = -\pi\left[e^{-r^2}\right]_0^\infty = \pi.$

   Hence, $\Gamma(1/2) = \sqrt{\pi}.$

5. $\Gamma(-1/2) = -2\sqrt{\pi}.$

   We have $\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}$. Substituting $\alpha = -1/2,$ we get

   $\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}.$

Beta function: The Beta function is defined by

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad m>0,\, n>0.$$

Some identities of Beta functions:

1. $\beta(m,n) = \beta(n,m)$

   Substitute $x = 1-t$ in the defintion of $\beta(m,n)$ and simplify.

2. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}(\theta)\cos^{2n-1}(\theta)\,d\theta = 2\int_0^{\pi/2} \sin^{2n-1}(\theta)\cos^{2m-1}(\theta)\,d\theta.$

   (substitute $x = \sin^2\theta$ in the defintion of $\beta(m,n)$ and simplify).

3. $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

   (substitute $x = t/(1+t)$ in the defintion of $\beta(m,n)$ and simplify).

4. Relationship between Beta and Gamma functions:

   Prove that $\quad \beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, m>0, n>0.$

   We can prove this result using double integrals and change of variables. We have

   $\Gamma(m) = \int_0^\infty x^{m-1}e^{-x}dx = 2\int_0^\infty u^{2m-1}e^{-u^2}du, \quad (\text{set } x = u^2)$

   $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx = 2\int_0^\infty v^{2n-1}e^{-v^2}dv, \quad (\text{set } x = v^2)$

   $\Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^\infty u^{2m-1}v^{2n-1}e^{-(u^2+v^2)}dudv.$

   Changing to polar coordinates, $u = r\cos\theta,\ v = r\sin\theta,$ we get

   $$\begin{aligned}\Gamma(m)\Gamma(n) &= 4\int_0^{\pi/2}\int_0^\infty \cos^{2m-1}(\theta)\sin^{2n-1}(\theta)r^{2m+2n-1}e^{-r^2}drd\theta \\ &= 4\left[\int_0^\infty r^{2m+2n-1}e^{-r^2}dr\right]\left[\int_0^{\pi/2}\cos^{2m-1}(\theta)\sin^{2n-1}(\theta)d\theta\right] \\ &= 2\beta(m,n)\int_0^\infty r^{2m+2n-1}e^{-r^2}dr, \quad (\text{using Eq. (1.76)})\end{aligned}$$

   We also have

   $\Gamma(m+n) = \int_0^\infty x^{m+n-1}e^{-x}dx = 2\int_0^\infty r^{2m+2n-1}e^{-r^2}dr, \quad (\text{set } x = r^2)$

   Combining the two results, we obtain

   $$\Gamma(m)\Gamma(n) = \beta(m,n)\Gamma(m+n), \quad \text{or} \quad \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

5. $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1).$

   We have

   $$\begin{aligned}\beta(m+1,n) &= 2\int_0^{\pi/2}\sin^{2m+1}(\theta)\cos^{2n-1}(\theta)d\theta = 2\int_0^{\pi/2}\sin^{2m-1}(\theta)\sin^2\theta\cos^{2n-1}(\theta)d\theta \\ &= 2\int_0^{\pi/2}\sin^{2m-1}(\theta)\cos^{2n-1}(\theta)(1-\cos^2\theta)d\theta \\ &= 2\int_0^{\pi/2}\sin^{2m-1}(\theta)\cos^{2n-1}(\theta)d\theta - 2\int_0^{\pi/2}\sin^{2m-1}(\theta)\cos^{2n+1}(\theta)d\theta \\ &= \beta(m,n) - \beta(m,n+1).\end{aligned}$$

   Therefore, $\quad \beta(m,n) = \beta(m+1,n) + \beta(m,n+1).$

Example: Given that $\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$,

$$\text{show that} \quad \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}.$$

Solution: Let $\frac{x}{1+x} = y$. Solving for x, we get $x = \frac{y}{1-y}$ and $dx = \frac{1}{(1-y)^2} dy$.

Then, $I = \int_0^\infty \frac{x^{p-1}}{1+x} dx = \int_0^1 y^{p-1}(1-y)^{-p} dy = \beta(p, 1-p) = \frac{\Gamma(p)\Gamma(1-p)}{\Gamma(1)}$.

Hence, the result.

Example: Compute (a) $\Gamma(4.5)$ (b) $\Gamma(-3.5)$ (c) $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ (d) $\beta\left(\frac{5}{2}, \frac{3}{2}\right)$ (e) $\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}$.

a. Using $\Gamma(p+1) = p\Gamma(p)$

$$\begin{aligned}
\Gamma(4.5) &= \Gamma(3.5+1) = 3.5\Gamma(3.5) = (3.5)(2.5)\Gamma(2.5) \\
&= (3.5)(2.5)(1.5)\Gamma(1.5) = (3.5)(2.5)(1.5)(0.5)\Gamma(0.5) \\
&= 6.5625/\sqrt{\pi} = 11.62875
\end{aligned}$$

b. Using $\Gamma(p) = \frac{\Gamma(p+1)}{p}$

$$\begin{aligned}
\Gamma(-3.5) &= \frac{\Gamma(-3.5+1)}{-3.5} = \frac{\Gamma(-2.5)}{-3.5} \\
&= \frac{\Gamma(-2.5+1)}{(-2.5)(-3.5)} = \frac{\Gamma(-1.5)}{(-2.5)(-3.5)} \\
&= \frac{\Gamma(-0.5)}{(-1.5)(-2.5)(-3.5)} \\
&= \frac{\Gamma(0.5)}{(-0.5)(-1.5)(-2.5)(-3.5)} \\
&= \frac{\sqrt{\pi}}{(0.5)(1.5)(2.5)(3.5)} \\
&= \frac{\sqrt{\pi}}{13.125} = 0.270019
\end{aligned}$$

c. Using $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$$\begin{aligned}
\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) &= \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) \\
&= \frac{\pi}{\sin\frac{\pi}{4}} = \sqrt{2}\pi = 4.444
\end{aligned}$$

d.

$$\begin{aligned}
\beta\left(\frac{5}{2},\frac{3}{2}\right) &= \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{3}{2}\right)} \\
&= \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)} \\
&= \frac{\left(\frac{3}{4}\sqrt{\pi}\right)\left(\frac{1}{2}\sqrt{\pi}\right)}{6} \\
&= \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{6} \\
&= \frac{3\pi}{48} \\
&= \frac{\pi}{16}
\end{aligned}$$

$$\boxed{\beta\left(\frac{5}{2},\frac{3}{2}\right) = \frac{\pi}{16}}$$

e.

$$\begin{aligned}
\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} &= \frac{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{n!} \\
&= \frac{(2n-1)(2n-3)\cdots 1}{2^n n!}\sqrt{\pi}
\end{aligned}$$

$$\boxed{\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} = \frac{(2n-1)(2n-3)\cdots 1}{2^n n!}\sqrt{\pi}}$$

Example: Evaluate $I = \int_0^\infty x^4 e^{-x^4}\,dx$.
Solution: Put $x^4 = t,\ 4x^3dx = dt,\ dx = \frac{1}{4}t^{-\frac{3}{4}}dt$

$$\begin{aligned}
I &= \int_0^\infty t\cdot e^{-t}\cdot t^{-\frac{3}{4}}dt\cdot\frac{1}{4} \\
&= \frac{1}{4}\int_0^\infty e^{-t}\cdot t^{\frac{1}{4}}dt \\
&= \frac{1}{4}\Gamma\left(1+\frac{1}{4}\right) \\
&= \frac{1}{4}\Gamma\left(\frac{5}{4}\right)
\end{aligned}$$

Example: Evaluate $I = \int_0^1 x^{1/3} \ln\left(\frac{1}{x}\right) dx$.
Solution: Let $t = \ln\left(\frac{1}{x}\right) \implies x = e^{-t}$, so $dx = -e^{-t}dt$.
Change of limits:

- When $x = 0$, $t \to \infty$
- When $x = 1$, $t = 0$

$$\begin{aligned}
I &= \int_{x=0}^{x=1} x^{1/3} \ln\left(\frac{1}{x}\right) dx \\
&= \int_{t=\infty}^{t=0} (e^{-t})^{1/3} \cdot t \cdot (-e^{-t}) dt \\
&= -\int_{t=\infty}^{t=0} t \cdot e^{-t/3} \cdot e^{-t} dt \\
&= -\int_{t=\infty}^{t=0} t \cdot e^{-4t/3} dt \\
&= \int_{t=0}^{t=\infty} t \cdot e^{-4t/3} dt
\end{aligned}$$

Let $y = \frac{4t}{3} \implies t = \frac{3y}{4},\ dt = \frac{3}{4}dy$.

$$\begin{aligned}
I &= \int_{y=0}^{y=\infty} \frac{3y}{4} e^{-y} \cdot \frac{3}{4} dy \\
&= \frac{9}{16} \int_0^\infty y e^{-y} dy \\
&= \frac{9}{16} \Gamma(2) \\
&= \frac{9}{16} \cdot 1! \\
&= \frac{9}{16}
\end{aligned}$$

Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{3\sqrt{\sin 8x}}{\sqrt{\cos x}}\,dx.$
Answer:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{3\sqrt{\sin 8x}}{\sqrt{\cos x}}\,dx &= \int_0^{\frac{\pi}{2}} \sin^{\frac{8}{3}} x \cdot \cos^{-\frac{1}{2}} x\,dx \\
&= \frac{1}{2}\,\mathrm{B}\left(\frac{\frac{8}{3}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right) \\
&= \frac{1}{2}\,\mathrm{B}\left(\frac{11}{6}, \frac{1}{4}\right) \\
&= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{11}{6}+\frac{1}{4}\right)} \\
&= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{4}\right)}{\frac{13}{12}\cdot\frac{1}{12}\Gamma\left(\frac{1}{12}\right)} \\
&= \frac{60}{13} \cdot \frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{12}\right)}
\end{aligned}$$

Evaluate $I = \int_0^{\infty} a^{-bx^2}\,dx.$
Solution:

$$\text{Put } a^{-bx^2} = e^{-t}, \text{ so } -bx^2 \ln a = -t.$$

$$\Rightarrow 2bx\ln a\,dx = dt, \quad x = \left(\frac{t}{b\ln a}\right)^{1/2}$$

$$\text{So} \quad dx = \frac{t^{-1/2}\,dt}{(2b\ln a)^{1/2}}$$

$$\begin{aligned}
I &= \int_0^{\infty} e^{-t}\cdot t^{-1/2}\frac{dt}{(2b\ln a)^{1/2}} \\
&= \frac{1}{(2b\ln a)^{1/2}}\int_0^{\infty} e^{-t}t^{-1/2}dt \\
&= \frac{1}{(2b\ln a)^{1/2}}\cdot\Gamma\left(1-\frac{1}{2}\right) \\
&= \frac{1}{(2b\ln a)^{1/2}}\cdot\Gamma\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{\pi}}{(2b\ln a)^{1/2}}
\end{aligned}$$

Example 7: Evaluate

$$I = \left[\int_0^{\infty} x\,e^{-x^8}dx\right] \times \left[\int_0^{\infty} x^2\,e^{-x^4}dx\right].$$

Solution: Let $I = I_1 \times I_2$.

For I_1:
Put $x^8 = t$ in I_1, so $x = t^{1/8}$, $dx = \frac{1}{8}t^{-7/8}dt$.

$$\begin{aligned}
I_1 = \int_0^\infty x\,e^{-x^8}dx &= \int_0^\infty t^{1/8}e^{-t}\cdot\frac{1}{8}t^{-7/8}dt \\
&= \frac{1}{8}\int_0^\infty t^{(1/8)-(7/8)}e^{-t}dt \\
&= \frac{1}{8}\int_0^\infty t^{-3/4}e^{-t}dt \\
&= \frac{1}{8}\Gamma\left(1-\frac{3}{4}\right) \\
&= \frac{1}{8}\Gamma\left(\frac{1}{4}\right)
\end{aligned}$$

For I_2:
Put $x^4 = t$ in I_2, so $x = t^{1/4}$, $dx = \frac{1}{4}t^{-3/4}dt$.

$$\begin{aligned}
I_2 = \int_0^\infty x^2e^{-x^4}dx &= \int_0^\infty (t^{1/4})^2e^{-t}\cdot\frac{1}{4}t^{-3/4}dt \\
&= \int_0^\infty t^{1/2}e^{-t}\cdot\frac{1}{4}t^{-3/4}dt \\
&= \frac{1}{4}\int_0^\infty t^{1/2-3/4}e^{-t}dt \\
&= \frac{1}{4}\int_0^\infty t^{-1/4}e^{-t}dt \\
&= \frac{1}{4}\Gamma\left(1-\frac{1}{4}\right) \\
&= \frac{1}{4}\Gamma\left(\frac{3}{4}\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
I = I_1\cdot I_2 &= \frac{1}{8}\Gamma\left(\frac{1}{4}\right)\cdot\frac{1}{4}\Gamma\left(\frac{3}{4}\right) \\
&= \frac{1}{32}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)
\end{aligned}$$

Recall that $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$, so

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\frac{\sqrt{2}}{2}} = \sqrt{2}\pi$$

Therefore,

$$I = \frac{1}{32}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{32}\sqrt{2}\pi = \frac{\sqrt{2}\pi}{32}$$

Example: Show that $\int_0^\infty \frac{x^2 dx}{(1+x^4)^3} = \frac{5\pi\sqrt{2}}{128}$.
Solution:

$$\text{Put } x = \sqrt{\tan\theta}, \quad dx = \frac{1}{2}\frac{1}{\sqrt{\tan\theta}} \cdot \sec^2\theta \, d\theta$$

$$\begin{aligned}
\int_0^\infty \frac{x^2 dx}{(1+x^4)^3} &= \int_0^{\frac{\pi}{2}} \frac{\tan\theta \cdot \frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta}{(1+\tan^2\theta)^3} d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\tan\theta)^{1/2}\sec^{-4}\theta \, d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{1/2}\theta\cos^{7/2}\theta \, d\theta
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \cdot \beta\left(\frac{1+1/2}{2}, \frac{1+7/2}{2}\right) \\
&= \frac{1}{4} \cdot \beta\left(\frac{3}{4}, \frac{9}{4}\right) \\
&= \frac{1}{4} \cdot \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{9}{4}\right)}{\Gamma(3)} \\
&= \frac{1}{4} \cdot \frac{\Gamma\left(\frac{3}{4}\right) \cdot \frac{5}{4} \cdot \frac{1}{4}\Gamma\left(\frac{1}{4}\right)}{2} \\
&= \frac{1}{4} \cdot \frac{5}{16} \cdot \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) \\
&= \frac{5}{128}\pi\sqrt{2}
\end{aligned}$$

since $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi$.

Example: Prove that

$$\int_0^{\frac{\pi}{2}} \frac{\cos^{2m-1}\theta \cdot \sin^{2n-1}\theta\, d\theta}{(a\cos^2\theta + b\sin^2\theta)^{m+n}} = \frac{\beta(m,n)}{2a^m b^n}.$$

Solution:

$$\text{Put } \tan\theta = t, \quad d\theta = \cos^2\theta\, dt, \quad \sin\theta = t\cos\theta$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos^{2m-1}\theta \cdot \sin^{2n-1}\theta\, d\theta}{(a\cos^2\theta + b\sin^2\theta)^{m+n}}$$
$$= \int_0^{\infty} \frac{\cos^{2m-1}\theta \cdot t^{2n-1} \cdot \cos^{2n-1}\theta \cdot \cos^2\theta\, dt}{(a\cos^2\theta + bt^2\cos^2\theta)^{m+n}}$$
$$= \int_0^{\infty} \frac{\cos^{2m+2n}\theta \cdot t^{2n-1} dt}{\cos^{2m+2n}\theta (a+bt^2)^{m+n}}$$
$$= \int_0^{\infty} \frac{t^{2n-1} dt}{(a+bt^2)^{m+n}}$$

$$\text{Put } \sqrt{b}t = \sqrt{a}y \implies t = \frac{\sqrt{a}}{\sqrt{b}}y, \quad dt = \frac{\sqrt{a}}{\sqrt{b}}dy$$

$$= \int_0^{\infty} \frac{\left(\frac{\sqrt{a}}{\sqrt{b}}y\right)^{2n-1} \cdot \frac{\sqrt{a}}{\sqrt{b}}dy}{(a + b\left(\frac{\sqrt{a}}{\sqrt{b}}y\right)^2)^{m+n}}$$
$$= \frac{a^n}{b^n}\int_0^{\infty} \frac{y^{2n-1}dy}{(a+ay^2)^{m+n}}$$
$$= \frac{a^n}{b^n}\int_0^{\infty} \frac{y^{2n-1}dy}{a^{m+n}(1+y^2)^{m+n}}$$
$$= \frac{1}{a^m b^n}\int_0^{\infty} \frac{y^{2n-1}dy}{(1+y^2)^{m+n}}$$

$$\text{Let } y^2 = x, \quad y = x^{1/2}, \quad dy = \frac{1}{2}x^{-1/2}dx$$
$$= \frac{1}{2a^m b^n}\int_0^{\infty} \frac{x^{n-1}dx}{(1+x)^{m+n}}$$
$$= \frac{1}{2a^m b^n}\mathrm{B}(n,m)$$
$$= \frac{\beta(m,n)}{2a^m b^n}$$

Example: Evaluate

$$I = \int_0^1 x^{3/2}(1-x^2)^{5/2}dx.$$

Solution:

$$\begin{aligned}
I &= \frac{1}{q}\beta\left(\frac{p+1}{q}, r+1\right) \\
&= \frac{1}{2}\beta\left(\frac{3/2+1}{2}, \frac{5}{2}+1\right) \\
&= \frac{1}{2}\beta\left(\frac{5}{4}, \frac{7}{2}\right) \\
&= \frac{1}{2}\cdot\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{5}{4}+\frac{7}{2}\right)} \\
&= \frac{1}{2}\cdot\frac{\Gamma\left(\frac{5}{4}\right)\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{19}{4}\right)} \\
&= \frac{1}{24}\Gamma\left(\frac{1}{4}\right)\cdot\frac{15}{4}\cdot\frac{11}{4}\cdot\frac{7}{4}\cdot\frac{3}{4}\Gamma\left(\frac{3}{4}\right) \\
&= \frac{4\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{121\Gamma\left(\frac{3}{4}\right)}
\end{aligned}$$

Bessel's differential equation

The boundary value problems (such as the one-dimensional heat equation) with cylindrical symmetry (independent of θ) reduce to two ordinary differential equations by the separation of variables technique. One of them is the most important differential equation known as the **Bessel's differential equation**:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-p^2)y=0$$

or equivalently,

$$x^2y''+xy'+(x^2-p^2)y=0 \tag{1}$$

Here, p, which is a given constant (not necessarily an integer), is known as the order of Bessel's equation.

Bessel's Functions (Cylindrical Functions)

Bessel's functions (Cylindrical functions) are series solutions of the Bessel differential equation (1) obtained by the Frobenius method.

Assume that p is real and non-negative. Assume the series solution of (1) as

$$y(x)=\sum_{m=0}^{\infty}a_mx^{m+r} \qquad (a_0\neq 0) \tag{2}$$

To determine the unknown coefficients a_m and power (exponent) r, substitute (2) in (1), we get

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_mx^{m+r}+\sum_{m=0}^{\infty}(m+r)a_mx^{m+r}+\sum_{m=0}^{\infty}a_mx^{m+r+2}-p^2\sum_{m=0}^{\infty}a_mx^{m+r}=0$$

Now equate the sum of the coefficients of x^{s+r} to zero.

For $s=0$ and $s=1$, the contribution comes from the first, second, and fourth series (not from the third series because it starts with x^{r+2}). For $s\geq 2$, all four terms contribute. Thus, the sum of the coefficients of powers of r, $r+1$, and $s+r$ are respectively given by

$$r(r-1)a_0+ra_0-p^2a_0=0 \qquad (s=0) \tag{4}$$

$$(r+1)ra_1+(r+1)a_1-p^2a_1=0 \qquad (s=1) \tag{5}$$

$$(s+r)(s+r-1)a_s+(s+r)a_s+a_{s-2}-p^2a_s=0 \qquad (s=2,3,\ldots) \tag{6}$$

Solving (4), we get the indicial equation

$$(r+p)(r-p)=0 \tag{7}$$

Solutions of (7) are $r_1=p$ $(p\geq 0)$ and $r_2=-p$.

We consider the following cases:

Case 1: $r_1=p$

With $r_1=p$, Equation (5) becomes $(2p+1)a_1=0$, so $a_1=0$.

Rewrite (6) as

$$(s+r+p)(s+r-p)a_s+a_{s-2}=0$$

Substituting $r = p$, this becomes

$$s(s+2p)a_s + a_{s-2} = 0 \tag{8}$$

or

$$a_s = -\frac{a_{s-2}}{s(s+2p)}$$

For $s = 3$,

$$a_3 = -\frac{a_1}{3(3+2p)}$$

Since $a_1 = 0$ and $p \geq 0$, then $a_3 = 0$. Thus from (8) it follows that

$$a_3 = 0, \quad a_5 = 0, \quad a_7 = 0, \quad \text{etc.}$$

i.e., all coefficients with odd subscripts are zero. Rewriting (8) with $s = 2m$, we have

$$2m(2m+2p)a_{2m} + a_{2m-2} = 0$$

Solving,

$$a_{2m} = -\frac{1}{2^2 m(m+p)} a_{2m-2}, \quad m = 1, 2, \ldots$$

Thus,

$$a_2 = -\frac{a_0}{2^2(1+p)}$$

$$a_4 = -\frac{a_2}{2^2 \cdot 2(2+p)} = \frac{a_0}{2^4 2!(p+1)(p+2)}$$

In general,

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(p+1)(p+2)\cdots(p+m)}, \quad m = 1, 2, \ldots \tag{9}$$

a_0, which is arbitrary, may be taken as

$$a_0 = \frac{1}{2^p \Gamma(p+1)}$$

Then,

$$a_2 = -\frac{a_0}{2^2(p+1)} = -\frac{1}{2^2 \cdot 2^p (p+1)\Gamma(p+1)} = -\frac{1}{2^{2+p}\Gamma(p+2)}$$

since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

Similarly,

$$a_4 = -\frac{a_2}{2^2 \cdot 2(p+2)} = \frac{1}{2^2 \cdot 2^2 \cdot 2^p \cdot 2!(p+2)\Gamma(p+2)} = \frac{1}{2^{4+p} 2!\Gamma(p+3)}$$

In general,

$$a_{2m} = \frac{(-1)^m}{2^{2m+p} m!\Gamma(p+m+1)}, \quad m = 1, 2, \ldots \tag{10}$$

By substituting these coefficients from (10) in (2) and observing that $a_1 = a_3 = a_5 = \cdots = 0$, a particular solution of the Bessel's Equation (1) is obtained as

$$J_p(x) = x^p \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+p} m!\Gamma(p+m+1)} \tag{11}$$

(11) is known as the Bessel's function of the first kind of order p, which converges for all x (by ratio test).

Case 2: For $r_2 = -p$

By replacing p by $-p$ in (11), we get a second linearly independent solution of (1) as

$$J_{-p}(x) = x^{-p} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-p} m! \Gamma(m-p+1)} \tag{12}$$

Hence the general solution of Bessel's Equation (1) for all $x \neq 0$ is

$$y(x) = c_1 J_p(x) + c_2 J_{-p}(x) \tag{13}$$

provided p is not an integer.

Linear Dependence of Bessel's Functions: J_n and J_{-n}

Assume that $p = n$ where n is an integer. Then from (11), we get

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m! \Gamma(n+m+1)}$$

Since $\Gamma(n+1) = n!$, we have $\Gamma(n+m+1) = (n+m)!$.

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m! n! (m+n)!} \tag{14}$$

Replacing p by $-n$ in (11), we get

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m! \Gamma(m-n+1)} \tag{15}$$

When $m-n+1 \leq 0$ or $m \leq (n-1)$, the gamma function of zero or negative integers is infinite. Therefore for $m = 0$ to $n-1$, the coefficients in (15) become zero. So m starts at n. Thus

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m! (m-n)!}$$

since $\Gamma(m-n+1) = (m-n)!$.

Put $m - n = s$ then s varies from 0 to ∞.

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n} x^{2(s+n)-n}}{2^{2(s+n)-n} (s+n)! s!}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s+n}}{2^{2s+n} s! (s+n)!}$$

$$J_{-n}(x) = (-1)^n J_n(x). \tag{16}$$

Recurrence relations (No proof)

1. $[x^v J_v(x)]' = x^v J_{v-1}(x)$.
2. $[x^{-v} J_v(x)]' = -x^{-v} J_{v+1}(x)$.
3. $xJ_v'(x) = xJ_{v-1}(x) - vJ_v(x)$.
4. $xJ_v'(x) = vJ_v(x) - xJ_{v+1}(x)$.
5. $2J_v'(x) = J_{v-1}(x) - J_{v+1}(x)$.
6. $2vJ_v(x) = x\left[J_{v-1}(x) + J_{v+1}(x)\right]$.

Elementary Bessel Functions

Bessel's functions J_p of orders $p = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$ are elementary and can be expressed in terms of sines and cosines and powers of x.

Result 1: $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.

Proof: With $p = \frac{1}{2}$, (11) reduces to

$$J_{\frac{1}{2}}(x) = \sqrt{x} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\frac{1}{2}} m! \Gamma\left(m + \frac{3}{2}\right)}$$

Now

$$\Gamma\left(m + \frac{3}{2}\right) = \left(m + \frac{1}{2}\right)\left(m - \frac{1}{2}\right)\left(m - \frac{3}{2}\right) \cdots \frac{3}{2}\Gamma\left(\frac{1}{2}\right)$$
$$= (2m+1)(2m-1)(2m-3) \cdots 3 \cdot 1 \cdot \sqrt{\pi}/2^m$$

Also

$$2^{2m+1} m! = 2^{m+1} \cdot 2^m m! = 2^{m+1}(2m)(2m-2) \cdots 4 \cdot 2$$

Thus

$$2^{2m+1} m! \cdot \Gamma\left(m + \frac{3}{2}\right) = \left[2^{m+1} \cdot 2m \cdot (2m-2) \cdots 4 \cdot 2\right]\left[(2m+1)(2m-1) \cdots 3 \cdot 1\right] \cdot 2^{-(m+1)} \sqrt{\pi}$$

$$= (2m+1)! \sqrt{\pi}$$

Then

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-1}}{(2m+1)! \sqrt{\pi}} = \sqrt{\frac{2}{\pi x}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = \sqrt{\frac{2}{\pi x}} \sin x.$$

Result 2: In the recurrence relation I, put $p = \frac{1}{2}$ then

$$\frac{d}{dx}\left\{\sqrt{x} J_{\frac{1}{2}}(x)\right\} = \sqrt{x} J_{-\frac{1}{2}}(x)$$

$$\frac{d}{dx}\left\{\sqrt{\frac{2}{\pi x}} \sin x\right\} = \sqrt{x} J_{-\frac{1}{2}}(x)$$

$$\sqrt{\frac{2}{\pi}}\cos x = \sqrt{x}J_{-\frac{1}{2}}(x)$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$$

Result 3:

$$J_{-\frac{1}{2}}(x) + J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x)$$

$$\text{or} \quad J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

Using result (1) and (2) for $J_{\frac{1}{2}}$ and $J_{-\frac{1}{2}}$, we get

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

Similarly with $p = -\frac{1}{2}$ in recurrence relation VI

Result 4:

$$J_{-\frac{3}{2}}(x) = -\frac{1}{x}J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

Example: Evaluate $\int J_5(x)\,dx$.

Solution: Putting n = 4 in

$$\int J_{p+1}(x)\,dx = \int J_{p-1}(x)\,dx - 2J_p(x),$$

we get

$$\int J_5(x)\,dx = \int J_3(x)\,dx - 2J_4(x) \tag{1}$$

Again with $p = 2$

$$\int J_3(x)\,dx = \int J_1(x)\,dx - 2J_2(x) \tag{2}$$

Also we know that

$$\int J_1(x)\,dx = -J_0(x) + c \tag{3}$$

Substituting (2) and (3) in (1)

$$\int J_5(x)\,dx = [-J_0(x) + c] - 2J_2(x) - 2J_4(x)$$

Example: Evaluate $\int x^2 J_1(x)\,dx$.

Solution: Put p = 2 in

$$\int x^p J_{p-1}(x)\,dx = x^p J_p(x) + c$$

Then

$$\int x^2 J_1(x)\,dx = x^2 J_2(x) + c$$

But

$$J_2(x) = \left[\frac{2}{x}J_1(x) - J_0(x)\right]$$

So,

$$\int x^2 J_1(x)\,dx = x^2\left[\frac{2}{x}J_1(x) - J_0(x)\right] + c = 2xJ_1(x) - x^2 J_0(x) + c.$$

Note: $\int J_0(x)\,dx$ can not be integrated but its values are tabulated.

Generating function for Bessel function

The generating function for the Bessel function is

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n. \tag{1}$$

Proof: If $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$ is the generating function of Bessel function, then the coefficients of different powers of t in the expansion of (1) are the Bessel's functions of different integral orders.

Consider

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{\frac{xt}{2}} \cdot e^{-\frac{x}{2t}}$$

Expanding in series, we get

$$= \left[1 + \frac{xt}{2} + \frac{1}{2!}\left(\frac{xt}{2}\right)^2 + \frac{1}{3!}\left(\frac{xt}{2}\right)^3 + \cdots\right]$$

$$\times\left[1 - \frac{x}{2t} + \frac{1}{2!}\left(\frac{x}{2t}\right)^2 - \frac{1}{3!}\left(\frac{x}{2t}\right)^3 + \cdots\right] \tag{2}$$

Case 1: n = 0.

The coefficient of $t^0 = 1$ in the expansion (2) is

$$1 - \left(\frac{x}{2}\right)^2 + \frac{1}{2!^2}\left(\frac{x}{2}\right)^4 - \frac{1}{3!^2}\left(\frac{x}{2}\right)^6 + \frac{1}{4!^2}\left(\frac{x}{2}\right)^8 - \cdots$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m} = J_0(x). \tag{3}$$

Case 2: Positive powers of $t : t^n$

The coefficient of t^n in the above expansion (2) is

$$\frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} + \cdots$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m}{m!(n+m)!}\left(\frac{x}{2}\right)^{n+2m}$$

$$= J_n(x). \tag{4}$$

Case 3: Negative powers of $t : t^{-n}$

$\$textThecoefficientoft^{-n}$ in the expansion (2) is

$$\frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n + \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+1} + \frac{1}{2!(n+2)!}2(-1)^{n+2}\left(\frac{x}{2}\right)^{n+2} + \cdots$$

$$= (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!} \left(\frac{x}{2}\right)^{n+2m}$$

$$= (-1)^n J_n(x) = J_{-n}(x) \tag{5}$$

Thus from (3), (4), and (5), we have

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n.$$

Jacobi Series

Proof: We have

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

Expanding the summation, we get,

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = J_0(x)+tJ_1(x)+t^2J_2(x)+t^3J_3(x)+\cdots+t^{-1}J_{-1}(x)+t^{-2}J_{-2}(x)+t^{-3}J_{-3}(x)+\cdots$$

When n is an integer,

$$J_{-n}(x) = (-1)^n J_n(x)$$

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = J_0(x) + (t-t^{-1})J_1(x) + (t^2+t^{-2})J_2(x) + (t^3-t^{-3})J_3(x) + \cdots \tag{1}$$

Let $t = \cos\theta + i\sin\theta$, then $\dfrac{1}{t} = \cos\theta - i\sin\theta$
Therefore,

$$t + \frac{1}{t} = 2\cos\theta; \qquad t - \frac{1}{t} = 2i\sin\theta$$

and,

$$t^n + \frac{1}{t^n} = 2\cos n\theta; \qquad t^n - \frac{1}{t^n} = 2i\sin n\theta$$

Using equation (1) we get,

$$e^{xi\sin\theta} = J_0(x) + 2(J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots) + 2i(J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots)$$

i.e.,

$$\cos(x\sin\theta)+i\sin(x\sin\theta) = J_0(x)+2(J_2(x)\cos 2\theta+J_4(x)\cos 4\theta+\cdots)+2i(J_1(x)\sin\theta+J_3(x)\sin 3\theta+\cdots)$$

Equating real and imaginary parts, we get,

$$\boxed{\begin{aligned}\cos(x\sin\theta) &= J_0(x) + 2\left(J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots\right)\\ \sin(x\sin\theta) &= 2\left(J_1(x)\sin\theta + J_3(x)\sin 3\theta + J_5(x)\sin 5\theta + \cdots\right)\end{aligned}}$$

Bessel's integral formula

Prove that

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$$

Proof: Consider the Jacobi series

$$\cos(x\sin\theta) = J_0(x) + 2\left(J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots\right) \qquad \cdots (1)$$

$$\sin(x\sin\theta) = 2\left(J_1(x)\sin\theta + J_3(x)\sin 3\theta + J_5(x)\sin 5\theta + \cdots\right) \qquad \cdots (2)$$

$$(1) \times \cos n\theta \implies$$

$$\cos(x\sin\theta)\cos n\theta = J_0(x)\cos n\theta + 2\left(J_2(x)\cos 2\theta\cos n\theta + J_4(x)\cos 4\theta\cos n\theta + \cdots\right)$$

where n is an even integer

Integrating between the limits 0 to π, we get,

$$\int_0^{\pi} \cos(x\sin\theta)\cos n\theta\, d\theta = \int_0^{\pi} J_0\cos n\theta\, d\theta + 2\int_0^{\pi}\left(J_2\cos 2\theta\cos n\theta + J_4\cos 4\theta\cos n\theta + \cdots\right) d\theta$$

$$= 0 + 2J_n\frac{\pi}{2} \qquad \because \int_0^{\pi}\cos n\theta\cos m\theta\, d\theta = \begin{cases} \frac{\pi}{2} & \text{when } n = m, \\ 0 & \text{otherwise} \end{cases}$$

$$\implies \int_0^{\pi}\cos(x\sin\theta)\cos n\theta\, d\theta = J_n\pi \qquad \cdots (3)$$

$(2) \times \sin n\theta \implies$

$$\sin(x\sin\theta)\sin n\theta = 2\left(J_1\sin\theta\sin n\theta + J_3\sin 3\theta\sin n\theta + J_5\sin 5\theta\sin n\theta + \cdots\right)$$

where n is an even integer
Integrating between the limits 0 to π, we get,

$$\int_0^{\pi}\sin(x\sin\theta)\sin n\theta\, d\theta = 2\int_0^{\pi}\left(J_1\sin\theta\sin n\theta + J_3\sin 3\theta\sin n\theta + J_5\sin 5\theta\sin n\theta + \cdots\right) d\theta$$

$$\implies \int_0^{\pi}\sin(x\sin\theta)\sin n\theta\, d\theta = 0 \qquad \cdots (4)$$

Adding (3) and (4), we get,

$$\int_0^{\pi}\left[\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta\right] d\theta = J_n\pi$$

$$\text{i.e.,} \quad \int_0^{\pi}\cos(x\sin\theta - n\theta)d\theta = J_n\pi$$

Therefore,

$$\boxed{J_n = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta - n\theta)d\theta}$$