ENGINEERING PHYSICS

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Barrier Potential

Class # 25

- Barrier potentials of finite widths
- Matter wave incident on a barrier potential $E<V_o$
- Solutions of the SWE
- Interpretation of the wave functions
- Probabilities of barrier penetration

Barrier potential problem statement

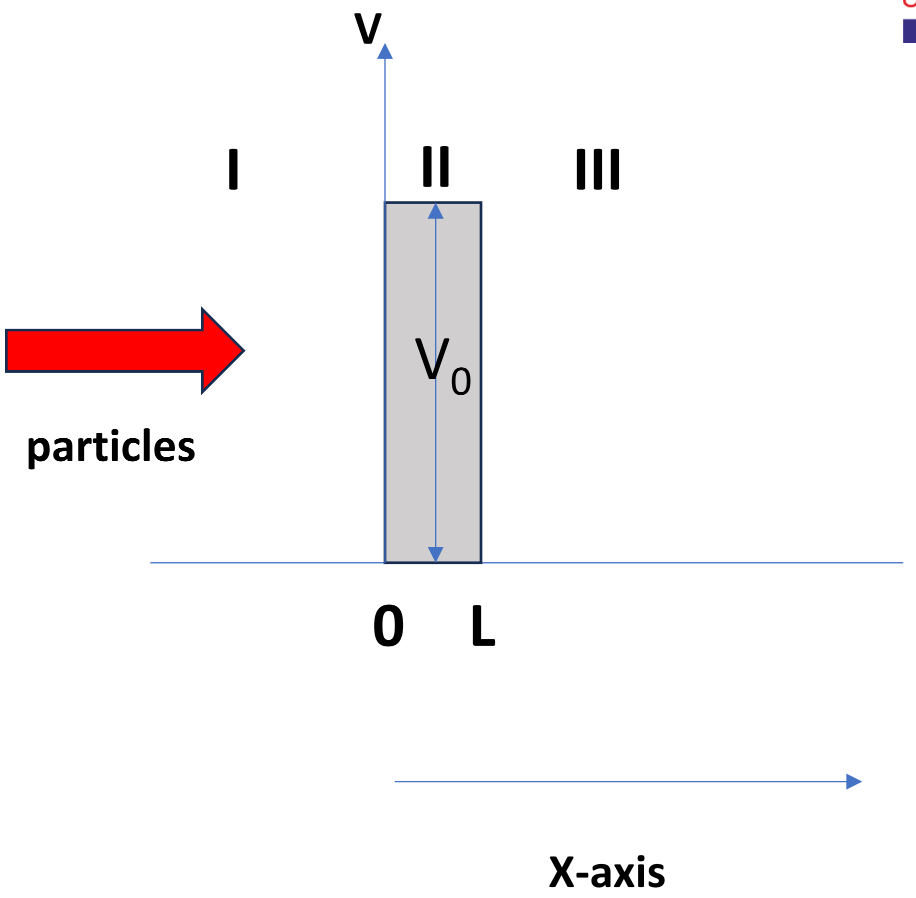
- *A 1D rectangular barrier potential is defined by*

  Region I $V = 0 \quad for\, x < 0$

  Region II $V = V_o\, for\, 0 < x < L$

  Region III $V = 0 \quad for\, x > L$

- *A particle of mass m and energy $E<V_o$ approaching the potential barrier from region I*
- *The three - part solution of the SWE will be wavefunctions of regions I, II and III.*



Barrier Potential – wave functions

Region I $\quad x < 0 \quad V = 0$

Standard free particle solution

$$\frac{d^2\psi_I(x)}{dx^2} + \frac{2m}{\hbar^2}E\psi_I(x) = 0$$

$$\frac{d^2\psi_I(x)}{dx^2} + k_I^2\psi_I(x) = 0$$

Where $k_I = \sqrt{\frac{2mE}{\hbar^2}}$

The solution is

$$\psi_I(x) = Ae^{ik_Ix} + Be^{-ik_Ix}$$

Barrier Potential – wave functions

Region II $0 < x < L$ $\quad V = V_o > E \ (E - V_o)$ *is negative*

- *The SWE reduces to*

$$\frac{d^2\psi_{II}(x)}{dx^2} + \frac{2m}{\hbar^2}(E - V_o)\psi_{II}(x) = 0$$

$$\frac{d^2\psi_{II}(x)}{dx^2} - \alpha^2\psi_{II}(x) = 0$$

where $\alpha = \sqrt{\frac{2m(V_o - E)}{\hbar^2}}$

And the solution is $\psi_{II}(x) = Ce^{\alpha x} + De^{-\alpha x}$

Barrier potential – wave functions

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Region III $x < L \quad V = 0$

Standard free particle solution

$$\frac{d^2\psi_{III}(x)}{dx^2} + \frac{2m}{\hbar^2}E\psi_{III}(x) = 0$$

$$\frac{d^2\psi_{III}(x)}{dx^2} + k_{III}^2\psi_{III}(x) = 0$$

Where $k_{III} = \sqrt{\frac{2mE}{\hbar^2}}$

The solution is

$$\psi_{III}(x) = Ge^{ik_{III}x} + He^{-ik_{III}x}$$

Barrier potential – acceptable wave functions

Region I

$$\psi_I(x) = Ae^{ik_I x} + Be^{-ik_I x}$$

(incident wave) (reflected wave)

Region II

$$\psi_{II}(x) = Ce^{\alpha x} + De^{-\alpha x}$$

Both parts of the solution is acceptable here

Barrier potential – acceptable wave functions

Region III $\psi_{III}(x) = Ge^{ik_{III}x} + He^{-ik_{III}x}$

The first term in RHS is the transmitted wave and the second term is the reflected wave in region III. Now reflection can only happen if the potential in region III undergoes a change. In the situation described there is no change in V. It remans V = 0 up to infinity .

Hence the second term in RHS cannot be allowed. We do this by making H = 0.

Thus, acceptable solution in region III is $\psi_{III}(x) = Ge^{ik_{III}x}$

Barrier potential – acceptable wave functions

Summarising,

Acceptable solutions

Region I $\psi_I(x) = Ae^{ik_Ix} + Be^{-ik_Ix}$

Region II $\psi_{II}(x) = Ce^{\alpha x} + De^{-\alpha x}$

Region III $\psi_{III}(x) = Ge^{ik_{III}x}$

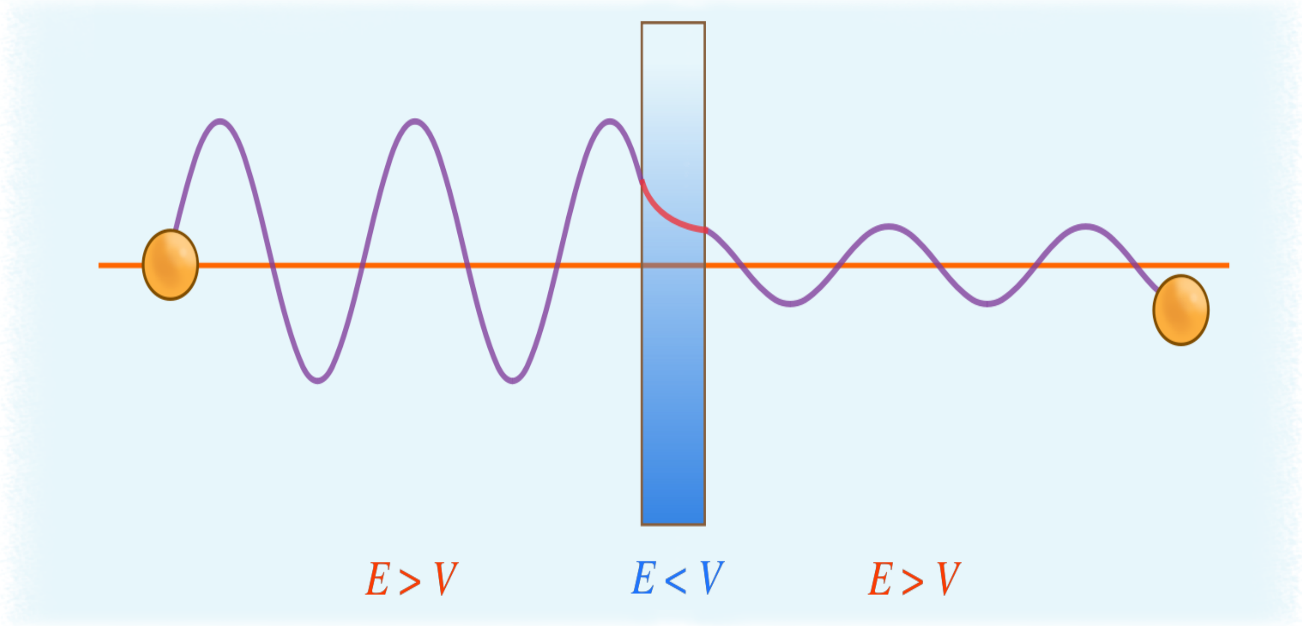

Image courtesy: Chemistry LibreTexts

Barrier potential – probability densities

As we now know ψ in all the three regions, it is natural to compute the probability densities, which by definition is $\psi^*\psi$

Thus for

Region I $\psi_I^*\psi_I$

Region II $\psi_{II}^*\psi_{II} = \psi_{II}^2$

Region III $\psi_{III}^*\psi_{III}$

To get the expressions for the probability densities we need to link the coefficients

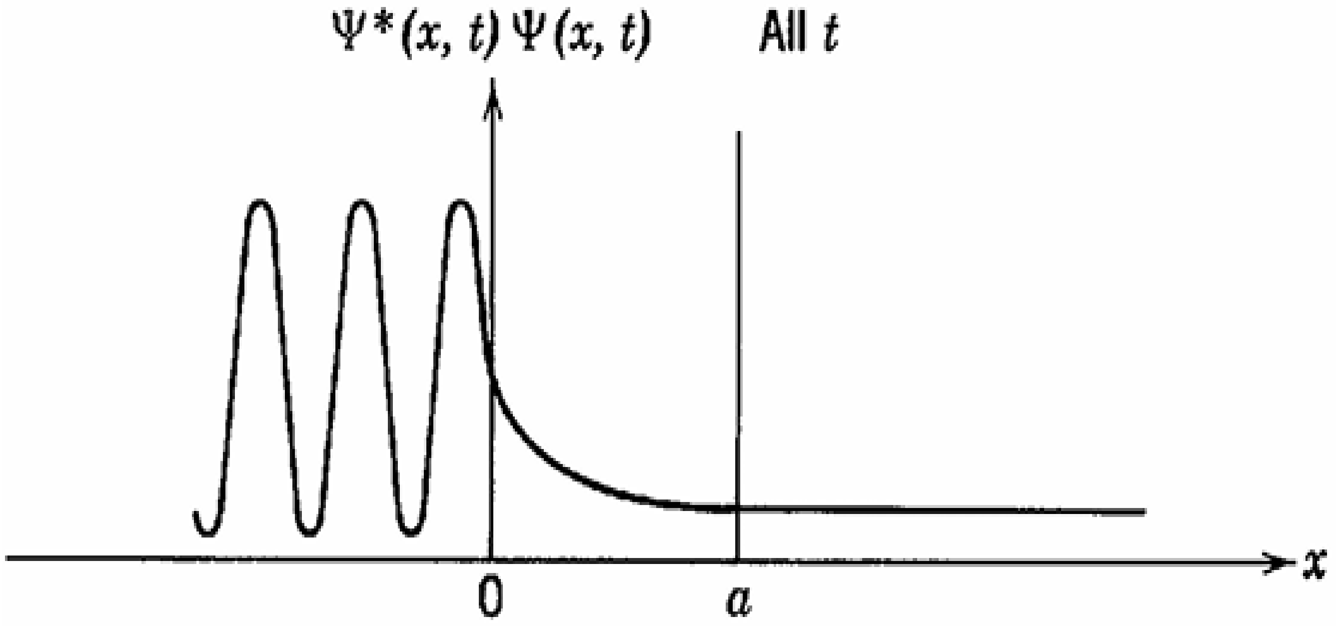

Image courtesy: quantum-physics-2nd-Eisberg_Resnick

Barrier potential- Boundary conditions

The coefficients A, B, C, D and G can be connected by solving the boundary conditions given below

Boundary conditions

Continuity of wave functions

$$\psi_I = \psi_{II}\ at\ x = 0 \qquad and \qquad \psi_{II} = \psi_{III}\ \ at\ x = L$$

Continuity of derivatives

$$\frac{d\psi_I}{dx} = \frac{d\psi_{II}}{dx}\ \ at\ x = 0 \quad and \quad \frac{d\psi_{II}}{dx} = \frac{d\psi_{III}}{dx}\ \ at\ x = L$$

Barrier potential- Boundary conditions

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By applying the boundary conditions we get the following equations

1. $A + B = C + D$ *for* $\psi_I = \psi_{II}\ at\ x = 0$

2. $ik_I(A - B) = \alpha(C - D)$ for $\frac{d\psi_I}{dx} = \frac{d\psi_{II}}{dx}\ \ at\ x = 0$

3. $Ce^{\alpha L} + De^{-\alpha L} = Ge^{ik_{III}L}$ for $\psi_{II} = \psi_{III}\ \ at\ x = L$

4. $\alpha\left(Ce^{\alpha L} - De^{-\alpha L}\right) = ik_{III}Ge^{ik_{III}L}$ for $\frac{d\psi_{II}}{dx} = \frac{d\psi_{III}}{dx}\ \ at\ x = L$

remember that $k_I = k_{III}$ *which we can simply take as k*

Barrier potential- Boundary conditions

Then

1. $A + B = C + D$
2. $ik(A - B) = \alpha(C - D)$
3. $Ce^{\alpha L} + De^{-\alpha L} = Ge^{ikL}$
4. $\alpha\left(Ce^{\alpha L} - De^{-\alpha L}\right) = ikGe^{ikL}$

By solving these we can express B, C, D and G in terms of A

To get A we follow the normalization procedure.

Barrier potential- Boundary conditions

An obvious question that comes to mind based on the following:

We have ψ_I describing what happens in region I.

We have ψ_{II} which tells us that something is going on in region II

We have ψ_{III} which is non-zero and describing a wave!!

Do particles appear on the other side of the barrier even though they do not have energy to go over the barrier? If true, then it is so counterintuitive and further how many of them would manage to do so?

We will address this issue in the next class!

Barrier potential- quiz

When particles of energy E strike a potential barrier of height V_0 (> E) then which of the following is true at the boundary x = 0?

1. All particles will get reflected
2. No particles will get reflected
3. Few particles will get reflected
4. Most particles will get reflected

THANK YOU

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