PES University, Bangalore

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Department of Science and Humanities

Engineering Mathematics - I (UE25MA141A)

Notes

Unit - 2: Higher-Order Differential Equations

Linear Differential Equations

Linear differential equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus the general linear differential equation of the nth order is of the form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + p_n y = X,$$

where $p_1, p_2, \ldots, p_n$ and X are functions of x only.

Linear differential equations with constant co-efficients are of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = X$$

where $k_1, k_2, \ldots, k_n$ are constants. Such equations are most important in the study of electro-mechanical vibrations and other engineering problems.

Theorem

If y_1, y_2 are only two solutions of the equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = 0 \tag{1}$$

then $c_1 y_1 + c_2 y_2$ $(= u)$ is also its solution.

Since $y = y_1$ and $y = y_2$ are solutions of (1),

$$\frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1} y_1}{dx^{n-1}} + k_2 \frac{d^{n-2} y_1}{dx^{n-2}} + \cdots + k_n y_1 = 0 \tag{2}$$

$$\frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1} y_2}{dx^{n-1}} + k_2 \frac{d^{n-2} y_2}{dx^{n-2}} + \cdots + k_n y_2 = 0 \tag{3}$$

If c_1, c_2 be two arbitrary constants, then

$$\begin{aligned}
&\frac{d^n (c_1 y_1 + c_2 y_2)}{dx^n} + k_1 \frac{d^{n-1} (c_1 y_1 + c_2 y_2)}{dx^{n-1}} + \cdots + k_n (c_1 y_1 + c_2 y_2) \\
&= c_1 \left(\frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + k_n y_1 \right) + c_2 \left(\frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots + k_n y_2 \right) \\
&= c_1(0) + c_2(0) = 0 \qquad \text{[By (2) and (3)]}
\end{aligned}$$

i.e.,

$$\frac{d^n u}{dx^n} + k_1 \frac{d^{n-1} u}{dx^{n-1}} + \cdots + k_n u = 0 \tag{4}$$

This proves the theorem.

(2) Since the general solution of a differential equation of the nth order contains n arbitrary constants, it follows, from above, that if $y_1, y_2, y_3, \ldots, y_n$ are n independent solutions of (1), then $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ $(= u)$ is its complete solution.

(3) If $y = v$ be any particular solution of

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = X \tag{5}$$

then

$$\frac{d^n v}{dx^n} + k_1 \frac{d^{n-1} v}{dx^{n-1}} + k_2 \frac{d^{n-2} v}{dx^{n-2}} + \cdots + k_n v = X \tag{6}$$

Adding (4) and (6), we have

$$\frac{d^n (u+v)}{dx^n} + k_1 \frac{d^{n-1} (u+v)}{dx^{n-1}} + \cdots + k_n (u+v) = X$$

This shows that $y = u + v$ is the complete solution of (5).

The part u is called the complementary function (C.F.) and the part v is called the particular integral (P.I.) of (5).

$$\therefore \text{the complete solution (C.S.) of (5) is } y = \text{C.F.} + \text{P.I.}$$

Thus in order to solve the question (5), we have to first find the C.F., i.e., the complete solution of (1), and then the P.I., i.e., a particular solution of (5).

Operator D

Denoting $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}$ etc. by D, D^2, D^3 etc., so that

$$\frac{dy}{dx} = Dy, \quad \frac{d^2 y}{dx^2} = D^2 y, \quad \frac{d^3 y}{dx^3} = D^3 y \text{ etc.,}$$

the equation (5) above can be written in the *symbolic form* $(D^n + k_1 D^{n-1} + \cdots + k_n) y = X$, i.e.,

$$f(D) y = X,$$

where $f(D) = D^n + k_1 D^{n-1} + \cdots + k_n$, i.e., a polynomial in D.

Thus the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity, i.e., $f(D)$ can be factorised by ordinary rules of algebra and the factors may be taken in any order. For instance

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} - 3y = (D^2 + 2D - 3)y = (D+3)(D-1)y \text{ or } (D-1)(D+3)y.$$

Rules for finding complementary function

To solve the equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = 0 \tag{1}$$

where k's are constants.

The equation (1) in symbolic form is

$$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n) y = 0 \tag{2}$$

Its symbolic co-efficient equated to zero i.e.

$$D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n = 0$$

is called the *auxiliary equation (A.E.)*. Let $m_1, m_2, \ldots, m_n$ be its roots.

Case I. *If all the roots be real and different*, then (2) is equivalent to

$$(D - m_1)(D - m_2) \cdots (D - m_n) y = 0 \tag{3}$$

Now (3) will be satisfied by the solution of $(D - m_1) y = 0$, i.e., by $\frac{dy}{dx} - m_1 y = 0$. This is a Leibnitz's linear and I.F. $= e^{-m_1 x}$

$$\therefore \text{ its solution is } y e^{-m_1 x} = c_1, \text{ i.e., } y = c_1 e^{m_1 x}$$

Similarly, since the factors in (3) can be taken in any order, it will be satisfied by the solutions of $(D - m_1) y = 0$, $(D - m_2) y = 0$ etc. i.e., by $y = c_1 e^{m_1 x}, y = c_2 e^{m_2 x}$ etc.

Thus the complete solution of (1) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$ (4)

Case II. *If two roots are equal* (i.e., $m_1 = m_2$), then (4) becomes

$$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

It has only $n - 1$ arbitrary constants and is, therefore, not the complete solution of (1). In this case, we proceed as follows:

The part of the complete solution corresponding to the repeated root is the complete solution of $(D - m_1)^2 y = 0$

Putting $(D - m_1) y = z$, it becomes $(D - m_1) z = 0$ or $\frac{dz}{dx} - m_1 z = 0$

This is a Leibnitz's linear in z and I.F. $= e^{-m_1 x}$ $\therefore$ its solution is $z e^{-m_1 x} = c_1$ or $z = c_1 e^{m_1 x}$

$(D - m_1) y = z = c_1 e^{m_1 x}$ or $\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$ (5)

Its I.F. being $e^{-m_1 x}$, the solution of (5) is

$$y e^{-m_1 x} = \int c_1 dx + c_2 = c_1 x + c_2 \quad \text{or} \quad y = (c_1 x + c_2) e^{m_1 x}$$

Thus the complete solution of (1) is $y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$

If, however, the A.E. has three equal roots (i.e., $m_1 = m_2 = m_3$), then the complete solution is

$$y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$$

Case III. *If one pair of roots be imaginary*, i.e., $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the complete solution is

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

$$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

$$= e^{\alpha x}(c_1 \cos\beta x + i \sin\beta x) + c_2(\cos\beta x - i\sin\beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

$$= e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

where $C_1 = c_1 + c_2$, $C_2 = i(c_1 - c_2)$ [*by Euler's Theorem*, $e^{i\theta} = \cos\theta + i\sin\theta$]

Case IV. *If two points of imaginary roots be equal* i.e., $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$, then by case II, the complete solution is

$$y = e^{\alpha x}\left[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x\right] + \cdots + c_n e^{m_n x}$$

Example: Find the solution of the differential equation $y'' - y' - 6y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^2 - m - 6 = 0, \quad \text{or} \quad (m-3)(m+2) = 0, \quad \text{or} \quad m = -2,\, 3.$$

The two linearly independent solutions are e^{3x} and e^{-2x}. The general solution is

$$y(x) = Ae^{3x} + Be^{-2x}.$$

Example: Solve the initial value problem

$$4y'' - 8y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 3.$$

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$4m^2 - 8m + 3 = 0, \quad \text{or} \quad m = 1/2,\, 3/2.$$

Hence, the linearly independent solutions are $e^{x/2}$ and $e^{3x/2}$. The general solution is

$$y(x) = Ae^{(3x)/2} + Be^{x/2}.$$

Substituting the initial conditions, we get

$$y(0) = 1 = A + B, \qquad y'(0) = 3 = \frac{3A}{2} + \frac{B}{2}.$$

Solving the above equations, we get $A = 5/2$ and $B = -3/2$. The solution of the initial value problem is

$$y(x) = \left[5e^{(3x)/2} - 3e^{x/2}\right]/2.$$

Example: Find the solution of the differential equation $4y'' + 4y' + y = 0$.

Solution Assume a solution of the form $y = e^{mx}$. The characteristic equation is given by

$$4m^2 + 4m + 1 = 0, \quad \text{or} \quad (2m+1)^2 = 0, \quad \text{or} \quad m = -\frac{1}{2},\, -\frac{1}{2},$$

which is a repeated root. Hence, the general solution is $y(x) = (A + Bx)e^{-x/2}$.

Example: Solve the initial value problem

$$y'' + 6y' + 9y = 0, \quad y(0) = 2, \quad y'(0) = 3.$$

Solution Assume a solution of the form $y = e^{mx}$. The characteristic equation is given by

$$m^2 + 6m + 9 = 0, \quad \text{or} \quad (m+3)^2 = 0, \quad \text{or} \quad m = -3, \ -3,$$

which is a repeated root. The general solution is $y(x) = (A + Bx)e^{-3x}$. Substituting in the initial conditions, we get

$$y(0) = 2 = A, \qquad y' = Be^{-3x} - 3(A + Bx)e^{-3x}, \qquad y'(0) = 3 = B - 3A.$$

The solution is $A = 2$, $B = 9$. The solution of the given initial value problem is

$$y(x) = (2 + 9x)e^{-3x}.$$

Example: Find the solution of the differential equation $y'' + 2y' + 2y = 0$.

Solution Assume a solution of the form $y = e^{mx}$. The characteristic equation is given by

$$m^2 + 2m + 2 = 0, \quad \text{or} \quad m = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i = p \pm iq.$$

The general solution is

$$y(x) = (A \cos qx + B \sin qx)e^{px} = (A \cos x + B \sin x)e^{-x}.$$

Example: Find the solution of the initial value problem

$$y'' + 4y' + 13y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution Assume a solution of the form $y = e^{mx}$. The characteristic equation is given by

$$m^2 + 4m + 13 = 0, \quad \text{or} \quad m = \frac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i = p \pm iq.$$

The general solution is given by

$$y(x) = [A \cos qx + B \sin qx]e^{px} = [A \cos 3x + B \sin 3x]e^{-2x}.$$

Substituting in the initial conditions, we obtain

$$y(0) = 0 = A,$$

$$y'(x) = Be^{-2x}(3 \cos 3x - 2 \sin 3x), \qquad y'(0) = 1 = 3B, \quad \text{or} \quad B = 1/3.$$

The solution of the initial value problem is

$$y(x) = \frac{e^{-2x} \sin 3x}{3}.$$

Inverse Operator

(1) Definition. $\frac{1}{f(D)}X$ is that function of x, not containing arbitrary constants, which when operated upon by $f(D)$ gives X.

i.e.,

$$f(D)\left[\frac{1}{f(D)}X\right] = X$$

Thus $\frac{1}{f(D)}X$ satisfies the equation $f(D)y = X$ and is, therefore, its particular integral.

Obviously, $f(D)$ and $1/f(D)$ are inverse operators.

(2)

$$\frac{1}{D}X = \int X dx$$

Let

$$\frac{1}{D}X = y$$

Operating by D,

$$D\frac{1}{D}X = Dy \quad \text{i.e.,} \quad X = \frac{dy}{dx}$$

Integrating both sides w.r.t. x, $y = \int X dx$, no constant being added as (i) does not contain any constant.

Thus

$$\frac{1}{D}X = \int X dx.$$

(3)

$$\frac{1}{D-a}X = e^{ax}\int e^{-ax}X dx.$$

Let

$$\frac{1}{D-a}X = y$$

Operating by $D-a$, $(D-a)\frac{1}{D-a}X = (D-a)y$.

or

$$\frac{dy}{dx} - ay = X$$

which is a Leibnitz's linear equation.

I.F. being e^{-ax}, its solution is

$$ye^{-ax} = \int Xe^{-ax}dx,$$

no constant being added as (ii) doesn't contain any constant.

Thus

$$\frac{1}{D-a}X = y = e^{ax}\int Xe^{-ax}dx.$$

Rules for finding particular integral

Consider the equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = X$$

which is symbolic form of $(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n)y = X$.

$$\therefore \quad \text{P.I.} = \frac{1}{D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n} X.$$

Case I. When $X = e^{ax}$

Since

$$De^{ax} = ae^{ax}$$
$$D^2 e^{ax} = a^2 e^{ax}$$
$$\vdots$$
$$D^r e^{ax} = a^r e^{ax}$$

$$\therefore \quad (D^n + k_1 D^{n-1} + \cdots + k_n) e^{ax} = (a^n + k_1 a^{n-1} + \cdots + k_n) e^{ax}, \quad \text{i.e.,} \quad f(D)e^{ax} = f(a)e^{ax}$$

Operating on both sides by $\frac{1}{f(D)}$,

$$\frac{1}{f(D)} f(D) e^{ax} = \frac{1}{f(D)} f(a) e^{ax} \quad \text{or} \quad e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$$

$\therefore$ dividing by $f(a)$,

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad \text{provided } f(a) \neq 0 \tag{1}$$

If $f(a) = 0$, the above rule fails and we proceed further.
Since a is a root of A.E. $f(D) = D^n + k_1 D^{n-1} + \cdots + k_n = 0$.
$\therefore D - a$ is a factor of $f(D)$. Suppose $f(D) = (D-a)\phi(D)$, where $\phi(a) \neq 0$. Then

$$\frac{1}{f(D)} = \frac{1}{(D-a)\phi(D)} = \frac{1}{\phi(a)} \frac{1}{D-a} \quad \text{(By (1))}$$

$$\frac{1}{D-a} e^{ax} = e^{ax} \int e^{-ax} e^{ax} dx = e^{ax} \int dx = x e^{ax}$$

i.e.,

$$\frac{1}{f(D)} e^{ax} = \frac{1}{\phi(a)} x e^{ax}$$

But $\phi(a) = f'(a)$, so

$$\frac{1}{f(D)} e^{ax} = \frac{x}{f'(a)} e^{ax} \quad \text{provided } f'(a) \neq 0 \tag{2}$$

If $f'(a) = 0$, then applying (2) again, we get

$$\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}, \quad \text{provided } f''(a) \neq 0 \tag{3}$$

and so on.

Example: *Find the P.I. of* $(D^2+5D+6)y = e^x$.
Solution.

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2+5D+6}e^x \quad [\text{Put } D=1]\\ &= \frac{1}{1^2+5\cdot 1+6}e^x = \frac{1}{12}e^x\end{aligned}$$

Example: *Find the P.I. of* $(D+2)(D-1)^2 y = e^{2x} + 2\sinh x$.
Solution.

$$\text{P.I.} = \frac{1}{(D+2)(D-1)^2}[e^{2x}+2\sinh x] = \frac{1}{(D+2)(D-1)^2}e^{2x} + \frac{1}{(D+2)(D-1)^2}(e^x - e^{-x})$$

Let us evaluate each of these terms separately.

$$\begin{aligned}\frac{1}{(D+2)(D-1)^2}e^{2x} &= \frac{1}{(D-1)^2}\frac{1}{D+2}e^{2x}\\ &= \frac{1}{(D-1)^2}\frac{1}{2+2}e^{2x} = \frac{1}{(D-1)^2}\frac{1}{4}e^{2x}\\ &= \frac{1}{4}\frac{1}{(D-1)^2}e^{2x}\\ &= \frac{1}{4}\cdot\frac{1}{(2-1)^2}e^{2x} = \frac{1}{4}\cdot 1e^{2x} = \frac{1}{4}e^{2x}\end{aligned}$$

$$\begin{aligned}\frac{1}{(D+2)(D-1)^2}e^{x} &= \frac{1}{(D-1)^2}\frac{1}{D+2}e^{x}\\ &= \frac{1}{(D-1)^2}\frac{1}{1+2}e^{x} = \frac{1}{(D-1)^2}\frac{1}{3}e^{x}\\ &= \frac{1}{3}\frac{1}{(1-1)^2}e^{x}\end{aligned}$$

Since $(1-1)^2 = 0$, we use the rule for repeated roots:

$$\frac{1}{(D-1)^2}e^x = x^2e^x/2$$

So,

$$\frac{1}{(D+2)(D-1)^2}e^{x} = \frac{1}{3}\cdot\frac{x^2}{2}e^x = \frac{x^2}{6}e^x$$

$$\begin{aligned}\frac{1}{(D+2)(D-1)^2}e^{-x} &= \frac{1}{(D-1)^2}\frac{1}{D+2}e^{-x}\\ &= \frac{1}{(D-1)^2}\frac{1}{-1+2}e^{-x} = \frac{1}{(D-1)^2}\frac{1}{1}e^{-x}\\ &= \frac{1}{(D-1)^2}e^{-x}\end{aligned}$$

Since $D=-1$, $(-1-1)^2 = 4$,

$$\frac{1}{(D-1)^2}e^{-x} = \frac{1}{4}e^{-x}$$

Hence,

$$\text{P.I.} = \frac{e^{2x}}{9} + \frac{x^2}{6}e^x + \frac{1}{4}e^{-x}$$

Case II. When $X = \sin(ax+b)$ or $\cos(ax+b)$.

Since

$$D\sin(ax+b) = a\cos(ax+b)$$
$$D^2\sin(ax+b) = -a^2\sin(ax+b)$$
$$D^3\sin(ax+b) = -a^3\cos(ax+b)$$

i.e.,

$$D^4\sin(ax+b) = a^4\sin(ax+b)$$
$$D^{2r}\sin(ax+b) = (-a^2)^r\sin(ax+b)$$
$$D^{2r+1}\sin(ax+b) = (-a^2)^r a\cos(ax+b)$$

In general,

$$f(D)\sin(ax+b) = f(-a^2)\sin(ax+b)$$

Operating on both sides by $1/f(D^2)$,

$$\frac{1}{f(D^2)}f(D^2)\sin(ax+b) = \frac{1}{f(D^2)}f(-a^2)\sin(ax+b)$$

or

$$\sin(ax+b) = \frac{f(-a^2)}{f(D^2)}\sin(ax+b)$$

$\therefore$ Dividing by $f(-a^2)$,

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b), \quad \text{provided } f(-a^2) \neq 0 \tag{4}$$

If $f(-a^2) = 0$, the above rule fails and we proceed further.

Since $\cos(ax+b) + i\sin(ax+b) = e^{i(ax+b)}$ [Euler's theorem],

$$\frac{1}{f(D^2)}\sin(ax+b) = \text{I.P. of } \frac{1}{f(D^2)}\Im\left[e^{i(ax+b)}\right]$$
$$= \Im\left[\frac{1}{f(D^2)}e^{i(ax+b)}\right]$$
$$= \Im\left[\frac{1}{f(-a^2)}e^{i(ax+b)}\right] = \frac{1}{f'(-a^2)}x\sin(ax+b), \quad \text{provided } f'(-a^2) \neq 0 \tag{5}$$

and so on.

Similarly,

$$\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b), \quad \text{provided } f(-a^2) \neq 0$$

If $f(-a^2) = 0$,

$$\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f'(-a^2)}x\cos(ax+b), \quad \text{provided } f'(-a^2) \neq 0$$

and so on.

Example: *Find the P.I. of* $(D^3+1)y = \cos(2x-1)$.
Solution.

$$\text{P.I.} = \frac{1}{D^3+1}\cos(2x-1)$$

$$= \frac{1}{D^3+1}\cos(2x-1) \qquad [\text{Put } D^2 = -2^2 = -4]$$

$$= \frac{1}{D(-4)+1}\cos(2x-1)$$

$$= \frac{(1+4D)}{(1-4D)(1+4D)}\cos(2x-1) \qquad [\text{Multiply and divide by } 1+4D]$$

$$= (1+4D)\frac{1}{1-16D^2}\cos(2x-1) \qquad [\text{Put } D^2 = -2^2 = -4]$$

$$= (1+4D)\frac{1}{65}\cos(2x-1)$$

$$= \frac{1}{65}[\cos(2x-1) + 4D\cos(2x-1)]$$

$$= \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)]$$

Example: *Find the P.I. of* $\frac{d^3y}{dx^3} + \frac{dy}{dx} = \sin 2x$.
Solution. Given equation in *symbolic form* is $(D^3+4D)y = \sin 2x$.

$$\therefore \text{P.I.} = \frac{1}{D(D^2+4)}\sin 2x$$

$$= \frac{1}{D(D^2+4)}\sin 2x \qquad [\because D^2+4 = 0 \text{ for } D^2 = -2^2;\ \text{Apply (5) 477}]$$

$$= x\frac{1}{3D^2+4}\sin 2x$$

$$= x\frac{1}{3(-4)+4}\sin 2x = -\frac{x}{8}\sin 2x$$

Case III. When $X = x^m$.
Here

$$\text{P.I.} = \frac{1}{f(D)}x^m = [f(D)]^{-1}x^m.$$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since the $(m+1)$th and higher derivatives of x^m are zero, we need not consider terms beyond D^m.

Example 13.10. *Find the P.I. of* $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2+2x+4$.
Solution. Given equation in *symbolic form* is $(D^2+D)y = x^2+2x+4$.

$$\therefore \text{P.I.} = \frac{1}{D(D+1)}(x^2+2x+4) = \frac{1}{D}(1+D)^{-1}(x^2+2x+4)$$

$$= \frac{1}{D}(1-D+D^2-\ldots)(x^2+2x+4)$$

$$= \frac{1}{D}[x^2+2x+4-(2x+2)+2]$$

$$= \int(x^2+4)dx = \frac{x^3}{3} + 4x.$$

Case IV. When $X = e^{ax}V$, V being a function of x.
If u is a function of x, then

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u = e^{ax}(D+a)u$$

$$D^2(e^{ax}u) = a^2e^{ax}u + 2ae^{ax}Du + e^{ax}D^2u = e^{ax}(D+a)^2u$$

and in general,

$$D^n(e^{ax}u) = e^{ax}(D+a)^n u$$

$$f(D)(e^{ax}u) = e^{ax}f(D+a)u$$

Operating both sides by $1/f(D)$,

$$\frac{1}{f(D)}f(D)(e^{ax}u) = \frac{1}{f(D)}[e^{ax}f(D+a)u]$$

$$e^{ax}u = \frac{1}{f(D)}[e^{ax}f(D+a)u]$$

Now put $f(D+a)u = V$, i.e., $u = \frac{1}{f(D+a)}V$, so that $e^{ax}\frac{1}{f(D+a)}V = \frac{1}{f(D)}(e^{ax}V)$
i.e.,

$$\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V \tag{6}$$

Example: *Find P.I. of* $(D^2 - 2D + 4)y = e^x \cos x$.
Solution.

$$\text{P.I.} = \frac{1}{D^2 - 2D + 4}e^x \cos x \qquad [\text{Replace } D \text{ by } D+1]$$

$$= \frac{e^x}{(D+1)^2 - 2(D+1) + 4}\cos x = e^x\frac{1}{D^2+3}\cos x \qquad [\text{Put } D^2 = -1^2 = -1]$$

$$= e^x\frac{1}{1+3}\cos x = \frac{1}{4}e^x \cos x$$

Case V. When X is any other function of x.
Here

$$\text{P.I.} = \frac{1}{f(D)}X.$$

If

$$f(D) = (D-m_1)(D-m_2)\ldots(D-m_n),$$

resolving into partial fractions,

$$\frac{1}{f(D)} = \frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \cdots + \frac{A_n}{D-m_n}$$

$$\therefore \quad \text{P.I.} = \left[\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \cdots + \frac{A_n}{D-m_n}\right]X$$

$$= A_1\frac{1}{D-m_1}X + A_2\frac{1}{D-m_2}X + \cdots + A_n\frac{1}{D-m_n}X$$

$$= A_1e^{m_1x}\int Xe^{-m_1x}dx + A_2e^{m_2x}\int Xe^{-m_2x}dx + \cdots + A_ne^{m_nx}\int Xe^{-m_nx}dx$$

Working rule to solve the equation

Let $D^n y + k_1 D^{n-1} y + \cdots + k_n y = X(x)$.

Step I. Complementary function (C.F.)

1. Write the auxiliary equation (A.E.)

$$D^n + k_1 D^{n-1} + \cdots + k_n = 0,$$

and find its n roots $m_1, \ldots, m_n$ (real or complex, repeated or distinct).

2. Then

$$y_{\text{C.F.}}(x) = \begin{cases} c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}, & \text{all } m_i \text{ real and distinct,} \\ (c_1 + c_2 x) e^{mx} + \cdots, & \text{if } m \text{ is a double root,} \\ e^{\alpha x}\big[A\cos(\beta x) + B\sin(\beta x)\big] + \cdots, & \text{if } m = \alpha \pm i\beta, \\ \vdots & \end{cases}$$

Step II. Particular integral (P.I.)

$$y_{\text{P.I.}}(x) = \frac{1}{D^n + k_1 D^{n-1} + \cdots + k_n} X(x).$$

Compute according to five cases:

Case (i) $X = e^{ax}$ with $f(D) = D^n + \cdots + k_n$.

$$y_{\text{P.I.}} = \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}, \quad f(a) \neq 0.$$

If $f(a) = 0$, then a is a root of multiplicity r of the A.E. and

$$y_{\text{P.I.}} = \frac{x^r e^{ax}}{f^{(r)}(a)}, \quad f^{(r)}(a) \neq 0.$$

Case (ii) $X = \sin(bx)$ or $\cos(bx)$. Set $D^2 = -b^2$:

$$y_{\text{P.I.}} = \frac{1}{f(D)} \sin(bx) = \frac{1}{f(-b^2)} \sin(bx), \quad f(-b^2) \neq 0,$$

etc. If $f(-b^2) = 0$ of order r, multiply by x^r and divide by the r-th derivative $f^{(r)}(-b^2)$.

Case (iii) $X = x^m$ (m a nonnegative integer). Expand

$$\frac{1}{f(D)} = [f(D)]^{-1} = a_0 + a_1 D + \cdots + a_m D^m, \quad a_i \text{ by binomial expansion,}$$

then apply term-by-term to x^m (higher derivatives vanish beyond order m).

Case (iv) $X = e^{ax}V(x)$.

$$y_{\text{P.I.}} = \frac{1}{f(D)}\left[e^{ax}V(x)\right] = e^{ax}\,\frac{1}{f(D+a)}\,V(x),$$

then reduce to cases (i)–(iii) on $V(x)$.

(v) When X is any function of x.

$$\text{P.I.} = \frac{1}{f(D)}X$$

Resolve $\dfrac{1}{f(D)}$ into partial fractions and operate each partial fraction on X remembering that

$$\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx.$$

Step III. To find the complete solution
Then the C.S. is $y = \text{C.F.} + \text{P.I.}$

Solve the following differential equations

1. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = xe^{3x} + \sin 2x$**.**
Solution. Given equation in *symbolic form* is $(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$
(i) To find C.F.
Its A.E. is $D^2 - 3D + 2 = 0$ or $(D-2)(D-1) = 0$ whence $D = 1, 2$.
Thus C.F. $= c_1e^x + c_2e^{2x}$
(ii) To find P.I.

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2-3D+2}(xe^{3x} + \sin 2x) = \frac{1}{D^2-3D+2}(e^{3x}\cdot x) + \frac{1}{D^2-3D+2}(\sin 2x)\\
&= e^{3x}\cdot\frac{1}{(D+3)^2-3(D+3)+2}(x) + \frac{1}{D^2-3D+2}(\sin 2x)\\
&= e^{3x}\cdot\frac{1}{D^2+3D+2}(x) + \frac{1}{4-3D+2}(\sin 2x)\\
&= e^{3x}\cdot\frac{1}{D^2+3D+2}(x) - \frac{3D-2}{9D^2-4}(\sin 2x)\\
&= \frac{e^{3x}}{2}\left[1+\frac{3D+D^2}{2}\right]^{-1}x - \frac{3D-2}{9(-4)-4}(\sin 2x)\\
&= \frac{e^{3x}}{2}\left(1-\frac{3D}{2}+\cdots\right)x + \frac{1}{40}(6\cos 2x - 2\sin 2x)\\
&= \frac{e^{3x}}{2}\left(x-\frac{3}{2}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)
\end{aligned}$$

(iii) Hence the C.S. is

$$y = c_1 e^x + c_2 e^{2x} + e^{3x}\left(\frac{x}{2} - \frac{3}{4}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$$

2. $(D^4 + 2D^2 + 1)y = x^2 \cos x$**.**
Solution. (i) **To find C.F.**
Its A.E. is $(D^2+1)^2 = 0$ whose roots are $D = \pm i, \pm i$.

$$\therefore \text{C.F.} = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x.$$

(ii) **To find P.I.**

$$\text{P.I.} = \frac{1}{(D^2+1)^2} x^2 \cos x = \frac{1}{(D^2+1)^2} x^2 \text{ Re.P. of } e^{ix}$$

$$= \text{Re.P. of } \frac{1}{(D^2+1)^2} x^2 e^{ix}$$

$$= \text{Re.P. of } e^{ix} \frac{1}{(D+i)^2} x^2$$

$$= \text{Re.P. of } e^{ix} \frac{1}{(D+2i)D} x^2$$

$$= \text{Re.P. of } e^{ix}\left[\frac{1}{4D^2} - \frac{i}{2}\frac{1}{D^2}\right] x^2$$

$$= \text{Re.P. of } e^{ix}\left[\frac{1}{4}\int x^2 dx^2 - \frac{i}{2}\int x^2 dx^2\right]$$

$$= \frac{1}{4}\text{Re.P. of } e^{ix}\left[x^4 - ix^3 - 3x^2\right]$$

$$= \frac{1}{48}\text{Re.P. of } \left[(\cos x + i\sin x)(x^4 - ix^3 - 3x^2)\right]$$

(iii) Hence the C.S. is

$$y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x + \frac{1}{48}\left[4x^3 \sin x - x^2(x^2 - 9)\cos x\right]$$

3. $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$**.**
Solution. (i) **To find C.F.**
Its A.E. is $D^2 - 4D + 4 = 0$, i.e., $(D-2)^2 = 0$. $\quad \therefore D = 2, 2$

$$\text{C.F.} = (c_1 + c_2 x)e^{2x}$$

(ii) **To find P.I.**

$$\text{P.I.} = \frac{1}{(D-2)^2}(8x^2 e^{2x}\sin 2x) = 8e^{2x}\frac{1}{(D+2-2)^2}(x^2 \sin 2x) = 8e^{2x}\frac{1}{D}\int x^2 \sin 2x \, dx$$

$$= 8e^{2x}\frac{1}{D}\left[x^2\left(-\frac{\cos 2x}{2}\right) - \int 2x\left(-\frac{\cos 2x}{2}\right)dx\right]$$

$$= 8e^{2x}\frac{1}{D}\left[-\frac{x^2}{2}\cos 2x + \int x\cos 2x \, dx\right]$$

$$= 8e^{2x}\frac{1}{D}\left[-\frac{x^2}{2}\cos 2x + x\frac{\sin 2x}{2} - \int\frac{\sin 2x}{2}dx\right]$$

$$= 8e^{2x}\frac{1}{D}\left[-\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{\cos 2x}{4}\right]$$

$$= 8e^{2x}\int\left[-\frac{x^2}{2}\sin 2x - \frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x\right]dx$$

$$= 8e^{2x}\left[\frac{1}{8}x^2\sin 2x + \frac{1}{8}x\cos 2x - \frac{1}{16}\sin 2x\right]$$

$$= e^{2x}\left[x^2\sin 2x + x\cos 2x - \frac{1}{2}\sin 2x\right]$$

Hence the C.S. is

$$y = e^{2x}[c_1 + c_2x + (3 - 2x^2)\sin 2x - 4x\cos 2x]$$

4. $\frac{d^2y}{dx^2} + a^2y = \sec ax$**.**

Solution. Given equation in *symbolic form* is $(D^2 + a^2)y = \sec ax$.

(i) **To find C.F.**

Its A.E. is $D^2 + a^2 = 0$. $D = \pm ia$.

Thus C.F. $= c_1\cos ax + c_2\sin ax$.

(ii) **To find P.I.**

$$\text{P.I.} = \frac{1}{D^2 + a^2}\sec ax = \frac{1}{(D + ia)(D - ia)}\sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right]\sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia}\sec ax - \frac{1}{D + ia}\sec ax\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\int e^{-iax}\sec ax\,dx - e^{-iax}\int e^{iax}\sec ax\,dx\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\left(x + \frac{i}{a}\log\cos ax\right) - e^{-iax}\left(x - \frac{i}{a}\log\cos ax\right)\right]$$

$$= \frac{x}{a^2}\sin ax + \frac{1}{a^2}\cos ax\log\cos ax$$

(iii) Hence the C.S. is

$$y = c_1\cos ax + c_2\sin ax + \frac{1}{a^2}x\sin ax + \frac{1}{a^2}\cos ax\log\cos ax$$

Linear Differential Equations with variable coefficients

Case-1: Cauchy's Homogeneous Linear Differential Equations

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X \tag{1}$$

where a_i's are constants and X is a function of x, is called Cauchy's homogeneous linear equation.

Such equations can be reduced to linear differential equations with constant coefficients by the substitution

$$x = e^z \quad \text{or} \quad z = \log x$$

so that

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} \qquad \text{or} \qquad x\frac{dy}{dx} = \frac{dy}{dz} = Dy, \quad \text{where} \quad D = \frac{d}{dz}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) \\ &= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) \\ &= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx} \\ &= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} \qquad \left(\because \frac{dz}{dx} = \frac{1}{x}\right) \end{aligned}$$

or

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = D^2y - Dy = D(D-1)y$$

Similarly,

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y \quad \text{and so on.}$$

Substituting these values in equation (1), we get a linear differential equation with constant coefficients, which can be solved by the methods already discussed.

Example: Solve

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right).$$

Solution. Given equation is a Cauchy's homogeneous linear equation.

Put

$$x = e^z \quad \text{i.e.,} \quad z = \log x$$

so that

$$x\frac{dy}{dx} = Dy, \quad x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y, \quad \text{where } D = \frac{d}{dz}$$

Substituting these values in the given equation, it reduces to

$$[D(D-1)(D-2) + 2D(D-1) + 2]y = 10(e^z + e^{-z})$$

or

$$(D^3 - D^2 + 2)y = 10(e^z + e^{-z})$$

which is a linear equation with constant co-efficients.

Its A.E. is $m^3 - m^2 + 2 = 0$ or $(m+1)(m^2 - 2m + 2) = 0$

$$\therefore \quad m = -1, \quad \frac{2 \pm \sqrt{4-8}}{2} = -1, 1 \pm i$$

$$\text{C.F.} = c_1 e^{-z} + e^z(c_2 \cos z + c_3 \sin z) = \frac{c_1}{x} + x[c_2 \cos(\log x) + c_3 \sin(\log x)]$$

$$\text{P.I.} = 10 \cdot \frac{1}{D^3 - D^2 + 2}(e^z + e^{-z}) = 10\left(\frac{1}{D^3 - D^2 + 2}e^z + \frac{1}{D^3 - D^2 + 2}e^{-z}\right)$$

$$= 10\left(\frac{1}{1^3 - 1^2 + 2}e^z + \frac{1}{(-1)^3 - (-1)^2 + 2}e^{-z}\right)$$

$$= 10\left(\frac{1}{2}e^z + \frac{1}{3(-1)^2 - 2(-1)}e^{-z}\right)$$

$$= 5e^z + 2e^{-z} = 5x + \frac{2}{x}\log x$$

Hence the C.S. is

$$y = \frac{c_1}{x} + x[c_2 \cos(\log x) + c_3 \sin(\log x)] + 5x + \frac{2}{x}\log x.$$

Example: Solve

$$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x.$$

Solution. Given equation is a Cauchy's homogeneous linear equation.
Put

$$x = e^z \quad \text{i.e.,} \quad z = \log x \quad \text{so that} \quad x\frac{dy}{dx} = Dy, \quad x^2\frac{d^2y}{dx^2} = D(D-1)y, \quad \text{where } D \equiv \frac{d}{dz}.$$

Substituting these values in the given equation, it reduces to

$$[D(D-1) - D - 3]y = ze^{2z} \quad \text{or} \quad (D^2 - 2D - 3)y = ze^{2z}$$

which is a linear equation with constant co-efficients.

Its A.E. is $m^2 - 2m - 3 = 0$ or $(m-3)(m+1) = 0$

$$\therefore \quad m = 3, -1$$

$$\text{C.F.} = c_1 e^{3z} + c_2 e^{-z} = c_1 x^3 + \frac{c_2}{x}$$

$$\text{P.I.} = \frac{1}{D^2 - 2D - 3}(e^{2z} \cdot z) = e^{2z} \cdot \frac{1}{(D+2)^2 - 2(D+2) - 3}z = e^{2z} \cdot \frac{1}{D^2 + 2D - 3}z$$

$$= e^{2z} \cdot \frac{1}{-3}\left[1 - \frac{2D}{3} + \frac{D^2}{3}\right]^{-1} z$$

$$= -\frac{1}{3}e^{2z}\left[1 + \frac{2D}{3} + \frac{D^2}{3} + \cdots\right] z = -\frac{1}{3}e^{2z}\left(z + \frac{2}{3}\right) = -\frac{x^2}{3}\left(\log x + \frac{2}{3}\right)$$

Hence the C.S. is

$$y = c_1 x^3 + \frac{c_2}{x} - \frac{x^2}{3}\left(\log x + \frac{2}{3}\right).$$

Case-2: Legendre's Homogeneous Linear Differential Equations

An equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = X \qquad (1)$$

where a_i's are constants and X is a function of x, is called Legendre's linear equation.

Such equations can be reduced to linear differential equations with constant co-efficients, by the substitution $a + bx = e^z$ i.e., $z = \log(a + bx)$ so that

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{b}{a+bx}\frac{dy}{dz}$$

or

$$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz} = bDy, \quad \text{where } D = \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{b}{a+bx}\frac{dy}{dz}\right) = -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\frac{d^2y}{dz^2}\frac{dz}{dx} = -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b^2}{(a+bx)^2}\frac{d^2y}{dz^2}$$

$$= \frac{b^2}{(a+bx)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

or

$$(a+bx)^2\frac{d^2y}{dx^2} = b^2(D^2y - Dy) = b^2D(D-1)y$$

Similarly,

$$(a+bx)^3\frac{d^3y}{dx^3} = b^3D(D-1)(D-2)y$$

Substituting these values in equation (1), we get a linear differential equation with constant co-efficients, which can be solved by the methods already discussed.

Example: Solve

$$(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1.$$

Solution. Given equation is a Legendre's linear equation.

Put

$$3x + 2 = e^z \quad \text{i.e.,} \quad z = \log(3x+2)$$

so that

$$(3x+2)\frac{dy}{dx} = 3Dy, \qquad (3x+2)^2\frac{d^2y}{dx^2} = 3^2D(D-1)y, \qquad \text{where } D = \frac{d}{dz}.$$

Substituting these values in the given equation, it reduces to

$$[3^2D(D-1) + 3\cdot 3D - 36]y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$$

or

$$9(D^2-4)y = \frac{1}{3}e^{2z} - \frac{1}{3}$$

or

$$(D^2-4)y = \frac{1}{27}(e^{2z}-1)$$

which is a linear equation with constant co-efficients.
Its A.E. is $m^2 - 4 = 0 \implies m = \pm 2$

$$\text{C.F.} = c_1e^{2z} + c_2e^{-2z} = c_1(3x+2)^2 + c_2(3x+2)^{-2}$$

$$\text{P.I.} = \frac{1}{27}\cdot\frac{1}{D^2-4}(e^{2z}-1) = \frac{1}{27}\left[\frac{1}{D^2-4}e^{2z} - \frac{1}{D^2-4}e^{0z}\right]$$

$$= \frac{1}{27}\left[\frac{z}{4}e^{2z} - \frac{1}{4}\right] = \frac{1}{108}\left[ze^{2z}+1\right] = \frac{1}{108}\left[(3x+2)^2\log(3x+2)+1\right]$$

Hence, the C.S. is

$$y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}\left[(3x+2)^2\log(3x+2)+1\right].$$

Applications of Linear Differential Equations

1. Simple Electric Circuit

Consider the simple electric circuit consisting of an inductance L [measured in Henry's], a resistance R [measured in ohms], and a capacitance C [measured in Farads]. An electromotive force E [measured in volts and represented by O] is applied. Usually the source of electric energy is battery or generator. If the source of energy is battery, then E is a constant. If the source of energy is generator, then E is a function of time t (in seconds).

The current i passing through the circuit is measured in amperes and the charge (or the quantity of electricity) q on the capacitor is measured in Coulombs.

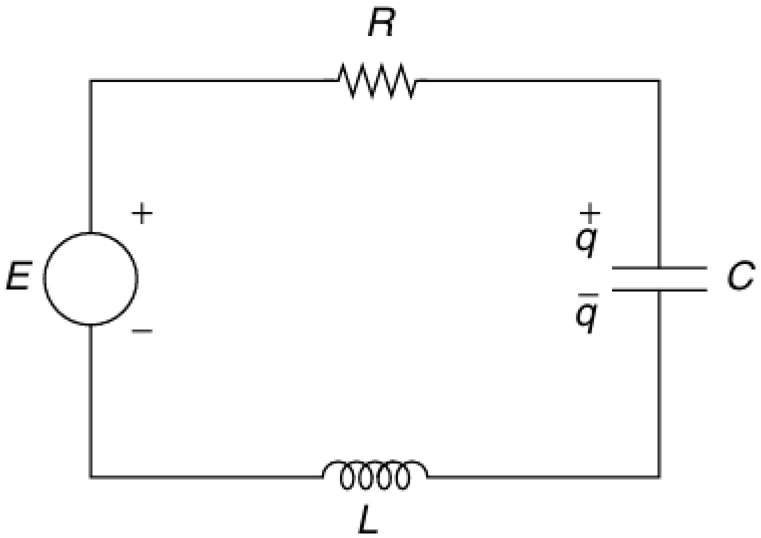

Figure 1: Simple RLC Circuit

Basic Results

1. A resistor opposes the current by producing a drop in e.m.f of magnitude E.

   By Ohm's law, the voltage drop across a resistance R is given by $E_R = Ri$.

2. An inductor opposes any change in the current by producing a drop in e.m.f of magnitude E_i.

   By law of Faraday, the voltage drop across an inductance L is given by $E_L = L\dfrac{di}{dt}$.

3. A capacitor stores energy. By experimental law the voltage drop across a capacitor C is given by
$$E_C = \frac{q}{C},$$
   where q is the electrical charge in the condenser.

4. Current i is the rate of flow of electricity or rate of flow of positive charge q.

   That is
$$i = \frac{dq}{dt} \implies q = \int i\,dt \implies E_c = \frac{1}{C}\int i\,dt.$$

We shall now find the linear differential equations (first or second order) that govern the flow of electricity in simple circuits in terms of its four elements L, C, R, E.

1. **L-C-R Series Circuit**

   Let i [i.e., $i(t)$] be the current flowing in the circuit at any time t (seconds). Kirchoff's voltage law states that "in a closed circuit, the sum of the voltage drops across each element of the circuit is equal to the impressed voltage".

   $\therefore$ the differential equation satisfied by i is

$$E_L + E_C + E_R = E,$$

   where E is a function of t.

$$\Rightarrow \qquad L\frac{di}{dt} + \frac{q}{C} + Ri = E$$

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E \tag{1}$$

   We may regard i or q as the dependent variable depending on the problems.

   If i is considered as the dependent variable, then we eliminate q by differentiating (1) w.r.t t and substituting i for $\frac{dq}{dt}$.

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}\frac{dq}{dt} = \frac{dE}{dt}$$

$$\Rightarrow \qquad L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dE}{dt}$$

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dE}{dt} \tag{2}$$

   If q is considered as dependent variable, then we eliminate i, replace i by $\frac{dq}{dt}$ in (1):

$$L\frac{d}{dt}\left(\frac{dq}{dt}\right) + R\frac{dq}{dt} + \frac{1}{C}q = E$$

$$\Rightarrow \qquad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E \tag{3}$$

2. **L-R Series Circuit**

   Suppose the circuit does not contain capacitor, then the differential equation (1) becomes

$$L\frac{di}{dt} + Ri = E$$

$$L\frac{di}{dt} + Ri = E \tag{4}$$

3. **C-R Series Circuit**

   Suppose the circuit does not contain inductance L, then the differential equation (1) becomes

$$Ri + \frac{q}{C} = E$$

   Differentiating w.r.t t,

$$R\frac{di}{dt} + \frac{1}{C}\frac{dq}{dt} = \frac{dE}{dt}$$

   But $i = \frac{dq}{dt}$ $\quad\therefore$

$$R\frac{di}{dt} + \frac{i}{C} = \frac{dE}{dt} \implies \frac{di}{dt} + \frac{i}{RC} = \frac{1}{R}\frac{dE}{dt}.$$

Problem:

A series circuit contains a resistor of $10\,\Omega$, an inductor of $2\,\text{H}$, and a capacitor of $0.01\,\text{F}$. A voltage source $E(t)$ is connected in series with these elements. Let $q(t)$ be the charge on the capacitor at time t and $i(t)$ the current in the circuit. Write the differential equation that governs the charge $q(t)$ in the circuit.

Solution:

By Kirchhoff's voltage law, the sum of the voltage drops across all elements is equal to the applied voltage:

$$E_L + E_R + E_C = E(t)$$

The voltage drops across each element are:

$$E_L = L\frac{di}{dt}$$
$$E_R = Ri$$
$$E_C = \frac{q}{C}$$

Given:

$$L = 2\,\text{H}, \quad R = 10\,\Omega, \quad C = 0.01\,\text{F}$$

Also, the current $i(t)$ is the rate of change of charge:

$$i = \frac{dq}{dt}$$

Substitute these into the Kirchhoff's law equation:

$$2\frac{di}{dt} + 10i + \frac{q}{0.01} = E(t)$$

Replace i with $\frac{dq}{dt}$:

$$2\frac{d}{dt}\left(\frac{dq}{dt}\right) + 10\frac{dq}{dt} + 100q = E(t)$$

$$2\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 100q = E(t)$$

Final Answer:

The required differential equation for the charge $q(t)$ is:

$$2\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 100q = E(t)$$

Problem:

A series circuit contains a resistor of $10\,\Omega$, an inductor of $2\,\text{H}$, and a capacitor of $0.01\,\text{F}$. A voltage source $E(t) = 50\sin(5t)$ is connected in series with these elements. Let $q(t)$ be the charge on the capacitor at time t and $i(t)$ the current in the circuit. Write and solve the differential equation that governs the charge $q(t)$ in the circuit, given that initially the charge and current are zero.

Solution:

By Kirchhoff's voltage law,

$$E_L + E_R + E_C = E(t)$$

where

$$E_L = L\frac{di}{dt}, \quad E_R = Ri, \quad E_C = \frac{q}{C}$$

Given:

$$L = 2\,\text{H}, \quad R = 10\,\Omega, \quad C = 0.01\,\text{F}, \quad E(t) = 50\sin(5t)$$

Also, $i = \frac{dq}{dt}$.

Substitute into the equation:

$$2\frac{di}{dt} + 10i + \frac{q}{0.01} = 50\sin(5t)$$

Replace i with $\frac{dq}{dt}$:

$$2\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 100q = 50\sin(5t)$$

This is a nonhomogeneous linear differential equation with constant coefficients.

Step 1: Solve the homogeneous equation

$$2\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 100q = 0$$

The characteristic equation is:

$$2m^2 + 10m + 100 = 0$$

$$m^2 + 5m + 50 = 0$$

$$m = \frac{-5 \pm \sqrt{25 - 200}}{2} = \frac{-5 \pm i\sqrt{175}}{2} = \frac{-5}{2} \pm \frac{\sqrt{175}}{2}i$$

$$\sqrt{175} = 5\sqrt{7}$$

So,

$$m = -\frac{5}{2} \pm \frac{5\sqrt{7}}{2}i$$

The complementary function (CF) is:

$$q_h(t) = e^{-\frac{5}{2}t}\left[C_1\cos\left(\frac{5\sqrt{7}}{2}t\right) + C_2\sin\left(\frac{5\sqrt{7}}{2}t\right)\right]$$

Step 2: Find a particular solution (PI)
Assume a particular solution of the form:

$$q_p(t) = A\sin(5t) + B\cos(5t)$$

Compute derivatives:

$$q_p' = 5A\cos(5t) - 5B\sin(5t)$$
$$q_p'' = -25A\sin(5t) - 25B\cos(5t)$$

Substitute into the differential equation:

$$2q_p'' + 10q_p' + 100q_p = 50\sin(5t)$$

$$2[-25A\sin(5t) - 25B\cos(5t)] + 10[5A\cos(5t) - 5B\sin(5t)] + 100[A\sin(5t) + B\cos(5t)] = 50\sin(5t)$$

$$-50A\sin(5t) - 50B\cos(5t) + 50A\cos(5t) - 50B\sin(5t) + 100A\sin(5t) + 100B\cos(5t) = 50\sin(5t)$$

Group $\sin(5t)$ and $\cos(5t)$ terms:

$$\sin(5t): \quad -50A - 50B + 100A = (50A - 50B)$$

$$\cos(5t): \quad -50B + 50A + 100B = (50A + 50B)$$

But let's check carefully:

$$\sin(5t): \quad -50A - 50B + 100A = 50A - 50B$$

$$\cos(5t): \quad -50B + 50A + 100B = 50A + 50B$$

Wait, let's expand correctly:

$$-50A\sin(5t) - 50B\cos(5t) + 50A\cos(5t) - 50B\sin(5t) + 100A\sin(5t) + 100B\cos(5t)$$

$$= (-50A - 50B + 100A)\sin(5t) + (-50B + 50A + 100B)\cos(5t)$$
$$= (50A - 50B)\sin(5t) + (50A + 50B)\cos(5t)$$

Set equal to $50\sin(5t)$:

$$(50A - 50B)\sin(5t) + (50A + 50B)\cos(5t) = 50\sin(5t)$$

Equate coefficients:

$$50A - 50B = 50 \implies A - B = 1$$
$$50A + 50B = 0 \implies A + B = 0 \implies B = -A$$

Substitute $B = -A$ into $A - B = 1$:

$$A - (-A) = 1 \implies 2A = 1 \implies A = \frac{1}{2}, \quad B = -\frac{1}{2}$$

So,

$$q_p(t) = \frac{1}{2}\sin(5t) - \frac{1}{2}\cos(5t)$$

Step 3: General solution

$$q(t) = q_h(t) + q_p(t)$$

$$q(t) = e^{-\frac{5}{2}t}\left[C_1\cos\left(\frac{5\sqrt{7}}{2}t\right) + C_2\sin\left(\frac{5\sqrt{7}}{2}t\right)\right] + \frac{1}{2}\sin(5t) - \frac{1}{2}\cos(5t)$$

Step 4: Apply initial conditions
Given $q(0) = 0$, $i(0) = q'(0) = 0$.
First, compute $q(0)$:

$$q(0) = e^0[C_1 \cdot 1 + C_2 \cdot 0] + 0 - \frac{1}{2} = C_1 - \frac{1}{2} = 0 \implies C_1 = \frac{1}{2}$$

Now, compute $q'(t)$:

$$q'(t) = \frac{d}{dt}\left[e^{-\frac{5}{2}t}\left(C_1\cos\left(\frac{5\sqrt{7}}{2}t\right) + C_2\sin\left(\frac{5\sqrt{7}}{2}t\right)\right)\right] + \frac{5}{2}\cos(5t) + \frac{5}{2}\sin(5t)$$

Let's use the product rule for the first term:

$$\frac{d}{dt}\left[e^{-\frac{5}{2}t}f(t)\right] = e^{-\frac{5}{2}t}f'(t) - \frac{5}{2}e^{-\frac{5}{2}t}f(t)$$

where $f(t) = C_1\cos\left(\frac{5\sqrt{7}}{2}t\right) + C_2\sin\left(\frac{5\sqrt{7}}{2}t\right)$

So,

$$f'(t) = -C_1\frac{5\sqrt{7}}{2}\sin\left(\frac{5\sqrt{7}}{2}t\right) + C_2\frac{5\sqrt{7}}{2}\cos\left(\frac{5\sqrt{7}}{2}t\right)$$

At $t = 0$:

$$q'(0) = e^0\left[-C_1 \cdot 0 + C_2 \cdot \frac{5\sqrt{7}}{2} \cdot 1\right] - \frac{5}{2}e^0[C_1 \cdot 1 + C_2 \cdot 0] + 5 \cdot \frac{1}{2} \cdot 1 + 5 \cdot \frac{1}{2} \cdot 0$$

$$= C_2\frac{5\sqrt{7}}{2} - \frac{5}{2}C_1 + \frac{5}{2}$$

Set $q'(0) = 0$ and $C_1 = \frac{1}{2}$:

$$0 = C_2\frac{5\sqrt{7}}{2} - \frac{5}{2} \cdot \frac{1}{2} + \frac{5}{2} = C_2\frac{5\sqrt{7}}{2} - \frac{5}{4} + \frac{5}{2} = C_2\frac{5\sqrt{7}}{2} + \frac{5}{4}$$

$$C_2\frac{5\sqrt{7}}{2} = -\frac{5}{4} \implies C_2 = -\frac{5}{4} \cdot \frac{2}{5\sqrt{7}} = -\frac{1}{2\sqrt{7}}$$

Final Answer:

$$\boxed{q(t) = e^{-\frac{5}{2}t}\left[\frac{1}{2}\cos\left(\frac{5\sqrt{7}}{2}t\right) - \frac{1}{2\sqrt{7}}\sin\left(\frac{5\sqrt{7}}{2}t\right)\right] + \frac{1}{2}\sin(5t) - \frac{1}{2}\cos(5t)}$$

Simple Harmonic Motion

A particle is said to be in simple harmonic motion if the acceleration of the particle is proportional to its displacement, i.e.,

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\boxed{\ddot{x} + \omega^2 x = 0} \tag{1}$$

where $x(t)$ is the displacement of the particle at any time t, from a fixed reference point O.

2. Oscillations of a spring

Airplanes, bridges, ships, machines, and cars are all examples of vibrating mechanical systems. The simplest example is a mass-spring system, which has a coil spring of natural length L hanging vertically from a fixed support (like a ceiling or beam). If we attach a constant mass m to the end of the spring, the spring stretches to a new length $(L+e)$ and comes to rest. This new position is called the static equilibrium position, where $e > 0$ is the amount the spring stretches because of the mass.

If we move the mass from this equilibrium position or give it an initial velocity (either up or down), the mass will start to move. Since the motion is vertical, we take downward as the positive direction. To find the displacement $x(t)$ of the mass from the equilibrium position at any time t, we use Newton's second law and Hooke's law. The mass m experiences the following forces:

(a) The weight of the mass, mg, acting downward.

(b) The spring's restoring force, $-k(x(t) + e)$, which tries to bring the mass back to its original position.

(c) A damping (or frictional) force from the surrounding medium, $-c\dot{x}(t)$, which resists the motion.

(d) Any external force applied to the mass, $F(t)$.

The differential equation (D.E.) describing the motion of the mass is obtained by Newton's second law as

$$m\ddot{x}(t) = mg - k(x(t) + e) - c\dot{x}(t) + F(t)$$

Here $k > 0$ is the spring constant (or stiffness of the spring), $c \geq 0$ is the damping constant, and g is the gravitational constant. Since the force on the mass exerted by the spring must be equal and opposite to the gravitational force on the mass, we have $ke = mg$. Thus, the D.E. modeling the motion of the mass is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

which is a second order linear non-homogeneous equation with constant coefficients. The displacement (or motion) of the mass at any time t is $x(t)$, which is the solution of D.E. (1). Let us consider three important cases of D.E. (1) referred to as free motion, damped motion, and forced motion.

Free, Undamped Oscillations of a Spring

In the absence of external force ($F(t) = 0$) and neglecting the damping force ($c = 0$), D.E. (1) reduces to

$$m\ddot{x} + kx = 0 \tag{2}$$

which is the *harmonic oscillator equation.* Putting $\omega^2 = \dfrac{k}{m}$, the equation (2) takes the form

$$\ddot{x} + \omega^2 x = 0$$

whose general solution is sinusoidal, given by

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t \tag{3}$$

Introducing $c_1 = A\cos\phi$, $c_2 = -A\sin\phi$, equation (3) can be rewritten as

$$x = A\cos\phi\cos\omega t - A\sin\phi\sin\omega t = A\cos(\omega t + \phi)$$

i.e.,

$$\boxed{x(t) = A\cos(\omega t + \phi)} \tag{4}$$

where $A = \sqrt{c_1^2 + c_2^2}$, $\tan\phi = -\dfrac{c_2}{c_1}$. The constant A is called the *amplitude* of the motion and gives the maximum (positive) displacement of the mass from its equilibrium position. Thus the free, undamped motion of the mass is a simple harmonic motion, which is periodic. The *period* of motion is the time interval between two successive maxima and is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

The *natural frequency* (or simply frequency) of the motion (or harmonic oscillator) is the reciprocal of the period, which gives the number of oscillations per second. Thus, the natural frequency is the undamped frequency (i.e., frequency of the system without damping).

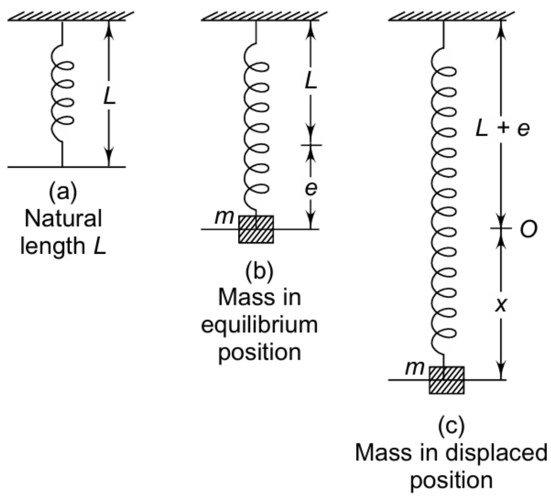

Figure 2: Forced damped mass-spring system

Example: A mass of 1 kg is attached to a spring with spring constant $4\,\mathrm{N/m}$. The system is in a medium that provides a damping force proportional to velocity with damping constant $2\,\mathrm{N\cdot s/m}$. An external force $F(t) = 10\cos(3t)$ acts on the mass. Let $x(t)$ be the displacement from equilibrium.

Solution:

1. **List the parameters:**

$$m = 1, \quad k = 4, \quad c = 2, \quad F(t) = 10\cos(3t)$$

2. **Write the forces:**

$$\text{Spring force: } -kx = -4x$$
$$\text{Damping force: } -c\dot{x} = -2\dot{x}$$
$$\text{External force: } F(t) = 10\cos(3t)$$

3. **Apply Newton's Second Law:**

$$m\ddot{x} = -kx - c\dot{x} + F(t)$$

Substitute the values:

$$1\ddot{x} = -4x - 2\dot{x} + 10\cos(3t)$$

4. **Rearrange to standard form:**

$$\ddot{x} + 2\dot{x} + 4x = 10\cos(3t)$$

Given the equation:

$$\ddot{x} + 2\dot{x} + 4x = 10\cos(3t)$$

1. Solve the Homogeneous Equation:

$$\ddot{x} + 2\dot{x} + 4x = 0$$

The characteristic equation is:

$$m^2 + 2m + 4 = 0$$
$$m = -1 \pm i\sqrt{3}$$

So the complementary function is:

$$x_h(t) = e^{-t}(C_1\cos(\sqrt{3}t) + C_2\sin(\sqrt{3}t))$$

2. Find a Particular Solution:
Try $x_p(t) = A\cos(3t) + B\sin(3t)$.
Compute derivatives:

$$\dot{x}_p = -3A\sin(3t) + 3B\cos(3t)$$

$$\ddot{x}_p = -9A\cos(3t) - 9B\sin(3t)$$

Substitute into the equation:

$$[-9A\cos(3t) - 9B\sin(3t)] + 2[-3A\sin(3t) + 3B\cos(3t)] + 4[A\cos(3t) + B\sin(3t)] = 10\cos(3t)$$

Group terms:

$$(-5A + 6B)\cos(3t) + (-5B - 6A)\sin(3t) = 10\cos(3t)$$

Set coefficients equal:

$$-5A + 6B = 10$$

$$-5B - 6A = 0$$

From the second equation: $B = -\frac{6}{5}A$
Substitute into the first:

$$-5A + 6\left(-\frac{6}{5}A\right) = 10 \implies -5A - \frac{36}{5}A = 10 \implies -\frac{61A}{5} = 10 \implies A = -\frac{50}{61}$$

$$B = -\frac{6}{5}A = \frac{60}{61}$$

So the particular solution is:

$$x_p(t) = -\frac{50}{61}\cos(3t) + \frac{60}{61}\sin(3t)$$

3. General Solution:

$$\boxed{x(t) = e^{-t}\left[C_1\cos(\sqrt{3}t) + C_2\sin(\sqrt{3}t)\right] - \frac{50}{61}\cos(3t) + \frac{60}{61}\sin(3t)}$$