ENGINEERING MATHEMATICS - I

Unit - 2: Higher Order Differential Equations

Department of Science and Humanities

Contents

Problem-1

A particle of mass m moves in a straight line under the action of a force mn^2x which is always directed towards a fixed point O on the line. The resistance to the motion is $2\lambda mn\,\dot{x}$. Initially $x(0) = 0$, $\dot{x}(0) = x_0$. Here $0 < \lambda < 1$. Determine the displacement $x(t)$.

Solution: We begin by writing down Newton's second law. The forces on the particle of mass m are:

- A restoring force of magnitude $m\,n^2\,x$ directed toward the origin,
- A "resistive" force of magnitude $2\lambda\,m\,n\,\dot{x}$ opposing the motion,

so that by Newton's second law,

$$m\,\ddot{x} + 2\lambda\,m\,n\,\dot{x} + m\,n^2\,x = 0 \quad \Longrightarrow \quad \ddot{x} + 2\lambda\,n\,\dot{x} + n^2\,x = 0.$$

Here $0 < \lambda < 1$ and the initial conditions are

$$x(0) = 0, \qquad \dot{x}(0) = x_0.$$

Problem-1 (contd.)

The characteristic equation is

$$r^2 + 2\lambda\, n\, r + n^2 = 0,$$

with roots

$$r = -\lambda n \pm i\, n\sqrt{1-\lambda^2}.$$

Thus the general solution is

$$x(t) = e^{-\lambda nt}\big(A\cos(\omega t) + B\sin(\omega t)\big), \quad \omega = n\sqrt{1-\lambda^2}.$$

Apply $x(0) = 0$ to get $A = 0$. Next,

$$\dot{x}(t) = e^{-\lambda nt}\Big[-\lambda n(A\cos\omega t + B\sin\omega t) + (-A\,\omega\sin\omega t + B\,\omega\cos\omega t)\Big],$$

so $\dot{x}(0) = B\,\omega = x_0$ gives $B = \frac{x_0}{\omega} = \frac{x_0}{n\sqrt{1-\lambda^2}}$. Therefore, the displacement $x(t)$ is given by

$$\boxed{x(t) = \frac{x_0}{n\sqrt{1-\lambda^2}}\; e^{-\lambda nt}\; \sin\big(n\sqrt{1-\lambda^2}\; t\big).}$$

Problem-2

An 8 lb weight is placed at one end of a spring suspended from the ceiling. The weight is raised to 5 inches above the equilibrium position and left free. Assuming the spring constant is 12 lb/ft, find:

1. The equation of motion.
2. The displacement function $x(t)$.
3. The amplitude of the motion.
4. The period T.
5. The frequency f.
6. The maximum velocity.

Answer: We consider an 8 lb weight suspended from a spring of constant $k = 12$ lb/ft, with no damping. Let $x(t)$ be the displacement (in feet) measured downward from the equilibrium position.

1. Compute the mass.

$$m = \frac{\text{weight}}{g} = \frac{8 \text{ lb}}{32 \text{ ft/s}^2} = \tfrac{1}{4} \text{ slug}.$$

2. Write the equation of motion.

By Newton's law,

$$m\,\ddot{x} + \underbrace{k\,x}_{\text{spring force}} = 0 \quad \Longrightarrow \quad \tfrac{1}{4}\,\ddot{x} + 12\,x = 0.$$

Multiply both sides by 4:

$$\ddot{x} + 48\,x = 0.$$

3. Form the characteristic (auxiliary) equation.
Replace $\ddot{x}$ by r^2:

$$r^2 + 48 = 0.$$

4. Solve for the roots r.

$$r^2 = -48 \implies r = \pm\sqrt{-48} = \pm 4\, i\sqrt{3}.$$

Thus the angular frequency is

$$\omega = 4\sqrt{3}.$$

5. Write the general solution. For complex-conjugate roots $r = \pm i\omega$, the solution is

$$x(t) = C_1 \cos(\omega t) \; + \; C_2 \sin(\omega t), \qquad \omega = 4\sqrt{3}.$$

Problem-2 (contd.)

From here, we use the initial conditions $x(0)$ and $\dot{x}(0)$ to determine C_1 and C_2. We take the downward direction as positive x. The mass is raised 5 inches *above* the equilibrium position, i.e. upward, which is negative in our coordinate system. Hence

$$x(0) = -5 \text{ in} = -\frac{5}{12} \text{ ft}.$$

Since $\dot{x}(0) = 0$ (released from rest), we have

$$x(0) = C_1 \cdot 1 + C_2 \cdot 0 = C_1 \quad \Longrightarrow \quad C_1 = -\frac{5}{12}.$$

We have

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t), \quad \omega = 4\sqrt{3}, \quad C_1 = -\tfrac{5}{12}.$$

Problem-2 (contd.)

Differentiate:

$$\dot{x}(t) = \frac{d}{dt}\Big[C_1 \cos(\omega t) + C_2 \sin(\omega t)\Big] = -C_1\,\omega\ \sin(\omega t) + C_2\,\omega\ \cos(\omega t).$$

Apply the initial condition $\dot{x}(0) = 0$:

$$\dot{x}(0) = -C_1\,\omega\ \sin(0) + C_2\,\omega\ \cos(0) = 0 + C_2\,\omega = 0 \implies C_2 = 0.$$

Hence the particular solution is

$$x(t) = C_1 \cos(\omega t) = -\frac{5}{12} \cos\big(4\sqrt{3}\,t\big) = \frac{5}{12}\ \sin\Big(4\sqrt{3}\,t - \frac{\pi}{2}\Big).$$

Problem-2 (contd.)

Also,

$$\text{Amplitude } A = \frac{5}{12},\ T = \frac{2\pi}{\omega} = \frac{\pi}{2\sqrt{3}},\ f = \frac{1}{T} = \frac{2\sqrt{3}}{\pi},\ v_{\max} = A\,\omega = \frac{5}{\sqrt{3}}.$$

Problem-3

Assuming $Q = 0$ and $I = 0$ at $t = 0$, in an RLC circuit having a source of voltage

$$E(t) = 155\sin(377t),$$

with $R = 100\,\Omega$, $L = 0.1\,\text{H}$, and $C = 10^{-3}\,\text{F}$, determine the current $i(t)$ at any instant of time.

Problem-3 (contd.)

For a series RLC circuit with source

$$E(t) = 155\,\sin(377t),$$

and

$$R = 100\ \Omega, \quad L = 0.1\ \mathrm{H}, \quad C = 10^{-3}\ \mathrm{F},$$

Kirchhoff's voltage law in terms of the charge $q(t)$ gives

$$L\,\ddot{q} + R\,\dot{q} + \frac{1}{C}\,q \;=\; E(t), \qquad i(t) = \dot{q}(t).$$

Substituting the numerical values:

$$0.1\,\ddot{q} + 100\,\dot{q} + 1000\,q = 155\,\sin(377t).$$

Dividing through by 0.1:

$$\ddot{q} + 1000\,\dot{q} + 10000\,q = 1550\,\sin(377t),$$

with initial conditions

$$q(0) = 0, \quad \dot{q}(0) = i(0) = 0.$$

Problem-3 (contd.)

Solve $r^2 + 1000\,r + 10000 = 0 \;\Rightarrow r_{1,2} = -500 \pm 200\sqrt{6}$
$r_1 \approx -10.1021,\; r_2 \approx -989.8979)$.

Assume

$$q_p(t) = A\sin(377t) + B\cos(377t).$$

Substituting into the ODE yields

$$A \approx -0.00128331, \quad B \approx -0.00366164.$$

Hence

$$i_p(t) = \dot{q}_p(t) = 377A\cos(377t) - 377B\sin(377t)$$

$$\Rightarrow i_p(t) = -0.483808\cos(377t) + 1.38044\sin(377t).$$

Problem-3 (contd.)

The general solution is

$$i(t) = i_h(t) + i_p(t) = C_1\, e^{-10.1021\, t} + C_2\, e^{-989.898\, t} \\ - 0.483808 \cos(377t) + 1.38044 \sin(377t)$$

Using initial conditions, we find

$$C_1 = -0.0423597, \quad C_2 = +0.526168.$$

Therefore,

$$i(t) = -0.0423597\, e^{-10.1021\, t} + 0.526168\, e^{-989.898\, t} \\ -0.483808 \cos(377t) + 1.38044 \sin(377t)$$