ENGINEERING MATHEMATICS I

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Class content

- Bessel differential equation of order n
- **Series Solution of Bessel's Differential Equation**
- **Bessel Function**

Bessel Differential Equation

The second order linear ordinary differential equation in the form,

$$x^2 y'' + xy' + \left(x^2 - n^2\right) y = 0 \quad ----- \quad (1)$$

is known as Bessel differential equation.

Here n is the order of the Bessel differential equation.

Series Solution of Bessel Differential Equation

Let, $y = \sum_{r=0}^{\infty} a_r x^{r+k}$ ------(2)

where, k is a positive real number a_r 's are constants and $a_0 \neq 0$

Then, $\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (r+k) x^{r+k-1}$ ------(3)

and $\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (r+k)(r+k-1) x^{r+k-2}$ ------(4)

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Series Solution of Bessel Differential Equation

Substituting equation (2), (3) and (4) in equation (1) we get,

$$x^2\sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k-2} + x\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}$$

$$+ \quad (x^2-n^2)\sum_{r=0}^{\infty}a_r x^{r+k} = 0$$

$$\Rightarrow\sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k} + \sum_{r=0}^{\infty}a_r(r+k)x^{r+k}$$

$$+\sum_{r=0}^{\infty}a_r x^{r+k+2} - n^2\sum_{r=0}^{\infty}a_r x^{r+k} = 0$$

Series Solution of Bessel Differential Equation

$$\Rightarrow \sum_{r=0}^{\infty} a_r (r+k)\{r+k-1+1\} x^{r+k} - n^2 \sum_{r=0}^{\infty} a_r x^{r+k} + \sum_{r=0}^{\infty} a_r x^{r+k+2} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (r+k)^2 x^{r+k} - n^2 \sum_{r=0}^{\infty} a_r x^{r+k} + \sum_{r=0}^{\infty} a_r x^{r+k+2} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r \left\{(r+k)^2 - n^2\right\} x^{r+k} + \sum_{r=0}^{\infty} a_r x^{r+k+2} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r \left\{(r+k)^2 - n^2\right\} x^{r+k} + \sum_{r=2}^{\infty} a_{r-2} x^{r+k} = 0$$

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Series Solution of Bessel Differential Equation

$$\Rightarrow a_0\left\{k^2-n^2\right\}x^k+a_1\left\{(k+1)^2-n^2\right\}x^{k+1}$$

$$+\sum_{r=2}^{\infty}a_r\left\{(r+k)^2-n^2\right\}x^{r+k}+\sum_{r=2}^{\infty}a_{r-2}x^{r+k}=0$$

$$\Rightarrow a_0\left\{k^2-n^2\right\}x^k \;+a_1\left\{(k+1)^2-n^2\right\}x^{k+1}$$

$$+\sum_{r=2}^{\infty}\left[a_r\left\{(r+k)^2-n^2\right\}+a_{r-2}\right]x^{r+k}=0$$

Equating the co-efficients of x^k to zero, we get,

$$a_0(k^2-n^2)=0 \quad \Rightarrow k^2-n^2=0 \quad (\because a_0\neq 0)$$

$$\Rightarrow \boxed{k=\pm n}$$

Equating the co-efficients of x^{k+1} to zero,

we get, $a_1\left((k+1)^2-n^2\right)=0$

$$\Rightarrow \boxed{a_1=0}$$

By equating the higher powers of x to zero,

we get,

$$\boxed{a_r=-\frac{a_{r-2}}{(r+k)^2-n^2}} \quad ----------(5)$$

for $r=2,3,4,5\ldots\ldots$

Series Solution of Bessel Differential Equation

Let us consider the two cases:

case $(i)\, k = n$:

Put $k = n$ in equation (5)

Then we get, $a_r = -\dfrac{a_{r-2}}{r^2 + 2nr}$ --------(6)

Put $r = 2, 3, 4, \ldots\ldots$ in (6) we get,

$$a_2 = -\frac{a_0}{2^2 + 4n} = -\frac{a_0}{2^2(n+1)}$$

$$\therefore a_2 = -\frac{a_0}{2^2(n+1)} \text{----------} (7)$$

$$\left(\because a_r = -\frac{a_{r-2}}{(r+k)^2 - n^2} \right)$$

Series Solution of Bessel Differential Equation

$$a_3 = -\frac{a_1}{3^2 + 6n} = 0 \quad (\because a_1 = 0) \qquad \left(\text{we have, } a_r = -\frac{a_{r-2}}{r^2 + 2nr} \right)$$

$$a_4 = -\frac{a_2}{4^2 + 8n}$$

$$= -\frac{1}{8(n+2)}\left(-\frac{a_0}{2^2(n+1)}\right) \qquad \left(\because a_2 = -\frac{a_0}{2^2(n+1)}\right)$$

$$\therefore a_4 = \frac{a_0}{32(n+1)(n+2)} \quad ---------(8)$$

Series Solution of Bessel Differential Equation

$$a_5 = -\frac{a_3}{5^2 + 10n} = 0$$

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We have, by (2),

$$y = a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + a_3 x^{k+3} + a_4 x^{k+4} \dots\dots\dots$$

substituting a_r 's in (2), we get,

$$y_1 = a_0 x^n + 0 \cdot x^{n+1} - \frac{a_0}{4(n+1)} x^{n+2} + 0 \cdot x^{n+3} + \frac{a_0}{32(n+1)(n+2)} x^{n+4} + \dots\dots\dots$$

Series Solution of Bessel Differential Equation

$$\text{i.e. } y_1 = a_0\left\{x^n - \frac{1}{4(n+1)}x^{n+2} + \frac{1}{32(n+1)(n+2)}x^{n+4} + \dots\dots\dots\right\}$$

To standardize the result ,let us take,

$$a_0 = \frac{1}{2^n\Gamma(n+1)}$$ then we get,

$$y_1 = \frac{1}{2^n\Gamma(n+1)}\left\{x^n - \frac{1}{4(n+1)}x^{n+2} + \frac{1}{32(n+1)(n+2)}x^{n+4} + \dots\dots\dots\right\}$$

$$= \frac{1}{2^n\Gamma(n+1)}x^n - \frac{1}{4(n+1)2^n\Gamma(n+1)}x^{n+2} + \frac{1}{32(n+1)(n+2)2^n\Gamma(n+1)}x^{n+4} + \dots\dots\dots$$

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Series Solution of Bessel Differential Equation

$$y_1 = \frac{1}{2^n\Gamma(n+1)}x^n - \frac{1}{2^2\cdot 2^n\Gamma(n+2)}x^{n+2} + \frac{1}{2\cdot 2^4\cdot 2^n\Gamma(n+3)}x^{n+4} + \ldots.$$

$$= \frac{1}{2^n\Gamma(n+1)}x^n - \frac{1}{2^{n+2}\Gamma(n+2)}x^{n+2} + \frac{1}{2}\frac{1}{2^{n+4}\Gamma(n+3)}x^{n+4} + \ldots.$$

$$= \frac{(-1)^0}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n + \frac{(-1)^1}{\Gamma(n+2)}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2}\frac{(-1)^2}{\Gamma(n+3)}\left(\frac{x}{2}\right)^{n+4} + \ldots.$$

$$= \frac{(-1)^0}{0!\Gamma(n+1)}\left(\frac{x}{2}\right)^{n+2\cdot 0} + \frac{(-1)^1}{1!\Gamma(n+2)}\left(\frac{x}{2}\right)^{n+2\cdot 1} + \frac{1}{2!}\frac{(-1)^2}{\Gamma(n+3)}\left(\frac{x}{2}\right)^{n+2\cdot 2} + \ldots.$$

$$\boxed{y_1 = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1)\,r!}} \quad -------(9)$$

Series Solution of Bessel Differential Equation

- Equation (9) Is called power series solution of Bessel differential equation.

- It is called Bessel function of first kind of order n.

- It is denoted by $J_n(x)$

Thus, $$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

Series Solution of Bessel Differential Equation

case $(ii)\, k = -n:$

Put $k = -n$ in equation (5)

Then we get,

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{\Gamma(-n+r+1)\, r!}$$

Series Solution of Bessel Differential Equation

Thus the complete solution of Bessel differential equation is given by

$$\boxed{y = C_1 J_n(x) + C_2 J_{-n}(x)} \quad \left(C_1 \text{ and } C_2 \text{ are arbitrary constants}\right)$$

provided n is a non integer.

THANK YOU

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