ENGINEERING MATHEMATICS - I

Unit - 3: Partial Differential Equations

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Contents

Lagrange's Linear Equation

1. The linear first-order partial differential equation of the form
$$P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z) \tag{1}$$
is called the *Lagrange's equation* in two independent variables x, y
2. **Theorem:** The general solution of the equation $Pp + Qq = R$ is given by $\phi(u,v) = 0$, where ϕ is an arbitrary function and $u(x,y,z) = c_1$, $v(x,y,z) = c_2$ are two linearly independent solutions of the equations
$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{2}$$
(These equations are called the *auxiliary* or *subsidiary equations*.)

Working Rule to Solve Lagrange's Linear Equation

To solve the equation

$$Pp + Qq = R$$

1. Form the auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

2. Solve the auxiliary equations by the method of grouping or the method of multipliers or both to get two independent solutions $u = a$ and $v = b$, where a, b are arbitrary constants
3. $\phi(u, v) = 0$ is the general solution of the equation $Pp + Qq = R$

Method of Grouping in Lagrange's Equation

1. The method of grouping involves pairing two of these ratios at a time to form ordinary differential equations which can be integrated to find the solution
2. Group two ratios at a time:

$$\frac{dx}{P} = \frac{dy}{Q}$$
$$\frac{dy}{Q} = \frac{dz}{R}$$
$$\frac{dx}{P} = \frac{dz}{R}$$

3. Integrate each group to obtain two independent solutions
4. The general solution is given by $\phi(u, v) = 0$, where u and v are the two independent solutions obtained

Example:

Suppose the auxiliary equations are:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Grouping $\frac{dx}{x} = \frac{dy}{y}$:

$$\int \frac{dx}{x} = \int \frac{dy}{y} \implies \ln x = \ln y + c_1 \implies \frac{x}{y} = c_2$$

Grouping $\frac{dx}{x} = \frac{dz}{z}$:

$$\int \frac{dx}{x} = \int \frac{dz}{z} \implies \ln x = \ln z + c_3 \implies \frac{x}{z} = c_4$$

The general solution is:

$$F\left(\frac{x}{y}, \frac{x}{z}\right) = 0$$

Method of Multipliers in Lagrange's Equation

1. Let the auxiliary equations be $\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$
2. Let l, m, n be constants or functions of x, y, z. Then by the property of ratio and proportion, we have

$$Each\ Ratio = \frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lp + mQ + nR}$$

3. l, m, n are chosen in such a way that $lP + mQ + nR = 0$. Thus

$$ldx + mdy + ndz = 0$$

4. Solve this differential equation, and the solution is $u = c_1$
5. Similarly, choose another set of multipliers (l_1, m_1, n_1) and the second solution is $v = C_2$
6. Required general solution is $\phi(u, v) = 0$, where u and v are the two independent solutions

Example:

Suppose the auxiliary equations are:

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$$

Choose multipliers $l = 1, m = 1, n = 1$. Then,

$$Each\,Ratio = \frac{dx+dy+dz}{y-z+z-x+x-y} = \frac{dx+dy+dz}{0}$$

This gives $dx + dy + dz = 0$. On integration, it gives $x + y + z = C_1$.

Differential Concept in Lagrange's Equation

1. Given the auxiliary equations: $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
2. Sometimes, instead of grouping or using multipliers directly, we use the **differential concept** to form a linear combination of differentials that is easier to integrate
3. It is essentially the same as the method of multipliers, but emphasizes the use of differentials and their integration
4. Multipliers may be chosen (more than once) such that the numerator $ldx + mdy + ndz$ is an exact differential of the denominator $lP + mQ + nR$
5. Finally, $\frac{ldx+mdy+ndz}{lP+mQ+nR}$ is combined with a fraction of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ to get an integral