

ENGINEERING MATHEMATICS - I

Unit - 3: Partial Differential Equations

Department of Science and Humanities





1 Method of Separation of Variables

Method of Separation of Variables



- It is based on the idea that a solution can be written as a product of functions, each depending on only one independent variable
- Separation of variables provides closed-form solutions for many fundamental partial differential equations—the one-dimensional heat equation, the wave equation, and Laplace's equation—by reducing each PDE to a set of ordinary differential equations

- 1 For a P.D.E. in the function u of two independent variables x and y , assume that the required solution is separable, i.e.,

$$u(x, y) = X(x)Y(y), \quad (1)$$

where $X(x)$ is a function of x alone and $Y(y)$ is a function of y alone

- 2 Substitution of u from (1) and its derivatives reduces the P.D.E. to the form

$$f(X, X', X'', \dots) = g(Y, Y', Y'', \dots) \quad (2)$$

which is separable in X and Y

- 3 Since the L.H.S. of (2) is a function of x alone and the R.H.S. of (2) is a function of y alone, then (2) must be equal to a common constant, say k



- ① Thus (2) reduces to

$$f(X, X', X'', \dots) = k \quad (3)$$

$$g(Y, Y', Y'', \dots) = k \quad (4)$$

- ② The determination of a solution to P.D.E. reduces to the determination of solutions to two O.D.E.s (with appropriate conditions).

Example

Use the separation of variables technique to solve

$$3u_x + 2u_y = 0 \quad \text{with} \quad u(x, 0) = 4e^{-x}.$$

Solution: Assume $u(x, y) = X(x)Y(y)$.

Then the P.D.E. becomes $3X'Y + 2XY' = 0$ or

$$\frac{X'}{X} = -\frac{2}{3} \frac{Y'}{Y} = k \quad (\text{constant})$$

Solving $X' - kX = 0$, we get $X(x) = c_1 e^{kx}$.

Similarly, $Y' + \frac{3}{2}kY = 0$, so $Y(y) = c_2 e^{-\frac{3}{2}ky}$.

So $u(x, y) = X(x)Y(y) = c_1 e^{kx} \cdot c_2 e^{-\frac{3}{2}ky} = ce^{kx} e^{-\frac{3}{2}ky} = ce^{k(x - \frac{3}{2}y)}$.

Given that $4e^{-x} = u(x, 0) = X(x)Y(0) = ce^{kx}$,

Thus $c = 4$, $k = -1$.

Hence, the required solution is

$$u(x, y) = 4e^{-\frac{1}{2}(2x-3y)}$$



Example

Use separation of variables to solve

$$5u_x - 4u_y = 0, \quad u(x, 0) = 3e^{2x}.$$

Solution. Assume $u(x, y) = X(x)Y(y)$. Then,

$$u_x = X'(x)Y(y), \quad u_y = X(x)Y'(y), \text{ so } 5X'Y - 4XY' = 0 \\ \implies \frac{X'}{X} = \frac{4}{5} \frac{Y'}{Y} = k.$$

Solve ODEs.

$$X'/X = k \Rightarrow X(x) = Ae^{kx},$$

$$\frac{Y'}{Y} = \frac{5}{4}k \Rightarrow Y(y) = Be^{\frac{5}{4}ky}.$$

Combine.

$$u(x, y) = AB e^{kx} e^{\frac{5}{4}ky} = C e^{k(x + \frac{5}{4}y)}.$$

Initial condition.

$$u(x, 0) = C e^{kx} = 3e^{2x} \implies C = 3, \quad k = 2.$$

Final answer. $u(x, y) = 3e^{2x + \frac{5}{2}y}.$

