ENGINEERING MATHEMATICS - I

Unit - 3: Partial Differential Equations

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P.I. when $f(x,y) = x^m y^n$, or a polynomial in x, y

We write the particular integral as

$$z = [F(D, D')]^{-1} x^m y^n$$

We expand $[F(D, D')]^{-1}$ as an infinite series and operate on $x^m y^n$. If $F(D, D')$ does not contain the constant term, that is, a term independent of D and D', then we expand $[F(D, D')]^{-1}$ in powers of $(D')^{-1}D$ if $m < n$ and in powers of $D^{-1}D'$ if $m > n$

Find the particular integral of the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x^2 + y^2$$

Solution: We have $[D^2 - (D')^2]z = x^2 + y^2$
We write $[D^2 - (D')^2] = D^2\left[1 - (D^{-1})^2(D')^2\right]$

$$\therefore [D^2 - (D')^2]^{-1} = D^{-2}\left[1 - (D^{-1})^2(D')^2\right]^{-1}$$

$$\Rightarrow [D^2 - (D')^2]^{-1} = D^{-2}\left[1 + (D^{-1})^2(D')^2 + (D^{-1})^4(D')^4 + \dots\right]$$

Therefore, the particular integral is given by

$$\begin{aligned}
PI &= [D^2 - (D')^2]^{-1}(x^2+y^2) \\
&= D^{-2}\left[1 + (D^{-1})^2(D')^2 + (D^{-1})^4(D')^4 + \dots\right](x^2+y^2) \\
&= D^{-2}\left[(x^2+y^2) + (D^{-1})^2(D')^2(x^2+y^2)\right] \\
&= D^{-2}\left[(x^2+y^2) + (D^{-1})^2(2)\right] \\
&= D^{-2}[x^2+y^2+x^2] \\
&= D^{-2}[2x^2+y^2] \\
&= \frac{x^4}{6} + \frac{x^2y^2}{2}. \quad \textbf{On integrating twice w.r.t. } x
\end{aligned}$$

We may also write $[D^2 - (D')^2] = -(D')^2[1 - (D')^{-2}D^2]$.
In this case, we obtain a different form of the particular integral as

$$z = -\frac{y^2}{6}[3x^2+y^2]$$

Problem

Find the particular integral of the partial differential equation

$$[D^2 + 2DD' + (D')^2]z = 3x + 2y, \quad \text{or} \quad [D + D']^2 z = 3x + 2y$$

We write

$$[D + D']^2 = [D(1 + D'D^{-1})]^2$$

and

$$[D + D']^2 = D^2(1 + D'D^{-1})^2$$

Problem (contd.)

The particular integral is given by

$$\begin{aligned}
z &= D^{-2}(1 + D'D^{-1})^{-2}(3x + 2y) \\
&= D^{-2}[1 - 2D'D^{-1} + 3(D'D^{-1})^2 - \ldots](3x + 2y) \\
&= D^{-2}[3x + 2y - 2D'D^{-1}(3x + 2y)] \\
&= D^{-2}[3x + 2y - 2D'(\frac{3x^2}{2} + 2xy)] \\
&= D^{-2}[3x + 2y - 4x] \\
&= D^{-2}[-x + 2y] \\
&= -\frac{x^3}{6} + x^2 y
\end{aligned}$$

We may also write

$$[D + D']^2 = [D'(1 + (D')^{-1}D)]^2$$

The particular integral is given by

$$\begin{aligned}
PI &= (D')^{-2}[1 + (D')^{-1}D]^{-2}(3x + 2y) \\
&= (D')^{-2}[1 - 2(D')^{-1}D + 3((D')^{-1}D)^2 - \dots](3x + 2y) \\
&= (D')^{-2}[3x + 2y - 2(D')^{-1}D(3x + 2y)] \\
&= (D')^{-2}[3x + 2y - 2(D')^{-1}(3)] \\
&= (D')^{-2}[3x + 2y - 6y] \\
&= (D')^{-2}[3x - 4y] \quad \textbf{On integrating twice w.r.t. } y \\
&= \frac{3xy^2}{2} - \frac{2y^3}{3}
\end{aligned}$$

Let $f(x,y)$ be not of any one of the forms as given in the above cases or the case of failure in Case 2

We use the following procedure:
Assume that $F(D, D')$ is reducible, that is, it can be factorized. Now consider one of the factors as

$$(a_1 D + b_1 D')\, z = a_1 p + b_1 q = f(x,y).$$

Since this is a Lagrange's equation, the auxiliary equations are

$$\frac{dx}{a_1} = \frac{dy}{b_1} = \frac{dz}{f(x,y)}.$$

The first two ratios give the solution

$$b_1\, x - a_1\, y = c, \quad c \text{ arbitrary constant.} \tag{1}$$

(Contd.)

Consider now the first and third ratios:

$$\frac{dx}{a_1} = \frac{dz}{f\left(x, y\right)} = \frac{dz}{f\left[x,\, (b_1 x - c)/a_1\right]},$$

or equivalently

$$a_1\, dz = f\Big[x,\, (b_1 x - c)/a_1\Big]\, dx.$$

Integrating, we get

$$a_1\, z = \int f\Big(x, \tfrac{b_1 x - c}{a_1}\Big)\, dx + c_1 = F(x, c) + c_1,$$

where c_1 is an arbitrary constant. After integration, we replace c by $b_1 x - a_1 y$ as given in Eq. 1. Since a particular integral is required, we set $c_1 = 0$.

Therefore,

$$z = \frac{1}{a_1} F\big(x,\ b_1 x - a_1 y\big)$$

We repeat the procedure for each factor to obtain the required particular integral.