PES University, Bangalore

(Established Under Karnataka Act 16 of 2013)

Department of Science and Humanities

Engineering Mathematics - I (UE25MA141A)

Notes

Unit - 3: Partial Differential Equations

Introduction

Let z be a dependent variable and x, y be independent variables, that is, $z = z(x, y)$. Then, we can define the first order partial derivatives $p = \partial z/\partial x = z_x$ and $q = \partial z/\partial y = z_y$, the second order partial derivatives $r = \partial^2 z/\partial x^2 = z_{xx}$, $s = \partial^2 z/\partial x\partial y = z_{xy}$, $t = \partial^2 z/\partial y^2 = z_{yy}$ etc., if they exist.

An equation containing x, y, z, p, q defines a first order partial differential equation, that is

$$f(x, y, z, p, q) = 0. \tag{1}$$

This equation is linear, if it is linear in p, q. An equation containing x, y, z, p, q, r, s, t defines a second order partial differential equation, that is

$$g(x, y, z, p, q, r, s, t) = 0. \tag{2}$$

This equation is linear, if it is linear in p, q, r, s, t. Most of the mathematical models describing the physical processes contain a partial differential equation or a system of partial differential equations. We have the following examples.

One-dimensional heat conduction equation: $u_t = u_{xx}$

Laplace equation: $u_{xx} + u_{yy} = 0.$

One-dimensional wave equation: $u_{tt} = u_{xx}.$

The solutions of partial differential equations are of the form $f(x, y, z) = c$, which represents a surface in three dimensions, where c is a constant. Therefore, we shall first discuss a few properties relating to surfaces and curves in space.

Formation of First and Second Order Equations: Elimination of arbitrary' functions

Partial differential equations can be obtained by eliminating arbitrary functions from a given equation. A first-order partial differential equation can be derived from an equation involving a single arbitrary function, while a second-order equation may arise from an expression containing two arbitrary functions. However, it is not always guaranteed that an nth-order partial differential equation can be formed from an equation involving n arbitrary functions. In some cases, higher-order partial derivatives may be required to eliminate all n arbitrary functions, leading to a partial differential equation of order higher than n. Moreover, such a resulting higher-order equation may not be unique.

Example: Eliminate the arbitrary function from $z = f(x^2 + y^2)$ to obtain a first order partial differential equation.

Solution we have $z = f(u)$, where $u = x^2 + y^2$. Differentiating partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = 2xz'(u), \quad \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = 2yz'(u).$$

Eliminating $z'(u)$, we get

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0.$$

Example: Obtain a second order partial differential equation by eliminating the arbitrary functions from

$$u = f(x + ct) + g(x - ct).$$

Solution Differentiating the given equation partially with respect to x and t, we obtain

$$\frac{\partial u}{\partial x} = f'(x + ct) + g'(x - ct), \quad \frac{\partial u}{\partial t} = cf'(x + ct) - cg'(x - ct),$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x + ct) + g''(x - ct), \quad \frac{\partial^2 u}{\partial t^2} = c^2 f''(x + ct) + c^2 g''(x - ct).$$

Hence, we obtain the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Consider now, the case when two functions $u = u(x, y, z)$, $v = v(x, y, z)$ are related through the relation $\phi(u, v) = 0$, where ϕ is some arbitrary function of u and v. Differentiating $\phi(u, v) = 0$ partially with respect to x and y, we get

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right] = 0$$

and

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right] = 0.$$

Since both $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ cannot be zero, non-trivial solution exists for the above equations. Hence, we have

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} \end{vmatrix} = 0$$

or

$$\begin{bmatrix} \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} \end{bmatrix} - \begin{bmatrix} \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} & \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} \end{bmatrix} = 0$$

or

$$\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right] + p\left[\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right] + q\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right] = 0. \quad (1)$$

Then, Eq. (1) can be written as a first order partial differential equation given by

$$p\frac{\partial(u, v)}{\partial(y, z)} + q\frac{\partial(u, v)}{\partial(z, x)} = \frac{\partial(u, v)}{\partial(x, y)}$$

or as

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z),$$

where

$$P(x,y,z) = \frac{\partial(u,v)}{\partial(y,z)}, \quad Q(x,y,z) = \frac{\partial(u,v)}{\partial(z,x)}, \quad R(x,y,z) = \frac{\partial(u,v)}{\partial(x,y)}$$

Example: Obtain the partial differential equation governing the equations

$$\phi(u,v) = 0, \quad u = xyz, \quad v = x + y + z.$$

Solution Differentiating $\phi(u,v) = 0$ partially with respect to x and y, we get

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right] = 0$$

and

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right] = 0$$

or

$$\frac{\partial \phi}{\partial u}[yz + xyp] + \frac{\partial \phi}{\partial v}[1 + p] = 0$$

and

$$\frac{\partial \phi}{\partial u}[xz + xyq] + \frac{\partial \phi}{\partial v}[1 + q] = 0.$$

Since both $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ are not zero, we get

$$(1+q)(yz + xyp) - (1+p)(xz + xyq) = 0$$

or

$$px(y-z) + qy(z-x) = z(x-y)$$

which is the required partial differential equation.

Example: Obtain the partial differential equation by eliminating the arbitrary function from the equation:

$$f(x+y+z,\ x^2+y^2+z^2) = 0.$$

Solution: Let us define:

$$u = x + y + z(x,y), \quad v = x^2 + y^2 + z(x,y)^2$$

Then the given equation becomes:

$$f(u,v) = 0$$

Differentiate partially with respect to x:

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = 0$$

Let $p = \frac{\partial z}{\partial x}$. Then:

$$\frac{\partial u}{\partial x} = 1 + p, \quad \frac{\partial v}{\partial x} = 2x + 2zp$$

Hence,

$$\frac{\partial f}{\partial u}(1 + p) + \frac{\partial f}{\partial v}(2x + 2zp) = 0 \tag{1}$$

Now differentiate partially with respect to y:

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = 0$$

Let $q = \frac{\partial z}{\partial y}$. Then:

$$\frac{\partial u}{\partial y} = 1 + q, \quad \frac{\partial v}{\partial y} = 2y + 2zq$$

So,

$$\frac{\partial f}{\partial u}(1 + q) + \frac{\partial f}{\partial v}(2y + 2zq) = 0 \tag{2}$$

Now eliminate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.
Multiply (1) by $(1 + q)$, and (2) by $(1 + p)$, then subtract:

$$(1+q)(1+p)\frac{\partial f}{\partial u} + (1+q)(2x+2zp)\frac{\partial f}{\partial v} - (1+p)(1+q)\frac{\partial f}{\partial u} - (1+p)(2y+2zq)\frac{\partial f}{\partial v} = 0$$

Cancelling common terms:

$$(1+q)(2x+2zp) - (1+p)(2y+2zq) = 0$$

Divide by 2:

$$(1+q)(x+zp) = (1+p)(y+zq)$$

Required PDE:

$$(1+q)(x+zp) = (1+p)(y+zq)$$

or in derivative notation,

$$\left(1 + \frac{\partial z}{\partial y}\right)\left(x + z\frac{\partial z}{\partial x}\right) = \left(1 + \frac{\partial z}{\partial x}\right)\left(y + z\frac{\partial z}{\partial y}\right)$$

Solution of First Order Equations

By eliminating the arbitrary constants a and b from a relation $f(x, y, z, a, b) = 0$, we can obtain a first order partial differential equation given by

$$F(x, y, z, p, q) = 0 \tag{1}$$

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. We have also shown that by eliminating the arbitrary function ϕ from the relation $\phi(u, v) = 0$, where $u = u(x, y, z)$, $v = v(x, y, z)$, we can obtain a first order partial differential equation as given in Eq. (1).

We have the following definitions:

Complete integral or complete solution Any relation of the form $f(x, y, z, a, b) = 0$ which contains two arbitrary constants and satisfies Eq.(1) is called a *complete integral* or a *complete solution* of Eq.(1).

General integral or general solution A relation of the form $\phi(u, v) = 0$, where ϕ is an arbitrary function of $u = u(x, y, z)$, $v = v(x, y, z)$ and satisfies Eq. (1) is called a *general integral* or a *general solution* of Eq. (1).

Particular integral or particular solution The solution obtained by determining the arbitrary constants in the complete integral or the arbitrary function in the general integral by using some specified condition is called a *particular integral* or a *particular solution*.

Lagrange's Equation

The linear first order partial differential equation of the form

$$P(x, y, z)\, p + Q(x, y, z)\, q = R(x, y, z) \tag{1}$$

is called the *Lagrange's equation* in two independent variables x, y.

Theorem The general solution of the equation $Pp + Qq = R$ is given by $\phi(u, v) = 0$ where ϕ is an arbitrary function and $u(x, y, z) = c_1$, $v(x, y, z) = c_2$ are two linearly independent solutions of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{2}$$

(These equations are called the *auxiliary* or *subsidiary equations.*)

Solution of the auxiliary equations

The solutions of the auxiliary equations (2) are of the form $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$, where c_1 and c_2 are arbitrary constants.

Methods for determining these solutions.

1. Using the methods of solution of ordinary differential equations, we solve taking any two pairs. This method is generally used when one of the variables is absent from one term of Eq. (2).

2. We solve one pair and if necessary use this solution to get the second solution.

3. We use multipliers and write each term in Eq. (2) equal to

$$\frac{a\,dx + b\,dy + c\,dz}{aP + bQ + cR}.$$

If possible, we can find some simple functions a, b, c such that $aP + bQ + cR = 0$. Then, the numerator gives $a\,dx + b\,dy + c\,dz = 0$ which can be integrated if the left hand side is an exact differential. Alternately, we may have the case when $aP + bQ + cR \neq 0$ and the numerator is the differential of $aP + bQ + cR$, that is,

$$a\,dx + b\,dy + c\,dz = d(aP + bQ + cR).$$

The solution of the Lagrange's equation $Pp + Qq = R$ is then given by $\phi(c_1, c_2) = 0$ or $\phi(u, v) = 0$. The solution can also be written as $c_2 = \phi(c_1)$ or $c_1 = \phi(c_2)$.

Example: $xp + yq = 3z$

Solution: This is a linear P.D.E. of first order $Pp + Qq = R$ with $P = x$, $Q = y$ and $R = 3z$. The Lagrange's auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z}$$

Integrating the first two equations (or fractions) $\frac{dx}{x} = \frac{dy}{y}$, we get

$$\ln x = \ln y + c_1 \quad \text{or} \quad \frac{x}{y} = c$$

Integrating the first and the last equations $\frac{dx}{x} = \frac{dz}{3z}$, we have

$$3 \ln x = \ln z + c_z \qquad \therefore x^3 = c_1 z$$

Thus the required solution is

$$x^3 = zf\left(\frac{x}{y}\right).$$

The general solution can also be written as

$$F\left(\frac{x^3}{z}, \frac{x}{y}\right) = 0.$$

Note: By integrating 2nd and 3rd equations $\frac{dy}{y} = \frac{dz}{3z}$, we also get $y^3 = c_2 z$ so the general solution is also given by

$$y^3 = zf\left(\frac{x}{y}\right).$$

Example: $yzp - xzq = xy$

Solution: Auxiliary equations are

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$$

From first and second fractions, we get

$$\frac{dx}{yz} = \frac{dy}{-xz}$$

or

$$\frac{dx}{y} = \frac{dy}{-x}$$

or

$$xdx + ydy = 0$$

Integrating $x^2 + y^2 = c_1$ From first and third fractions

$$\frac{dx}{yz} = \frac{dz}{xy}$$

or

$$\frac{dx}{z} = \frac{dz}{x}$$

Integrating $x^2 - z^2 = c_2$ Thus the general solution is

$$F(x^2 + y^2, x^2 - z^2) = 0.$$

Example: $z(z^2 + xy)(px - qy) = x^4$
Solution: Auxiliary equations are

$$\frac{dx}{xz(z^2 + xy)} = \frac{dy}{-yz(z^2 + xy)} = \frac{dz}{x^4}$$

From first and second fractions, we get

$$\frac{dx}{x} = \frac{dy}{-y}$$

on integration $xy = c_1$
From first and third fractions

$$x^3 dx = (z^3 + xyz)dz$$

using $xy = c_1$,

$$x^3 dx = (z^3 + c_1 z)dz$$

Integrating

$$\frac{x^4}{4} = \frac{z^4}{4} + c_1 \frac{z^2}{2} + c_2$$

or

$$x^4 - z^4 - 2c_1 z^2 = c_2$$

Substituting for c_1,

$$x^4 - z^4 - 2(xy)z^2 = c_2$$

The general solution is

$$F\left(xy, x^4 - z^4 - 2xyz^2\right) = 0$$

Example: $px(x + y) = qy(x + y) - (2x + 2y + z)(x - y)$
Solution: Auxiliary equations are

$$\frac{dx}{x(x + y)} = \frac{dy}{-y(x + y)} = \frac{dz}{-(x - y)(2x + 2y + z)}$$

From first two fractions, cancelling (x+y), we get

$$\frac{dx}{x} = -\frac{dy}{y} \quad \text{or} \quad d(\ln x) + d(\ln y) = c$$

which on integration gives $xy = c_1$

$$\frac{dx+dy}{x(x+y)-y(x+y)} = \frac{dx+dy}{(x+y)(x-y)} = \frac{dz}{-(x-y)(2x+2y+z)}$$

Cancelling the (x-y) term, we get

$$(2x+2y+z)(dx+dy)+(x+y)dz=0$$

or

$$(x+y+z)(dx+dy)+(x+y+z)(dx+dy)+(x+y)dz=0$$
$$((x+y+z)d(x+y)+(x+y)dz)=0$$

i.e.,

$$d\left((x+y)(x+y+z)\right)=0$$

Integrating $(x+y)(x+y+z)=c_2$

Thus the general solution is

$$F(xy,(x+y)(x+y+z))=0$$

Example: $(x^2-y^2-yz)p+(x^2-y^2-zx)q=z(x-y)$

Solution: Auxiliary equations are

$$\frac{dx}{x^2-y^2-yz} = \frac{dy}{x^2-y^2-zx} = \frac{dz}{z(x-y)}$$

$$dx-dy-dz = x^2-y^2-yz-(x^2-y^2-zx)$$
$$-z(x-y)=0$$

Integrating $x-y-z=c_1$

From first and second

$$\frac{x\,dx-y\,dy}{x^3-xy^2-x^2y-y^3} = \frac{dz}{z(x-y)}$$

or

$$\frac{x\,dx-y\,dy}{(x^2-y^2)(x-y)} = \frac{dz}{z(x-y)}$$

i.e.,

$$\frac{1}{2}d(\ln(x^2-y^2))=d(\ln z)$$

$$\therefore \quad \frac{(x^2-y^2)}{z^2}=c_2$$

$\therefore$ The general solution is

$$f(x-y-z,\frac{x^2-y^2}{z^2})=0$$

Applications of Partial Differential Equations

Many problems in fluid mechanics, solid mechanics, heat transfer, electromagnetic theory, and other areas of physics are described using Initial Boundary Value Problems (IBVPs). These problems involve partial differential equations (PDEs) along with initial conditions (which describe the state of the system at the starting time) and/or boundary conditions (which describe the state of the system at the edges or boundaries of the region being studied). The method of separation of variables is a powerful way to solve these IBVPs, especially when the PDE is linear and the boundary conditions are homogeneous (meaning they are set to zero or a constant value).

Separation of variables provides closed-form solutions for many fundamental partial differential equations—most notably the one-dimensional heat equation, the wave equation, and Laplace's equation—by reducing each PDE to a set of ordinary differential equations. Unlike the general solution of an ordinary differential equation, which typically involves arbitrary constants, the general solution of a PDE contains arbitrary functions that are then determined by the given initial and boundary conditions.

Method of separation of variables

Separation of variables is a powerful technique to solve P.D.E. For a P.D.E. in the function u of two independent variables x and y, assume that the required solution is separable, i.e.,

$$u(x,y) = X(x)Y(y) \tag{1}$$

where $X(x)$ is a function of x alone and $Y(y)$ is a function of y alone. Then substitution of u from (1) and its derivatives reduces the P.D.E. to the form

$$f(X, X', X'', \ldots) = g(Y, Y', Y'', \ldots) \tag{2}$$

which is separable in X and Y. Since the L.H.S. of (2) is a function of x alone and the R.H.S. of (2) is a function of y alone, then (2) must be equal to a common constant, say k. Thus (2) reduces to

$$f(X, X', X'', \ldots) = k \tag{3}$$

$$g(Y, Y', Y'', \ldots) = k \tag{4}$$

Thus the determination of solution to P.D.E. reduces to the determination of solutions to two O.D.E.s (with appropriate conditions).

Example: Use the separation of variables technique to solve

$$3u_x + 2u_y = 0 \quad \text{with} \quad u(x,0) = 4e^{-x}.$$

Solution: Assume $u(x,y) = X(x)Y(y)$. Then the P.D.E. becomes

$$3X'Y + 2XY' = 0$$

or

$$\frac{X'}{X} = -\frac{2}{3}\frac{Y'}{Y} = k \quad \text{(constant)}$$

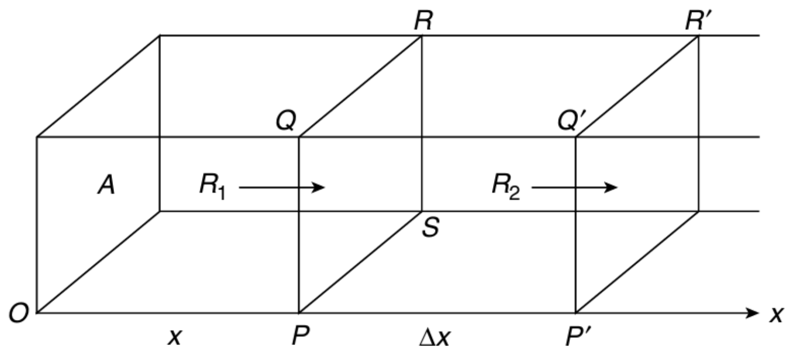
Solving $X' - kX = 0$, we get $X(x) = c_1 e^{kx}$.
Similarly, $Y' + \frac{3}{2}kY = 0$, so $Y(y) = c_2 e^{-\frac{3}{2}ky}$.
So $u(x,y) = X(x)Y(y) = c_1 e^{kx} \cdot c_2 e^{-\frac{3}{2}ky} = ce^{kx}e^{-\frac{3}{2}ky} = ce^{k(x-\frac{3}{2}y)}$.

Given that $4e^{-x} = u(x,0) = X(x)Y(0) = ce^{kx}$,
Thus $c = 4$, $k = -1$.
Hence, the required solution is

$$u(x,y) = 4e^{-\frac{1}{2}(2x-3y)}.$$

Derivation of one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$



Consider a long thin bar (or wire or rod) of constant cross-sectional area A and homogeneous conducting material. Let ρ be the density of the material, c be the specific heat and k be the thermal conductivity of the material. We assume that the surface of the bar is insulated so that the heat flow is along parallel lines which are perpendicular to the area A.

Choose one end of the bar as origin and the direction of heat flow as the positive x-axis.

Let $u(x,t)$ be the temperature at a distance x from 0. If Δu be the temperature change in the slab of thickness Δx of the bar, and time change Δt, then the quantity of heat in this slab is

(specific heat) $\times$ (mass of the element slab) $\times$ (change in temperature) $= c\big(A\rho\,\Delta x\big)\,\Delta u$.

Hence, the rate of change (i.e., increase) of heat in the slab at time t is

$$c\big(A\rho\,\Delta x\big) \lim_{\Delta t \to 0} \frac{\Delta u}{\Delta t} = c\big(A\rho\,\Delta x\big)\,\frac{\partial u}{\partial t}.$$

Let R_1 be the rate of inflow of heat at x in the slab and R_2 be the rate of outflow of heat at $x + \Delta x$. Then

$$c\big(A\rho\,\Delta x\big)\,\frac{\partial u}{\partial t} = R_1 - R_2, \tag{5}$$

where by Fourier's law

$$R_1 = -\,kA\left(\frac{\partial u}{\partial x}\right)_x \quad \text{and} \quad R_2 = -\,kA\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}.$$

The negative sign reflects the fact that heat flows from regions of higher temperature to regions of lower temperature.

Since $\dfrac{\partial u}{\partial t}$ is negative and R_1, R_2 are positive, the rate of increase of heat at time t is

$$R_1 - R_2 = kA\left[\left(\tfrac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\tfrac{\partial u}{\partial x}\right)_x\right]. \tag{6}$$

Combining (5) and (6), we get

$$c\big(A\rho\,\Delta x\big)\,\frac{\partial u}{\partial t} = kA\left[\left(\tfrac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\tfrac{\partial u}{\partial x}\right)_x\right].$$

Hence

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho}\,\frac{\left(\tfrac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\tfrac{\partial u}{\partial x}\right)_x}{\Delta x}.$$

As $\Delta x \to 0$, the difference quotient becomes a second derivative,

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho}\,\frac{\partial^2 u}{\partial x^2},$$

where $\dfrac{k}{c\rho}$ is the thermal diffusivity of the material. Setting

$$\alpha^2 = \frac{k}{c\rho},$$

we obtain the one-dimensional heat equation

$$\boxed{\frac{\partial u}{\partial t} = \alpha^2\,\frac{\partial^2 u}{\partial x^2}.}$$

Solution of the heat equation by the method of separation of variables

The heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Let

$$u = XT \tag{2}$$

where X is a function of x only and T is a function of t only, be a solution of (1).

Then

$$\frac{\partial u}{\partial t} = XT' \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''T$$

Substituting in (1), we have

$$XT' = \alpha^2 X''T$$

Separating the variables, we get

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} \tag{3}$$

Now the L.H.S. of (3) is a function of x only and the R.H.S. is a function of t only. Since x and t are independent variables, this equation can hold only when both sides reduce to a constant, say k. The equation (3) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \qquad \text{and} \qquad \frac{dT}{dt} - k\alpha^2 T = 0 \tag{4}$$

Solving equations (4), we get
(i) When k is positive and $= p^2$, say

$$X = c_1 e^{px} + c_2 e^{-px}, \quad T = c_3 e^{\alpha^2 p^2 t}$$

$$u = (c_1 e^{px} + c_2 e^{-px}) \cdot c_3 e^{\alpha^2 p^2 t} \tag{5}$$

(ii) When k is negative and $= -p^2$, say

$$X = c_1 \cos px + c_2 \sin px, \quad T = c_3 e^{-\alpha^2 p^2 t}$$

$$u = (c_1 \cos px + c_2 \sin px) \cdot c_3 e^{-\alpha^2 p^2 t} \tag{6}$$

(iii) When $k = 0$

$$X = c_1 x + c_2, \quad T = c_3$$

$$u = (c_1 x + c_2) c_3 \tag{7}$$

Thus the various possible solutions of the heat equation (1) are:

$$u = (c_1 e^{px} + c_2 e^{-px}) \cdot c_3 e^{\alpha^2 p^2 t} \tag{8}$$

$$u = (c_1 \cos px + c_2 \sin px) \cdot c_3 e^{-\alpha^2 p^2 t} \tag{9}$$

$$u = (c_1 x + c_2) c_3 \tag{10}$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. Since u decreases as time t increases, the only suitable solution of the heat equation is

$$u = (c_1 \cos px + c_2 \sin px) c_3 e^{-\alpha^2 p^2 t}$$

Higher-Order Linear Equations with Constant Coefficients

We have already learned how to solve higher order linear ordinary differential equations with constant coefficients. These same methods can also be used to solve higher order linear partial differential equations with constant coefficients. For example, consider the following partial differential equation:

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + A_{n-1} \frac{\partial^n z}{\partial x \partial y^{n-1}} + A_n \frac{\partial^n z}{\partial y^n} = f(x, y) \tag{1}$$

where A_i, $i = 0, 1, \ldots, n$ are constants.

Denote

$$p = \frac{\partial z}{\partial x} = Dz, \quad q = \frac{\partial z}{\partial y} = D'z, \quad D^2 z = \frac{\partial^2 z}{\partial x^2}, \quad DD'z = \frac{\partial^2 z}{\partial x \partial y}$$

etc.

Then, Eq. (16.64) can be written as

$$F(D, D')z = \sum_{r=0}^{n} A_r D^{n-r} (D')^r z = f(x, y). \tag{2}$$

As in the case of ordinary differential equations, the general solution of Eq. (2) can be written as

$$z = (\text{complementary function}) + (\text{particular integral})$$

where the complementary function is the solution of the homogeneous equation $F(D, D')z = 0$ and the particular integral is a solution satisfying the non-homogeneous equation (2) and does not contain any arbitrary constants.

Complementary function

Consider the linear, homogeneous equation

$$F(D, D')z = 0. \tag{3}$$

If $F(D, D')$ can be factorised into linear factors of the form $(a_iD + b_iD' + c_i)$, then $F(D, D')$ is said to be *reducible*. Otherwise, it is said to be *irreducible*. $F(D, D')$ may contain some reducible and some irreducible factors. We have the following examples.

$$(D^2 - D'^2)z = (D - D')(D + D')z = 0 \tag{4}$$

$$(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = (D - D')(D - 2D')(D - 3D')z = 0 \tag{5}$$

$$(2D^2 - D')z = 0 \tag{6}$$

$$(D^3 + D^2D' + D'^3)z = 0 \tag{7}$$

$$(D^3 - D^2D' + DD' - D'^2)z = (D - D')(D^2 + D'^2)z = 0 \tag{8}$$

Eqs. (4) and (5) are the reducible forms, Eqs. (6) and (7) are irreducible forms and Eq. (8) has one reducible and one irreducible factor.

Note that we will discuss only the equations which are in "reducible form". We consider the following cases.

Distinct roots

1. Consider the solution of the partial differential equation with one linear factor, that is, the solution of

$$(a_iD + b_iD' + c_i)z = 0, \quad \text{or} \quad a_ip + b_iq = -c_iz. \tag{9}$$

where k_2, k_3 are arbitrary constants and $k_2 = e^{k_3/a_i}$.

The solution of Eq. (9) is given by

$$\boxed{z = e^{-(c_ix)/a_i}\, \phi_i(b_ix - a_iy)} \tag{10}$$

Some particular cases:

a) If the linear factor is of the form $(a_iD + b_iD')$, then Eq. (10) simplifies to

$$\boxed{z = \phi_i(b_ix - a_iy)} \tag{11}$$

(b) If the linear factor is of the form $(b_iD' + c_i)$, then we have the differential equation as $(b_iD' + c_i)z = 0$ or $b_iq_i + c_iz = 0$. The solutions of the auxiliary equations are

$$b_ix = k_1 \quad \text{and} \quad z = k_2e^{-(c_iy)/b_i}$$

Hence, the solution of Eq. (9), when $a_i = 0$, is given by

$$k_2 = \phi_i(k_1), \quad \text{or} \quad ze^{(c_i y)/b_i} = \phi_i(b_i x)$$

or

$$\boxed{z = e^{-(c_i y)/b_i}\,\phi_i(b_i x)} \tag{12}$$

(c) Similarly, if the linear factor is of the form $(a_i D + c_i)$, then the solution of Eq. (9), when $b_i = 0$ is given by

$$\boxed{z = e^{-(c_i x)/a_i}\,\phi_i(a_i y)} \tag{13}$$

The complementary function is given by

$$z = e^{-(c_1 x)/a_1}\phi_1(b_1 x - a_1 y) + e^{-(c_2 x)/a_2}\phi_2(b_2 x - a_2 y) + \ldots + e^{-(c_n x)/a_n}\phi_n(b_n x - a_n y)$$

$$\Rightarrow \boxed{z = \sum_{i=1}^{n} e^{-(c_i x)/a_i}\,\phi_i(b_i x - a_i y)} \tag{14}$$

Similarly, the complementary function in the other cases can be written.

Note: If all terms in $F(D, D')$ have the same total degree, then we can set $m = D/D'$ and look for factors of the equation using m.

Example Find the solutions of the following partial differential equations

$$\begin{aligned}
&\text{(i)} && (D^2 - D'^2)z = 0,\\
&\text{(ii)} && (D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0,\\
&\text{(iii)} && 3r + 7s + 2t + 7p + 4q + 2z = 0,\\
&\text{(iv)} && 2r - s - t - p + q = 0.
\end{aligned}$$

(i) We have $(D^2 - D'^2)z = (D + D')(D - D')z = 0$.

For the factor $D - D'$, we have $a_1 = 1$, $b_1 = -1$.
For the factor $D + D'$, we have $a_2 = 1$, $b_2 = 1$.
Therefore, using Eq. (11), we obtain the solution as

$$z = \phi_1^*(-x - y) + \phi_2(x - y) = \phi_1(x + y) + \phi_2(x - y).$$

(ii) We have

$$(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = (D - D')(D - 2D')(D - 3D')z = 0.$$

For the factor $D - D'$, we have $a_1 = 1$, $b_1 = -1$.
For the factor $D - 2D'$, we have $a_2 = 1$, $b_2 = -2$.
For the factor $D - 3D'$, we have $a_3 = 1$, $b_3 = -3$.

If we set $(D/D') = m$, we get the equation as $m^3 - 6m^2 + 11m - 6 = 0$, whose roots are $m = 1, 2, 3$. Hence, the factors are $D - D'$, $D - 2D'$, $D - 3D'$.

Using Eq. (11), the solution can be written as

$$\begin{aligned}
z &= \phi_1^*(-x - y) + \phi_2^*(-2x - y) + \phi_3^*(-3x - y)\\
&= \phi_1(x + y) + \phi_2(2x + y) + \phi_3(3x + y).
\end{aligned}$$

The aforementioned note does not apply to the following two problems.

(iii) We have

$$(3D^2 + 7DD' + 2D'^2 + 7D + 4D' + 2)z = (3D + D' + 1)(D + 2D' + 2)z = 0.$$

For the factor $3D + D' + 1$, we have $a_1 = 3$, $b_1 = 1$, $c_1 = 1$.
For the factor $D + 2D' + 2$, we have $a_2 = 1$, $b_2 = 2$, $c_2 = 2$.
Using Eq. (10), the solution can be written as

$$z = e^{-x/3}\phi_1(x - 3y) + e^{-2x}\phi_2(2x - y).$$

(iv) We have

$$(2D^2 - DD' - D'^2 - D + D')z = (D - D')(2D + D' - 1)z = 0.$$

For the factor $D - D'$, we have $a_1 = 1$, $b_1 = -1$, $c_1 = 0$.
For the factor $2D + D' - 1$, we have $a_2 = 2$, $b_2 = 1$, $c_2 = -1$.
Using Eqs. (10) and (11), the solution can be written as

$$\begin{aligned} z &= \phi_1^*(-x - y) + e^{x/2}\phi_2(x - 2y) \\ &= \phi_1(x + y) + e^{x/2}\phi_2(x - 2y). \end{aligned}$$

Multiple factors

Let a factor be of multiplicity 2, that is, we have either of the factors

$$(a_iD + b_iD' + c_i)^2, \quad \text{or} \quad (a_iD + c_i)^2, \quad \text{or} \quad (b_iD' + c_i)^2.$$

1. For the equation $(a_iD + b_iD' + c_i)^2 z = 0$, the general solution is given by

$$\boxed{z = e^{-(c_ix)/a_i}\left[x\,\phi_i(b_ix - a_iy) + \psi_i(b_ix - a_iy)\right]}$$

2. For the equation $(a_iD + c_i)^2 z = 0$, the general solution is given by

$$\boxed{z = e^{-(c_ix)/a_i}\left[x\,\phi_i(a_iy) + \psi_i(a_iy)\right]}$$

3. For the equation $(b_iD' + c_i)^2 z = 0$, the general solution is

$$\boxed{z = e^{-(c_iy)/b_i}\left[y\,\phi_i(b_ix) + \psi_i(b_ix)\right]}$$

4. If the factor is of multiplicity m, say $(a_1D + b_1D' + c_1)^m$, then the general solution is

$$\boxed{z = e^{-(c_1x)/a_1}\left[\psi_1(b_1x - a_1y) + x\psi_2(b_1x - a_1y) + \ldots + x^{m-1}\psi_m(b_1x - a_1y)\right]}$$

Similarly, we can write the general solution in the other cases.

Example: Find the solutions of the following partial differential equations

(i) $(D^4 - 2D^2D'^2 + D'^4)z = 0$,

(ii) $(4D^3 - 3DD'^2 + D'^3)z = 0$.

Solution

(i) We have

$$(D^4 - 2D^2D'^2 + D'^4)z = (D^2 - D'^2)^2 z = (D - D')^2(D + D')^2 z = 0.$$

For the factor $(D - D')^2$, we have $a_1 = 1$, $b_1 = -1$, $c_1 = 0$.
For the factor $(D + D')^2$, we have $a_2 = 1$, $b_2 = 1$, $c_2 = 0$.
If we set $m = D/D'$, we get $m^4 - 2m^2 + 1 = 0$. The roots are $m^2 = 1$, or $m = \pm 1$ which are double roots. Hence, the factors are $(D - D')^2$ and $(D + D')^2$.

Therefore, the general solution as

$$z = [x\phi_1^*(-x - y) + \phi_2^*(-x - y)] + [x\psi_1(-x - y) + \psi_2(-x - y)]$$

or

$$z = x\phi_1(x + y) + \phi_2(x + y) + x\psi_1(x - y) + \psi_2(x - y).$$

(ii) We have

$$(4D^3 - 3DD'^2 + D'^3)z = (D + D')(4D^2 - 4DD' + D'^2)z$$

$$= (D + D')(2D - D')^2 z = 0.$$

For the factor $D + D'$, we have $a_1 = 1$, $b_1 = 1$, $c_1 = 0$.
For the factor $(2D - D')^2$, we have $a_2 = 2$, $b_2 = -1$, $c_2 = 0$.
The general solution as

$$z = \phi_1(x - y) + x\psi_1^*(x - 2y) + \psi_2^*(-x - 2y)$$

or

$$z = \phi_1(x - y) + x\psi_1(x + 2y) + \psi_2(x + 2y).$$

Particular integral

Consider the non-homogeneous equation

$$F(D, D')z = \sum_{r=0}^{n} A_r D^{n-r}(D')^r z = f(x, y).$$

The particular integral is now written as

$$z = [F(D, D')^{-1}]f(x, y).$$

We consider the following cases:

1. When $f(x, y) = e^{ax+by}$. The particular integral in this case is

$$z = [F(D, D')^{-1}]e^{ax+by} = \frac{1}{F(a, b)}e^{ax+by}, \quad \text{if } F(a, b) \neq 0.$$

If $F(a, b) = 0$, then we write the particular integral as

$$z = \phi(x, y)e^{ax+by},$$

where

$$\phi(x, y) = [F(D + a, D' + b)]^{-1}(1).$$

We expand the operator $[F(D + a, D' + b)]^{-1}$ in an infinite series symbolically, and determine $\phi(x, y)$. Then, the particular integral is given by $z = \phi(x, y)e^{ax+by}$. Note that D^{-1} and $(D')^{-1}$ mean integral with respect to x and y respectively, keeping the other variable as constant.

Example: Find the general solution of the partial differential equation

$$[2D^2 - DD' - (D')^2 + D - D']z = e^{2x+3y}.$$

Solution: We write

$$F(D, D')z = [2D^2 - DD' - (D')^2 + D - D']z = [(2D + D' + 1)(D - D')]z = e^{2x+3y}.$$

The homogeneous equation is

$$(2D + D' + 1)(D - D')z = 0$$

For the factor $D - D'$, we have $a_1 = 1$, $b_1 = -1$, $c_1 = 0$.
For the factor $2D + D' + 1$, we have $a_2 = 2$, $b_2 = 1$, $c_2 = 1$.
Therefore, the complementary function as

$$z = e^{-x/2}\phi_1(x - 2y) + \phi_2(x + y).$$

Since $F(2, 3) = (4 + 3 + 1)(2 - 3) = 8 \times (-1) = -8 \neq 0$, we obtain the particular integral as

$$z = \frac{1}{F(2, 3)}e^{2x+3y} = -\frac{1}{8}e^{2x+3y}.$$

Therefore, the general solution of the given differential equation is given by

$$z = e^{-x/2}\phi_1(x-2y) + \phi_2(x+y) - \frac{1}{8}e^{2x+3y}.$$

Example: Find a particular integral of the differential equation

$$(4D^2 + 3DD' - D'^2 - D - D')z = 3e^{(x+2y)/2}.$$

Solution: We have

$$F(D, D') = 4D^2 + 3DD' - D'^2 - D - D'$$

and

$$F\left(\frac{1}{2}, 1\right) = 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{2}\right)(1) - 1 - \frac{1}{2} - 1 = 0.$$

We write the particular integral as $z = \phi(x, y)e^{(x+2y)/2}$, where $\phi(x, y)$ is a function to be determined.

The given equation is: $F(D, D')[z] = 3e^{(x+2y)/2}$
$\Rightarrow F\left(D + \frac{1}{2}, D' + 1\right) \cdot \phi(x, y)e^{(x+2y)/2} = 3e^{(x+2y)/2}$
$\Rightarrow F\left(D + \frac{1}{2}, D' + 1\right) \cdot \phi(x, y) = 3$
Therefore,

$$\phi(x, y) = \left[F\left(D + \frac{1}{2}, D' + 1\right)\right]^{-1}(3)$$

Now,

$$F\left(D + \frac{1}{2}, D' + 1\right) = 4\left(D + \frac{1}{2}\right)^2 + 3\left(D + \frac{1}{2}\right)(D'+1) - (D'+1)^2 - (D+\frac{1}{2}) - (D'+1)$$

$$= 6D - \frac{3}{2}D' + 3DD' + 4D^2 - (D')^2$$

$$= 6D\left[1 - \frac{1}{4}D'D^{-1} + \frac{1}{6}\{3D' + 4D - D'(D')^2\}\right]$$

Therefore,

$$\phi(x, y) = \left[F\left(D + \frac{1}{2}, D' + 1\right)\right]^{-1}(3)$$

becomes

$$\phi(x, y) = \frac{1}{2}D^{-1}\left[1 - \frac{1}{4}D'D^{-1} + \frac{1}{6}\{3D' + 4D - D'(D')^2\}\right]^{-1}(1)$$

$$= \frac{1}{2}D^{-1}(1) = \frac{x}{2}.$$

Hence, the particular integral is given by

$$z = \frac{x}{2}e^{(x+2y)/2}.$$

Note: We have

$$D[z] = \frac{\partial}{\partial x}\left[\phi(x,y)e^{(x+2y)/2}\right] = \left(\frac{\partial \phi}{\partial x}\right)e^{(x+2y)/2} + \phi(x,y)\cdot\frac{1}{2}e^{(x+2y)/2}$$

So,

$$D[z] = \left(D + \frac{1}{2}\right)\phi(x,y)\cdot e^{(x+2y)/2}$$

Similarly,

$$D'[z] = (D'+1)\phi(x,y)\cdot e^{(x+2y)/2}$$

When you apply the full operator $F(D, D')$ to z, every D becomes $D+\frac{1}{2}$ and every D' becomes $D'+1$ when acting on $\phi(x,y)$, and you factor out the exponential:

$$F(D,D')[z] = F(D,D')[\phi(x,y)e^{(x+2y)/2}] = e^{(x+2y)/2}F\left(D+\frac{1}{2}, D'+1\right)\phi(x,y)$$

2. When $f(x,y) = \sin(ax+by)$, or $f(x,y) = \cos(ax+by)$

   The particular integral in this case is

$$z = \frac{\sin(ax+by)}{F(-a^2,-ab,-b^2)}, \quad \text{if } F(-a^2,-ab,-b^2) \neq 0.$$

Similarly, if $f(x,y) = \cos(ax+by)$, we get the particular integral as

$$z = \frac{\cos(ax+by)}{F(-a^2,-ab,-b^2)}, \quad \text{if } F(-a^2,-ab,-b^2) \neq 0.$$

If $F(-a^2,-ab,-b^2) = 0$, then we shall follow the procedure as given in Case 4.

Example: Find the particular integral of the differential equation

$$2\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin(x-2y).$$

Solution We have

$$F(D^2, DD', (D')^2)z = [2D^2 - 3DD' + (D')^2]z = \sin(x-2y).$$

We have the right hand side as $\sin(ax+by)$, where $a=1,\ b=-2.$

Hence,

$$F(D^2, DD', (D')^2)\sin(x-2y) = [2(-1)-3(2)+(-4)]\sin(x-2y) = -12\sin(x-2y).$$

The particular integral is given by

$$z = \frac{\sin(x-2y)}{F(-a^2,-ab,-b^2)} = -\frac{1}{12}\sin(x-2y).$$

3. When $F(x,y) = x^m y^n$, or a polynomial in x, y.

We write the particular integral as

$$z = [F(D, D')]^{-1} x^m y^n$$

We expand $[F(D,D')]^{-1}$ as an infinite series and operate on $x^m y^n$. If $F(D, D')$ does not contain the constant term, that is, a term independent of D and D', then we expand $[F(D,D')]^{-1}$ in powers of $(D')^{-1}D$ if $m < n$ and in powers of $D^{-1}D'$ if $m > n$.

Find the particular integrals of the following partial differential equations:

(a) $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = x^2 + y^2,$

(b) $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 3x + 2y.$

Solution:

(a)

We have

$$[D^2 - (D')^2]z = x^2 + y^2.$$

We write

$$[D^2 - (D')^2] = D^2\left[1 - (D^{-1})^2(D')^2\right]$$

and

$$[D^2 - (D')^2]^{-1} = D^{-2}\left[1 - (D^{-1})^2(D')^2\right]^{-1} = D^{-2}\left[1 + (D^{-1})^2(D')^2 + (D^{-1})^4(D')^4 + \ldots\right].$$

Therefore, the particular integral is given by

$$\begin{aligned} z &= [D^2 - (D')^2]^{-1}(x^2 + y^2) \\ &= D^{-2}\left[1 + (D^{-1})^2(D')^2 + (D^{-1})^4(D')^4 + \ldots\right](x^2 + y^2) \\ &= D^{-2}\left[(x^2 + y^2) + (D^{-1})^2(D')^2(x^2 + y^2)\right] \\ &= D^{-2}[x^2 + y^2 + 2] \\ &= D^{-2}[2x^2 + y^2] \\ &= \frac{x^4}{6} + \frac{x^2 y^2}{2}. \end{aligned}$$

We may also write

$$[D^2 - (D')^2] = -(D')^2[1 - (D')^{-2}D^2].$$

In this case, we obtain a different form of the particular integral as

$$z = -\frac{y^2}{6}[3x^2 + y^2].$$

(b)

We have

$$[D^2 + 2DD' + (D')^2]z = 3x + 2y, \quad \text{or} \quad [D + D']^2 z = 3x + 2y.$$

We write

$$[D + D']^2 = [D(1 + D'D^{-1})]^2$$

and

$$[D + D']^2 = D^2(1 + D'D^{-1})^2.$$

The particular integral is given by

$$\begin{aligned} z &= D^{-2}(1 + D'D^{-1})^{-2}(3x + 2y) \\ &= D^{-2}[1 - 2D'D^{-1} + 3(D'D^{-1})^2 - \ldots](3x + 2y) \\ &= D^{-2}[3x + 2y - 2D'D^{-1}(3x + 2y)] \\ &= D^{-2}[3x + 2y - 4x] \\ &= D^{-2}[-x + 2y] \\ &= -\frac{x^3}{6} + x^2 y. \end{aligned}$$

We may also write

$$[D + D']^2 = [D'(1 + (D')^{-1}D)]^2$$

The particular integral is given by

$$\begin{aligned} z &= (D')^{-2}[1 + (D')^{-1}D]^{-2}(3x + 2y) \\ &= (D')^{-2}[1 - 2(D')^{-1}D + 3((D')^{-1}D)^2 - \ldots](3x + 2y) \\ &= (D')^{-2}[3x + 2y - 2(D')^{-1}D(3x + 2y)] \\ &= (D')^{-2}[3x + 2y - 6y] \\ &= (D')^{-2}[3x - 4y] \\ &= \frac{3xy^2}{2} - \frac{2y^3}{3}. \end{aligned}$$

4. Let $f(x, y)$ be not of any one of the forms as given in the above cases or the case of failure in Case 2.

   We use the following procedure:

   Assume that $F(D, D')$ is reducible, that is, it can be factorised. Now consider one of the factors as

$$(a_1 D + b_1 D')\, z = a_1 p + b_1 q = f(x, y).$$

   Since this is a Lagrange's equation, the auxiliary equations are

$$\frac{dx}{a_1} = \frac{dy}{b_1} = \frac{dz}{f(x, y)}.$$

   The first two ratios give the solution

$$b_1\, x - a_1\, y = c, \quad c \text{ arbitrary constant.} \tag{7}$$

Consider now the first and third ratios:

$$\frac{dx}{a_1} = \frac{dz}{f\big(x,y\big)} = \frac{dz}{f\big[x,\,(b_1x-c)/a_1\big]},$$

or equivalently

$$a_1\,dz = f\Big[x,\,(b_1x-c)/a_1\Big]\,dx.$$

Integrating, we get

$$a_1\,z = \int f\Big(x,\tfrac{b_1x-c}{a_1}\Big)\,dx + c_1 = F(x,c) + c_1,$$

where c_1 is an arbitrary constant. After integration, we replace c by $b_1x - a_1y$ as given in Eq. 5. Since a particular integral is required, we set $c_1 = 0$.

Therefore,

$$z = \frac{1}{a_1}\,F\big(x,\;b_1x - a_1y\big)$$

We repeat the procedure for each factor to obtain the required particular integral.

Example: Find the solution of the differential equation $[2D^2+5DD'+3(D')^2]z = ye^x$.

Solution We write

$$[2D^2 + 5DD' + 3(D')^2]z = (2D + 3D')(D + D')z = ye^x.$$

The complementary function as

$$z = \phi_1(3x - 2y) + \phi_2(x - y).$$

The particular integral is given by

$$z = (2D + 3D')^{-1}(D + D')^{-1}(ye^x).$$

We first obtain $(D + D')^{-1}(ye^x)$ as in case 4. For the sake of completeness, we repeat the procedure used in this case. Denote

$$u = (D + D')^{-1}(ye^x) \quad \text{or} \quad (D + D')u = ye^x.$$

The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{ye^x}.$$

The first two terms give $y = x + c$. Using the first and third terms, we get

$$\frac{dx}{1} = \frac{du}{(x + c)e^x}$$

and

$$u = \int (x + c)e^x dx = (x + c - 1)e^x = (y - 1)e^x.$$

Now, denote

$$z = (2D + 3D')^{-1}u = (2D + 3D')^{-1}(y - 1)e^x$$

or

$$(2D + 3D')z = (y - 1)e^x.$$

The auxiliary equations are

$$\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{(y-1)e^x}.$$

The first two terms give $2y = 3x + c_1$. The first and third terms give

$$\frac{dx}{2} = \frac{dz}{[(3x + c_1)/2 - 1]e^x}$$

and

$$z = \frac{1}{4}\int (3x + c_1 - 2)e^x dx = \frac{1}{4}(3x + c_1 - 5)e^x = \frac{1}{4}(2y - 5)e^x$$

which is the required particular integral. The general solution of the differential equation is

$$z = \phi_1(3x - 2y) + \phi_2(x - y) + \frac{1}{4}(2y - 5)e^x.$$