PES University, Bangalore

(Established Under Karnataka Act 16 of 2013)

Department of Science and Humanities

Engineering Mathematics - I (UE25MA141A)

Assignment

Unit - 4: Special Functions

Beta and Gamma Functions

1. Prove that $\Gamma(2p)\sqrt{\pi} = 2^{2p-1}\Gamma(p)\Gamma\left(p+\frac{1}{2}\right)$.
2. Prove that $\beta\left(p, \frac{1}{2}\right) = 2^{2p-1}\beta(p,p)$.
3. $\int_0^1 x^m(\log x)^n dx = (-1)^n \frac{\Gamma(n+1)}{(m+n)^{n+1}}$, where n is a positive integer and $m > -1$.
4. $\int_0^1 x^p(1-x^q)^r dx = \frac{1}{q}\beta\left(\frac{p+1}{q}, r+1\right)$.
5. $\int_0^\infty e^{-ax}x^{n-1}dx = \frac{\Gamma(n)}{a^n}$, where a and n are positive constants.

Evaluate the following integrals using beta and gamma functions:

6. $I = \int_0^\infty (x^2+4)e^{-2x^2}dx$.
   Answer: $\frac{17}{8}\sqrt{\frac{\pi}{2}}$.
7. $I = \int_0^1 \frac{1}{\sqrt{-\log x}}dx$.
   Answer: $\sqrt{\pi}$.
8. $I = \int_0^\infty 3^{-4x^2}dx$.
   Answer: $\frac{\sqrt{\pi}}{4\sqrt{\log 3}}$.
9. $I = \int_0^1 x^4(1-x)^3 dx$.
   Answer: $\frac{1}{280}$.
10. $I = \int_0^1 x^2(1-x^5)^{-1/2}dx$.
    Answer: $\frac{1}{5}\frac{\Gamma\left(\frac{3}{5}\right)\sqrt{\pi}}{\Gamma\left(\frac{11}{10}\right)}$.
11. $I = \int_0^1 x^2(1-x^3)^4 dx$.
    Answer: $\frac{1}{15}$.
12. $I = \int_0^{\pi/2} \frac{\sqrt{\sin 8x}}{\sqrt{\cos x}}dx$.
    Answer: $\frac{60}{13}\frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{11}{12}\right)}$.

13. $I = \int_0^2 (8 - x^3)^{-1/3} dx$.
Answer: $\frac{2\pi}{3\sqrt{3}}$.

14. $I = \int_0^2 \frac{x^2}{\sqrt{2-x}} dx$.
Answer: $\frac{64\sqrt{2}}{15}$.

15. Show that $\int_0^\infty \sqrt{x} e^{-x^2} dx * \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx = \frac{\pi}{2\sqrt{2}}$.

16. Prove that $\int_0^3 \frac{x^{3/2}}{\sqrt{3-x}} dx \times \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{432\pi}{35}$.

Bessel Functions

17. Prove that $J_0'(x) = -J_1(x)$.

18. Prove that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

19. Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$.
Answer: $\mathbf{J_5(x) = \left(\frac{384}{x^4} - \frac{72}{x^2} - 1\right) J_1 + \left(\frac{12}{x} - \frac{192}{x^3}\right) J_0}$.

20. Express $J_{-5/2}(x)$ in terms of sine and cosine functions.
Answer: $\mathbf{J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{3}{x} \sin x + \frac{3-x^2}{x^2} \cos x\right]}$.