ENGINEERING MATHEMATICS I

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Class content

Problems on Beta and Gamma functions

Recall

1. $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

2. $\Gamma(n+1) = n\Gamma(n)$

3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

4. $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

5. $\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

6. $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Problems on Beta and Gamma functions

1. Evaluate $\int_0^1 \frac{dx}{\sqrt{-\log x}}$

Solution: Let $-\log x = t$

$$\Rightarrow x = e^{-t}$$

$$\Rightarrow dx = -e^{-t}dt$$

Therefore $\displaystyle\int_0^1 \frac{dx}{\sqrt{-\log x}} = \int_\infty^0 \frac{-e^{-t}dt}{\sqrt{t}}$

$$= \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt$$

$$= \Gamma\left(\frac{1}{2}\right)$$

$$= \sqrt{\pi}$$

Problems on Beta and Gamma functions

2. Prove that $(m+1)^{n+1}(-1)^n\int_0^1 x^m(\log x)^n\,dx=\Gamma(n+1)$,

where n is positive integer and $m>-1$

Problems on Beta and Gamma functions

Proof: Consider, $\int_0^1 x^m (\log x)^n \, dx$

Take, $x = e^{-y}, \quad dx = -e^{-y} dy$

then, $\int_0^1 x^m (\log x)^n \, dx = \int_\infty^0 \left(e^{-y}\right)^m \left(\log e^{-y}\right)^n \left(-e^{-y}\right) dy$

$$= -\int_\infty^0 e^{-my} \, (-y)^n \, e^{-y} dy$$

$$= (-1)^n \int_0^\infty e^{-(m+1)y} y^n dy$$

Let, $(m+1)y = u, \; (m+1)dy = du$

$$\therefore \int_0^1 x^m (\log x)^n \, dx = (-1)^n \int_0^\infty e^{-u} \left(\frac{u}{m+1}\right)^n \frac{du}{(m+1)}$$

Problems on Beta and Gamma functions

$$=(-1)^n\int_0^\infty e^{-u}\,\frac{u^n}{(m+1)^n}\,\frac{du}{(m+1)}$$

$$=\frac{(-1)^n}{(m+1)^{n+1}}\int_0^\infty e^{-u}u^n\,du$$

$$=\frac{(-1)^n}{(m+1)^{n+1}}\Gamma(n+1)$$

Therefore, $(m+1)^{n+1}(-1)^n\int_0^1 x^m(\log x)^n\,dx=\Gamma(n+1),$

Problems on Beta and Gamma functions

3. Evaluate $\int_0^1 x^4 (1-x)^3 \, dx$

Problems on Beta and Gamma functions

Solution: $\int_0^1 x^4 (1-x)^3 \, dx = \int_0^1 x^{5-1} (1-x)^{4-1} \, dx$

$$= \beta(5,4)$$

$$= \frac{\Gamma(5)\Gamma(4)}{\Gamma(9)}$$

$$= \frac{4!3!}{8!}$$

$$= \frac{1}{280}$$

Problems on Beta and Gamma functions

4. Prove that $\int_0^1 x^m \left(1-x^n\right)^r dx = \frac{1}{n}\beta\left(\frac{m+1}{n}, r+1\right)$

Problems on Beta and Gamma functions

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Proof: Take, $x^n = y \Rightarrow nx^{n-1}dx = dy$

$$\text{LHS} = \int_0^1 x^m \left(1 - x^n\right)^r dx$$

$$= \int_0^1 y^{\frac{m}{n}} (1 - y)^r \frac{dy}{ny^{\frac{n-1}{n}}}$$

$$= \frac{1}{n} \int_0^1 y^{\frac{m-n+1}{n}} (1 - y)^r dy$$

$$= \frac{1}{n} \beta\left(\frac{m+1}{n}, r+1\right) = RHS$$

Problems on Beta and Gamma functions

5. Evaluate $\int_0^{2\pi} \sin^8 \theta d\theta$

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Solution: $$\int_0^{2\pi} \sin^8\theta d\theta = 4\int_0^{\frac{\pi}{2}} \sin^8\theta d\theta$$

$$= 4.\frac{1}{2}\beta\left(\frac{9}{2},\frac{1}{2}\right)$$

$$= 2\frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(5)}$$

$$= 2.\frac{\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{4!}$$

$$= \frac{35\pi}{2}$$

Problems on Beta and Gamma functions

6. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta$

Problems on Beta and Gamma functions

Solution: $$\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\,d\theta = \int_0^{\frac{\pi}{2}} \cot^{\frac{1}{2}}\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\cos\theta}{\sin\theta}\right)^{\frac{1}{2}} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\theta\cos^{\frac{1}{2}}\theta\,d\theta$$

$$= \frac{1}{2}\beta\left(\frac{1}{4},\frac{3}{4}\right)$$

Problems on Beta and Gamma functions

$$= \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)}$$

$$= \frac{1}{2}\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}$$

$$= \frac{\pi}{\sqrt{2}}$$

Problems on Beta and Gamma functions

7. Evaluate $\int_0^\infty \frac{dx}{1+x^4}$

Problems on Beta and Gamma functions

Solution: Take $x^2 = \tan\theta \Rightarrow 2x\,dx = \sec^2\theta\, d\theta$

$$\Rightarrow dx = \frac{\sec^2\theta\, d\theta}{2x}$$

Therefore, $dx = \dfrac{\sec^2\theta\, d\theta}{2\sqrt{\tan\theta}}$

Then $\displaystyle\int_0^{\infty} \frac{dx}{1+x^4} = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^2\theta}\,\frac{\sec^2\theta\, d\theta}{2\sqrt{\tan\theta}}$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\sec^2\theta}\,\frac{\sec^2\theta\, d\theta}{2\sqrt{\tan\theta}}$$

Problems on Beta and Gamma functions

$$= \int_0^{\frac{\pi}{2}} \frac{d\theta}{2\sqrt{\tan\theta}}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\tan\theta)^{-\frac{1}{2}}\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{\sin\theta}{\cos\theta}\right)^{-\frac{1}{2}} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\theta \cos^{\frac{1}{2}}\theta\, d\theta$$

Problems on Beta and Gamma functions

$$= \frac{1}{2}\frac{1}{2}\beta\left(\frac{1}{4},\frac{3}{4}\right)$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)}$$

$$= \frac{1}{4}\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}$$

$$= \frac{1}{4}\frac{\pi}{\left(\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{\pi}{2\sqrt{2}}$$

THANK YOU

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