ENGINEERING PHYSICS

Dr R Vasudevan Iyer

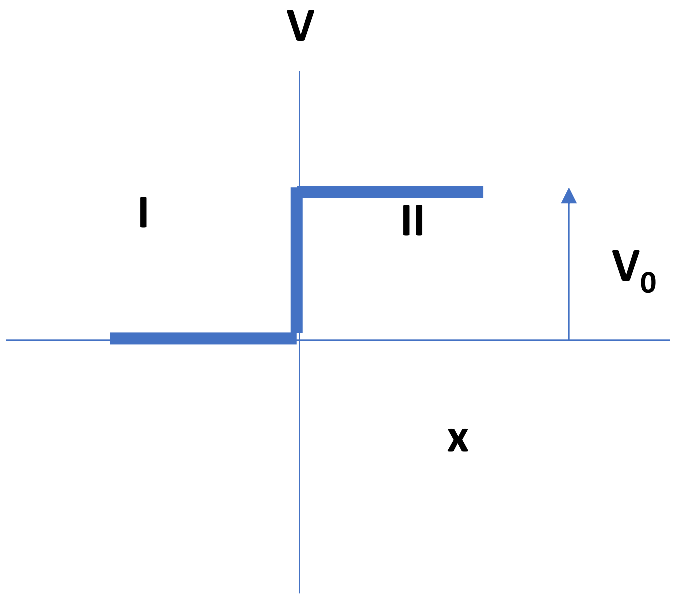
Department of Science and Humanities

ENGINEERING PHYSICS

Step Potential: solution of Schrödinger's wave equation for particle with E>Vo, Reflection and transmission coefficients

Class # 23

- Defining a step potential
- Region I, $-\infty < x < 0$, $V = 0$
- Region II, $0 < x < \infty$, $V = V_0$



The situation:

Particles of energy E ($> V_0$) are incident at the step, in region I

Step Potential: solution of Schrödinger’s wave equation for particle with E>Vo, Reflection and transmission coefficients

Realising a step potential

Metal wires separated by a gap and connected to opposite polarities of a battery

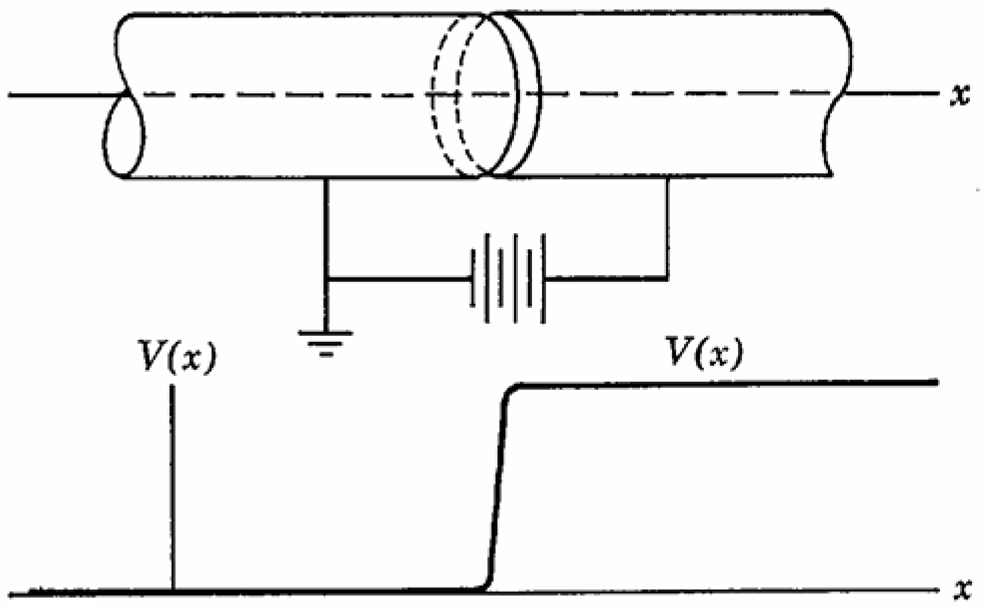
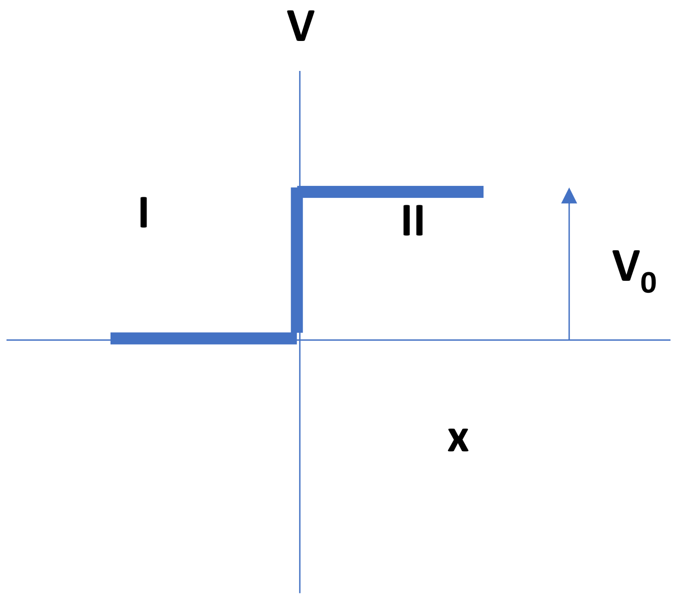


Image credit: quantum-physics-2nd-Eisberg_Resnick

Step Potential: solution of Schrödinger's wave equation for particle with E>Vo, Reflection and transmission coefficients

To get the solution we need to solve Schrodinger's equation for regions I and II. For region I, $\frac{d^2\psi_I(x)}{dx^2} + \frac{2m}{\hbar^2}E\psi_I(x) = 0$ *or* $\frac{d^2\psi_I(x)}{dx^2} + k_I^2\psi_I(x) = 0$, *where* $k_I = \sqrt{\frac{2mE}{\hbar^2}}$. *The solution is* $\psi_I(x) = Ae^{ik_Ix} + Be^{-ik_Ix}$



Ae^{ik_Ix} *is the incident wave and* Be^{-ik_Ix} *the reflected wave.*

Step Potential: solution of Schrödinger's wave equation for particle with E>Vo, Reflection and transmission coefficients

For region II, $\frac{d^2\psi_{II}(x)}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\psi_{II}(x) = 0$ *or* $\frac{d^2\psi_I(x)}{dx^2} + k_{II}^2\psi_I(x) = 0$, *where* $k_{II} = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$. *The solution is* $\psi_{II}(x) = Ce^{ik_{II}x} + De^{-ik_{II}x}$

$Ce^{ik_{II}x}$ *is the transmitted wave and* $De^{-ik_{II}x}$ *the reflected wave.*

As there is no change in the potential energy in region II there will be no reflected wave. Hence, we take D = 0

The accepted solution is $\psi_{II}(x) = Ce^{ik_{II}x}$

Step Potential: solution of Schrödinger's wave equation for particle with E>Vo, Reflection and transmission coefficients

Thus, the acceptable solutions are

Region I: $\psi_I(x) = Ae^{ik_I x} + Be^{-ik_I x}$ and For region II, $\psi_{II}(x) = Ce^{ik_{II} x}$

Boundary conditions: At x = 0, $\psi_I = \psi_{II}$ *and* $\frac{d\psi_I}{dx} = \frac{d\psi_{II}}{dx}$

Solving for these conditions we get

$A + B = C$ *and* $ik_I(A - B) = ik_{II}C$ *or* $k_I(A - B) = k_{II}C$

Step Potential: solution of Schrödinger's wave equation for particle with E>Vo, Reflection and transmission coefficients

Solving for B and C in terms of A we get $B = \frac{k_I - k_{II}}{k_I + k_{II}} A$ and $C = \frac{2k_I}{k_I + k_{II}} A$

The wave function for incident wave is $\psi_i(x) = Ae^{ik_I x}$ *and for the reflected wave it is* $\psi_r(x) = Be^{-ik_I x}$

The incident beam flux is given by $v_I \psi_i(x)^* \psi_i(x)$ *and the reflected beam flux is* $v_I \psi_r(x)^* \psi_r(x)$. *The reflection coefficient is then given by* $R = \frac{v_I \psi_r(x)^* \psi_r(x)}{v_I \psi_i(x)^* \psi_i(x)} = \frac{\psi_r(x)^* \psi_r(x)}{\psi_i(x)^* \psi_i(x)}$. *Now* $\psi_i(x) = Ae^{ik_I x}$ *and* $\psi_i^* = A^* e^{-ik_I x}$ *. Thus* $\psi_i(x)^* \psi_i(x) = A^* A$

Step Potential: solution of Schrödinger's wave equation for particle with E>Vo, Reflection and transmission coefficients

Also, $\psi_r(x) = Be^{-ik_I x}$ *and* $\psi_r^* = B^* e^{ik_I x}$ *. Thus* $\psi_r(x)^* \psi_r(x) = B^* B$

Hence $R = \frac{B^* B}{A^* A} = \left(\frac{B}{A}\right)^* \left(\frac{B}{A}\right) = \left|\frac{B}{A}\right|^2$*. The ratio* $\frac{B}{A} = \frac{k_I - k_{II}}{k_I + k_{II}}$*. As the ratio is not complex, we have* $R = \left(\frac{B}{A}\right)^2 = \left(\frac{k_I - k_{II}}{k_I + k_{II}}\right)^2$

Now the transmission coefficient is given by $T = \frac{transmission\ beam\ flux}{incident\ beam\ flux}$

Step Potential: solution of Schrödinger's wave equation for particle with E>Vo, Reflection and transmission coefficients

The transmission beam flux is given by $v_{II}\psi_t(x)^*\psi_t(x)$

Here, $\psi_t(x) = \psi_{II}(x) = Ce^{ik_{II}x}$*. The transmission beam flux is* $v_{II}C^*e^{-ik_{II}x}Ce^{ik_{II}x} =$

$v_{II}C^*C$*. Therefore* $T = \frac{v_{II}C^*C}{v_I A^*A} = \left(\frac{v_{II}}{v_I}\right)\left(\frac{C}{A}\right)^2$ *as the ratio C/A is real*

Now $v_I = \frac{p_I}{m} = \frac{\hbar k_I}{m}$ *and* $v_{II} = \frac{p_{II}}{m} = \frac{\hbar k_{II}}{m}$*, where* p_I *and* p_{II} *are the momenta of the*

particles in regions I and II respectively. Thus, $T = \frac{v_{II}C^*C}{v_I A^*A} = \left(\frac{k_{II}}{k_I}\right)\left(\frac{2k_I}{k_I+k_{II}}\right)^2 = \frac{4k_I k_{II}}{(k_I+k_{II})^2}$

Step Potential: solution of Schrödinger's wave equation for particle with E>Vo, Reflection and transmission coefficients

So, we have $R = \left(\frac{k_I - k_{II}}{k_I + k_{II}}\right)^2$ *and* $T = \frac{4k_I k_{II}}{(k_I + k_{II})^2}$. *What happens when we add R and T?*

$$R + T = \left(\frac{k_I - k_{II}}{k_I + k_{II}}\right)^2 + \frac{4k_I k_{II}}{(k_I + k_{II})^2} = \left(\frac{k_I + k_{II}}{k_I + k_{II}}\right)^2 = 1$$

Now the number of particles incident is N_i .

If nothing is lost, then a certain number will be reflected, N_r *and a certain number will be transmitted,* N_t. *Thus* $N_i = N_r + N_t$. *Then we have* $\frac{N_r}{N_i} + \frac{N_t}{N_i} = 1$

Step Potential: solution of Schrödinger's wave equation for particle with E>Vo, Reflection and transmission coefficients

However $R = \frac{N_r}{N_i}$ *and* $T = \frac{N_t}{N_i}$*. We thus see that R + T = 1 is a statement of*

CONSERVATION OF PARTICLES

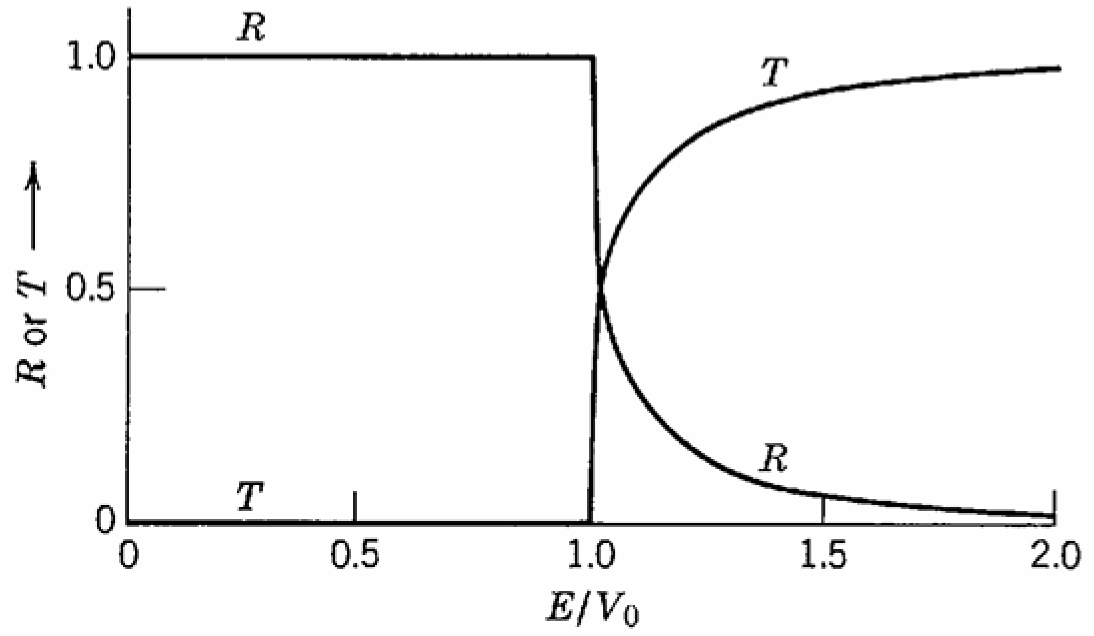
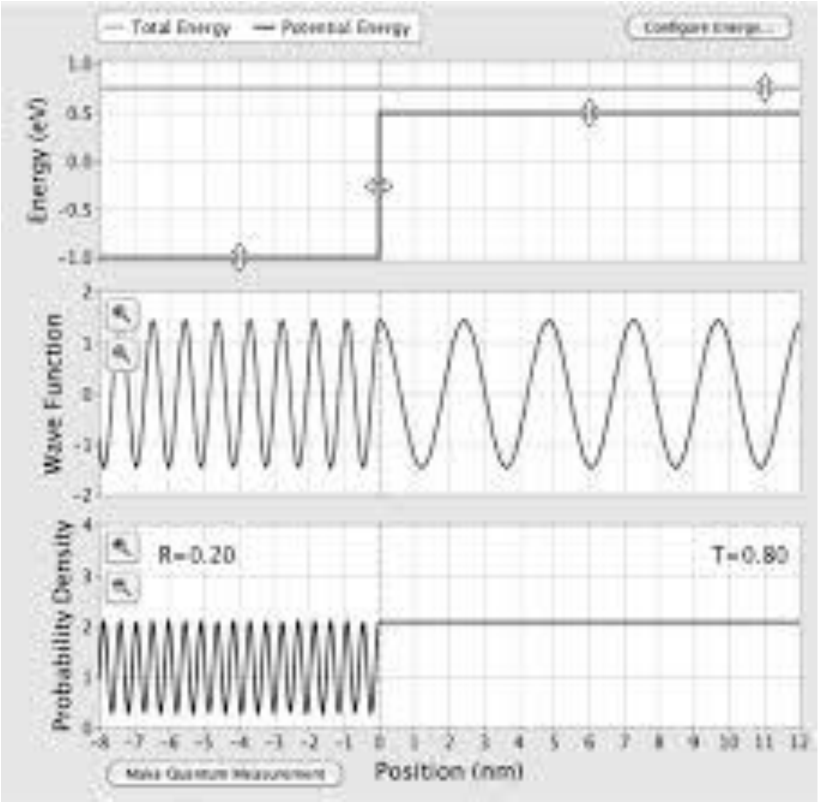


Image credit: quantum-physics-2nd-Eisberg_Resnick

Step Potential: solution of Schrödinger's wave equation for particle with E>Vo, Reflection and transmission coefficients

Eigen function plots



Step Potential: solution of Schrödinger's wave equation for particle with E>Vo, Reflection and transmission coefficients

Probability density plots with $k_I = 2k_{II}$

Associated eigen functions are

$$\psi_I^*\psi_I = \frac{16}{9}|A|^2\left[cos^2(kx) + \frac{sin^2 kx}{4}\right]$$ and

$$\psi_{II}^*\psi_{II} = \frac{16}{9}|A|^2$$

For x = 0 we get

$\psi_I^*\psi_I = \frac{16}{9}|A|^2$ as max value

For minimum value we use

$\psi_I^*\psi_I = |A|^2\left[\frac{8}{9} - \frac{4cos^2 2kx}{9}\right]$. At x = 0 we get

$$\psi_I^*\psi_I = \frac{4}{9}|A|^2$$

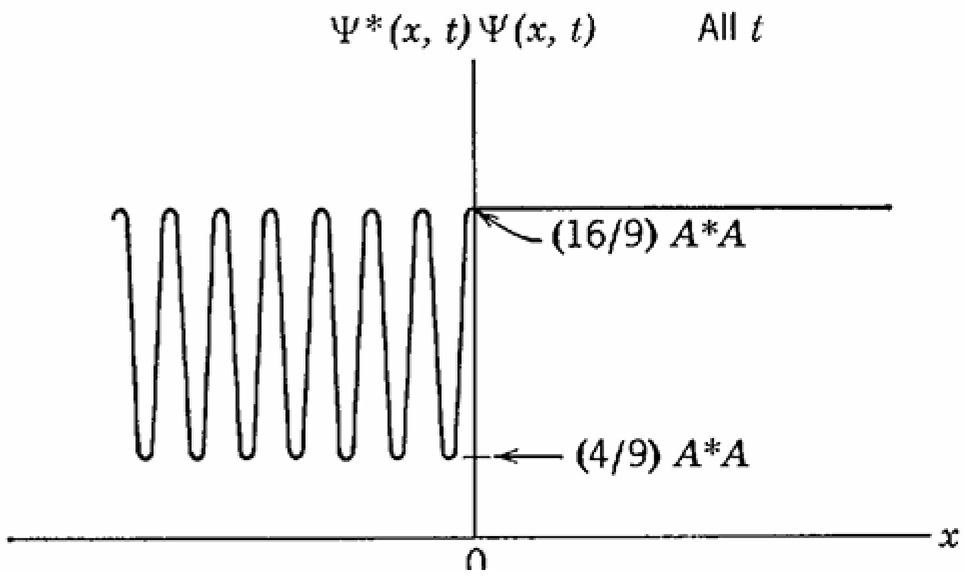


Image credit: quantum-physics-2nd-Eisberg_Resnick

Numericals….

1. Electrons with energies of 4.00 eV are incident on a potential step 3.0 eV high. Find the probability of reflection at x=0 and transmission for x>0

2. A proton with energy E is incident on a potential step of height 3.5eV. If the de Broglie wavelength of the particle after transmission is 1.228 nm, find the energy of the proton.

THANK YOU

Dr R Vasudevan Iyer

Professor, Department of Science and Humanities

rviyer@pes.edu