

ENGINEERING MATHEMATICS - I

Unit - 3: Partial Differential Equations

Department of Science and Humanities



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Formation of PDE by elimination of arbitrary functions



- ① Partial differential equations can be obtained by eliminating arbitrary functions from a given equation
- ② A first-order partial differential equation can be derived from an equation involving a single arbitrary function, while a second-order equation may arise from an expression containing two arbitrary functions
- ③ However, it is not always guaranteed that an n th-order partial differential equation can be formed from an equation involving n arbitrary functions
- ④ In some cases, higher-order partial derivatives may be required to eliminate all n arbitrary functions, leading to a partial differential equation of order higher than n . Moreover, such a resulting higher-order equation may not be unique

Example: Formation of a PDE by eliminating an arbitrary function

Eliminate the arbitrary function from $z = f(x + y)$ to obtain a first-order partial differential equation.

Answer: Let $u = x + y$. Then $z = f(u)$.

Differentiating with respect to x :

$$\frac{\partial z}{\partial x} = f'(u) \cdot \frac{\partial u}{\partial x} = f'(x + y) \cdot 1 = f'(x + y)$$

Differentiating with respect to y :

$$\frac{\partial z}{\partial y} = f'(u) \cdot \frac{\partial u}{\partial y} = f'(x + y) \cdot 1 = f'(x + y)$$

From above,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$$

So, the required PDE is:

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$$

Example: 2

Obtain a second-order partial differential equation by eliminating the arbitrary functions from

$$u = f(x + ct) + g(x - ct).$$



Solution Differentiating the given equation partially with respect to x and t , we obtain

$$\frac{\partial u}{\partial x} = f'(x + ct) + g'(x - ct), \quad \frac{\partial u}{\partial t} = cf'(x + ct) - cg'(x - ct),$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x + ct) + g''(x - ct), \quad \frac{\partial^2 u}{\partial t^2} = c^2 f''(x + ct) + c^2 g''(x - ct).$$

Hence, we obtain the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Example: 3

Given $z = f(x^2 + y)$, where f is an arbitrary function, form the partial differential equation by eliminating the arbitrary function

Solution:

- ① Differentiate with respect to x :

$$\frac{\partial z}{\partial x} = f'(x^2 + y) \cdot 2x$$

- ② Differentiate with respect to y :

$$\frac{\partial z}{\partial y} = f'(x^2 + y) \cdot 1 = f'(x^2 + y)$$

- ③ Eliminate f' by dividing the first equation by the second:

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{2x f'(x^2 + y)}{f'(x^2 + y)} = 2x$$

Required PDE:

$$\frac{\partial z}{\partial x} - 2x \frac{\partial z}{\partial y} = 0$$



Example: 4

Given $z = f(x^2 + y^2) + x$, where f is an arbitrary function, form the partial differential equation by eliminating the arbitrary function



Solution:

- ① Differentiate with respect to x : $\frac{\partial z}{\partial x} = f'(x^2 + y^2) \cdot 2x + 1$
- ② Differentiate with respect to y : $\frac{\partial z}{\partial y} = f'(x^2 + y^2) \cdot 2y$
- ③ Eliminate f' by solving the second equation for $f'(x^2 + y^2)$:

$$f'(x^2 + y^2) = \frac{1}{2y} \frac{\partial z}{\partial y} \quad (\text{for } y \neq 0)$$

Substitute into the first equation: $\frac{\partial z}{\partial x} = 2x \left(\frac{1}{2y} \frac{\partial z}{\partial y} \right) + 1 = \frac{x}{y} \frac{\partial z}{\partial y} + 1$

Required PDE:

$$\frac{\partial z}{\partial x} - \frac{x}{y} \frac{\partial z}{\partial y} = 1$$

Solutions of Partial Differential Equations



- A solution of a partial differential equation is a relation between the dependent and independent variables that satisfies the partial differential equation
- It is known that a partial differential equation is formed by eliminating arbitrary constants and arbitrary functions. These relations are solutions of the partial differential equation formed
- There are two types of solutions for partial differential equations:
 1. Solution containing arbitrary constants
 2. Solution containing arbitrary functions

Complete Solution & General Solution



- A solution of a partial differential equation which contains as many arbitrary constants as the number of independent variables is called the **complete solution**
- A solution of a partial differential equation, which contains as many arbitrary functions as the order of the equation, is called the **general solution**

Example

Suppose we have the relation:

$$z = f(x + y)$$



where f is an arbitrary (differentiable) function.
Now, $\frac{\partial z}{\partial x} = f'(x + y)$; $\frac{\partial z}{\partial y} = f'(x + y)$. Therefore,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$$

The partial differential equation: $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$

Conclusion:

The equation $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$ is a partial differential equation formed by eliminating the arbitrary function f from the original relation. The original relation $z = f(x + y)$ is a solution to this partial differential equation

Solution of First-Order Equations



- An equation containing x, y, z, p, q defines a first order partial differential equation, that is

$$F(x, y, z, p, q) = 0 \quad (1)$$

- **General solution:** A relation of the form $\phi(u, v) = 0$, where ϕ is an arbitrary function of $u = u(x, y, z)$, $v = v(x, y, z)$ and satisfies Eq. (1) is called a *general solution*
- **Particular solution:** The solution obtained by determining the arbitrary function in the general solution by using some specified condition is called a *particular solution*