PES University, Bangalore

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Department of Science and Humanities

Engineering Mathematics - I (UE25MA141A)

Assignment

Unit - 1: Partial Differentiation

Problems on partial derivatives

1. **Find the first order partial derivatives of the following:**

   (i) $u = \tan^{-1} \frac{x^2+y^2}{x+y}$ (ii) $u = \cos^{-1}\left(\frac{x}{y}\right)$.

   Answer: $\frac{\partial u}{\partial x} = \frac{x^2+2xy-y^2}{(x+y)^2+(x^2+y^2)^2}$; $\frac{\partial u}{\partial y} = \frac{x}{y\sqrt{y^2-x^2}}$.

2. If $u = x^y$, show that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

   Answer: $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left[x^{y-1}(y\log x + 1)\right]$.

3. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

   (i) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

   (ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 u}{\partial y \partial z} + 2\frac{\partial^2 u}{\partial z \partial x} + 2\frac{\partial^2 u}{\partial x \partial y} = -\frac{9}{(x+y+z)^2}$.

4. If $\theta = t^n e^{-\frac{r^2}{4t}}$, find the value of n which will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$.

   Answer: $\frac{r^2}{4t} - \frac{3}{2} = n + \frac{r^2}{4t} \implies n = -\frac{3}{2}$.

5. If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, then prove that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$.

Total derivative, Implicit function, Chain rule

6. If $x^3 + y^3 - 3axy = 0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
   Answer: $\frac{dy}{dx} = -\frac{3x^2-3ay}{3y^2-3ax} = \frac{ay-x^2}{y^2-ax}$
   $\frac{d^2y}{dx^2} = \frac{2a^3xy}{(ax-y^2)^3}$

7. Find $\frac{dy}{dx}$ when $(\cos x)^y = (\sin y)^x$.
   Answer: $\frac{dy}{dx} = \frac{y\tan x+\log\sin y}{\log\cos x - x\cot y}$

8. If $u = x\log(xy)$, where $x^3 + y^3 + 3xy = 1$.Find $\frac{du}{dx}$

   Answer: $\frac{du}{dx} = 1 + \log(xy) - \frac{x}{y}\cdot\frac{x^2+y}{y^2+x}$

9. If $u = x^3 + y^3$ where $x = a\cos t$, $y = b\sin t$, find $\frac{du}{dt}$ and verify the result.

   Answer: $\frac{du}{dt} = -3a^3\cos^2 t\sin t + 3b^3\sin^2 t\cos t$.

10. If $z = f(x, y)$ where $x = e^u\cos v$ and $y = e^u\sin v$, show that
    (i) $y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}$.
    (ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

11. If $u = u(y - z,\ z - x,\ x - y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

12. If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

13. If $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, show that
    $\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$.

Homogeneous Function and Euler's Theorem

14. Verify Euler's theorem for the function $z = \frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}$.
    Hindt z is homogeneous function of degree $-\frac{1}{6}$.

15. If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$.

16. If $u = \sin^{-1}\left[\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0$.

17. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that (i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$
    (ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 3u\sin u$.

18. If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, Prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u\cos 2u}{4\cos^3 u}$.

Taylor Series and Maclaurin Series for a function of two variables $f(x, y)$

19. Expand $e^x \sin y$ in ascending powers of x and y.
Answer: $e^x \sin y = 0 + [x \cdot 0 + y \cdot 1] + \frac{1}{2!}[x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0] + \frac{1}{3!}[x^3 \cdot 0 + 3x^2y \cdot 1 + 3xy^2 \cdot 0 + y^3 \cdot (-1)] + \ldots$
$\Rightarrow e^x \sin y = y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3 + \ldots$

20. Expand $\tan^{-1} \frac{y}{x}$ in the neighbourhood of $(1, 1)$ up to the second degree terms.Hence compute f(1.1, 0.9) approximately.

Answer: $\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + \left[(x-1) \cdot \left(-\frac{1}{2}\right) + (y-1) \cdot \frac{1}{2}\right]$
$+ \frac{1}{2!}\left[(x-1)^2 \cdot \left(-\frac{1}{2}\right) + 2(x-1)(y-1) \cdot 0 + (y-1)^2 \cdot \left(-\frac{1}{2}\right)\right]$
$- \frac{1}{12}\left[(x-1)^3 + 3(x-1)^2(y-1) - 3(x-1)(y-1)^2 - (y-1)^3\right] + \ldots$ $f(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}(0.1) + \frac{1}{2}(-0.1) + \frac{1}{4}(0.1)^2 - \frac{1}{4}(-0.1)^2 - \frac{1}{12}\left[(0.1)^3 + 3(0.1)^2(-0.1) - 3(0.1)(-0.1)^2 - (-0.1)^3\right] + \ldots = 0.6857$ approximately.

21. Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's Theorem.
$x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2).$

Maxima and Minima of a function of two variables $f(x, y)$

22. Test the function $f(x, y) = x^3y^2(6-x-y)$ for maxima and minima for points not at the origin.
Answer: (3, 2) is the only stationary point under consideration. f(x, y) has a maximum value a (3, 2).

23. Examine for minimum and maximum values: $f(x, y) = \sin x + \sin y + \sin(x+y)$.
Answer: $f(x, y)$ has a maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.
Maximum value $= f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}.$

Lagrange's method of undetermined multipliers

24. Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$.
Anser: P $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$ is at a minimum distance from $A(3, 4, 12)$ and the minimum distance $= 12$. Q $\left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$ is at a maximum distance from $A(3, 4, 12)$ and the maximum distance $= 14$.

25. Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.