ENGINEERING MATHEMATICS - I

Unit - 3: Partial Differential Equations

Department of Science and Humanities

PES UNIVERSITY

Contents

1. Particular Integral when $f(x, y) = e^{ax+by}$

2. Particular Integral when $f(x, y) = sin(ax + b)$ or $cos(ax + b)$

Particular Integral when $f(x, y) = e^{ax+by}$

Consider the non-homogeneous equation

$$F(D, D')z = f(x, y)$$

The particular integral is now written as

$$z = [F(D, D')^{-1}]f(x, y)$$

When $f(x,y) = e^{ax+by}$

The particular integral in this case is

$$z = [F(D,D')^{-1}]e^{ax+by} = \frac{1}{F(a,b)}e^{ax+by}, \quad \text{if } F(a,b) \neq 0.$$

If $F(a,b) = 0$, then we write the particular integral as

$$z = \phi(x,y)e^{ax+by},$$

where

$$\phi(x,y) = [F(D+a, D'+b)]^{-1}(1).$$

We expand the operator $[F(D+a, D'+b)]^{-1}$ in an infinite series symbolically, and determine $\phi(x,y)$. Then, the particular integral is given by $z = \phi(x,y)e^{ax+by}$. Note that D^{-1} and $(D')^{-1}$ mean integral with respect to x and y respectively, keeping the other variable as constant

Problem

Find the general solution of the partial differential equation

$$[2D^2 - DD' - (D')^2 + D - D']z = e^{2x+3y}.$$

Solution: We write

$$\begin{aligned} F(D, D')\, z &= \left[\, 2D^2 - D\, D' - (D')^2 + D - D' \right] z \\ &= \left[(2D + D' + 1)(D - D')\right] z = e^{2x+3y}\,. \end{aligned}$$

The homogeneous equation is

$$(2D + D' + 1)(D - D')z = 0$$

For the factor $D - D'$, we have $a_1 = 1$, $b_1 = -1$, $c_1 = 0$.
For the factor $2D + D' + 1$, we have $a_2 = 2$, $b_2 = 1$, $c_2 = 1$.
Therefore, the complementary function as

$$z = e^{-x/2}\phi_1(x - 2y) + \phi_2(x + y)$$

Since $F(2,3) = (4+3+1)(2-3) = 8 \times (-1) = -8 \neq 0$, we obtain the particular integral as

$$z = \frac{1}{F(2,3)} e^{2x+3y} = -\frac{1}{8} e^{2x+3y}.$$

Therefore, the general solution of the given differential equation is given by

$$z = e^{-x/2}\phi_1(x-2y) + \phi_2(x+y) - \frac{1}{8} e^{2x+3y}$$

Problem

Find a particular integral of the differential equation

$$(4D^2 + 3DD' - D'^2 - D - D')z = 3e^{(x+2y)/2}$$

Solution: We have

$$F(D, D') = 4D^2 + 3DD' - D'^2 - D - D'$$

and

$$F\left(\frac{1}{2}, 1\right) = 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{2}\right)(1) - 1 - \frac{1}{2} - 1 = 0.$$

We write the particular integral as $z = \phi(x, y)e^{(x+2y)/2}$, where $\phi(x, y)$ is a function to be determined.

The given equation is: $F(D, D')[z] = 3e^{(x+2y)/2}$

$\Rightarrow F\left(D + \frac{1}{2}, D' + 1\right) \cdot \phi(x, y)e^{(x+2y)/2} = 3e^{(x+2y)/2}$

$\Rightarrow F\left(D + \frac{1}{2}, D' + 1\right) \cdot \phi(x, y) = 3$

Therefore,

$$\phi(x, y) = \left[F\left(D + \frac{1}{2}, D' + 1\right)\right]^{-1} (3)$$

Problem (contd.)

$$F\left(D+\frac{1}{2},D'+1\right)=4\left(D+\frac{1}{2}\right)^2+3\left(D+\frac{1}{2}\right)(D'+1)-(D'+1)^2-$$
$$(D+\frac{1}{2})-(D'+1)$$

$$=6D-\frac{3}{2}D'+3DD'+4D^2-(D')^2$$
$$=6D\left[1-\frac{1}{4}D'D^{-1}+\frac{1}{6}\{3D'+4D-D'(D')^2\}\right]$$

Problem (contd.)

Therefore,

$$\phi(x,y) = \left[F\left(D+\frac{1}{2}, D'+1\right)\right]^{-1}(3)$$

becomes

$$\phi(x,y) = \frac{1}{2}D^{-1}\left[1-\frac{1}{4}D'D^{-1}+\frac{1}{6}\{3D'+4D-D'(D')^2\}\right]^{-1}(1)$$

$$= \frac{1}{2}D^{-1}(1) = \frac{x}{2}.$$

Hence, the particular integral is given by

$$z = \frac{x}{2}e^{(x+2y)/2}$$

P.I. when $f(x,y) = \sin(ax+by)$, or $f(x,y) = \cos(ax+by)$

The particular integral in this case is

$$z = \frac{\sin(ax+by)}{\left[F\left(D^2,\, DD',\, (D')^2\right)\right]}$$

$$\Rightarrow z = \frac{\sin(ax+by)}{F(-a^2,-ab,-b^2)}, \quad \text{if } F(-a^2,-ab,-b^2) \neq 0.$$

Similarly, if $f(x,y) = \cos(ax+by)$, we get the particular integral as

$$z = \frac{\cos(ax+by)}{F(-a^2,-ab,-b^2)}, \quad \text{if } F(-a^2,-ab,-b^2) \neq 0.$$

If $F(-a^2,-ab,-b^2) = 0$, then the particular integral is found by using Case 4.

Problem

Find the particular integral of the differential equation

$$2\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin(x-2y)$$

Solution We have

$$F(D^2, DD', (D')^2)z = [2D^2 - 3DD' + (D')^2]z = \sin(x-2y)$$

We have the right hand side as $\sin(ax+by)$, where $a=1$, $b=-2$
Hence,

$$\begin{aligned} F(D^2, DD', (D')^2)\sin(x-2y) &= [2(-1) - 3(2) + (-4)]\sin(x-2y) \\ &= -12\sin(x-2y) \end{aligned}$$

The particular integral is given by

$$z = \frac{\sin(x-2y)}{F(-a^2, -ab, -b^2)} = -\frac{1}{12}\sin(x-2y)$$