

ENGINEERING MATHEMATICS - I

Unit - 3: Partial Differential Equations

Department of Science and Humanities





- 1 Let $f(x, y)$ be not of any one of the forms, that is, e^{ax+by} or $\sin(ax + by)$ or $\cos(ax + by)$ or a polynomial case; or the case of failure in Case 2

Let $f(x, y)$ be not of any one of the standard forms, or the case of failure in Case 2



We use the following procedure:

Assume that $F(D, D')$ is reducible, that is, it can be factorized.

Now consider one of the factors as

$$(a_1 D + b_1 D') z = a_1 p + b_1 q = f(x, y).$$

Since this is a Lagrange's equation, the auxiliary equations are

$$\frac{dx}{a_1} = \frac{dy}{b_1} = \frac{dz}{f(x, y)}.$$

The first two ratios give the solution

$$b_1 x - a_1 y = c, \quad c \text{ arbitrary constant.} \quad (1)$$

Consider now the first and third ratios:

$$\frac{dx}{a_1} = \frac{dz}{f(x, y)} = \frac{dz}{f\left[x, (b_1x - c)/a_1\right]},$$

or equivalently

$$a_1 dz = f\left[x, (b_1x - c)/a_1\right] dx.$$

Integrating, we get

$$a_1 z = \int f\left(x, \frac{b_1x - c}{a_1}\right) dx + c_1 = F(x, c) + c_1,$$

where c_1 is an arbitrary constant. After integration, we replace c by $b_1x - a_1y$ as given in Eq. 1. Since a particular integral is required, we set $c_1 = 0$.

Therefore,

$$z = \frac{1}{a_1} F(x, b_1x - a_1y)$$

We repeat the procedure for each factor to obtain the required particular integral.

Problem



Find the general solution of the partial differential equation

$$\left[3D^2 + 5DD' + 2(D')^2 \right] z = x e^y$$

To find CF: The factors are

$$3D^2 + 5DD' + 2(D')^2 = (3D + 2D')(D + D')$$

For $3D + 2D'$, $a = 3, b = 2$. For $D + D'$, $a = b = 1$.

Hence, the complementary function is

$$z = \phi_1(2x - 3y) + \phi_2(x - y)$$

Problem (contd.)



We first obtain $(D + D')^{-1}(xe^y)$.

Denote

$$u = (D + D')^{-1}(xe^y) \quad \text{or} \quad (D + D')u = xe^y.$$

The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{xe^y}.$$

The first two terms give $\frac{dx}{1} = \frac{dy}{1}$ we get $y = x + c$.

Using the first and third terms, we get $\frac{dx}{1} = \frac{du}{xe^{x+c}}$

Thus,

$$u = \int x e^{x+c} dx = e^c \int x e^x dx = e^c \int x e^x dx = e^c (e^x - 1) = e^y (x - 1),$$

since $c = y - x$

Now, denote

$$z = (3D + 2D')^{-1} u = (3D + 2D')^{-1} (e^y (x - 1)),$$

or

$$(3D + 2D') z = e^y (x - 1)$$

Problem (contd.)



The auxiliary equations are

$$\frac{dx}{3} = \frac{dy}{2} = \frac{dz}{e^y(x-1)}$$

The first two terms give

$$3y = 2x + c_1$$

The first and third terms give

$$\frac{dx}{3} = \frac{dz}{e^y(x-1)} \Rightarrow \frac{dx}{3} = \frac{dz}{e^{\frac{c_1}{2}} e^x (x-1)},$$

since $y = \frac{2x+c_1}{2}$ and $e^y = e^{\frac{c_1}{2}} e^x$

Problem (contd.)

Hence,

$$z = \frac{1}{3}e^{\frac{c_1}{2}} \int e^x(e^x - 1)dx$$



Now,

$$I = \int (x - 1)e^x dx = \int xe^x dx - \int e^x dx = (xe^x - e^x) - e^x = (x - 2)e^x$$

Hence,

$$z = \frac{1}{3}e^{\frac{c_1}{2}} \int (x - 1)e^x dx = \frac{1}{3}e^{\frac{c_1}{2}} ((x - 2)e^x) = \frac{1}{3}e^{\frac{c_1}{2}} (x - 2)e^x$$

Hence,

$$z = \frac{1}{3}e^{\frac{3y-2x}{2}} (x - 2)e^x,$$

which is the required particular integral. The general solution is

$$z = \phi_1(2x - 3y) + \phi_2(x - y) + \frac{1}{3}e^{\frac{3y-2x}{2}} (x - 2)e^x$$