PES University, Bangalore

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Department of Science and Humanities

Engineering Mathematics - I
(UE25MA141A)

Question Bank

Unit - 1: Partial Differentiation

1. Find all the first-order partial derivatives of the following function:

$$w = \cos(x^2 + 2y) - e^{4x - yz^4} + y^3$$

Answer:

(a) **Partial derivative with respect to x:**

$$\frac{\partial w}{\partial x} = -2x \sin(x^2 + 2y) - 4e^{4x - yz^4}$$

(b) **Partial derivative with respect to y:**

$$\frac{\partial w}{\partial y} = -2 \sin(x^2 + 2y) + z^4 e^{4x - yz^4} + 3y^2$$

(c) **Partial derivative with respect to z:**

$$\frac{\partial w}{\partial z} = 4yz^3 e^{4x - yz^4}$$

2. Find all the first-order partial derivatives of the following function:

$$f(u, v) = u^2 \sin(u + v^3) - \sec(4u) \tan^{-1}(2v)$$

Answer:

(a) **Partial derivative with respect to u:**

$$\begin{aligned}\frac{\partial f}{\partial u} &= \frac{\partial}{\partial u}\left[u^2 \sin(u + v^3)\right] - \frac{\partial}{\partial u}\left[\sec(4u) \tan^{-1}(2v)\right] \\ &= 2u \sin(u + v^3) + u^2 \cos(u + v^3) - 4 \sec(4u) \tan(4u) \tan^{-1}(2v)\end{aligned}$$

(b) **Partial derivative with respect to v:**

$$\begin{aligned}\frac{\partial f}{\partial v} &= \frac{\partial}{\partial v}\left[u^2 \sin(u + v^3)\right] - \frac{\partial}{\partial v}\left[\sec(4u) \tan^{-1}(2v)\right] \\ &= 3u^2 v^2 \cos(u + v^3) - \frac{2 \sec(4u)}{1 + 4v^2}\end{aligned}$$

3. Find all the first-order partial derivatives of the following function:

$$f(u, v, p, t) = 8u^2 t^3 p - \sqrt{v} p^2 t^{-5} + 2u^2 t + 3p^4 - v$$

Answer:

(a) **Partial derivative with respect to u:**

$$\frac{\partial f}{\partial u} = 16ut^3p + 4ut$$

(b) **Partial derivative with respect to v:**

$$\frac{\partial f}{\partial v} = -\frac{p^2}{2\sqrt{v}t^5} - 1$$

(c) **Partial derivative with respect to p:**

$$\frac{\partial f}{\partial p} = 8u^2t^3 - 2p\sqrt{v}t^{-5} + 12p^3$$

(d) **Partial derivative with respect to t:**

$$\frac{\partial f}{\partial t} = 24u^2t^2p + 5\sqrt{v}p^2t^{-6} + 2u^2$$

4. Given the function $u = x^y$, show that: (i) $u_{xy} = u_{yx}$; (ii) $u_{xxy} = u_{xyx}$.

Answer

$$\begin{aligned} u_{xy} &= \frac{\partial}{\partial y}(u_x) = \frac{\partial}{\partial y}(yx^{y-1}) \\ &= x^{y-1} + yx^{y-1}\ln x \\ &= x^{y-1}(1 + y\ln x) \end{aligned}$$

$$\begin{aligned} u_{yx} &= \frac{\partial}{\partial x}(u_y) = \frac{\partial}{\partial x}(x^y \ln x) \\ &= yx^{y-1}\ln x + x^y \cdot \frac{1}{x} \\ &= x^{y-1}(y\ln x + 1) \end{aligned}$$

Part (ii): To prove $u_{xxy} = u_{xyx}$

$$\begin{aligned} u_{xxy} &= \frac{\partial}{\partial y}(u_{xx}) = \frac{\partial}{\partial y}(y(y-1)x^{y-2}) \\ &= (2y-1)x^{y-2} + y(y-1)x^{y-2}\ln x \\ &= x^{y-2}[(2y-1) + y(y-1)\ln x] \end{aligned}$$

$$
\begin{aligned}
u_{xyx} &= \frac{\partial}{\partial x}(u_{xy}) = \frac{\partial}{\partial x}(x^{y-1}(1+y\ln x)) \\
&= (y-1)x^{y-2}(1+y\ln x) + x^{y-1}\left(\frac{y}{x}\right) \\
&= x^{y-2}[(y-1)(1+y\ln x)+y] \\
&= x^{y-2}[y-1+y(y-1)\ln x+y] \\
&= x^{y-2}[(2y-1)+y(y-1)\ln x]
\end{aligned}
$$

5. If $u = \sin^{-1}(x-y)$, where $x = 3t$, $y = 4t^3$, then show that:

$$
\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}, -1 < t < 1
$$

6. Find $\frac{du}{dx}$ if: $u = \cos(x^2+y^2)$ and $a^2x^2 + b^2y^2 = c^2$

Answer:

$$
\begin{aligned}
\frac{du}{dx} &= -2x\sin(x^2+y^2) + \left(-2y\sin(x^2+y^2)\right)\left(-\frac{a^2x}{b^2y}\right) \\
&= -2x\sin(x^2+y^2) + \frac{2a^2x}{b^2}\sin(x^2+y^2) \\
&= 2x\sin(x^2+y^2)\left(-1+\frac{a^2}{b^2}\right) \\
&= -2x\sin(x^2+y^2)\left(1-\frac{a^2}{b^2}\right) \\
&= -\frac{2x}{b^2}\sin(x^2+y^2)(b^2-a^2)
\end{aligned}
$$

7. If z is a function of x and y, and

$$
\begin{aligned}
x &= u\cos\alpha - v\sin\alpha \\
y &= u\sin\alpha + v\cos\alpha
\end{aligned}
$$

then show that:

$$
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}
$$

8. If $z = f(x,y)$ where $x = u^2 - v^2$, $y = 2uv$, prove that:

$$
4(u^2+v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}
$$

9. Transform the partial differential equation $z_{xx} + 2z_{xy} + z_{yy} = 0$ by changing the independent variables using the transformation: $u = x - y; v = x + y$. Show the transformed equation in terms of the new variables u and v.

Answer

$$-2z_{uv} + 2z_{vu} + 4z_{vv} = 0$$

If we now assume $z_{uv} = z_{vu}$), then:

$$4z_{vv} = 0 \implies \boxed{z_{vv} = 0}$$

However, without this assumption, the most general form is:

$$\boxed{z_{vu} - z_{uv} + 2z_{vv} = 0}$$

10. Given $u = (x-y)(y-z)(z-x)$, prove that:

 (i) $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$

 (ii) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = u$

11. If $z = xf\left(\frac{y}{x}\right) + g\left(\frac{x}{y}\right)$, then show that:

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$$

12. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, then find the value of:

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

13. If $f(x, y) = \tan^{-1}(xy)$, compute an approximate value of $f(0.9, -1.2)$.

Answer

$$\begin{aligned} f(0.9, -1.2) &\approx -\frac{\pi}{4} + \left[-\frac{1}{2}(-0.1) + \frac{1}{2}(-0.2)\right] + \frac{1}{4}\left[(-0.1)^2 + (-0.2)^2\right] \\ &= -\frac{\pi}{4} + [0.05 - 0.1] + \frac{1}{4}[0.01 + 0.04] \\ &= -0.7854 - 0.05 + 0.0125 = -0.8229 \end{aligned}$$

14. Expand $\frac{1}{1+x-y}$ using Taylor's series up to second-degree terms.

$$f(x, y) \approx 1 + [x(-1) + y(1)] + \frac{1}{2}\left[x^2(2) + 2xy(-2) + y^2(2)\right]$$

Simplify:

$$f(x, y) \approx 1 - x + y + \frac{1}{2}\left(2x^2 - 4xy + 2y^2\right)$$

$$f(x, y) \approx 1 - x + y + x^2 - 2xy + y^2$$

15. Find the maximum and minimum values of the function:

$$f(x, y) = \sin x \sin y \sin(x + y), \quad 0 < x, y < \pi.$$

Answer

The critical points are: $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$.
Maximum value:

$$f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}$$

Minimum value:

$$f\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 \left(-\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{8}$$

16. Discuss the maxima and minima of the function:

$$f(x, y) = x^3y^2(1 - x - y)$$

where $x > 0$, $y > 0$, and $x + y < 1$.

Answer

The critical points are: $(0, 0)$, $(0, \frac{2}{3})$, $(\frac{3}{4}, 0)$, $(1, 0)$, $\left(\frac{1}{2}, \frac{1}{3}\right)$:
Maximum value:

$$f\left(\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$$

17. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Answer:

$$x = \pm\frac{a}{\sqrt{3}}; y = \pm\frac{b}{\sqrt{3}}, \quad z = \pm\frac{c}{\sqrt{3}}$$

$$V_{\max} = \frac{8abc}{3\sqrt{3}}$$

18. Divide the number 24 into three parts such that the continued product of the first, square of the second, and the cube of the third may be maximum.

Answer

The three parts are 4, 8, and 12, and their product is:

$$4 \times 8^2 \times 12^3 = 4 \times 64 \times 1728 = 442368$$

19. Find the maximum value of $x^m y^n z^p$ subject to the constraint $x + y + z = a$.

Answer

$$x = \frac{am}{m+n+p}; y = \frac{an}{m+n+p}; z = \frac{ap}{m+n+p}$$

The maximum value is:

$$f_{\max} = \frac{a^{m+n+p} m^m n^n p^p}{(m+n+p)^{m+n+p}}$$