ENGINEERING MATHEMATICS I

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Class content

Problems on Generating Functions and Jacobi Series

Problems on Generating Functions and Jacobi Series

- The generating function for the sequence of functions $f_n(x)$ is,

  $G(x,t) = \sum_{n=-\infty}^{\infty} f_n(x)t^n$ which generates $f_n(x)$.

  i.e., $f_n(x)$ appear as coefficients of various powers of t

  in the expansion of $G(x,t)$.

- Generating function for Bessel function of integral order is $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$

  i.e. $\sum_{n=-\infty}^{\infty} J_n(x)t^n = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$

- Jacobi series: $\cos(x\sin\theta) = J_0 + 2\left(J_2\cos 2\theta + J_4\cos 4\theta + ...\right)$

  $\sin(x\sin\theta) = 2\left(J_1\sin\theta + J_3\sin 3\theta + J_5\sin 5\theta.....\right)$

Problems on Generating Functions and Jacobi Series

1. Using generating functions prove that $J_n(-x) = (-1)^n J_n(x)$

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Ans: We have $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} t^n J_n(x)$

Replacing x by –x, we get,

$$e^{\frac{-x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} t^n J_n(-x) \quad \text{-----------------------}(1)$$

Thus by (1)

$$\sum_{-\infty}^{\infty} t^n J_n(-x) = e^{\frac{-x}{2}\left(t-\frac{1}{t}\right)}$$

$$= e^{\frac{x}{2}\left(-t-\left(-\frac{1}{t}\right)\right)}$$

$$= \sum_{-\infty}^{\infty} (-t)^n J_n(x)$$

$$= \sum_{-\infty}^{\infty} (-1)^n t^n J_n(x)$$

Thus, $\sum_{-\infty}^{\infty} t^n J_n(-x) = \sum_{-\infty}^{\infty} (-1)^n t^n J_n(x)$

Comparing on both sides, we get,

$$\boxed{J_n(-x) = (-1)^n J_n(x)}$$

2.Using generating functions prove that, $J_{-n}(x) = (-1)^n J_n(x)$ where n is an integer.

Problems on Generating Functions and Jacobi Series

We have $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} t^n J_n(x)$

Let $t = \frac{1}{u} \Rightarrow \frac{1}{t} = u$

Therefore $e^{\frac{x}{2}\left(\frac{1}{u}-u\right)} = \sum_{-\infty}^{\infty} \left(\frac{1}{u}\right)^n J_n(x)$

$$= \sum_{-\infty}^{\infty} u^{-n} J_n(x)$$

Put $-n = N$ then

$$e^{\frac{x}{2}\left(\frac{1}{u}-u\right)} = \sum_{-\infty}^{\infty} u^N J_{-N}(x) \quad ------(1)$$

Also $e^{\frac{x}{2}\left(\frac{1}{u}-u\right)} = e^{\frac{-x}{2}\left(-\frac{1}{u}+u\right)}$

$$= \sum_{-\infty}^{\infty} u^N J_N(-x) \quad ------(2)$$

By (1) and (2)

$$J_{-N}(x) = J_N(-x)$$

$$= (-1)^N J_N(x)$$

Therefore, $J_{-n}(x) = \left(-1\right)^n J_n(x)$

Problems on Generating Functions and Jacobi Series

3. Prove that $J_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos(x\cos\phi)\, d\phi$

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Ans: We Know that $\cos(x\sin\theta) = J_0 + 2\left(J_2\cos 2\theta + J_4\cos 4\theta + ...\right)$

Replace θ by $\dfrac{\pi}{2} - \phi$ we get,

$$\cos\left(x\sin(\frac{\pi}{2}-\varphi)\right) = J_0 + 2\left(J_2\cos 2(\frac{\pi}{2}-\varphi) + J_4\cos 4(\frac{\pi}{2}-\varphi) + ...\right)$$

$$\cos(x\cos\phi) = J_0 + 2\left(J_2(-\cos 2\varphi) + J_4\cos 4\varphi + ...\right)$$

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Integrating wrt ϕ between 0 to π

$$\int_0^{\pi}\cos(x\cos\phi)d\phi = \int_0^{\pi}\left(J_0 + 2\left(J_2(-\cos 2\phi) + J_4\cos 4\phi + ...\right)\right)d\phi$$

$$= J_0.\pi + 0 \qquad \left(\because \int_0^{\pi}\cos n\varphi\, d\varphi = 0\right)$$

$$\therefore J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\phi)\; d\phi$$

4. Prove that, $J_0^2 + 2J_1^2 + 2J_2^2 + \text{--------} = 1$

Problems on Generating Functions and Jacobi Series

Ans: $$\cos(x\sin\theta) = J_0 + 2\left(J_2\cos 2\theta + J_4\cos 4\theta + \dots\dots\dots\right) \quad --\rightarrow (1)$$

$$\sin(x\sin\theta) = 2\left(J_1\sin\theta + J_3\sin 3\theta + .J_5\sin 5\theta\dots\dots\dots\right) \quad --\rightarrow (2)$$

Squaring (1) and (2), we get,

$$\cos^2(x\sin\theta) = J_0^{\,2} + 4\left(J_2^{\,2}\cos^2 2\theta + J_4^{\,2}\cos^2 4\theta + \dots\dots\dots\right) + \dots\dots\dots$$

$$\sin^2(x\sin\theta) = 4\left(J_1^{\,2}\sin^2\theta + J_3^{\,2}\sin^2 3\theta + J_5^{\,2}\sin^2 5\theta\dots\dots\dots\right) + \dots\dots\dots$$

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$$\Rightarrow \int_0^{\pi} \cos^2(x\sin\theta)d\theta = \int_0^{\pi}\left[J_0^{\ 2} + 4\left(J_2^{\ 2}\cos^2 2\theta + J_4^{\ 2}\cos^2 4\theta + \ldots\ldots\ldots\right) + \ldots\ldots\ldots\right]d\theta$$

and

$$\int_0^{\pi} \sin^2(x\sin\theta)d\theta = 4\int_0^{\pi}\left[\left(J_1^{\ 2}\sin^2\theta + J_3^{\ 2}\sin^2 3\theta + J_5^{\ 2}\sin^2 5\theta\ldots\ldots\ldots\right) + \ldots\ldots\ldots\right]d\theta$$

Adding above two equations, we get,

$$\int_0^{\pi}\left[\sin^2(x\sin\theta)+\cos^2(x\sin\theta)\right]d\theta$$

$$=\int_0^{\pi}\left[J_0^{\,2}+4\left(J_1^{\,2}\sin^2\theta+J_2^{\,2}\cos^2 2\theta+J_3^{\,2}\sin^2 3\theta+J_4^{\,2}\cos^2 4\theta+.........\right)+.........\right]d\theta$$

$$\Rightarrow \pi = J_0^{\,2}\pi+4(J_1^{\,2}\frac{\pi}{2}+J_2^{\,2}\frac{\pi}{2}+J_3^{\,2}\frac{\pi}{2}+J_4^{\,2}\frac{\pi}{2}+------)$$

$$\Rightarrow 1=J_0^{\,2}+2(J_1^{\,2}+J_2^{\,2}+J_3^{\,2}+J_4^{\,2}+------)$$

Problems on Generating Functions and Jacobi Series

5. Prove that $\frac{x}{2} = J_1 + 3\,J_3 + 5\,J_5 + - - - - - - - -$

Problems on Generating Functions and Jacobi Series

Ans: $\sin(x\sin\theta) = 2\left(J_1\sin\theta + J_3\sin 3\theta + .J_5\sin 5\theta.........\right)$

Diffrentiating wrt θ

$$\cos(x\sin\theta)x\cos\theta = 2\left(J_1\cos\theta + J_3\cos 3\theta\cdot 3 + J_5\cos 5\theta\cdot 5 + ...\right)$$

put $\theta = 0$, we get

$$x = 2\left(J_1 + 3J_3 + 5J_5........\right)$$

$$\Rightarrow \frac{x}{2} = J_1 + 3J_3 + 5J_5...........$$

THANK YOU

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