



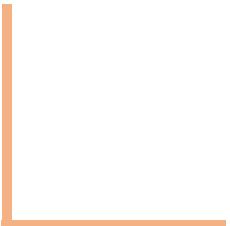
ENGINEERING MATHEMATICS I

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Class content

- ❖ Bessel differential equation of order n
- ❖ Series Solution of Bessel's Differential Equation
- ❖ Bessel Function



Bessel Differential Equation

The second order linear ordinary differential equation in the form,

$$x^2 y'' + xy' + (x^2 - n^2) y = 0 \quad \text{--- (1)}$$

is known as Bessel differential equation.

Here n is the order of the Bessel differential equation.

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Series Solution of Bessel Differential Equation

Let, $y = \sum_{r=0}^{\infty} a_r x^{r+k}$ ----- (2)

where, k is a positive real number a_r 's are constants and $a_0 \neq 0$

Then, $\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (r+k) x^{r+k-1}$ ----- (3)

and $\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (r+k)(r+k-1) x^{r+k-2}$ ----- (4)

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Series Solution of Bessel Differential Equation

Substituting equation (2), (3) and (4) in equation (1) we get,

$$x^2 \sum_{r=0}^{\infty} a_r (r+k)(r+k-1) x^{r+k-2} + x \sum_{r=0}^{\infty} a_r (r+k) x^{r+k-1}$$

$$+ (x^2 - n^2) \sum_{r=0}^{\infty} a_r x^{r+k} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (r+k)(r+k-1) x^{r+k} + \sum_{r=0}^{\infty} a_r (r+k) x^{r+k}$$

$$+ \sum_{r=0}^{\infty} a_r x^{r+k+2} - n^2 \sum_{r=0}^{\infty} a_r x^{r+k} = 0$$

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Series Solution of Bessel Differential Equation

$$\Rightarrow \sum_{r=0}^{\infty} a_r (r+k) \{r+k-1+1\} x^{r+k} - n^2 \sum_{r=0}^{\infty} a_r x^{r+k} + \sum_{r=0}^{\infty} a_r x^{r+k+2} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (r+k)^2 x^{r+k} - n^2 \sum_{r=0}^{\infty} a_r x^{r+k} + \sum_{r=0}^{\infty} a_r x^{r+k+2} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r \{(r+k)^2 - n^2\} x^{r+k} + \sum_{r=0}^{\infty} a_r x^{r+k+2} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r \{(r+k)^2 - n^2\} x^{r+k} + \sum_{r=2}^{\infty} a_{r-2} x^{r+k} = 0$$

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Series Solution of Bessel Differential Equation



$$\begin{aligned} & \Rightarrow a_0 \left\{ k^2 - n^2 \right\} x^k + a_1 \left\{ (k+1)^2 - n^2 \right\} x^{k+1} \\ & + \sum_{r=2}^{\infty} a_r \left\{ (r+k)^2 - n^2 \right\} x^{r+k} + \sum_{r=2}^{\infty} a_{r-2} x^{r+k} = 0 \\ & \Rightarrow a_0 \left\{ k^2 - n^2 \right\} x^k + a_1 \left\{ (k+1)^2 - n^2 \right\} x^{k+1} \\ & + \sum_{r=2}^{\infty} \left[a_r \left\{ (r+k)^2 - n^2 \right\} + a_{r-2} \right] x^{r+k} = 0 \end{aligned}$$

Equating the co-efficients of x^k to zero, we get,

$$a_0(k^2 - n^2) = 0 \implies k^2 - n^2 = 0 \quad (\because a_0 \neq 0)$$

$$\Rightarrow k = \pm n$$

Equating the co-efficients of x^{k+1} to zero,

we get, $a_1 \left((k+1)^2 - n^2 \right) = 0$

$$\Rightarrow a_1 = 0$$

By equating the higher powers of x to zero,

we get,

$$a_r = -\frac{a_{r-2}}{(r+k)^2 - n^2} \quad \text{--- (5)}$$

for $r = 2, 3, 4, 5, \dots$

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Series Solution of Bessel Differential Equation

Let us consider the two cases:

case (i) $k = n$:

Put $k = n$ in equation (5)

$$\text{Then we get, } a_r = -\frac{a_{r-2}}{r^2 + 2nr} \quad \dots \dots \dots \quad (6)$$

Put $r = 2, 3, 4, \dots \dots$ in (6) we get,

$$a_2 = -\frac{a_0}{2^2 + 4n} = -\frac{a_0}{2^2(n+1)}$$

$$\therefore a_2 = -\frac{a_0}{2^2(n+1)} \quad \dots \dots \dots \quad (7)$$

$$\left(\because a_r = -\frac{a_{r-2}}{(r+k)^2 - n^2} \right)$$

Series Solution of Bessel Differential Equation

$$a_3 = -\frac{a_1}{3^2 + 6n} = 0 \quad (\because a_1 = 0) \quad \left(\text{we have, } a_r = -\frac{a_{r-2}}{r^2 + 2nr} \right)$$

$$a_4 = -\frac{a_2}{4^2 + 8n}$$

$$= -\frac{1}{8(n+2)} \left(-\frac{a_0}{2^2(n+1)} \right) \quad \left(\because a_2 = -\frac{a_0}{2^2(n+1)} \right)$$

$$\therefore a_4 = \frac{a_0}{32(n+1)(n+2)} \quad \text{----- (8)}$$

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Series Solution of Bessel Differential Equation

$$a_5 = -\frac{a_3}{5^2 + 10n} = 0$$

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We have, by (2),

$$y = a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + a_3 x^{k+3} + a_4 x^{k+4} \dots \dots \dots$$

substituting a_r 's in (2), we get,

$$y_1 = a_0 x^n + 0 \cdot x^{n+1} - \frac{a_0}{4(n+1)} x^{n+2} + 0 \cdot x^{n+3} + \frac{a_0}{32(n+1)(n+2)} x^{n+4} + \dots \dots \dots$$

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Series Solution of Bessel Differential Equation

$$\text{i.e. } y_1 = a_0 \left\{ x^n - \frac{1}{4(n+1)} x^{n+2} + \frac{1}{32(n+1)(n+2)} x^{n+4} + \dots \right\}$$

To standardize the result ,let us take,

$$a_0 = \frac{1}{2^n \Gamma(n+1)} \text{ then we get,}$$

$$y_1 = \frac{1}{2^n \Gamma(n+1)} \left\{ x^n - \frac{1}{4(n+1)} x^{n+2} + \frac{1}{32(n+1)(n+2)} x^{n+4} + \dots \right\}$$

$$= \frac{1}{2^n \Gamma(n+1)} x^n - \frac{1}{4(n+1)2^n \Gamma(n+1)} x^{n+2} + \frac{1}{32(n+1)(n+2)2^n \Gamma(n+1)} x^{n+4} + \dots$$

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Series Solution of Bessel Differential Equation

$$\begin{aligned}y_1 &= \frac{1}{2^n \Gamma(n+1)} x^n - \frac{1}{2^2 \cdot 2^n \Gamma(n+2)} x^{n+2} + \frac{1}{2 \cdot 2^4 \cdot 2^n \Gamma(n+3)} x^{n+4} + \dots \\&= \frac{1}{2^n \Gamma(n+1)} x^n - \frac{1}{2^{n+2} \Gamma(n+2)} x^{n+2} + \frac{1}{2} \frac{1}{2^{n+4} \Gamma(n+3)} x^{n+4} + \dots \\&= \frac{(-1)^0}{\Gamma(n+1)} \left(\frac{x}{2}\right)^n + \frac{(-1)^1}{\Gamma(n+2)} \left(\frac{x}{2}\right)^{n+2} + \frac{1}{2} \frac{(-1)^2}{\Gamma(n+3)} \left(\frac{x}{2}\right)^{n+4} + \dots \\&= \frac{(-1)^0}{0! \Gamma(n+1)} \left(\frac{x}{2}\right)^{n+2 \cdot 0} + \frac{(-1)^1}{1! \Gamma(n+2)} \left(\frac{x}{2}\right)^{n+2 \cdot 1} + \frac{1}{2! \Gamma(n+3)} \left(\frac{x}{2}\right)^{n+2 \cdot 2} + \dots\end{aligned}$$

$$y_1 = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)r!} \quad ----- (9)$$

Series Solution of Bessel Differential Equation

- ❖ Equation (9) Is called power series solution of Bessel differential equation.
- ❖ It is called Bessel function of first kind of order n.
- ❖ It is denoted by $J_n(x)$

Thus,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

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Series Solution of Bessel Differential Equation

case (ii) $k = -n$:

Put $k = -n$ in equation (5)

Then we get,

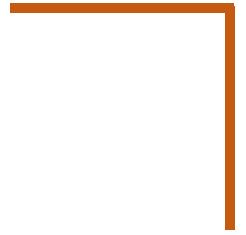
$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{\Gamma(-n+r+1)r!}$$

Series Solution of Bessel Differential Equation

Thus the complete solution of Bessel differential equation is given by

$$y = C_1 J_n(x) + C_2 J_{-n}(x) \quad (C_1 \text{ and } C_2 \text{ are arbitrary constants})$$

provided n is a non integer.



THANK YOU

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