# ENGINEERING MATHEMATICS I

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## Class content

Recurrence relations

# ENGINEERING MATHEMATICS I

## Some important recurrence relations

1. $$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$$

2. $$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$$

3. $$\frac{d}{dx} J_n(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$$

4. $$\frac{d}{dx} J_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

5. $$\frac{d}{dx} J_n(x) = \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right)$$

6. $$2n J_n(x) = x\left(J_{n-1}(x) + J_{n+1}(x)\right)$$

# Recurrence relation 1

$$\boxed{\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)}$$

Proof: Consider $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

Multiply by $x^n$ on both sides, we get,

$$x^n J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^n}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2n+2r}}{r!\Gamma(n+r+1)} \left(\frac{1}{2}\right)^{n+2r}$$

## Recurrence relation 1

Differentiating with respect to x, we get,

$$\frac{d}{dx}\left(x^n J_n(x)\right) = \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r)x^{2n+2r-1}}{r!\Gamma(n+r+1)} \left(\frac{1}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)x^{2n+2r-1}}{r!\Gamma(n+r+1)2^{n+2r-1}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)x^n}{r!(n+r)\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1}$$

## Recurrence relation 1

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)}{r!(n+r)\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= x^n J_{n-1}(x)$$

Thus, $\boxed{\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)}$

# ENGINEERING MATHEMATICS I

## Recurrence relation 2

$$\boxed{\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)}$$

Proof: Consider $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

Multiply by $x^{-n}$ on both sides, we get,

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{-n}}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

## Recurrence relation 2

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \, x^{2r}}{r!\Gamma(n+r+1)} \left(\frac{1}{2}\right)^{n+2r}$$

Differentiating with respect to x, we get,

$$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = \sum_{r=0}^{\infty} \frac{(-1)^r \, (2r) x^{2r-1}}{r!\Gamma(n+r+1)} \left(\frac{1}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \, r \, x^{2r-1}}{r!\Gamma(n+r+1) 2^{n+2r-1}}$$

## Recurrence relation 2

$$= \sum_{r=1}^{\infty} \frac{(-1)^r \, r \, x^{2r-1}}{r!\,\Gamma(n+r+1)2^{n+2r-1}}$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^r \; x^{2r-1}}{(r-1)!\,\Gamma(n+r+1)2^{n+2r-1}}$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^{(r-1+1)} \; x^{2(r-1+1)-1}}{(r-1)!\,\Gamma(n+r-1+1+1)2^{n+2(r-1+1)-1}}$$

## Recurrence relation 2

Put $s = r - 1$, then

$$\frac{d}{dx}\left(x^{-n}J_n(x)\right) = \sum_{s=0}^{\infty} \frac{(-1)^{s+1}\ x^{2(s+1)-1}}{s!\Gamma(n+s+2)2^{n+2(s+1)-1}}$$

$$= -x^{-n}\sum_{s=0}^{\infty} \frac{(-1)^{s}\ x^{2s+(n+1)}}{s!\Gamma((n+1)+s+1)2^{(n+1)+2s}}$$

$$= -x^{-n}J_{n+1}(x)$$

Thus $\boxed{\frac{d}{dx}\left(x^{-n}J_n(x)\right) = -x^{-n}J_{n+1}(x)}$

# Recurrence relation 3

$$\frac{d}{dx} J_n(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$$

Proof: By recurrence relation 1

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$$

$$\Rightarrow x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$$

$$\Rightarrow x^n \left\{J_n'(x) + n x^{-1} J_n(x)\right\} = x^n J_{n-1}(x)$$

$$\Rightarrow J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x)$$

Thus $$\boxed{\frac{d}{dx} J_n(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)}$$

# Recurrence relation 4

$$\boxed{\frac{d}{dx}J_n(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)}$$

Proof: By recurrence relation 2

$$\frac{d}{dx}\left(x^{-n}J_n(x)\right) = -x^{-n}J_{n+1}(x)$$

$$\Rightarrow x^{-n}J_n{}'(x) - nx^{-n-1}J_n(x) = -x^{-n}J_{n+1}(x)$$

$$\Rightarrow x^{-n}\left\{J_n{}'(x) - nx^{-1}J_n(x)\right\} = -x^{-n}J_{n+1}(x)$$

$$\Rightarrow J_n{}'(x) - \frac{n}{x}J_n(x) = -J_{n+1}(x)$$

Thus $$\boxed{\frac{d}{dx}J_n(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)}$$

## Recurrence relation 5

$$\frac{d}{dx} J_n(x) = \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right)$$

Proof: By recurrence relation 3

$$\frac{d}{dx} J_n(x) = J_{n-1}(x) - \frac{n}{x} J_n(x) \quad ----\rightarrow (1)$$

By recurrence relation 4

$$\frac{d}{dx} J_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x) \quad ----\rightarrow (2)$$

$$(1)+(2) \Rightarrow 2\frac{d}{dx} J_n(x) = J_{n-1}(x) - J_{n+1}(x)$$

i,.e $$\boxed{\frac{d}{dx} J_n(x) = \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right)}$$

## Recurrence relation 5

$$\boxed{2n\,J_n(x) = x\left(J_{n-1}(x) + J_{n+1}(x)\right)}$$

Proof: By recurrence relation 3

$$\frac{d}{dx}J_n(x) = J_{n-1}(x) - \frac{n}{x}J_n(x) \qquad ----\rightarrow (1)$$

By recurrence relation 4

$$\frac{d}{dx}J_n(x) = \frac{n}{x}J_n(x) - J_{n+1}(x) \qquad ----\rightarrow (2)$$

$$(1)-(2) \Rightarrow \; 0 = J_{n-1}(x) + J_{n+1}(x) - \frac{2n}{x}J_n(x)$$

i,.e $\boxed{2n\,J_n(x) = x\left(J_{n-1}(x) + J_{n+1}(x)\right)}$

# THANK YOU

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