ENGINEERING MATHEMATICS I

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Class content

- Definition of Beta function
- Properties of Beta function

Beta Function

- Beta function is a definite integral whose integrand depends on two variables.

- It is also known as Euler’s integral of First kind.

Definition

Beta function is defined as

$$\boxed{\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx \ \ where \quad m,n>0} \quad ------(1)$$

Alternate form of beta function

Put $x = \sin^2\theta,\ dx = 2\sin\theta\cos\theta d\theta$ in (1)

Then $\beta(m,n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} 2\sin\theta\cos\theta d\theta$

$$\therefore\ \beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$$

This is another form of $\beta(m,n)$.

Properties of beta functions

1. Symmetry: $\beta(m,n)=\beta(n,m)$

Proof: We have, by definition, $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$

Put $x=1-t$

Then, $\beta(m,n)=\int_1^0 (1-t)^{m-1}t^{n-1}\left(-dt\right)$

$$=\int_0^1 t^{n-1}(1-t)^{m-1}dt \quad =\beta(n,m)$$

Therefore, $\boxed{\beta(m,n)=\beta(n,m)}$

Properties of beta functions

Note: We have, $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$

But $\beta(m,n) = \beta(n,m)$

Therefore, $$\boxed{\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta d\theta}$$

Properties of beta functions

2. $\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$ where $p > -1$ and $q > -1$

Proof: We have, $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$

Put $2m-1 = p,\ 2n-1 = q$

$$\Rightarrow m = \frac{p+1}{2},\ n = \frac{q+1}{2}$$

Properties of beta functions

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Then, $\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)=2\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta d\theta$

Therefore,

$$\boxed{\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta d\theta=\frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)}$$

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Properties of beta functions

Note: We have, $\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

put, $q = 0$

Then, $\int_0^{\frac{\pi}{2}} \sin^p \theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{0+1}{2}\right)$

$$= \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{1}{2}\right)$$

Properties of beta functions

$$=\frac{1}{2}\beta\left(\frac{1}{2},\frac{p+1}{2}\right)$$

$$=\int_0^{\frac{\pi}{2}}\cos^p\theta d\theta \quad \left(\because \beta(m,n)=\beta(n,m)\right)$$

Thus,

$$\boxed{\int_0^{\frac{\pi}{2}}\sin^p\theta d\theta=\int_0^{\frac{\pi}{2}}\cos^p\theta d\theta=\frac{1}{2}\beta\left(\frac{p+1}{2},\frac{1}{2}\right)}$$

Properties of beta functions

3. As an improper integral $\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

Proof: We have, $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$

Put $x=\frac{1}{1+y}$ then $dx=-\frac{1}{(1+y)^2}dy$

Then, $\beta(m,n)=\int_\infty^0 \frac{1}{(1+y)^{m-1}}\left(1-\frac{1}{1+y}\right)^{n-1}\left(-\frac{1}{(1+y)^2}\right)dy$

Properties of beta functions

$$=\int_0^\infty \frac{1}{(1+y)^{m-1}}\left(\frac{y}{1+y}\right)^{n-1}\frac{1}{(1+y)^2}dy$$

$$=\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy$$

$$=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx \qquad (\text{dummy variable})$$

Therefore, $\boxed{\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx}$

Properties of beta function

4. $\beta(m,n)=\beta(m,n+1)+\beta(m+1,n)$

Proof: $$\text{RHS}=\int_0^1 x^{m-1}(1-x)^{n+1-1}\,dx+\int_0^1 x^{m+1-1}(1-x)^{n-1}\,dx$$

$$=\int_0^1 [x^{m-1}(1-x)^{n}\,dx+\int_0^1 x^{m}(1-x)^{n-1}]dx$$

$$=\int_0^1 (1-x)^{n-1}x^{m-1}[1-x+x]dx$$

$$=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx=\beta(m,n)=LHS$$

Thus $$\boxed{\beta(m,n)=\beta(m,n+1)+\beta(m+1,n)}$$

THANK YOU

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