

ENGINEERING MATHEMATICS - I

Unit - 3: Partial Differential Equations

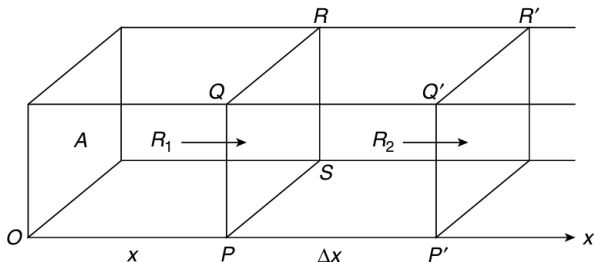
Department of Science and Humanities





- 1 Applications of PDE – Modelling of the one-dimensional heat equation

Modelling of the one-dimensional heat equation



Consider a long, thin bar (or rod) of constant cross-sectional area A and homogeneous conducting material. Let ρ be the density of the material, c be the specific heat, and k be the thermal conductivity of the material. We assume that the surface of the bar is insulated so that the heat flow is along parallel lines which are perpendicular to the area A .

Choose one end of the bar as origin and the direction of heat flow as the positive x -axis. Let $u(x, t)$ be the temperature at a distance x from 0. If Δu be the temperature change in the slab of thickness Δx of the bar, and time change Δt , then the quantity of heat in this slab is



$$\begin{aligned} & (\text{specific heat}) \times (\text{mass of the element slab}) \times (\text{change in temperature}) \\ &= c (A \rho \Delta x) \Delta u. \end{aligned}$$

Hence, the rate of change (i.e., increase) of heat in the slab at time t is

$$c(A\rho \Delta x) \lim_{\Delta t \rightarrow 0} \frac{\Delta u}{\Delta t} = c(A\rho \Delta x) \frac{\partial u}{\partial t}.$$

Let R_1 be the rate of inflow of heat at x in the slab and R_2 be the rate of outflow of heat at $x + \Delta x$. Then



$$c(A\rho \Delta x) \frac{\partial u}{\partial t} = R_1 - R_2, \quad (1)$$

where by Fourier's law

$$R_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x \quad \text{and} \quad R_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}.$$

The negative sign reflects the fact that heat flows from regions of higher temperature to regions of lower temperature.

Since $\frac{\partial u}{\partial t}$ is negative and R_1, R_2 are positive, the rate of increase of heat at time t is

$$R_1 - R_2 = kA \left[\left(\frac{\partial u}{\partial x} \right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x} \right)_x \right]. \quad (2)$$

Combining (1) and (2), we get

$$c(A\rho \Delta x) \frac{\partial u}{\partial t} = kA \left[\left(\frac{\partial u}{\partial x} \right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x} \right)_x \right].$$

Hence

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho} \frac{\left(\frac{\partial u}{\partial x} \right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x} \right)_x}{\Delta x}.$$

As $\Delta x \rightarrow 0$, the difference quotient becomes a second derivative,

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho} \frac{\partial^2 u}{\partial x^2},$$

where $\frac{k}{c\rho}$ is the thermal diffusivity of the material.



Setting

$$\alpha^2 = \frac{k}{c\rho},$$

we obtain the one-dimensional heat equation

$$\boxed{\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} .}$$