PES University, Bangalore

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Department of Science and Humanities

Engineering Mathematics - I (UE25MA141A)

Question and Answers

Unit - 4: Special Functions

1. Evaluate $\left(\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta\right) \times \left(\int_0^{\pi/2} \frac{1}{\sqrt{\tan\theta}}\, d\theta\right) = \frac{1}{4}\left[\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)\right]^2.$

   Answer: We evaluate:

$$I = \left(\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta\right) \times \left(\int_0^{\pi/2} \frac{1}{\sqrt{\tan\theta}}\, d\theta\right)$$

We write the integrals in terms of sine and cosine:

$$I_1 = \int_0^{\pi/2} \frac{\sin^{1/2}\theta}{\cos^{1/2}\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta = \beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$I_2 = \int_0^{\pi/2} \frac{\cos^{1/2}\theta}{\sin^{1/2}\theta}\, d\theta = \int_0^{\pi/2} \cos^{1/2}\theta \sin^{-1/2}\theta\, d\theta = \beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

By symmetry of the Beta function:

$$I = I_1 \cdot I_2 = \left[\beta\left(\frac{1}{4}, \frac{3}{4}\right)\right]^2$$

Now, recall the relation between Beta and Gamma functions:

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \Rightarrow B\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)} = \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

Hence:

$$I = \left[\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)\right]^2$$

Using the identity:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \Rightarrow \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \frac{\pi}{\frac{\sqrt{2}}{2}} = \pi\sqrt{2}$$

Thus:

$$I = \left(\pi\sqrt{2}\right)^2 = 2\pi^2$$

$$\boxed{\left(\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta\right) \cdot \left(\int_0^{\pi/2} \frac{1}{\sqrt{\tan\theta}}\, d\theta\right) = \left[\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)\right]^2 = 2\pi^2}$$

2. Show that $\left(\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta\right) \times \left(\int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta\right) = \pi.$

Answer:

$$I_1 = \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta, \qquad I_2 = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin\theta}}\, d\theta.$$

Recall the standard Beta function

$$\int_0^{\frac{\pi}{2}} \sin^{p-1}\theta \cos^{q-1}\theta\, d\theta = \tfrac{1}{2}\,\beta\big(\tfrac{p}{2}, \tfrac{q}{2}\big),$$

$$I_1 = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\, d\theta = \tfrac{1}{2}\,\beta\Big(\tfrac{\frac{3}{2}}{2}, \tfrac{1}{2}\Big) = \tfrac{1}{2}\,\beta\big(\tfrac{3}{4}, \tfrac{1}{2}\big),$$

$$I_2 = \int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\theta\, d\theta = \tfrac{1}{2}\,\beta\Big(\tfrac{\frac{1}{2}}{2}, \tfrac{1}{2}\Big) = \tfrac{1}{2}\,\beta\big(\tfrac{1}{4}, \tfrac{1}{2}\big).$$

Hence

$$I_1\, I_2 = \frac{1}{4}\,\beta\big(\tfrac{3}{4}, \tfrac{1}{2}\big)\,\beta\big(\tfrac{1}{4}, \tfrac{1}{2}\big).$$

Using the Beta–Gamma relation $\beta(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$, and noting $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$ and $\Gamma(\tfrac{5}{4}) = \tfrac{1}{4}\,\Gamma(\tfrac{1}{4})$, we get

$$B\big(\tfrac{3}{4}, \tfrac{1}{2}\big) = \frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{4})}, \quad B\big(\tfrac{1}{4}, \tfrac{1}{2}\big) = \frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}.$$

Therefore

$$\beta\big(\tfrac{3}{4}, \tfrac{1}{2}\big)\,\beta\big(\tfrac{1}{4}, \tfrac{1}{2}\big) = \frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{4})} \times \frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})} = \frac{\Gamma(\frac{1}{4})\,\big[\Gamma(\frac{1}{2})\big]^2}{\Gamma(\frac{5}{4})} = \frac{\Gamma(\frac{1}{4})\,\pi}{\frac{1}{4}\,\Gamma(\frac{1}{4})} = 4\pi.$$

Hence

$$I_1\, I_2 = \frac{1}{4} \times 4\pi = \pi.$$

This completes the proof that

$$\left(\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta\right) \times \left(\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin\theta}}\, d\theta\right) = \pi.$$

3. Show that $\int_0^\infty x^n e^{-a^2x^2}\,dx = \frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right)$. **Deduce that** $\int_0^\infty e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a}$.

Answer:

$$I_n \;=\; \int_0^\infty x^n e^{-a^2x^2}\,dx.$$

Set $t = a^2x^2$, so $x = \frac{\sqrt{t}}{a}$ and

$$dx = \frac{1}{2a}\,t^{-\frac{1}{2}}\,dt.$$

Then

$$x^n = \frac{t^{\frac{n}{2}}}{a^n}, \quad e^{-a^2x^2} = e^{-t},$$

and

$$I_n = \int_0^\infty \frac{t^{\frac{n}{2}}}{a^n}\,e^{-t}\,\frac{1}{2a}\,t^{-\frac{1}{2}}\,dt = \frac{1}{2\,a^{n+1}}\int_0^\infty t^{\frac{n+1}{2}-1}\,e^{-t}\,dt = \frac{1}{2\,a^{n+1}}\,\Gamma\Big(\frac{n+1}{2}\Big).$$

In particular, for $n = 0$,

$$\int_0^\infty e^{-a^2x^2}\,dx = I_0 = \frac{1}{2\,a^1}\,\Gamma\Big(\frac{1}{2}\Big) = \frac{1}{2a}\,\sqrt{\pi},$$

since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. $\square$