ENGINEERING PHYSICS

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Class #15

- *Definition of a state vector*
- *Basis and inner products*
- *Orthogonality and Orthonormality of state vectors*
- *Projection operators & their properties*

State Vectors

- In quantum mechanics, the **state** of a system is described by a **state vector** (also called a *ket*) written as: $|\,\psi\rangle$
- In the **wave function picture**, the state is described by $\psi(x)$, the probability amplitude in position space.
- In the **vector picture**, $|\,\psi\rangle$ is more general — the wave function is just its representation in a **chosen basis**:

$$\psi(x) = \langle x \,|\, \psi\rangle$$

Here, $|\,x\rangle$ is the position eigenstate.

- The state vector contains all measurable information about the system.

Basis vectors and Inner product

- In quantum mechanics, the state of a system is described by a state vector (ket) in a complex vector space called Hilbert space.
- To describe this state mathematically, we need a set of **basis vectors** — just like in regular geometry we need $\hat{\imath}, \hat{\jmath}, \hat{k}$ to describe any 3D vector.
- The inner product of two ket vectors $|\,u\rangle$ and $|\,v\rangle$is:

$$\langle\, u \mid v\,\rangle = (\langle u|)|v\rangle = (|u\rangle^{*})(|v\rangle) = \sum_{i} u_i^{*}\, v_i$$

It's a complex number that encodes the "overlap" between two states.

Orthogonal Basis and Norm

- Two vectors $|\, \boldsymbol{u} \rangle$ and $|\, \boldsymbol{v} \rangle$ are **orthogonal** if their inner product is zero: $\langle \boldsymbol{u} \mid \boldsymbol{v} \rangle = \boldsymbol{0}$

- This means the two states are completely distinct and share no probability amplitude overlap

- The norm (length) of a state vector $|\, \boldsymbol{\psi} \rangle$ is:

$$\parallel \boldsymbol{\psi} \parallel = \langle \boldsymbol{\psi} \mid \boldsymbol{\psi} \rangle$$

- If $\parallel \boldsymbol{\psi} \parallel = \boldsymbol{1}$, the vector is normalized.

Orthonormal Basis

A set of vectors $\{|\ e_1\rangle, |\ e_2\rangle, \ldots\ \}$ is an **orthonormal basis** if:

$$\langle e_i|e_j\rangle = \delta_{ij}$$

and the set spans the entire space, so any state can be written as:

$$|\psi\rangle = \sum_i c_i\ |e_i\rangle$$

With $\mathbf{c_i} = \langle e_i\ |\ \psi\rangle$.

Example

- If $|\, \boldsymbol{u}\rangle$ and $|\, \boldsymbol{v}\rangle$ are orthogonal unit vectors, then:

  $\langle u \mid v\rangle = 0, \langle u \mid u\rangle = 1, \langle v \mid v\rangle = 1$

  $$\langle\, \boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{u} - \boldsymbol{v}\,\rangle = ?$$

  Using linearity:

  $\langle u+v|u-v\rangle=\langle u|u\rangle-\langle u|v\rangle+\langle v|u\rangle-\langle v|v\rangle \quad = 1 - 0 + 0 - 1 = 0$

Projection Operators

- A **projection operator** projects a state vector onto a specific direction (state) in Hilbert space.

- If $|\,v\rangle$ is a normalized vector $(\langle v\,|\,v\rangle = 1)$, the projection operator onto $|v\rangle$ is: $P = |\,v\rangle\langle v\,|$

- When P acts on any state $|\,\psi\rangle$: $P\,|\,\psi\rangle = |\,v\rangle\langle v\,|\,\psi\rangle$

- This gives the component of $|\,\psi\rangle$ **along** $|\,v\rangle$.

Action of a Projection Operator on its Own State

- If $|\,v\rangle$ is normalized and $P = |\,v\rangle\langle v\,|$

$$P\,|\,v\rangle = |\,v\rangle\langle v\,|\,v\rangle = |\,v\rangle(1) = |\,v\rangle$$

- So projection operators leave their target state unchanged.

Test your understanding:

Q1. Which of the following is the correct representation of an orthonormal basis?

A. $\langle e_i | e_j \rangle = 0 \; for \; all \; i, j$

B. $\langle e_i | e_j \rangle = \delta_{ij}$

C. $\langle e_i | e_j \rangle = 1 \; for \; all \; i, j$

D. $\langle e_i | e_j \rangle = e_i + e_j$

Answer: B. $\langle e_i | e_j \rangle = \delta_{ij}$

Test your understanding:

Q2. A normalized state vector is written as:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

If a projection operator $P = |0\rangle\langle 0|$ acts on $|\psi\rangle$,what is the resulting state?

Answer: $\frac{1}{\sqrt{2}}|0\rangle$

THANK YOU

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