PES University, Bangalore

(Established Under Karnataka Act 16 of 2013)

Department of Science and Humanities

Engineering Mathematics - I (UE25MA141A)

Unit - 1: Partial Differential Equations: Q & A

1. Find all the first-order partial derivatives of the following function:

$$f(u,v) = u^2 \sin(u+v^3) - \sec(4u)\tan^{-1}(2v)$$

Answer:

The first-order partial derivatives of f are:

(a) **Partial derivative with respect to u:**

$$\begin{aligned}\frac{\partial f}{\partial u} &= \frac{\partial}{\partial u}\left[u^2 \sin(u+v^3)\right] - \frac{\partial}{\partial u}\left[\sec(4u)\tan^{-1}(2v)\right] \\ &= 2u\sin(u+v^3) + u^2\cos(u+v^3) - 4\sec(4u)\tan(4u)\tan^{-1}(2v)\end{aligned}$$

(b) **Partial derivative with respect to v:**

$$\begin{aligned}\frac{\partial f}{\partial v} &= \frac{\partial}{\partial v}\left[u^2 \sin(u+v^3)\right] - \frac{\partial}{\partial v}\left[\sec(4u)\tan^{-1}(2v)\right] \\ &= 3u^2v^2\cos(u+v^3) - \frac{2\sec(4u)}{1+4v^2}\end{aligned}$$

Derivation:

(a) For $\frac{\partial f}{\partial u}$:

- Differentiate $u^2 \sin(u+v^3)$ using the product rule:

$$\frac{\partial}{\partial u}\left(u^2\sin(u+v^3)\right) = 2u\sin(u+v^3) + u^2\cos(u+v^3)$$

- Differentiate $-\sec(4u)\tan^{-1}(2v)$ with respect to u:

$$\frac{\partial}{\partial u}\left(-\sec(4u)\tan^{-1}(2v)\right) = -4\sec(4u)\tan(4u)\tan^{-1}(2v)$$

(b) For $\frac{\partial f}{\partial v}$:

- Differentiate $u^2\sin(u+v^3)$ with respect to v:

$$\frac{\partial}{\partial v}\left(u^2\sin(u+v^3)\right) = 3u^2v^2\cos(u+v^3)$$

- Differentiate $-\sec(4u)\tan^{-1}(2v)$ with respect to v:

$$\frac{\partial}{\partial v}\left(-\sec(4u)\tan^{-1}(2v)\right) = -\frac{2\sec(4u)}{1+4v^2}$$

Total Derivative

2. If $u = \sin^{-1}(x-y)$, where $x = 3t$, $y = 4t^3$, then show that:

$$\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}, -1 < t < 1$$

Solution:

Given:

$$u = \sin^{-1}(x-y), \quad \text{where } x = 3t,\ y = 4t^3$$

Using the chain rule:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

Compute the partial derivatives:

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}}, \quad \frac{\partial u}{\partial y} = \frac{-1}{\sqrt{1-(x-y)^2}}$$

Compute the ordinary derivatives:

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 12t^2$$

Substitute into the chain rule:

$$\begin{aligned}\frac{du}{dt} &= \frac{1}{\sqrt{1-(x-y)^2}}(3) + \frac{-1}{\sqrt{1-(x-y)^2}}(12t^2)\\ &= \frac{3-12t^2}{\sqrt{1-(3t-4t^3)^2}}\end{aligned}$$

Simplify the denominator:

$$\begin{aligned}
1-(x-y)^2 &= 1-(3t-4t^3)^2 \\
&= 1-(9t^2-24t^4+16t^6) \\
&= 1-9t^2+24t^4-16t^6 \\
&= (1-t^2)(16t^4-8t^2+1)
\end{aligned}$$

Notice that:

$$\sqrt{(1-t^2)(16t^4-8t^2+1)} = \sqrt{1-t^2}\sqrt{(4t^2-1)^2} = \sqrt{1-t^2}(1-4t^2)$$

Thus:

$$\frac{du}{dt} = \frac{3(1-4t^2)}{\sqrt{1-t^2}(1-4t^2)} = \frac{3}{\sqrt{1-t^2}}, \quad \text{for } -1 < t < 1$$

3. If $z = f(x,y)$ where $x = u^2 - v^2$, $y = 2uv$, prove that:

$$4(u^2+v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$$

Solution:

Given $z = f(x,y)$ with the coordinate transformations:

$$x = u^2 - v^2, \quad y = 2uv$$

Step 1: First Partial Derivatives

Using the chain rule:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = 2u\frac{\partial z}{\partial x} + 2v\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -2v\frac{\partial z}{\partial x} + 2u\frac{\partial z}{\partial y}$$

Step 2: Second Partial Derivatives

For $\frac{\partial^2 z}{\partial u^2}$:

$$\begin{aligned}\frac{\partial^2 z}{\partial u^2} &= \frac{\partial}{\partial u}\left(2u\frac{\partial z}{\partial x} + 2v\frac{\partial z}{\partial y}\right)\\ &= 2\frac{\partial z}{\partial x} + 2u\left(2u\frac{\partial^2 z}{\partial x^2} + 2v\frac{\partial^2 z}{\partial y \partial x}\right)\\ &\quad + 2v\left(2u\frac{\partial^2 z}{\partial x \partial y} + 2v\frac{\partial^2 z}{\partial y^2}\right)\\ &= 2\frac{\partial z}{\partial x} + 4u^2\frac{\partial^2 z}{\partial x^2} + 8uv\frac{\partial^2 z}{\partial x \partial y} + 4v^2\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

For $\frac{\partial^2 z}{\partial v^2}$:

$$\begin{aligned}\frac{\partial^2 z}{\partial v^2} &= \frac{\partial}{\partial v}\left(-2v\frac{\partial z}{\partial x} + 2u\frac{\partial z}{\partial y}\right)\\ &= -2\frac{\partial z}{\partial x} - 2v\left(-2v\frac{\partial^2 z}{\partial x^2} + 2u\frac{\partial^2 z}{\partial y \partial x}\right)\\ &\quad + 2u\left(-2v\frac{\partial^2 z}{\partial x \partial y} + 2u\frac{\partial^2 z}{\partial y^2}\right)\\ &= -2\frac{\partial z}{\partial x} + 4v^2\frac{\partial^2 z}{\partial x^2} - 8uv\frac{\partial^2 z}{\partial x \partial y} + 4u^2\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

Step 3: Sum of Second Derivatives

Adding both second derivatives:

$$\begin{aligned}\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} &= 4(u^2+v^2)\frac{\partial^2 z}{\partial x^2} + 4(u^2+v^2)\frac{\partial^2 z}{\partial y^2}\\ &= 4(u^2+v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)\end{aligned}$$

Thus, we have proved:

$$4(u^2+v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$$

4. If $z = xf\left(\frac{y}{x}\right) + g\left(\frac{x}{y}\right)$, then show that:

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$$

Solution:

Given the function:

$$z = xf\left(\frac{y}{x}\right) + g\left(\frac{x}{y}\right)$$

Step 1: Verify Homogeneity

Let:

$$u = xf\left(\frac{y}{x}\right), \quad v = g\left(\frac{x}{y}\right)$$

1. For u:

$$u(tx, ty) = txf\left(\frac{ty}{tx}\right) = txf\left(\frac{y}{x}\right) = tu(x, y)$$

Thus, u is homogeneous of degree 1.

2. For v:

$$v(tx, ty) = g\left(\frac{tx}{ty}\right) = g\left(\frac{x}{y}\right) = t^0 v(x, y)$$

Thus, v is homogeneous of degree 0.

Step 2: Apply Euler's Theorem

For any homogeneous function ϕ of degree n:

$$x^2\frac{\partial^2 \phi}{\partial x^2} + 2xy\frac{\partial^2 \phi}{\partial x \partial y} + y^2\frac{\partial^2 \phi}{\partial y^2} = n(n-1)\phi$$

Applying to u (degree 1):

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 1(1-1)u = 0$$

Applying to v (degree 0):

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 0(0-1)v = 0$$

Step 3: Combine Results

Since $z = u + v$, we add the two equations:

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0 + 0 = 0$$

Final Result:

$$\boxed{x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0}$$

5. Expand $\frac{1}{1+x-y}$ using Taylor's series up to second-degree terms.

Taylor Series Expansion of $\frac{1}{1+x-y}$

We want to expand the function:

$$f(x,y) = \frac{1}{1+x-y}$$

using a Taylor series about the point $(0,0)$ up to second-degree terms.

Step 1: Compute $f(0,0)$

$$f(0,0) = \frac{1}{1+0-0} = 1$$

Step 2: Compute First-Order Partial Derivatives

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\frac{1}{1+x-y}\right) = -\frac{1}{(1+x-y)^2}$$
$$f_y(x,y) = \frac{\partial}{\partial y}\left(\frac{1}{1+x-y}\right) = \frac{1}{(1+x-y)^2}$$

At $(0,0)$:

$$f_x(0,0) = -1, \quad f_y(0,0) = 1$$

Step 3: Compute Second-Order Partial Derivatives

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(-\frac{1}{(1+x-y)^2}\right) = \frac{2}{(1+x-y)^3}$$
$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(-\frac{1}{(1+x-y)^2}\right) = -\frac{2}{(1+x-y)^3}$$
$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{1}{(1+x-y)^2}\right) = \frac{2}{(1+x-y)^3}$$

At $(0,0)$:

$$f_{xx}(0,0) = 2, \quad f_{xy}(0,0) = -2, \quad f_{yy}(0,0) = 2$$

Step 4: Construct the Taylor Series

The second-order Taylor expansion of $f(x, y)$ about $(0, 0)$ is:

$$f(x, y) \approx f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2!}\left[x^2 f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2 f_{yy}(0, 0)\right]$$

Substituting the computed values:

$$f(x, y) \approx 1 + [x(-1) + y(1)] + \frac{1}{2}\left[x^2(2) + 2xy(-2) + y^2(2)\right]$$

Simplify:

$$f(x, y) \approx 1 - x + y + \frac{1}{2}\left(2x^2 - 4xy + 2y^2\right)$$

$$f(x, y) \approx 1 - x + y + x^2 - 2xy + y^2$$

Final Answer

The second-order Taylor expansion of $\frac{1}{1+x-y}$ about $(0, 0)$ is:

$$\boxed{1 - x + y + x^2 - 2xy + y^2}$$

6. Discuss the maxima and minima of the function:

$$f(x, y) = x^3 y^2 (1 - x - y)$$

where $x > 0$, $y > 0$, and $x + y < 1$.

Solution

Given the function:

$$f(x, y) = x^3 y^2 (1 - x - y) = x^3 y^2 - x^4 y^2 - x^3 y^3$$

Partial Derivatives

First, we compute the partial derivatives:

$$f_x = 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3$$

$$f_y = 2x^3 y - 2x^4 y - 3x^3 y^2$$

Second Partial Derivatives

For classification of critical points:

$$
\begin{aligned}
f_{xx} &= 6xy^2 - 12x^2y^2 - 6xy^3 \\
f_{xy} &= 6x^2y - 8x^3y - 9x^2y^2 \\
f_{yy} &= 2x^3 - 2x^4 - 6x^3y
\end{aligned}
$$

Finding Critical Points

To find stationary points, solve $f_x = 0$ and $f_y = 0$:

$$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \Rightarrow x^2y^2(3 - 4x - 3y) = 0 \quad (1)$$
$$2x^3y - 2x^4y - 3x^3y^2 = 0 \Rightarrow x^3y(2 - 2x - 3y) = 0 \quad (2)$$

Solutions:

- $(0, 0)$ satisfies both equations
- Solving $3 - 4x - 3y = 0$ and $2 - 2x - 3y = 0$ gives $x = \frac{1}{2}$, $y = \frac{1}{3}$
- Other boundary cases:
  - When $x = 0$: $(0, \frac{2}{3})$
  - When $y = 0$: $(\frac{3}{4}, 0)$, $(1, 0)$

Classifying Critical Points

At $(0, 0)$ and boundary points:

For $(0, 0)$, $(0, \frac{2}{3})$, $(\frac{3}{4}, 0)$, and $(1, 0)$:

$$r = 0, \quad s = 0, \quad t = 0$$

Thus $rt - s^2 = 0$ (test inconclusive).

At $\left(\frac{1}{2}, \frac{1}{3}\right)$:

$$\begin{aligned}
r &= 6 \cdot \frac{1}{2} \cdot \left(\frac{1}{3}\right)^2 - 12 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{3}\right)^2 - 6 \cdot \frac{1}{2} \cdot \left(\frac{1}{3}\right)^3 = -\frac{1}{9} < 0 \\
s &= 6 \cdot \left(\frac{1}{2}\right)^2 \cdot \frac{1}{3} - 8 \cdot \left(\frac{1}{2}\right)^3 \cdot \frac{1}{3} - 9 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{3}\right)^2 = -\frac{1}{12} \\
t &= 2 \cdot \left(\frac{1}{2}\right)^3 - 2 \cdot \left(\frac{1}{2}\right)^4 - 6 \cdot \left(\frac{1}{2}\right)^3 \cdot \frac{1}{3} = -\frac{1}{8} \\
rt - s^2 &= \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2 = \frac{1}{144} > 0
\end{aligned}$$

Since $rt - s^2 > 0$ and $r < 0$, this is a local maximum.

Maximum value:

$$f\left(\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$$

7. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Solution

Given the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Step 1: Problem Setup

By the symmetry of the ellipsoid, the largest parallelepiped will have edges parallel to the coordinate axes and its center at the origin $(0, 0, 0)$.

Let $P(x, y, z)$ be the vertex of the parallelepiped lying on the ellipsoid. The dimensions of the parallelepiped will then be $2x$, $2y$, and $2z$ respectively.

Step 2: Volume Expression

The volume V of the parallelepiped is:

$$V = 2x \cdot 2y \cdot 2z = 8xyz$$

Step 3: Constraint Equation

The constraint is given by the ellipsoid equation:

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

Step 4: Lagrange Multipliers

We form the auxiliary function using Lagrange multiplier λ:

$$F(x, y, z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

Step 5: Partial Derivatives

Taking partial derivatives and setting them to zero:

$$\begin{aligned}
F_x &= 8yz + \frac{2\lambda x}{a^2} = 0 \quad \Rightarrow \quad 4yz = -\frac{\lambda x}{a^2} \\
F_y &= 8xz + \frac{2\lambda y}{b^2} = 0 \quad \Rightarrow \quad 4xz = -\frac{\lambda y}{b^2} \\
F_z &= 8xy + \frac{2\lambda z}{c^2} = 0 \quad \Rightarrow \quad 4xy = -\frac{\lambda z}{c^2}
\end{aligned}$$

Step 6: Solving the System

From the above equations, we get:

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Substituting into the constraint:

$$3\frac{x^2}{a^2} = 1 \quad \Rightarrow \quad x = \pm\frac{a}{\sqrt{3}}$$

Similarly:

$$y = \pm\frac{b}{\sqrt{3}}, \quad z = \pm\frac{c}{\sqrt{3}}$$

Step 7: Maximum Volume

There are 8 possible vertices. For maximum volume, we take the positive coordinates:

$$V_{\max} = 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right) = \frac{8abc}{3\sqrt{3}}$$