UE25MA141A: ENGINEERING MATHEMATICS - I

Unit - 1: Partial Differentiation

Department of Science and Humanities

PES UNIVERSITY

Composite functions-Problem 1

If $u = \sin(x^2 + y^2)$, where $x = r\cos t$, $y = r\sin t$, then find $\dfrac{\partial u}{\partial t}$.

Solution:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial u}{\partial x} = 2x\cos(x^2 + y^2), \quad \frac{\partial u}{\partial y} = 2y\cos(x^2 + y^2)$$

$$\frac{\partial x}{\partial t} = -rsint, \quad \frac{\partial y}{\partial t} = rcost$$

$$\frac{\partial u}{\partial t} = 2x\cos(x^2 + y^2)(-r\sin t) + 2y\cos(x^2 + y^2)(r\cos t)$$

Substitute $x = r\cos t, \; y = r\sin t$:

$$\frac{\partial u}{\partial t} = 2r\cos(r^2)\big[-r\cos t\sin t + r\sin t\cos t\big] = 0$$

$$\boxed{\frac{\partial u}{\partial t} = 0}$$

Composite functions-Problem 2

Find $\dfrac{\partial u}{\partial \theta}$ where $u = x^2y + yz^3,\ x = r\cos\theta,\ y = r\sin\theta,\ z = \theta.$

<u>Solution:</u>

$$u = x^2y + yz^3, \quad x = r\cos\theta,\ y = r\sin\theta,\ z = \theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta}$$

$$\frac{\partial u}{\partial x} = 2xy, \quad \frac{\partial u}{\partial y} = x^2 + z^3, \quad \frac{\partial u}{\partial z} = 3yz^2$$

$$\frac{\partial x}{\partial \theta} = -rsin\theta, \quad \frac{\partial y}{\partial \theta} = rcos\theta, \quad \frac{\partial z}{\partial \theta} = 1$$

$$\frac{\partial u}{\partial \theta} = (2xy)(-rsin\theta) + (x^2 + z^3)(rcos\theta) + (3yz^2)(1)$$

$$\boxed{\frac{\partial u}{\partial \theta} = r^3\cos\theta\,(3\cos^2\theta - 2)\ +\ r\left(3\theta^2\sin\theta + \theta^3\cos\theta\right)}$$

Composite functions-Problem 3

If $u = f(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$, then express $\dfrac{\partial u}{\partial x}$ in terms of $f'(r)$.

Solution:

$$\frac{\partial u}{\partial x} = \frac{du}{dr}\,\frac{\partial r}{\partial x},$$

$$\implies \frac{\partial u}{\partial x} = f'(r)\,\frac{\partial r}{\partial x},$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

$$\therefore \boxed{\frac{\partial u}{\partial x} = f'(r)\frac{x}{r}}$$

Composite functions-Problem 4

If $u = e^{x^2+y^2}$, $x = r\cos\theta$, $y = r\sin\theta$, then find $\dfrac{\partial u}{\partial \theta}$.

Solution:

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\ \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\ \frac{\partial y}{\partial \theta}$$

$$\frac{\partial u}{\partial x} = 2xe^{x^2+y^2}, \quad \frac{\partial u}{\partial y} = 2ye^{x^2+y^2}$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

Thus,

$$\frac{\partial u}{\partial \theta} = 2xe^{x^2+y^2}(-r\sin\theta) + 2ye^{x^2+y^2}(r\cos\theta)$$

Problem 4 Cont...

Substitute $x = r\cos\theta,\ y = r\sin\theta$:

$$\frac{\partial u}{\partial \theta} = 2re^{r^2}(-r\cos\theta\sin\theta + r\sin\theta\cos\theta) = 0$$

$$\therefore \boxed{\frac{du}{d\theta} = 0}$$

Composite functions-Problem 5

If $u = \tan^{-1}\left(\frac{y}{x}\right)$, $x = r\cos\theta$, $y = r\sin\theta$, then show that $\frac{\partial u}{\partial \theta} = 1$.

Solution:

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$u_x = \frac{-y}{x^2+y^2}, \qquad u_y = \frac{x}{x^2+y^2}$$

$$x_\theta = -r\sin\theta, \quad y_\theta = r\cos\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{-y}{x^2+y^2}(-r\sin\theta) + \frac{x}{x^2+y^2}(r\cos\theta)$$

Substitute $x = r\cos\theta,\ y = r\sin\theta$:

$$\frac{\partial u}{\partial \theta} = \frac{r^2(\sin^2\theta + \cos^2\theta)}{r^2} = 1$$

$$\boxed{\frac{\partial u}{\partial \theta} = 1}$$

If $u = \sin(xy) + y^3$, $x = e^s$, $y = s + t$, then find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$.

Solution:

$u_x = y\cos(xy),\ u_y = x\cos(xy) + 3y^2$

$\frac{\partial x}{\partial s} = e^s,\ \frac{\partial y}{\partial s} = 1,\ \frac{\partial x}{\partial t} = 0,\ \frac{\partial y}{\partial t} = 1,$

$\frac{\partial u}{\partial s} = y\cos(xy)e^s + (x\cos(xy) + 3y^2)(1),$

$\frac{\partial u}{\partial t} = (x\cos(xy) + 3y^2)(1)$

Substitute $x = e^s,\ y = s + t$:

$$\boxed{\frac{\partial u}{\partial s} = (s+t)\cos(e^s(s+t))e^s + e^s\cos(e^s(s+t)) + 3(s+t)^2}$$

$$\boxed{\frac{\partial u}{\partial t} = e^s\cos(e^s(s+t)) + 3(s+t)^2}$$

Composite functions-Problem 7

If $u = f(r,\theta)$, $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$, then find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$.

Solution:

$$\frac{\partial u}{\partial x} = f_r\frac{\partial r}{\partial x} + f_\theta\frac{\partial \theta}{\partial x}, \quad \frac{\partial u}{\partial y} = f_r\frac{\partial r}{\partial y} + f_\theta\frac{\partial \theta}{\partial y}$$

Derivatives:

$$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial \theta}{\partial x} = -\frac{y}{r^2},\ \frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$

Hence:

$$\boxed{\frac{\partial u}{\partial x} = f_r\frac{x}{r} - f_\theta\frac{y}{r^2}} \qquad \boxed{\frac{\partial u}{\partial y} = f_r\frac{y}{r} + f_\theta\frac{x}{r^2}}$$

THANK YOU