ENGINEERING MATHEMATICS I

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Class content

- Relation between Beta and Gamma functions
- Duplication formula

Relationship between beta and gamma functions

$$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Relationship between beta and gamma functions

Proof: By definition, $\Gamma(m) = 2\int_0^\infty e^{-x^2}\ x^{2m-1}dx$

and $\Gamma(n) = 2\int_0^\infty e^{-y^2}\ y^{2n-1}dy$

Therefore, $\Gamma(m)\Gamma(n) = 2\int_0^\infty e^{-x^2}\ x^{2m-1}dx \times 2\int_0^\infty e^{-y^2}\ y^{2n-1}dy$

$$= 4\int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)}\ x^{2m-1}y^{2n-1}dxdy$$

Relationship between beta and gamma functions

Put $x = r\cos\theta$ *and* $y = r\sin\theta$

then $dxdy = rdrd\theta$

$$\therefore \Gamma(m)\Gamma(n) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}\left(r\cos\theta\right)^{2m-1}\left(r\sin\theta\right)^{2n-1} rdrd\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta\sin^{2n-1}\theta d\theta \times 2\int_0^{\infty} e^{-r^2} r^{2m-1} r^{2n-1} rdr$$

Put $r^2 = t,\ rdr = \dfrac{dt}{2}$ in second integral, we get,

Relationship between beta and gamma functions

$$\Gamma(m)\Gamma(n) = \beta(m,n).2.\frac{1}{2}\int_0^\infty e^{-t} t^{m-\frac{1}{2}+n-\frac{1}{2}} dt$$

$$= \beta(m,n)\int_0^\infty e^{-t} t^{m+n-1} dt$$

$$= \beta(m,n)\Gamma(m+n)$$

Thus, $$\boxed{\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

Legendre duplication formula for gamma function

$$\sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)$$

Proof: We know that, $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$

Therefore, $\beta(m,m) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2m-1}\theta d\theta$

$$= 2\int_0^{\frac{\pi}{2}} \frac{\sin^{2m-1} 2\theta}{2^{2p-1}} d\theta$$

$$= \frac{2}{2^{2p-1}}\int_0^{\frac{\pi}{2}} \sin^{2m-1} 2\theta d\theta$$

Legendre duplication formula for gamma function

Let $2\theta = t \Rightarrow 2d\theta = dt$

Then $\beta(m,m) = \dfrac{2}{2^{2m-1}} \displaystyle\int_0^{\pi} \sin^{2m-1} t \, \frac{dt}{2}$

$$= \frac{1}{2^{2m-1}} \int_0^{\frac{\pi}{2}} \sin^{2m-1} t\,dt$$

$$= \frac{1}{2^{2m-1}} \beta\left(\frac{1}{2}, m\right)$$

Legendre duplication formula for gamma function

$$\text{i.e.}\quad \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \frac{\Gamma(\frac{1}{2})\Gamma(m)}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\Rightarrow \frac{\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \frac{\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\Rightarrow \sqrt{\pi}\,\Gamma(2m) = 2^{2p-1}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)$$

Note:

$$\Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi},0<n<1$$

Proof: We know that, $\Gamma(n)\Gamma(m)=\beta(n,m)\Gamma(m+n)$

Put $m=1-n$

$$\Rightarrow \Gamma(n)\Gamma(1-n)=\beta(n,1-n)\Gamma(1)$$

$$\Rightarrow \Gamma(n)\Gamma(1-n)=\beta(n,1-n)$$

Also we know that, $\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

Therefore

$$\boxed{\beta(n,1-n)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx=\frac{\pi}{\sin n\pi}}$$

THANK YOU

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