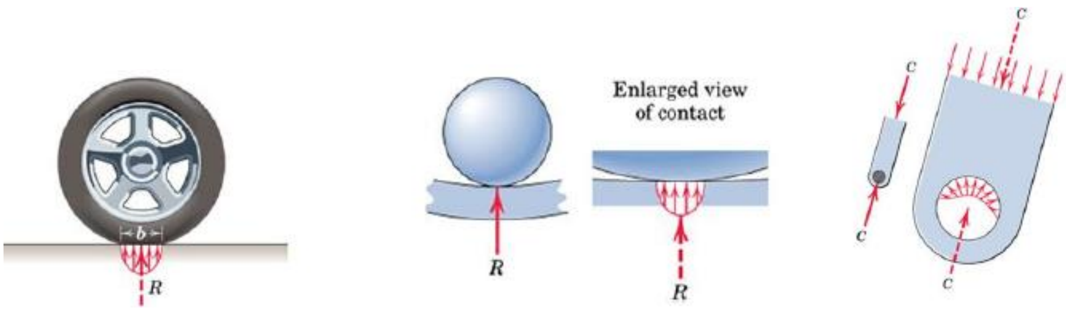
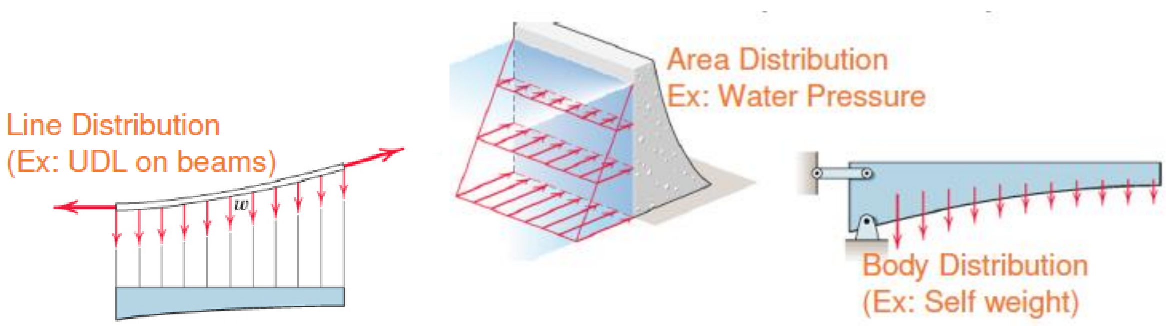
Concentrated Forces: If dimension of the contact area is negligible compared to other dimensions of the body then, the contact forces may be treated as Concentrated Forces.

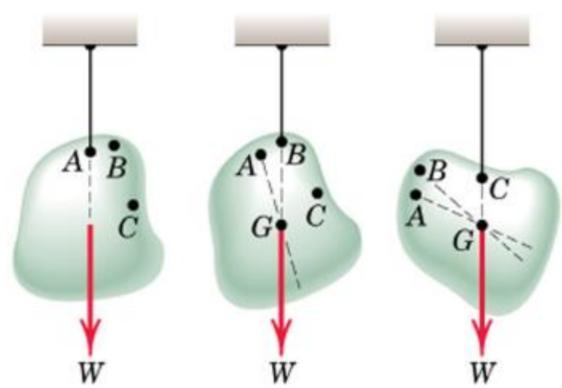


Distributed Forces: If forces are applied over a region whose dimension is not negligible compared with other pertinent dimensions then proper distribution of contact forces must be accounted for to know intensity of force at any location.



CENTER OF MASS

A body of mass m in equilibrium under the action of tension in the cord, and resultant W of the gravitational forces acting on all particles of the body and is collinear with the cord.



Suspend the body from different points on the body. Mark its position by drilling a hypothetical hole of negligible size along its line of action as shown in the fig.

These lines of action will be concurrent at a single point G. which is called the ***center of gravity*** of the body. And it is the point about which the ***entire weight of the body is said to be concentrated.***

Determining the Center of Gravity:

To determine mathematically the location of the center of gravity of any body, we apply ***the principle of moments*** to the parallel system of gravitational forces.
Moment of resultant gravitational force W about any axis equals sum of the moments about the same axis of the gravitational forces dW acting on all particles treated as infinitesimal elements.
Weight of the body $W= \int dW$
Moment of weight of an element (dW) @ y-axis = x dW
Sum of moments for all elements of body = $\int x \; dw$
From Principle of Moments: $\overline{x}\, W = \int x \, dW$.

$$\bar{x} = \frac{\int x \; dW}{W} \qquad \bar{y} = \frac{\int y \; dW}{W} \qquad \bar{z} = \frac{\int z \; dW}{W}$$

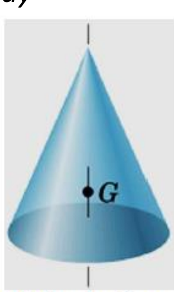
Numerator of these expressions represents the sum of the moments and Product of W and corresponding coordinate of G represents the moment of the sum that is Moment Principle.
With the substitution of *W= mg* and *dW = g dm*, the expressions for the coordinates of the center of gravity become

$$\bar{x} = \frac{\int x \; dm}{m} \qquad \bar{y} = \frac{\int y \; dm}{m} \qquad \bar{z} = \frac{\int z \; dm}{m} \quad \longrightarrow \quad (1)$$

The density ρ of a body is its mass per unit volume. Thus, the mass of a differential element of volume *dV* becomes $dm = \rho \; dV$.

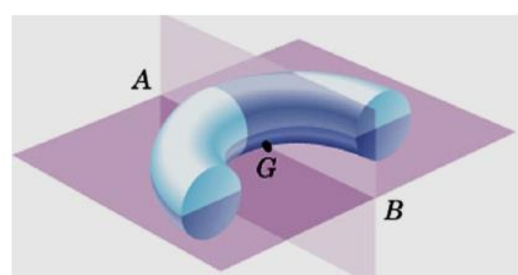
$$\bar{x} = \frac{\int x\rho \; dV}{\int \rho \; dV} \qquad \bar{y} = \frac{\int y\rho \; dV}{\int \rho \; dV} \qquad \bar{z} = \frac{\int z\rho \; dV}{\int \rho \; dV} \quad \longrightarrow \quad (2)$$

They define a unique point, which is a function of distribution of mass. This point ***is Center of Mass (CM).*** CM coincides with CG as long as gravity field is treated as uniform and parallel
CG or CM may lie outside the body. *CM always lie on a line or a plane of symmetry in a homogeneous body*



Right Circular Cone
CM on central axis

Half Right Circular Cone
CM on vertical plane of symmetry



Half Ring
CM on intersection of two planes of symmetry (line AB)

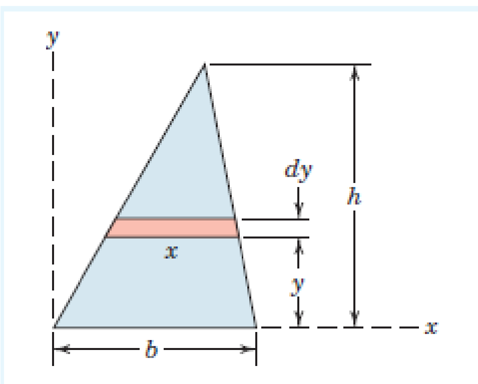
When a body of density ρ has a small but constant thickness *t*, The mass of an element becomes *dm*= ρ *t dA*. Again, if ρ and *t* are constant over the entire area, the coordinates of the center of mass of the body also become the coordinates of the centroid *C* of the surface area

$$\bar{x} = \frac{\int x\ dA}{A} \quad \bar{y} = \frac{\int y\, dA}{A} \quad \bar{z} = \frac{\int z\, dA}{A}$$

$$\bar{x} = \frac{\int x_c\ dA}{A} \quad \bar{y} = \frac{\int y_c\, dA}{A} \quad \bar{z} = \frac{\int z_c\ dA}{A}$$

The numerators in the equation are called the *first moments of area*.

CENTROID OF A TRIANGULAR AREA:



The *x*-axis is taken to coincide with the base. A differential strip of Area$dA = x\ dy$ is chosen. By similar triangles $\frac{x}{(h-y)} = \frac{b}{h}$

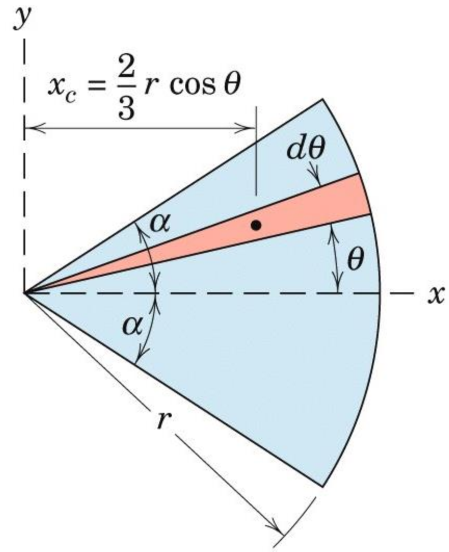
Applying the Equation $\bar{y} = \frac{\int y_c\, dA}{A}$ gives $[A\,\bar{y} = \int y_c dA]$

$$\frac{bh}{2}\,\bar{y} = \int_0^h y\frac{b(h-y)}{h}dy = \frac{bh^2}{6}$$

$$\bar{y} = \frac{h}{3}$$

With respect to base of the triangle.

CENTROID FOR AN AREA OF A CIRCULAR SECTOR:



The area may also be covered by swinging a triangle of differential area about the vertex and through the total angle of the sector. This triangle, shown in the illustration, has an area dA = (r/2)(r dϴ), where higher-order terms are neglected. From the centroid of the triangular element of area is two-thirds of its altitude from its vertex, so that the x-coordinate to the centroid of the element is

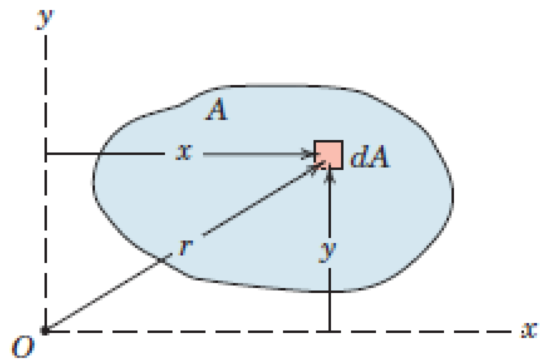
$$x_c = \frac{2r\cos\theta}{3}$$

$$\left[A\,\bar{x} = \int x_c\, dA\right] \qquad (r^2\alpha)\bar{x} = \int_{-\alpha}^{\alpha} \left(\frac{2r\cos\theta}{3}\right)\left(\frac{1}{2}r^2\, d\theta\right)$$

$$(r^2\alpha)\,\bar{x} = \frac{2}{3} r^3 \sin\alpha$$

$$\bar{x} = \frac{2}{3}\frac{r\sin\alpha}{\alpha}$$

MOMENT OF INERTIA



Consider the area *A* in the *x*-*y* plane, Figure. The moments of inertia of the element *dA* about the *x*- and *y*-axes are, by definition, $dI_x = y^2\,dA$ and, $dI_y = x^2\,dA$ respectively. The moments of inertia of *A* about the same axes are therefore

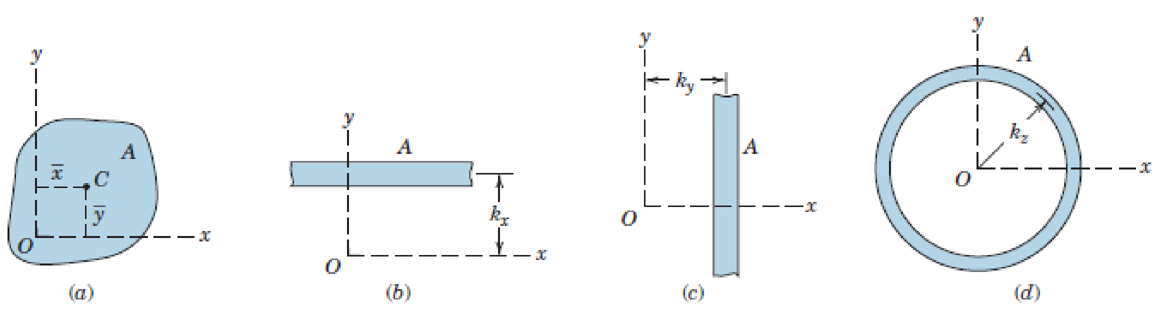
$$I_x = \int y^2\,dA$$

$$I_y = \int x^2\,dA$$

The moment of inertia of *dA* about the pole *O* (*z*-axis) is, by similar definition, $dI_z = r^2\,dA$. The moment of inertia of the entire area about *O* is

$$I_x = \int r^2\,dA$$

RADIUS OF GYRATION



Consider an area *A*, Figure(a), which has rectangular moments of inertia I_x and I_y about *O*. We now visualize this area as concentrated into a long narrow strip of area *A* a distance k_x from the *x*-axis, Figure(*b)*.

By definition the moment of inertia of the strip about the *x*-axis will be the same as that of the original area if ${k_x}^2 A = \overline{I_x}$. The distance k_x is called the *radius of gyration* of the area about the *x*-axis.

A similar relation for the *y*-axis is written by considering the area as concentrated into a narrow strip parallel to the *y*-axis as shown in Figure (*c)*.

$$I_x = {k_x}^2\,A$$

$$I_y = k_y{}^2 A$$

Also, if we visualize the area as concentrated into a narrow ring of radius k_z as shown in Figure (*d*), we may express the polar moment of inertia as $k_z^2 A = I_z$. In summary we write

$$I_x = k_x{}^2 A$$
$$I_y = k_y{}^2 A$$
$$I_z = k_z{}^2 A$$

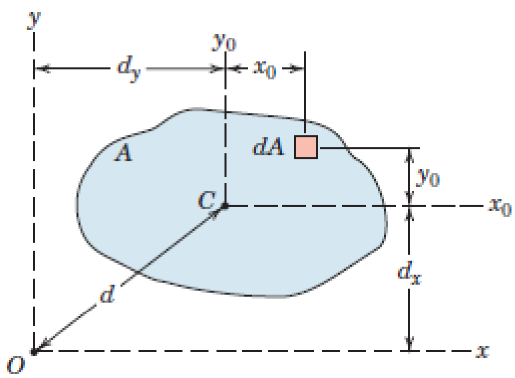
The radius of gyration, then, is a measure of the distribution of the area from the axis in question.

PERPENDICULAR AXIS THEOREM:

The moment of inertia (MI) of a plane area about an axis normal to the plane is equal to the sum of the moments of inertia about any two mutually perpendicular axes lying in the plane and passing through the given axis.

That means the Moment of Inertia $I_z = I_x + I_y$

PARALLEL AXIS THEOREM:



The moment of inertia of an area about a non-centroidal axis may be easily expressed in terms of the moment of inertia about a parallel centroidal axis. In Figure the x_0-y_0 axes pass through the centroid *C* of the area. Let us now determine the moments of inertia of the area about the parallel *x*-*y* axes. By definition, the moment of inertia of the element *dA* about the *x*-axis is

$$dI_x = (y_o + d_x)^2\, dA$$

Expanding and integrating give us

$$I_x = \int y_o{}^2\, dA + 2\, d_x \int y_o\, dA + d_x{}^2 \int dA$$

We see that the first integral is by definition the moment of inertia $\bar{I}_x$ about the centroidal x_0-axis. The second integral is zero, since $\int y_0\, dA = A\bar{y}_0$ and $\bar{y}_0$ is automatically zero with the centroid on the x_0-axis. The third term is simply $Ad_x{}^2$. Thus, the expression for I_x and the similar expression for I_y become

$$I_x = \bar{I}_x + Ad_x^2$$
$$I_y = \bar{I}_y + Ad_y^2$$
Eq (1)

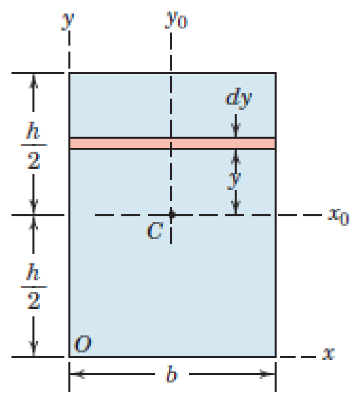
By equation $I_z = I_x + I_y$, the sum of above two equations gives

$$I_z = \bar{I}_z + Ad^2$$ Eq (2)

Equations (1) and (2) are called *parallel-axis theorems*. Two points in particular should be noted. First, the axes between which the transfer is made *must be parallel*, and second, one of the axes *must pass through the centroid of the area*.

If a transfer is desired between two parallel axes neither of which passes through the centroid, it is first necessary to transfer from one axis to the parallel centroidal axis and then to transfer from the centroidal axis to the second axis.

MOMENT OF INERTIA OF A RECTANGULAR AREA ABOUT ITS CENTROIDAL X_O, Y_O AND POLAR Z_O AXIS.



Solution. For the calculation of the moment of inertia $\bar{I}_x$ about the x_0-axis, a horizontal strip of area $b\,dy$ is chosen so that all elements of the strip have the same y-coordinate. Thus,

$$\left[I_x = \int y^2\,dA\right] \qquad \bar{I}_x = \int_{-h/2}^{h/2} y^2 b\,dy = \tfrac{1}{12}bh^3$$

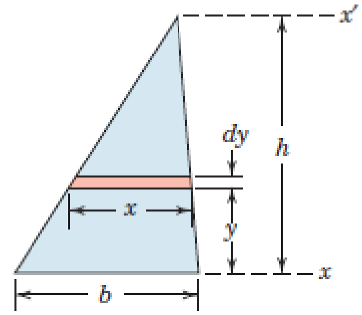
By interchange of symbols, the moment of inertia about the centroidal y_0-axis is

$$\bar{I}_y = \tfrac{1}{12}hb^3$$

The centroidal polar moment of inertia is

$$[\bar{I}_z = \bar{I}_x + \bar{I}_y] \qquad \bar{I}_z = \tfrac{1}{12}(bh^3 + hb^3) = \tfrac{1}{12}A(b^2 + h^2)$$

MOMENT OF INERTIA OF A TRIANGLE ABOUT AN AXIS PASSING THROUGH ITS BASE, CENTROID AND ITS VERTEX.



Solution. A strip of area parallel to the base is selected as shown in the figure, and it has the area $dA = x\,dy = [(h-y)b/h]\,dy$. By definition

$$[I_x = \int y^2\,dA] \qquad I_x = \int_0^h y^2 \frac{h-y}{h} b\,dy = b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = \frac{bh^3}{12}$$

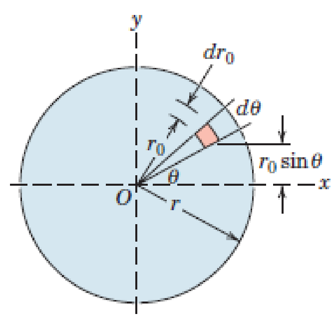
By the parallel-axis theorem the moment of inertia $\bar{I}$ about an axis through the centroid, a distance $h/3$ above the x-axis, is

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36}$$

A transfer from the centroidal axis to the x'-axis through the vertex gives

$$[I = \bar{I} + Ad^2] \qquad I_{x'} = \frac{bh^3}{36} + \left(\frac{bh}{2}\right)\left(\frac{2h}{3}\right)^2 = \frac{bh^3}{4}$$

MOMENT OF INERTIA OF A CIRCLE ABOUT ITS CENTOIDAL X, Y AND POLAR Z AXIS



Solution. A differential element of area in the form of a circular ring may be used for the calculation of the moment of inertia about the polar z-axis through O since all elements of the ring are equidistant from O. The elemental area is $dA = 2\pi r_0\,dr_0$, and thus,

$$[I_z = \int r^2\,dA] \qquad I_z = \int_0^r r_0^2(2\pi r_0\,dr_0) = \frac{\pi r^4}{2} = \tfrac{1}{2}Ar^2$$

The polar radius of gyration is

$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_z = \frac{r}{\sqrt{2}}$$

By symmetry $I_x = I_y$, so that from Eq.

$$[I_z = I_x + I_y] \qquad I_x = \frac{1}{2}I_z = \frac{\pi r^4}{4} = \frac{1}{4}Ar^2$$

The radius of gyration about the diametral axis is

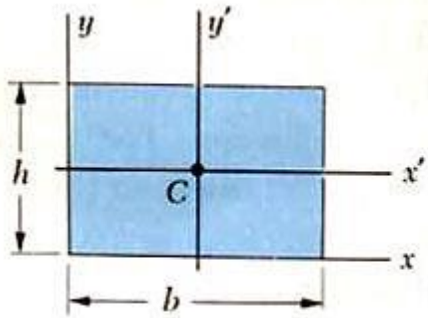
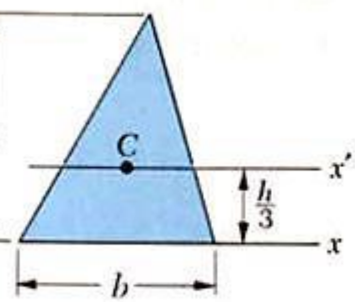
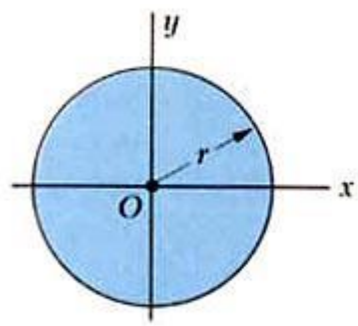
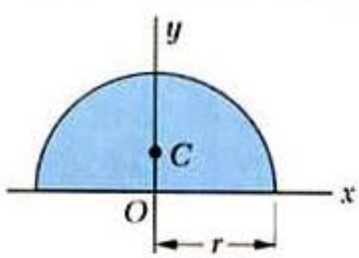
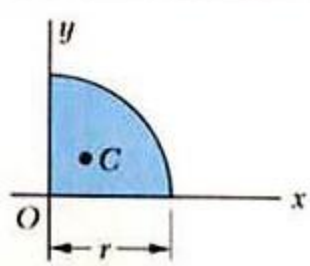
$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_x = \frac{r}{2}$$

The foregoing determination of I_x is the simplest possible. The result may also be obtained by direct integration, using the element of area $dA = r_0\, dr_0\, d\theta$ shown in the lower figure. By definition

$$\left[I_x = \int y^2\, dA\right] \qquad I_x = \int_0^{2\pi}\int_0^r (r_0 \sin\theta)^2 r_0\, dr_0\, d\theta$$

$$= \int_0^{2\pi} \frac{r^4 \sin^2\theta}{4}\, d\theta$$

$$= \frac{r^4}{4}\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{\pi r^4}{4}$$

<table>
<tr><td>Rectangle</td><td></td><td>$\bar{I}_{x'} = \frac{1}{12}bh^3$
$\bar{I}_{y'} = \frac{1}{12}b^3h$
$I_x = \frac{1}{3}bh^3$
$I_y = \frac{1}{3}b^3h$
$J_C = \frac{1}{12}bh(b^2 + h^2)$</td></tr>
<tr><td>Triangle</td><td></td><td>$\bar{I}_{x'} = \frac{1}{36}bh^3$
$I_x = \frac{1}{12}bh^3$</td></tr>
<tr><td>Circle</td><td></td><td>$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$
$J_O = \frac{1}{2}\pi r^4$</td></tr>
<tr><td>Semicircle</td><td></td><td>$I_x = I_y = \frac{1}{8}\pi r^4$
$J_O = \frac{1}{4}\pi r^4$</td></tr>
<tr><td>Quarter circle</td><td></td><td>$I_x = I_y = \frac{1}{16}\pi r^4$
$J_O = \frac{1}{8}\pi r^4$</td></tr>
</table>

Reference: **Engineering Mechanics** *Meriam & Kraige, Wiley Publication -8th Edition*