ENGINEERING MATHEMATICS - I

Unit - 3: Partial Differential Equations

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Contents

Solution of the heat equation

The heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Let

$$u = XT \tag{2}$$

where X is a function of x only and T is a function of t only, be a solution of (1).

Then

$$\frac{\partial u}{\partial t} = XT' \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''T$$

Substituting in (1), we have

$$XT' = c^2 X''T$$

Separating the variables, we get

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} \tag{3}$$

Now the L.H.S. of (3) is a function of x only and the R.H.S. is a function of t only. Since x and t are independent variables, this equation can hold only when both sides reduce to a constant, say k. The equation (3) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \qquad \text{and} \qquad \frac{dT}{dt} - kc^2T = 0 \tag{4}$$

Solution of the heat equation (contd.)

Solving equations (4), we get
(i) When k is positive and $= p^2$, say

$$X = c_1 e^{px} + c_2 e^{-px}, \quad T = c_3 e^{c^2 p^2 t}$$

$$u = (c_1 e^{px} + c_2 e^{-px}) \cdot c_3 e^{c^2 p^2 t} \tag{5}$$

(ii) When k is negative and $= -p^2$, say

$$X = c_1 \cos px + c_2 \sin px, \quad T = c_3 e^{-c^2 p^2 t}$$

$$u = (c_1 \cos px + c_2 \sin px) \cdot c_3 e^{-c^2 p^2 t} \tag{6}$$

(iii) When $k = 0$

$$X = c_1 x + c_2, \quad T = c_3$$

$$u = (c_1 x + c_2) c_3 \tag{7}$$

Solution of the heat equation (contd.)

Thus the various possible solutions of the heat equation (1) are:

$$u = (c_1 e^{px} + c_2 e^{-px}) \cdot c_3 e^{c^2 p^2 t} \tag{5}$$

$$u = (c_1 \cos px + c_2 \sin px) \cdot c_3 e^{-c^2 p^2 t} \tag{6}$$

$$u = (c_1 x + c_2) c_3 \tag{7}$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. Since u decreases as time t increases, the only suitable solution of the heat equation is

$$u = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2 p^2 t}$$