

# ENGINEERING MATHEMATICS - I

## Unit - 3: Partial Differential Equations

Department of Science and Humanities





- 1 Formation of PDE by elimination of arbitrary functions
- 2 Solutions of Partial Differential Equations
- 3 Solution of First-Order Equations

# Formation of PDE by elimination of arbitrary functions



- ➊ Partial differential equations can be obtained by eliminating arbitrary functions from a given equation
- ➋ A first-order partial differential equation can be derived from an equation involving a single arbitrary function, while a second-order equation may arise from an expression containing two arbitrary functions
- ➌ However, it is not always guaranteed that an  $n$ th-order partial differential equation can be formed from an equation involving  $n$  arbitrary functions
- ➍ In some cases, higher-order partial derivatives may be required to eliminate all  $n$  arbitrary functions, leading to a partial differential equation of order higher than  $n$ . Moreover, such a resulting higher-order equation may not be unique

## Example: Formation of a PDE by eliminating an arbitrary function

Eliminate the arbitrary function from  $z = f(x + y)$  to obtain a first-order partial differential equation.

**Answer:** Let  $u = x + y$ . Then  $z = f(u)$ .

Differentiating with respect to  $x$ :

$$\frac{\partial z}{\partial x} = f'(u) \cdot \frac{\partial u}{\partial x} = f'(x + y) \cdot 1 = f'(x + y)$$

Differentiating with respect to  $y$ :

$$\frac{\partial z}{\partial y} = f'(u) \cdot \frac{\partial u}{\partial y} = f'(x + y) \cdot 1 = f'(x + y)$$

From above,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$$

So, the required PDE is:

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$$



## Example: 2

Obtain a second-order partial differential equation by eliminating the arbitrary functions from

$$u = f(x + ct) + g(x - ct).$$



**Solution** Differentiating the given equation partially with respect to  $x$  and  $t$ , we obtain

$$\frac{\partial u}{\partial x} = f'(x + ct) + g'(x - ct), \quad \frac{\partial u}{\partial t} = cf'(x + ct) - cg'(x - ct),$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x + ct) + g''(x - ct), \quad \frac{\partial^2 u}{\partial t^2} = c^2 f''(x + ct) + c^2 g''(x - ct).$$

Hence, we obtain the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

## Example: 3

Given  $z = f(x^2 + y)$ , where  $f$  is an arbitrary function, form the partial differential equation by eliminating the arbitrary function



**Solution:**

- ① Differentiate with respect to  $x$ :

$$\frac{\partial z}{\partial x} = f'(x^2 + y) \cdot 2x$$

- ② Differentiate with respect to  $y$ :

$$\frac{\partial z}{\partial y} = f'(x^2 + y) \cdot 1 = f'(x^2 + y)$$

- ③ Eliminate  $f'$  by dividing the first equation by the second:

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{2x f'(x^2 + y)}{f'(x^2 + y)} = 2x$$

**Required PDE:**

$$\frac{\partial z}{\partial x} - 2x \frac{\partial z}{\partial y} = 0$$

## Example: 4

Given  $z = f(x^2 + y^2) + x$ , where  $f$  is an arbitrary function, form the partial differential equation by eliminating the arbitrary function



**Solution:**

- 1 Differentiate with respect to  $x$ :  $\frac{\partial z}{\partial x} = f'(x^2 + y^2) \cdot 2x + 1$
- 2 Differentiate with respect to  $y$ :  $\frac{\partial z}{\partial y} = f'(x^2 + y^2) \cdot 2y$
- 3 Eliminate  $f'$  by solving the second equation for  $f'(x^2 + y^2)$ :

$$f'(x^2 + y^2) = \frac{1}{2y} \frac{\partial z}{\partial y} \quad (\text{for } y \neq 0)$$

$$\text{Substitute into the first equation: } \frac{\partial z}{\partial x} = 2x \left( \frac{1}{2y} \frac{\partial z}{\partial y} \right) + 1 = \frac{x}{y} \frac{\partial z}{\partial y} + 1$$

**Required PDE:**

$$\frac{\partial z}{\partial x} - \frac{x}{y} \frac{\partial z}{\partial y} = 1$$



- A solution of a partial differential equation is a relation between the dependent and independent variables that satisfies the partial differential equation
- It is known that a partial differential equation is formed by eliminating arbitrary constants and arbitrary functions. These relations are solutions of the partial differential equation formed
- There are two types of solutions for partial differential equations:
  1. Solution containing arbitrary constants
  2. Solution containing arbitrary functions



- A solution of a partial differential equation which contains as many arbitrary constants as the number of independent variables is called the **complete solution**
- A solution of a partial differential equation, which contains as many arbitrary functions as the order of the equation, is called the **general solution**

# Example

Suppose we have the relation:

$$z = f(x + y)$$

where  $f$  is an arbitrary (differentiable) function.

Now,  $\frac{\partial z}{\partial x} = f'(x + y)$ ;  $\frac{\partial z}{\partial y} = f'(x + y)$ . Therefore,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$$

The partial differential equation:  $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$

**Conclusion:**

The equation  $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$  is a partial differential equation formed by eliminating the arbitrary function  $f$  from the original relation. The original relation  $z = f(x + y)$  is a solution to this partial differential equation





- An equation containing  $x, y, z, p, q$  defines a first order partial differential equation, that is

$$F(x, y, z, p, q) = 0 \quad (1)$$

- **General solution:** A relation of the form  $\phi(u, v) = 0$ , where  $\phi$  is an arbitrary function of  $u = u(x, y, z)$ ,  $v = v(x, y, z)$  and satisfies Eq. (1) is called a *general solution*
- **Particular solution:** The solution obtained by determining the arbitrary function in the general solution by using some specified condition is called a *particular solution*