ENGINEERING MATHEMATICS - I

Unit - 3: Partial Differential Equations

Department of Science and Humanities

PES UNIVERSITY

Contents

Higher-Order Linear Equations with Constant Coefficients

- We have already learned how to solve higher-order linear ordinary differential equations with constant coefficients. These same methods can also be used to solve higher-order linear partial differential equations with constant coefficients
- For example, consider the following partial differential equation

$$\begin{aligned} A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots \\ \cdots + A_{n-1} \frac{\partial^n z}{\partial x\,\partial y^{n-1}} + A_n \frac{\partial^n z}{\partial y^n} = f(x, y)\,. \end{aligned} \tag{1}$$

where A_i, $i = 0, 1, \ldots, n$ are constants.

Higher-Order Linear Equations with Constant Coefficients (contd.)

Denote

$$p = \frac{\partial z}{\partial x} = Dz, \quad q = \frac{\partial z}{\partial y} = D'z, \quad D^2 z = \frac{\partial^2 z}{\partial x^2}, \quad DD'z = \frac{\partial^2 z}{\partial x \partial y}$$

etc.
Then, Eq. (1) can be written as

$$F(D, D')z = \sum_{r=0}^{n} A_r D^{n-r} (D')^r z = f(x, y). \tag{2}$$

As in the case of ordinary differential equations, the general solution of Eq. (2) can be written as

$$z = (\text{complementary function}) + (\text{particular integral}),$$

where the complementary function is the solution of the homogeneous equation $F(D, D')z = 0$ and the particular integral is a solution satisfying the non-homogeneous equation (2) and does not contain any arbitrary constants.

Consider the linear, homogeneous equation

$$F(D, D')z = 0. \tag{3}$$

If $F(D, D')$ can be factorised into linear factors of the form

$$(a_i D + b_i D' + c_i),$$

then $F(D, D')$ is said to be *reducible*. Otherwise, it is said to be *irreducible*

Complementary Function (contd.)

$F(D, D')$ may contain some reducible and some irreducible factors. We have the following examples.

$$(D^2 - D'^2)z = (D - D')(D + D')z = 0 \tag{4}$$

$$(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = (D - D')(D - 2D')(D - 3D')z = 0 \tag{5}$$

$$(2D^2 - D')z = 0 \tag{6}$$

$$(D^3 + D^2D' + D'^3)z = 0 \tag{7}$$

$$(D^3 - D^2D' + DD' - D'^2)z = (D - D')(D^2 + D'^2)z = 0 \tag{8}$$

Eqs. (4) and (5) are the reducible forms, Eqs. (6) and (7) are irreducible forms and Eq. (8) has one reducible and one irreducible factor.
Note that we will discuss only the equations which are in "reducible form".

Distinct Roots

Consider the solution of the partial differential equation with one linear factor, that is, the solution of

$$(a_i D + b_i D' + c_i)z = 0, \quad \text{or} \quad a_i p + b_i q = -c_i z. \tag{9}$$

The solution of Eq. (9) is given by

$$\boxed{z = e^{-(c_i x)/a_i}\, \phi_i(b_i x - a_i y)} \tag{10}$$

Some particular cases:

a) If the linear factor is of the form $(a_i D + b_i D')$, then Eq.(10) simplifies to

$$\boxed{z = \phi_i(b_i x - a_i y)} \tag{11}$$

(b) If the linear factor is of the form $(b_i D' + c_i)$, then we have the differential equation as $(b_i D' + c_i)z = 0$ or $b_i q_i + c_i z = 0$. The solutions of the auxiliary equations are

$$b_i x = k_1 \quad \text{and} \quad z = k_2 e^{-(c_i y)/b_i}$$

Hence, the solution of Eq. (9), when $a_i = 0$, is given by

$$k_2 = \phi_i(k_1), \quad \text{or} \quad z e^{(c_i y)/b_i} = \phi_i(b_i x)$$

or

$$\boxed{z = e^{-(c_i y)/b_i}\, \phi_i(b_i x)} \tag{12}$$

(c) Similarly, if the linear factor is of the form $(a_i D + c_i)$, then the solution of Eq.(9), when $b_i = 0$, is given by

$$\boxed{z = e^{-\frac{c_i x}{a_i}} \phi_i\big(a_i y\big)} \tag{13}$$

The complementary function is given by

$$z = e^{-\frac{c_1 x}{a_1}} \phi_1\big(b_1 x - a_1 y\big) + e^{-\frac{c_2 x}{a_2}} \phi_2\big(b_2 x - a_2 y\big) + \cdots + e^{-\frac{c_n x}{a_n}} \phi_n\big(b_n x - a_n y\big)$$

Similarly, the complementary function in the other cases can be written.

Note: If all terms in $F(D, D')$ have the same total degree, then we can set $m = D/D'$ and look for factors of the equation using m.

Example: Find the solutions of the following PDE's

$$\begin{aligned} &\text{(i)} \quad (D^2 - D'^2)z = 0, \\ &\text{(ii)} \quad (D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0, \\ &\text{(iii)} \quad 3r + 7s + 2t + 7p + 4q + 2z = 0, \\ &\text{(iv)} \quad 2r - s - t - p + q = 0. \end{aligned}$$

(i) We have $(D^2 - D'^2)z = (D + D')(D - D')z = 0$.
For the factor $D - D'$, we have $a_1 = 1$, $b_1 = -1$.
For the factor $D + D'$, we have $a_2 = 1$, $b_2 = 1$.
Therefore, using Eq. (11), we obtain the solution as

$$z = \phi_1^*(-x - y) + \phi_2(x - y) = \phi_1(x + y) + \phi_2(x - y).$$

(ii) We have

$$(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z$$
$$\Rightarrow (D - D')(D - 2D')(D - 3D')z = 0.$$

For the factor $D - D'$, we have $a_1 = 1$, $b_1 = -1$.
For the factor $D - 2D'$, we have $a_2 = 1$, $b_2 = -2$.
For the factor $D - 3D'$, we have $a_3 = 1$, $b_3 = -3$.
If we set $(D/D') = m$, we get the equation as $m^3 - 6m^2 + 11m - 6 = 0$, whose roots are $m = 1, 2, 3$. Hence, the factors are $D - D'$, $D - 2D'$, $D - 3D'$.
Using Eq. (11), the solution can be written as

$$z = \phi_1^*(-x - y) + \phi_2^*(-2x - y) + \phi_3^*(-3x - y)$$
$$= \phi_1(x + y) + \phi_2(2x + y) + \phi_3(3x + y).$$

The aforementioned note does not apply to the following two problems.
(iii) We have

$$(3D^2+7DD'+2D'^2+7D+4D'+2)z=(3D+D'+1)(D+2D'+2)z=0.$$

For the factor $3D+D'+1$, we have $a_1=3$, $b_1=1$, $c_1=1$.
For the factor $D+2D'+2$, we have $a_2=1$, $b_2=2$, $c_2=2$.
Using Eq. (10), the solution can be written as

$$z=e^{-x/3}\phi_1(x-3y)+e^{-2x}\phi_2(2x-y).$$

(iv) We have

$$(2D^2 - DD' - D'^2 - D + D')z = (D - D')(2D + D' - 1)z = 0.$$

For the factor $D - D'$, we have $a_1 = 1$, $b_1 = -1$, $c_1 = 0$.
For the factor $2D + D' - 1$, we have $a_2 = 2$, $b_2 = 1$, $c_2 = -1$.
Using Eqs. (10) and (11), the solution can be written as

$$z = \phi_1^*(-x - y) + e^{x/2}\phi_2(x - 2y)$$

$$= \phi_1(x + y) + e^{x/2}\phi_2(x - 2y).$$

Multiple Factors

Let a factor be of multiplicity 2, that is, we have either of the factors

$$(a_iD + b_iD' + c_i)^2, \quad \text{or} \quad (a_iD + c_i)^2, \quad \text{or} \quad (b_iD' + c_i)^2$$

1. For the equation $(a_iD + b_iD' + c_i)^2 z = 0$, the general solution is given by

$$\boxed{z = e^{-(c_ix)/a_i}\left[x\,\phi_i(b_ix - a_iy) + \psi_i(b_ix - a_iy)\right]}$$

2. For the equation $(a_iD + c_i)^2 z = 0$, the general solution is given by

$$\boxed{z = e^{-(c_ix)/a_i}\left[x\,\phi_i(a_iy) + \psi_i(a_iy)\right]}$$

Multiple Factors (contd.)

1. For the equation $(b_i D' + c_i)^2 z = 0$, the general solution is

$$\boxed{z = e^{-(c_i y)/b_i}\left[y\,\phi_i(b_i x) + \psi_i(b_i x)\right]}$$

2. If the factor is of multiplicity m, say $(a_1 D + b_1 D' + c_1)^m$, then the general solution is

$$\boxed{\begin{aligned} z = e^{-\frac{c_1 x}{a_1}}\Big[&\psi_1(b_1 x - a_1 y) + x\,\psi_2(b_1 x - a_1 y) + \dots \\ &+ x^{m-1}\,\psi_m(b_1 x - a_1 y)\Big]\end{aligned}}$$

Similarly, we can write the general solution in the other cases.