ENGINEERING MATHEMATICS I

Kavyashree

Department of Science and Humanities

Class content

- Generating Functions
- Generating function for Bessel function of integral order
- Jacobi Series

Generating Functions

The generating function for the sequence of functions $f_n(x)$ is,

$$G(x,t) = \sum_{-\infty}^{\infty} f_n(x)\, t^n$$

which generates $f_n(x)$.

i.e., $f_n(x)$ appear as coefficients of various powers of t.

ENGINEERING MATHEMATICS I

Generating Functions

Prove that $\sum_{n=-\infty}^{\infty} J_n(x)t^n = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$

OR

Prove that generating functions for Bessel function

of integral order is $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$

Generating Functions

Proof: Consider $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{\frac{xt}{2}} e^{-\frac{x}{2t}}$

$$= \sum_{m=0}^{\infty} \left(\frac{xt}{2}\right)^m \frac{1}{m!} \sum_{n=0}^{\infty} \left(-\frac{x}{2t}\right)^n \frac{1}{n!}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{x}{2t}\right)^m \left(-\frac{xt}{2}\right)^n \frac{1}{m!}\frac{1}{n!}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^m \left(\frac{1}{t}\right)\left(\frac{x}{2}\right)^n t^n (-1)^n \frac{1}{m!}\frac{1}{n!}$$

Generating Functions

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^{m+n} t^{m-n}\,\frac{1}{m!}\,\frac{1}{n!}$$

Let $m-n=i, \quad \Rightarrow \quad m=n+i$

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{i=-\infty}^{\infty}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^{i+2n} t^{i}\,\frac{1}{(n+i)!}\,\frac{1}{n!}$$

$$= \sum_{i=-\infty}^{\infty}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\,\frac{1}{(n+i)!}\left(\frac{x}{2}\right)^{i+2n}\right\}t^{i}$$

Generating Functions

$$\Rightarrow \quad e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{i=-\infty}^{\infty}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{(n+i)!}\left(\frac{x}{2}\right)^{i+2n}\right\}t^i$$

$$= \sum_{i=-\infty}^{\infty}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{\Gamma(n+i+1)}\left(\frac{x}{2}\right)^{i+2n}\right\}t^i$$

$$= \sum_{i=-\infty}^{\infty} J_i(x)t^i$$

Thus $\boxed{\sum_{n=-\infty}^{\infty} J_n(x)t^n = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}$

Jacobi Series

$$\cos(x\sin\theta) = J_0 + 2\left(J_2\cos 2\theta + J_4\cos 4\theta + \ldots\ldots\ldots\right)$$

$$\sin(x\sin\theta) = 2\left(J_1\sin\theta + J_3\sin 3\theta + .J_5\sin 5\theta\ldots\ldots\ldots\right)$$

Jacobi Series

Proof: We Know that $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$

Expanding the summation, we get,

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + ----$$
$$+ t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + ----$$

When n is an integer, $J_{-n}(x) = (-1)^n J_n(x)$

Jacobi Series

Therefore,

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = J_0(x) + (t - t^{-1})J_1(x) + (t^2 + t^{-2})J_2(x) \quad --\rightarrow (1)$$
$$+ (t^3 - t^{-3})J_3(x) + \cdots\cdots$$

Let $t = \cos\theta + i\sin\theta$, then $\frac{1}{t} = \cos\theta - i\sin\theta$

Therefore, $t + \frac{1}{t} = 2\cos\theta$; $t - \frac{1}{t} = 2i\sin\theta$

and, $t^n + \frac{1}{t^n} = 2\cos n\theta$; $t^n - \frac{1}{t^n} = 2i\sin n\theta$

Jacobi Series

Using equation (1) we get,

$$e^{\frac{1}{2}x(2i\sin\theta)} = J_0(x) + 2(J_2\cos 2\theta + J_4\cos 4\theta + ------) + 2i(J_1\sin\theta + J_3\sin 3\theta + ------)$$

i.e., $$\cos(x\sin\theta) + i\sin(x\sin\theta) = J_0(x) + 2(J_2\cos 2\theta + J_4\cos 4\theta + --) + 2i(J_1\sin\theta + J_3\sin 3\theta + ---)$$

Equating real and imaginary parts, we get,

$$\boxed{\begin{aligned}\cos(x\sin\theta) &= J_0 + 2\left(J_2\cos 2\theta + J_4\cos 4\theta + ...\right)\\ \sin(x\sin\theta) &= 2\left(J_1\sin\theta + J_3\sin 3\theta + .J_5\sin 5\theta.........\right)\end{aligned}}$$

THANK YOU

Kavyashree

Department of Science and Humanities

Email: kavyashree@pes.edu

Phone:9008455661; Extn:714