Stochastic Optimization with CPLEX

Juxihong Julaiti, Ph.D. Candidate in Operations Research and Industrial Engineering
Pennsylvania State University
Dec, 2018

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In [1]:
```python
import numpy as np
import matplotlib.pyplot as plt
import random
random.seed(a=600)
```

New-vendor Problem

original problem:

$$max_x \; E[f(x)]$$
$$st:$$
$$f(x) = c_s \hat{s} + c_d \hat{s}' - c_p x$$
$$0 \leq \hat{s} \leq \min\{x, \hat{d}\}$$
$$0 \leq \hat{s}' \leq \max\{0, x - \hat{d}\}$$

where $f(x)$ is the profit of purchasing x units; c_s is the unit selling price; c_d is the unit discounted selling price; c_p is the unit purchasing cost; $\hat{s}$ is sales; $\hat{s}'$ is discounted sales;

issue:

$E[f(x)] = \int_0^\infty xf(x)dx$ => hard to solve

relaxation:

Assume the uncertainty set U has a finite number of elements. And the problem can be reformulated into a big LP problem:

$$max_x \; \sum_{u \in U} p_u \; [c_s s_u + c_d s'_u - c_p x]$$
$$st:$$
$$0 \leq s_u \leq \min\{x, d_u\} \; , \forall u \in U$$
$$0 \leq s'_u \leq \max\{0, x - d_u\} \; , \forall u \in U$$

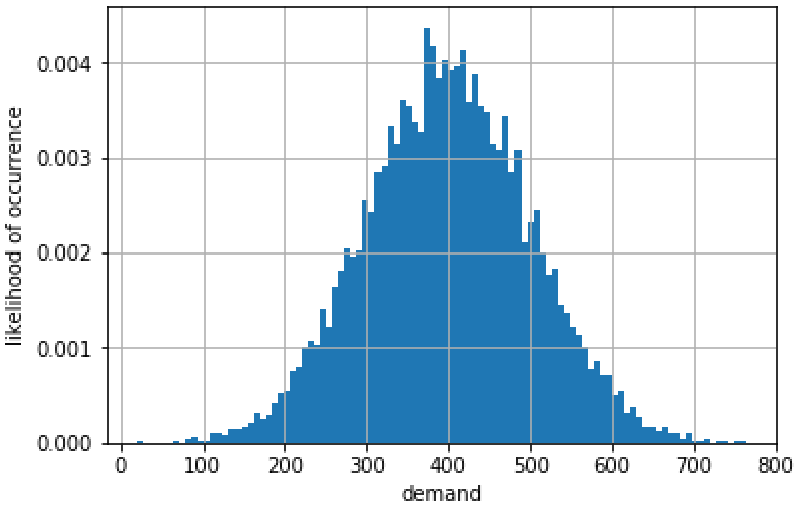
reformulate the problem:

$$max_x \; \sum_{u \in U} p_u \; [c_s s_u + c_d s'_u - c_p x]$$
$$st:$$
$$s_u + s'_u = x \; , \forall u \in U$$
$$0 \leq s_u \leq \alpha_u \; , \forall u \in U$$
$$\alpha_u \leq x \; , \forall u \in U$$
$$\alpha_u \leq d_u \; , \forall u \in U$$
$$0 \leq s'_u \leq \beta_u \; , \forall u \in U$$
$$\beta_u \geq 0 \; , \forall u \in U$$
$$\beta_u \geq x - d_u \; , \forall u \in U$$

Demands

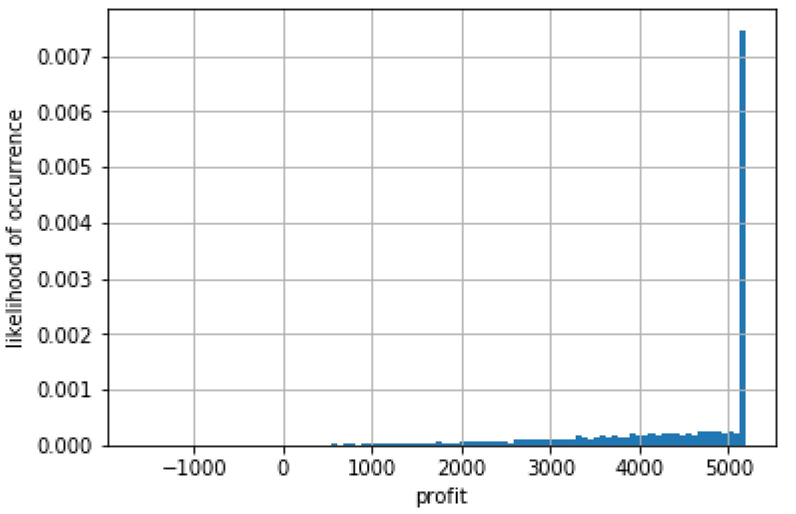
generating random demands from a normal distribution

In [2]:
```
sigma   = 100
mu      = 400
samples = 10000
demand  = [max(random.normalvariate(mu,sigma),0) for i in range(samples)]

Cs=15
Cd=-3
Cp=2

plt.hist(demand,bins=100,density=True)
plt.grid(True)
plt.ylabel("likelihood of occurrence")
plt.xlabel("demand")
plt.show()
```



what if x= μ

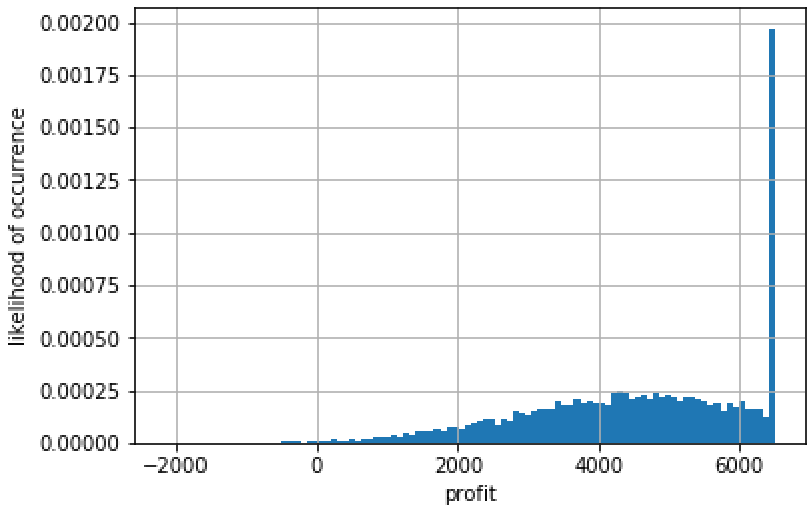
In [3]:
```
x=mu
profit=[Cs*min(x,demand[u])+Cd*max(0,x-demand[u])-Cp*x for u in range(samples)]
plt.hist(profit,bins=100,density=True)
plt.grid(True)
plt.ylabel("likelihood of occurrence")
plt.xlabel("profit")
plt.show()
print("order quantity:{0:.2f}|mean profit:{1:.2f}|std:{2:.2f} | worst case:{3:.2f}".format(x,np.mean(profit),np.std(profi
profit_array=np.array(profit)
print(100*len(profit_array[profit_array<=4000])/len(profit_array),"% of cases profit is less than 4000" )
```




```
order quantity:400.00|mean profit:4472.32|std:1053.03 | worst case:-1641.30
25.77 % of cases profit is less than 4000
```

what if x=$\mu + \sigma$

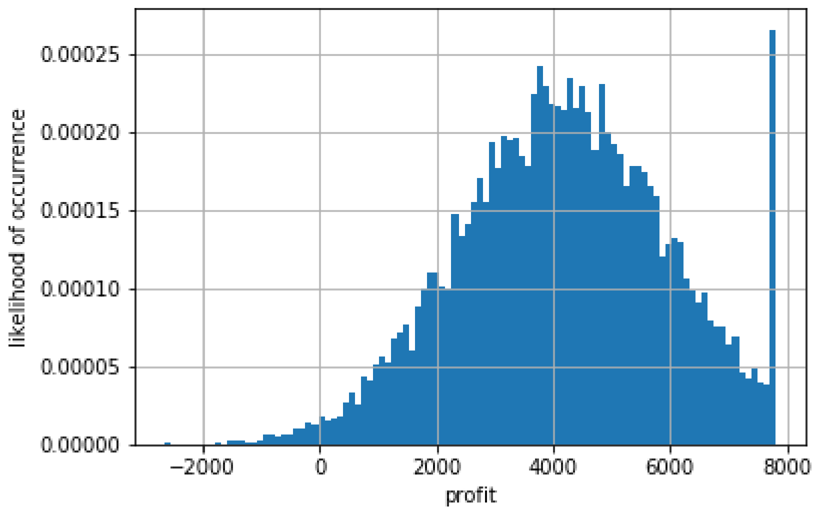
In [4]:
```
x=mu+sigma
profit=[Cs*min(x,demand[u])+Cd*max(0,x-demand[u])-Cp*x for u in range(samples)]
plt.hist(profit,bins=100,density=True)
plt.grid(True)
plt.ylabel("likelihood of occurrence")
plt.xlabel("profit")
plt.show()
print("order quantity:{0:.2f}|mean profit:{1:.2f}|std:{2:.2f} | worst case:{3:.2f}".format(x,np.mean(profit),np.std(profi
profit_array=np.array(profit)
print(100*len(profit_array[profit_array<=4000])/len(profit_array),"% of cases profit is less than 4000" )
```




```
order quantity:500.00|mean profit:4535.67|std:1562.59 | worst case:-2141.30
35.25 % of cases profit is less than 4000
```

what if x=$\mu + 2\sigma$

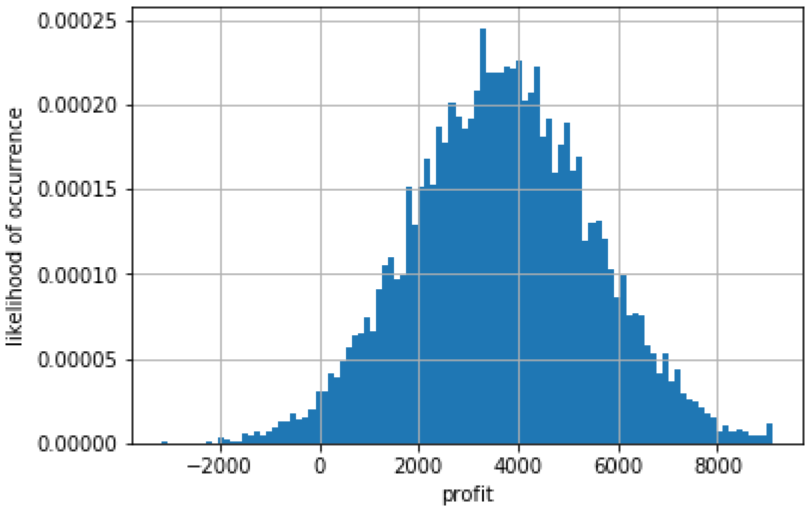
In [5]:
```
x=mu+2*sigma
profit=[Cs*min(x,demand[u])+Cd*max(0,x-demand[u])-Cp*x for u in range(samples)]
plt.hist(profit,bins=100,density=True)
plt.grid(True)
plt.ylabel("likelihood of occurrence")
plt.xlabel("profit")
plt.show()
print("order quantity:{0:.2f}|mean profit:{1:.2f}|std:{2:.2f} | worst case:{3:.2f}".format(x,np.mean(profit),np.std(profi
profit_array=np.array(profit)
print(100*len(profit_array[profit_array<=4000])/len(profit_array),"% of cases profit is less than 4000" )
```




```
order quantity:600.00|mean profit:4172.72|std:1771.04 | worst case:-2641.30
46.0 % of cases profit is less than 4000
```

what if x=$\mu + 3\sigma$

In [6]:
```
x=mu+3*sigma
profit=[Cs*min(x,demand[u])+Cd*max(0,x-demand[u])-Cp*x for u in range(samples)]
plt.hist(profit,bins=100,density=True)
plt.grid(True)
plt.ylabel("likelihood of occurrence")
plt.xlabel("profit")
plt.show()
print("order quantity:{0:.2f}|mean profit:{1:.2f}|std:{2:.2f} | worst case:{3:.2f}".format(x,np.mean(profit),np.std(profi
profit_array=np.array(profit)
print(100*len(profit_array[profit_array<=4000])/len(profit_array),"% of cases profit is less than 4000" )
```




```
order quantity:700.00|mean profit:3687.02|std:1804.15 | worst case:-3141.30
57.09 % of cases profit is less than 4000
```

Case I: the original problem

$max_x \;\; \sum_{u \in U} p_u \; [c_s s_u + c_d s'_u - c_p x]$

$st:$

$s_u + s'_u = x \;, \forall u \in U$

$0 \leq s_u \leq \alpha_u \;, \forall u \in U$

$\alpha_u \leq x \;, \forall u \in U$

$\alpha_u \leq d_u \;, \forall u \in U$

$0 \leq s'_u \leq \beta_u \;, \forall u \in U$

$\beta_u \geq 0 \;, \forall u \in U$

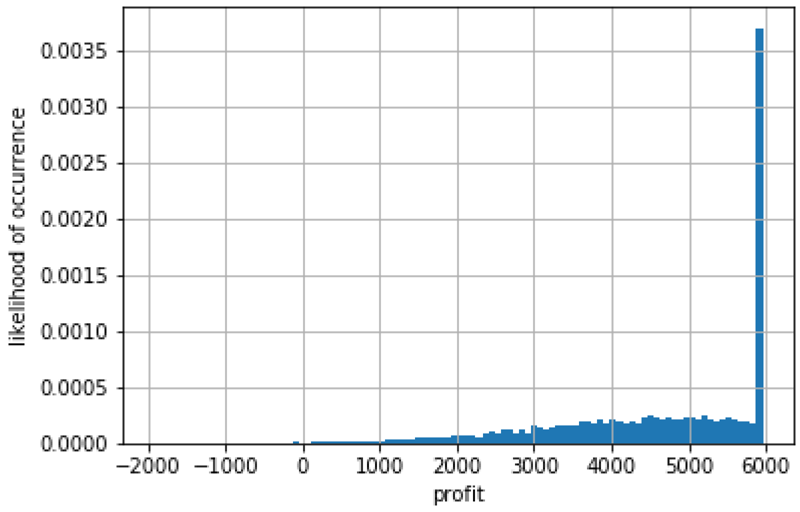
$\beta_u \geq x - d_u \;, \forall u \in U$

In [7]:
```
from docplex.mp.model import Model
```

In [8]:
```
md1 = Model(name='news-vendor case 1')
x = md1.continuous_var(name="x",lb=0)
s = {(u):md1.continuous_var(name="s_{0}".format(u),lb=0) for u in range(samples)}
sp = {(u):md1.continuous_var(name="sp_{0}".format(u),lb=0) for u in range(samples)}
alpha = {(u):md1.continuous_var(name="alpha_{0}".format(u),lb=0) for u in range(samples)}
beta = {(u):md1.continuous_var(name="beta_{0}".format(u),lb=0) for u in range(samples)}

profit=(1.0/samples)*md1.sum(Cs*s[u]+Cd*sp[u]-Cp*x for u in range(samples))
md1.maximize(profit)

for u in range(samples):
    md1.add_constraint(s[u]+sp[u]==x)
    md1.add_constraint(s[u]<=alpha[u])
    md1.add_constraint(alpha[u]<=x)
    md1.add_constraint(alpha[u]<=demand[u])
    md1.add_constraint(sp[u]<=beta[u])
    md1.add_constraint(beta[u]>=x-demand[u])

md1.print_information()
```

```
Model: news-vendor case 1
 - number of variables: 40001
   - binary=0, integer=0, continuous=40001
 - number of constraints: 60000
   - linear=60000
 - parameters: defaults
```

In [9]:
```
md1.solve()
x=md1.solution.as_dict()['x']
profit=[Cs*min(x,demand[u])+Cd*max(0,x-demand[u])-Cp*x for u in range(samples)]
plt.hist(profit,bins=100,density=True)
plt.grid(True)
plt.ylabel("likelihood of occurrence")
plt.xlabel("profit")
plt.show()
print("order quantity:{0:.2f}|mean profit:{1:.2f}|std:{2:.2f} | worst case:{3:.2f}".format(x,np.mean(profit),np.std(profi
profit_array=np.array(profit)
print(100*len(profit_array[profit_array<=4000])/len(profit_array),"% of cases profit is less than 4000" )
```




```
order quantity:458.49|mean profit:4582.56|std:1384.24 | worst case:-1933.75
31.25 % of cases profit is less than 4000
```

solution from the optimization model has a high mean and lower std.

Case II: maximize the worst case

$max_x \; w$

$st:$

$w \leq c_s s_u + c_d s'_u - c_p x \, , \forall u \in U$

$s_u + s'_u = x \, , \forall u \in U$

$0 \leq s_u \leq \alpha_u \, , \forall u \in U$

$\alpha_u \leq x \, , \forall u \in U$

$\alpha_u \leq d_u \, , \forall u \in U$

$0 \leq s'_u \leq \beta_u \, , \forall u \in U$

$\beta_u \geq 0 \, , \forall u \in U$

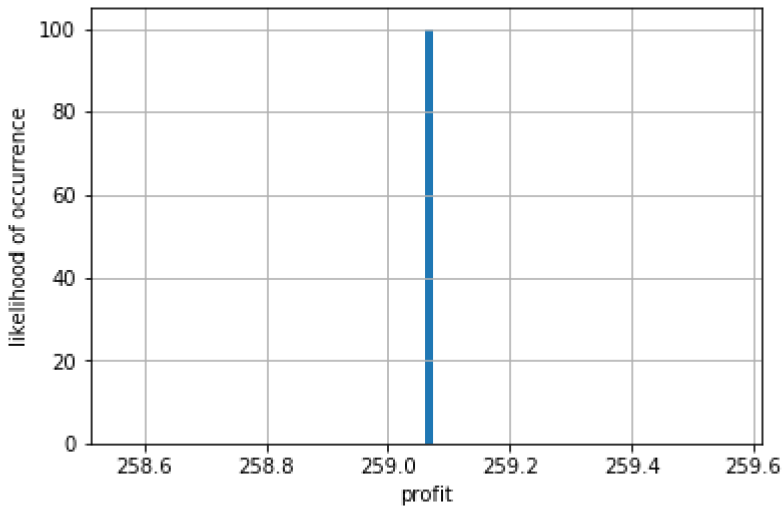
$\beta_u \geq x - d_u \, , \forall u \in U$

In [10]:
```
md2 = Model(name='news-vendor case 2')
w = md2.continuous_var(name="w",lb=-md2.infinity)
x = md2.continuous_var(name="x",lb=0)
s = {(u):md2.continuous_var(name="s_{0}".format(u),lb=0) for u in range(samples)}
sp = {(u):md2.continuous_var(name="sp_{0}".format(u),lb=0) for u in range(samples)}
alpha = {(u):md2.continuous_var(name="alpha_{0}".format(u),lb=0) for u in range(samples)}
beta = {(u):md2.continuous_var(name="beta_{0}".format(u),lb=0) for u in range(samples)}

md2.maximize(w)

for u in range(samples):
    md2.add_constraint(w<=Cs*s[u]+Cd*sp[u]-Cp*x)
    md2.add_constraint(s[u]+sp[u]==x)
    md2.add_constraint(s[u]<=alpha[u])
    md2.add_constraint(alpha[u]<=x)
    md2.add_constraint(alpha[u]<=demand[u])
    md2.add_constraint(sp[u]<=beta[u])
    md2.add_constraint(beta[u]>=x-demand[u])

md2.print_information()
```

```
Model: news-vendor case 2
 - number of variables: 40002
   - binary=0, integer=0, continuous=40002
 - number of constraints: 70000
   - linear=70000
 - parameters: defaults
```

In [11]:
```
md2.solve()
md2.report()
x=md2.solution.as_dict()['x']
profit=[Cs*min(x,demand[u])+Cd*max(0,x-demand[u])-Cp*x for u in range(samples)]
plt.hist(profit,bins=100,density=True)
plt.grid(True)
plt.ylabel("likelihood of occurrence")
plt.xlabel("profit")
plt.show()
print("order quantity:{0:.2f}|mean profit:{1:.2f}|std:{2:.2f} | worst case:{3:.2f}".format(x,np.mean(profit),np.std(profi
profit_array=np.array(profit)
print(100*len(profit_array[profit_array<=4000])/len(profit_array),"% of cases profit is less than 4000" )
```

```
* model news-vendor case 2 solved with objective = 259.063
```




```
order quantity:19.93|mean profit:259.06|std:0.00 | worst case:259.06
100.0 % of cases profit is less than 4000
```

Case III: chance constraints

80% of time has a profit more than 4000

$$
\begin{aligned}
& max_x \;\; \sum_{u \in U} p_u \; [c_s s_u + c_d s'_u - c_p x] \\
& st: \\
& \sum_{u \in U} \theta_u \geq 0.80 * |U| \, , \forall u \in U \\
& c_s s_u + c_d s'_u - c_p x \geq 4000 * \theta_u - M * (1 - \theta_u) \, , \forall u \in U \\
& s_u + s'_u = x \, , \forall u \in U \\
& 0 \leq s_u \leq \alpha_u \, , \forall u \in U \\
& \alpha_u \leq x \, , \forall u \in U \\
& \alpha_u \leq d_u \, , \forall u \in U \\
& 0 \leq s'_u \leq \beta_u \, , \forall u \in U \\
& \beta_u \geq 0 \, , \forall u \in U \\
& \beta_u \geq x - d_u \, , \forall u \in U
\end{aligned}
$$

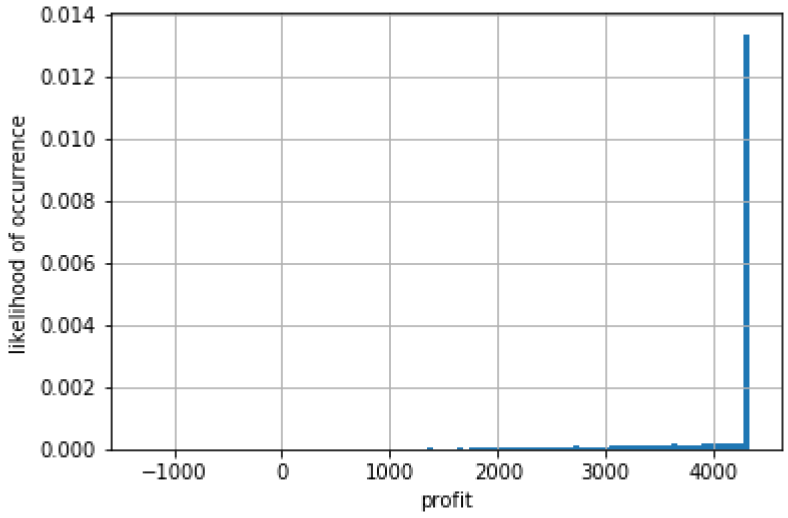
In [12]:
```
md3 = Model(name='news-vendor chance constraint')
x = md3.continuous_var(name="x",lb=0)
s = {(u):md3.continuous_var(name="s_{0}".format(u),lb=0) for u in range(samples)}
sp = {(u):md3.continuous_var(name="sp_{0}".format(u),lb=0) for u in range(samples)}
alpha = {(u):md3.continuous_var(name="alpha_{0}".format(u),lb=0) for u in range(samples)}
beta = {(u):md3.continuous_var(name="beta_{0}".format(u),lb=0) for u in range(samples)}

theta= {(u):md3.binary_var(name="theta_{0}".format(u)) for u in range(samples)}

profit=(1.0/samples)*md3.sum(Cs*s[u]+Cd*sp[u]-Cp*x for u in range(samples))
md3.maximize(profit)


md3.add_constraint(md3.sum(theta[u] for u in range(samples))>=0.80*samples)

for u in range(samples):
    md3.add_constraint(Cs*s[u]+Cd*sp[u]-Cp*x>=4000*theta[u]-(9999)*(1-theta[u]))
    md3.add_constraint(s[u]+sp[u]==x)
    md3.add_constraint(s[u]<=alpha[u])
    md3.add_constraint(alpha[u]<=x)
    md3.add_constraint(alpha[u]<=demand[u])
    md3.add_constraint(sp[u]<=beta[u])
    md3.add_constraint(beta[u]>=x-demand[u])

md3.print_information()
```

```
Model: news-vendor chance constraint
 - number of variables: 50001
   - binary=10000, integer=0, continuous=40001
 - number of constraints: 70001
   - linear=70001
 - parameters: defaults
```

In [13]:
```
md3.solve()
md3.report()
```

```
* model news-vendor chance constraint solved with objective = 4058.269
```

In [14]:
```
x=md3.solution.as_dict()['x']
profit=[Cs*min(x,demand[u])+Cd*max(0,x-demand[u])-Cp*x for u in range(samples)]
plt.hist(profit,bins=100,density=True)
plt.grid(True)
plt.ylabel("likelihood of occurrence")
plt.xlabel("profit")
plt.show()
print("order quantity:{0:.2f}|mean profit:{1:.2f}|std:{2:.2f} | worst case:{3:.2f}".format(x,np.mean(profit),np.std(profi
profit_array=np.array(profit)
print(100*len(profit_array[profit_array<=4000])/len(profit_array),"% of cases profit is less than 4000" )
```




```
order quantity:333.51|mean profit:4058.27|std:646.31 | worst case:-1308.84
20.01 % of cases profit is less than 4000
```

In [ ]: