RSA

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CYBER CHALLENGE.IT

CYBERSECURITY NATIONAL LABORATORY

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Goal

- Give the definition of public-key encryption
- Present the RSA public-key cryptosystem
- Show how to choose the parameters

Prerequisites

- Lectures:
  - *CR_0.1 - Number Theory and modular arithmetic*
  - *CR_1.1 – Introduction to cryptography and classical ciphers*
  - *CR_2.1 – Key Exchange & DH*

Outline

- Introduction to public-key cryptography
- RSA textbook scheme
- RSA-CRT
- Choose parameters

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Recall: Cryptography

- Two main methods:
  - *Symmetric key* - Single key
  - *Public key* - Double key

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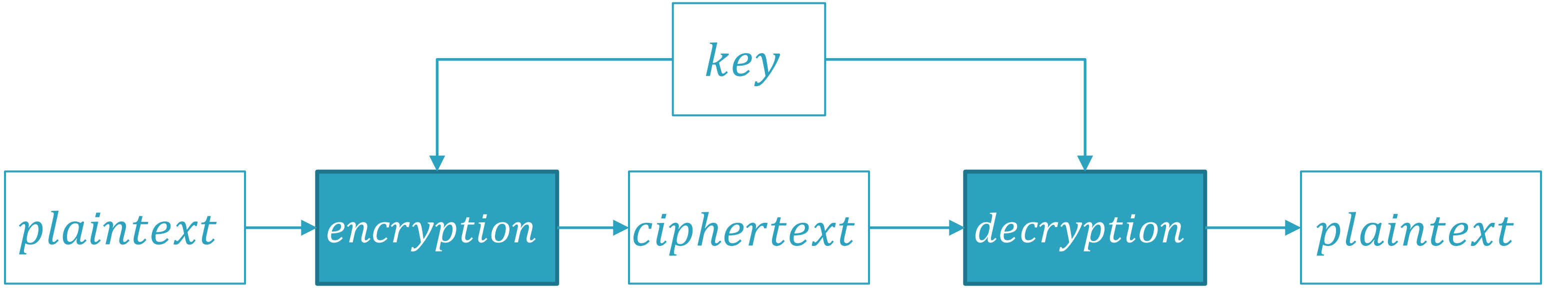
Seen in:
- *CR_1.2 – XOR cipher*
- *CR_1.3 – Block ciphers*
- *CR_1.4 – Stream ciphers*

Recall: Symmetric key cryptography

- Requires that both sender and recipient know the same key
- An issue is how they do share it without meeting



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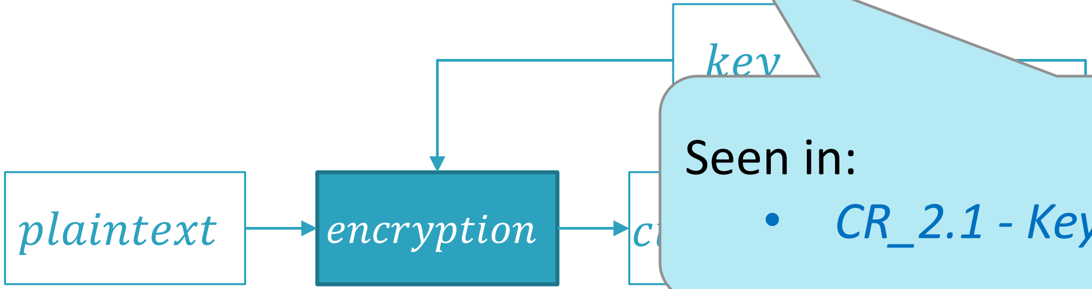

Seen in:

- *CR_2.1 - Key Exchange & DH*

Recall: public-key Encryption

- Require the use of two keys:
  - a public key, which can be known by anyone and can be used to encrypt messages and verify signatures
  - a corresponding private key, known only to the recipient, used to decrypt messages and to sign them
- Rely an asymmetric exchange: whoever encodes messages or verifies signatures cannot decode messages or create signatures
- Also called {double key, asymmetric} encryption

Asymmetric Encryption

- Developed to address two important issues:
  - key distribution: ensuring secure communications with a personal key without depending on a key distribution center and without trusting the behavior of others
  - digital signatures: verify that a message comes intact from the declared sender
- Complements rather than replaces symmetric encryption
- Is based on properties guaranteed by number theory rather than on the use of permutations and substitutions

Asymmetric vs symmetric encryption

- Symmetric encryption: same algorithm used to encrypt and decrypt with the secret same key. Key and algorithm are shared by sender and receiver
  - Almost impossible to decrypt a message if only the algorithm and the cipher text are known
- Asymmetric encryption: same algorithm used to encrypt and decrypt, but two keys are used: one to encrypt, the other to decrypt. Sender and receiver must each have a key that pairs with the other (not the same)
  - Almost impossible to decrypt a message if only the algorithm, the cipher text, and one of the keys are known

Principles of asymmetric cryptography

- A distinction is made between the keys of the subjects:
  - public key: publicly disclosed by the subject
  - private key: kept secret by the subject
- It must be computationally difficult to derive the decryption key knowing the algorithm and the encryption key
- The two keys can be (complementarily) used for encryption/decryption
- Encryption with public key guarantees confidentiality
- Encryption with private key guarantees authentication
- With an appropriate mix we can also guarantee messages integrity

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"trap-door one-way functions", seen in:

- *CR_0.1 – Number theory and modular arithmetic*

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RSA

- The most famous asymmetric encryption algorithm is RSA, from the authors **R**ivest, **S**hamir, **A**dleman, developed in 1977 and still used in practice today
  - Based on exponentiation of integers modulo n. Very large integers are used (typically 2048 bits)
  - Encryption and decryption are single modular exponentiation operation: exponentiation is easy (requires $O((\log n)^3)$ operations)
  - Security is guaranteed by the cost of factoring large numbers: factoring is difficult (requires $O\left(e^{(\log n)(\log \log n)}\right)$operations)

RSA Key Generation

- A user generates a pair of public/private keys as follows:
  - Randomly chooses two prime numbers: p, q
  - Computes $n = p \times q$ and $\phi(n) = (p-1) \times (q-1)$ Euler's totient
  - Randomly chooses the public key e such that:
    - $1 < e < \phi(n)$ with e and $\phi(n)$ coprime ($\gcd(e, \phi(n)) = 1$)
  - Determines the private key d by solving the equation:
    - $(e \times d)\ mod\ \phi(n) = 1$ with $0 \leq d \leq n$ ($d = e^{-1}\ mod\ \phi(n)$)
  - Share public key $k_{pub} = \{e, n\}$ and keeps private key $k_{priv} = \{d, n\}$

RSA encryption and decryption

- **Public Key** - $k_{pub} = \{e, n\}$ - **Private Key** - key $k_{priv} = \{d, n\}$
- To encrypt a message M, the sender:
  - Gets the recipient's public key $k_{pub} = \{e, n\}$
  - Computes $C = M^e \; mod \; n$, with $0 \leq M < n$
- To decipher the ciphertext C, the recipient:
  - Uses his private key $k_{priv} = \{d, n\}$
  - Computes $M = C^d \; mod \; n$
- The "magic" is due to the fact that $(M^e)^d \; (mod \; n) = M$

Why RSA Works

- Euler's theorem:
  - $a^{\phi(n)}\ mod\ n = 1$ if $\gcd(a, n) = 1$
- In RSA we have:
  - $n = p \times q$ and $\phi(n) = (p-1) \times (q-1)$
- The keys in the pair (e, d) are inverses $mod\ \phi(n)$
  - $e \times d = k \times \phi(n) + 1$ for some integer value of k

Why RSA Works

- Euler's theorem:
  - $a^{\phi(n)}\ mod\ n = 1$ if $\gcd(a, n) = 1$
- Thus, working modulo n, $C^d =$
  - $= M^{e \times d}$ (since $C = M^e$)
  - $= M^{k \times \phi(n)+1}$ since (e, d) are inverses $mod\ \phi(n)$
  - $= M^1 \times \left(M^{\phi(n)}\right)^k$ with simple arithmetic
  - $= M^1 \times 1^k$ for the Euler's theorem
  - $= M^1 = M$

An example in RSA - Key Setup

- Select two primes: $p = 17$ and $q = 11$
- Compute $n = p \times q = 17 \times 11 = 187$
- Compute $\phi(n) = (p-1) \times (q-1) = 16 \times 10 = 160$
- Select e: $GCD(e, 160) = 1; e = 7$
- Determine $d < 160$ such that $(d \times e)\, mod\, 160 = 1$
  - We have $d = 23$ as $23 \times 7 = 161 = 160 + 1$
- Publish public key $k_{pub} = \{e = 7, n = 187\}$
- Keep private key secret $k_{priv} = \{d = 23, n = 187\}$

An example in RSA - En/Decryption

- Public key = $\{e = 7, n = 187\}$
- Private key = $\{d = 23, n = 187\}$
- Given $M = 88$ $(88 < 187)$
- Cipher M:
  - $C = M^e \, mod \, n = 88^7 \, mod \, 187 = 11$
- Decipher C:
  - $M = C^d \, mod \, n = 11^{23} \, mod \, 187 = 88$

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RSA-CRT

- In practice, the public exponent e is chosen to optimize the (encryption) exponentiation operation
- The private exponent d is unfortunately not as convenient as the public one
- We can use the Chinese Remained Theorem (CRT) to optimize also the decryption function

RSA-CRT – Key generation

- Generate all the parameters from RSA textbook schema as seen before (assuming $p > q$)
- Precompute (just once) the following private values:
  - $d_p = d \ (mod \ p - 1)$
  - $d_q = d \ (mod \ q - 1)$
  - $q_{inv} = q^{-1} \ (mod \ p)$

RSA-CRT – Decryption

- To decrypt a ciphertext c we have to execute:
  - $m_1 = c^{d_p} \ (mod\ p)$
  - $m_2 = c^{d_q} \ (mod\ p)$
  - $h = q_{inv} \times (m_1 - m_2) \ (mod\ p)$
  - $m = m_2 + h \times q \ (mod\ n)$
- In practice RSA-CRT is four time more efficiently than the standard decryption operation $m = c^d \ (mod\ n)$
- This thanks to the exponent used (d_p and d_q) which are smaller than d

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Choose parameters

- Choose the correct parameters is essential for the security of RSA
- Some example recommendations for the primes:
  - Primes p and q should be at least 1024 bits
  - Same primes should not be reused for different keys
  - Values of p and q should not be too close to each others

Choose parameters

- The public exponent $e = 65537$ is commonly choose
- This is mainly due to the exponentiation by squaring algorithm used to evaluate $c = m^e \ (mod\ n)$:
- Smaller public exponent (e.g., $e = 3$) are vulnerable

What can go wrong?

- In the next lecture we will present different vulnerabilities, mainly due to some bad choice in the parameters
- For example:
  - How to factorize public modulo
  - Standard attacks in vulnerable scenarios
  - Attacks with RSA oracles

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