Department of mathematics

MA102 Mathematics II

Academic Year 2015 - 2016

Tutorial & Additional Problem Set - 2

Date of Tutorial Class: August 11, 2015

SECTION - A (for Tutorial – 2)

1. Under the mapping $f(z) = \dfrac{1}{z}$, find the image of each of the following regions in the complex plane.

   (i) $\{z : |z| = r\}$, (ii) $\{z : -\pi/2 < Arg(z) < \pi/2\}$, (ii) $\{z : |z-1| = 1\}$.

2. Find (i) $\displaystyle\lim_{z\to\infty} \frac{4z^2}{(z-1)^2}$, (ii) $\displaystyle\lim_{z\to 1} \frac{1}{(z-1)^3}$, (iii) $\displaystyle\lim_{z\to\infty} \frac{z^2+1}{z-1}$.

3. The following functions are defined for $z \neq 0$. Which of these functions can be defined at $z = 0$ so that they become continuous at $z = 0$.

   (a) $\dfrac{\Re(z)}{|z|}$ (b) $\dfrac{z}{|z|}$ (c) $\dfrac{z\Re(z)}{|z|}$ (d) $\dfrac{\Re(z^2)}{|z|^2}$ (e) $\dfrac{z^2}{|z|}$

4. Let $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ for $z = x+iy \neq 0$ and $f(0) = 0$. Show that $f(z)$ is continuous at origin but $f'(0)$ does not exist.

5. Show that $g(z) = (3x^2 + 2x - 3y^2 - 1) + \; i\,(6xy + 2y)$ is analytic at all points in the complex plane. Write this function in terms of z.

6. If $f(z)$ is analytic in a domain D, then show that $\overline{f(\overline{z})}$ is analytic in a domain $\overline{D} = \{z \in \mathbb{C} \; : \; \overline{z} \in D\}$.

7. Let $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ be the Laplacian operator. If $f(z)$ is analytic in $\mathbb{C}$, then prove the following:

   (i) $\nabla^2\{|f(z)|^2\} = 4\,|f'(z)|^2$ (ii) $\nabla^2\,\{\log|f(z)|\} = 0$

8. Find the analytic function $f(z) = u(x,\, y) + \; i\; v(x,\, y)$ given the following:
   (First verify that they are harmonic functions)
   (a) $u(x,\, y) = y^3 - 3x^2y$ (b) $u(x,\, y) - v(x,\, y) = (x-y)(x^2 + 4xy + y^2)$

SECTION - B (Supplementary Problems)

(These are for your own practice. They will not be discussed in Tutorials.)

9. Under the mapping $f(z) = z^2$, find the image of the region $\{z : Re(z) > 0, Im(z) > 0\}$ in the complex plane.

10. Let $f(z) = z^2/|z|^2$.

    (a) Find the limit of $f(z)$ as $z = (x+iy) \to 0$ along the line $y = x$.

    (b) Find the limit of $f(z)$ as $z = (x+iy) \to 0$ along the line $y = 2x$.

    (c) Find the limit of $f(z)$ as $z = (x+iy) \to 0$ along the path $y = x^2$.

    (d) What can you conclude about the limit of $f(z)$ as $z \to 0$.

11. Using definition show that (i) $\lim_{z \to 0} \frac{|z|^2}{z} = 0$, (ii) $\lim_{z \to i}(z^2 + 4) = 3$.

12. Let $f(z) = \left(x^3 y(y - i\,x)\right) / \left(x^6 + y^2\right)$ for $z = x + iy \neq 0$ and $f(0) = 0$. Examine whether the function $f(z)$ is differentiable at $z = 0$? Check whether $f(z)$ satisfies the Cauchy-Riemann (CR) equations at $z = 0$?

13. Show that the function $f(z) = xy + \ i\,y$ is continuous everywhere, but not analytic in $\mathbb{C}$.

14. Let $f(z) = \left(x^{(4/3)} y^{(5/3)} + i\,x^{(5/3)} y^{(4/3)}\right) / \left(x^2 + y^2\right)$ for $z = x + iy \neq 0$ and $f(0) = 0$. Show that CR equations hold true at $z = 0$ but that f is not differentiable at this point.

15. Let $f(z) = z^3$. For $z_1 = 1$ and $z_2 = i$, show that there does not exist any point c on the line $y = 1 - x$, joining z_1 and z_2, such that $\frac{f(z_1) - f(z_2)}{z_1 - z_2} = f'(c)$
(i.e., Mean Value Theorem does not extend to complex derivatives.)

16. Let a function $f(z) = u(x,\ y) + i\,v(x,\ y)$ be analytic in a domain D. Prove that $f(z)$ must be constant in D if any one of the following holds:
(a) $f'(z) = 0$ throughout in D
(b) $f(z)$ is real-valued for all z in D
(c) $f(z)$ is purely imaginary valued for all z in D
(d) $\overline{f(z)}$ is analytic in D
(e) $|f(z)|$ is constant in D
(f) $\arg(f(z))$ is constant in D
(g) $\Re(f(z)) = u(x,\ y)$ is constant in D
(h) $\Im(f(z)) = v(x,\ y)$ is constant in D
(i) $\Re(f(z)) = \Im(f(z))$ for all $z \in D$
(j) $\Re(f(z)) = (\Im(f(z)))^2$ for all $z \in D$
(k) $k_1 \Re(f(z)) + k_2 \Im(f(z)) = k_3$ for all $z \in D$, where $k_1,\ k_2,\ k_3$ are constants.
(l) $v(x,\ y)$ is a harmonic conjugate of $u(x,\ y)$ in D and also $u(x,\ y)$ is harmonic conjugate of $v(x,\ y)$ in D.

17. Let $f(z) = z^{(1/2)} = r^{(1/2)}(\cos(\theta/2) + \ i\ \sin(\theta/2))$ where $r > 0$ and $-\pi < \theta < \pi$. Verify Cauchy-Riemann equations in polar form for $f(z)$.

18. Let v be a harmonic conjugate of u. Show that $h = u^2 - v^2$ is a harmonic function.