MA 201: Partial Differential Equations
Lecture - 1

Let us recall the general form of the nth order ordinary differential equation(ODE):

$$F(x, y(x), y'(x), y''(x), \cdots, y^{(n)}(x)) = 0, \tag{1}$$

where $y'(x) = \frac{dy}{dx},\ y''(x) = \frac{d^2y}{dx^2}, \cdots, y^{(n)}(x) = \frac{d^ny}{dx^n}$.

Facts:

- In an ODE, there is only one independent variable x so that all the derivatives appearing in the equation are ordinary derivatives of the unknown function $y(x)$.
- The order of an ODE is the order of the highest derivative that occurs in the equation.
- Equation (1) is linear if F is linear in $y, y', y'', \ldots, y^{(n)}$, with the coefficients depending on the independent variable x.

Definition (Partial Diff. Equations)

A partial differential equation (PDE) for a function $u(x_1, x_2, \ldots, x_n)$ $(n \geq 2)$ is a relation of the form

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_1 x_1}, u_{x_1 x_2}, \ldots,) = 0, \tag{2}$$

where F is a given function of the independent variables $x_1, x_2, \ldots, x_n$; of the unknown function u and of a finite number of its partial derivatives.

Definition (Solution of a PDE)

A function $\phi(x_1, \ldots, x_n)$ is a solution to (2) if ϕ and its partial derivatives appearing in (2) satisfy (2) identically for $x_1, \ldots, x_n$ in some region $\Omega \subset \mathbb{R}^n$.

The order of an equation: The order of a PDE is the order of the highest derivative appearing in the equation. If the highest derivative is of order m, then the equation is said to be order m.

$$u_t - u_{xx} = f(x, t) \text{ (second-order equation)}$$
$$u_t + u_{xxx} + u_{xxxx} = 0 \text{ (fourth-order equation)}$$

Examples.

- $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},\ k \in \mathbb{R}^+$: 2nd order equation in two variables x and t.
- $\left(\frac{\partial u}{\partial x}\right)^3 + \frac{\partial u}{\partial t} = 0$: 1st order third degree equation in two variables x and t.
- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} = 0$: 1st order equation in three variables x, y and t.
- $\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial u}{\partial t} = f(x, y, t)$: 3rd order equation in three variables x, y and t.
- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$: 2nd order equation in three variables x, y and z.

Definition (Classification)

A PDE of order m is said to be

- LINEAR if F is linear in the unknown function u and its partial derivatives, with coefficients depending only on the independent variables $x_1, x_2, \ldots, x_n$.

  Ex. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xu + y, \quad y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = xyu + x.$

- QUASI-LINEAR if it is linear in the derivatives of order m with coefficients that depend on $x_1, x_2, \ldots, x_n$ and the derivatives of order $< m$.

  Ex. $(u^2 - xy)\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$

- SEMI-LINEAR if it is quasi-linear and the coefficients of derivatives of order m are functions of the independent variables only.

  Ex. $(x^2 + y^2)\frac{\partial u}{\partial t} + 2xy\frac{\partial u}{\partial x} = 0.$

- NONLINEAR if it is not linear in the derivatives of order m, e.g., $xu_x^2 + yu_y^2 = k$.

Example (Some well-known PDEs)

- The Laplace's equation in n dimensions:

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \text{ (second-order, linear, homogeneous)}$$

- The Poisson equation:

$$\Delta u = f \text{ (second-order, linear, nonhomogeneous)}$$

- The heat equation:

$$\frac{\partial u}{\partial t} - k\Delta u = 0 \ \ (k = \text{const.} > 0) \text{ (second-order, linear, homogeneous)}$$

- The wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2\Delta u = 0 \ \ (c = \text{const.} > 0) \text{ (second-order, linear, homogeneous)}$$

- The Transport equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \text{ (first-order, linear, homogeneous)}$$

- The Burger's equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \text{ (first-order, quasilinear, homogeneous)}$$

What type of equations we will be studying?

- Linear, quasi-linear, and nonlinear first-order PDEs involving two independent variables.
- Linear second-order PDEs in two independent variables.

First-order PDEs: A first order PDE in two independent variables x, y and the dependent variable u can be written in the form

$$F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0. \tag{3}$$

For convenience, set

$$p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y}.$$

Equation (3) then takes the form

$$F(x, y, u, p, q) = 0. \tag{4}$$

First-order PDEs arise in many applications, such as

- Transport of material in a fluid flow.
- Propagation of wave-fronts in optics.

Classification of first-order PDEs

- If (3) is of the form

$$\boxed{a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y)u + d(x,y)}$$

  then it is a **linear** first-order PDE.

- If (3) has the form

$$\boxed{a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y,u)}$$

  then it is a **semilinear** first-order PDE. It is linear in the leading (highest-order) terms $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. However, it need not be linear in u.

- If (3) has the form

$$\boxed{a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)}$$

  then it a **quasi-linear** PDE. Here F is linear in $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ with the coefficients a, b and c depending on x and y as well as u.

- If F is not linear in the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, then (3) is said to be **nonlinear**.

$$\text{Linear PDE} \subsetneq \text{Semi-linear PDE} \subsetneq \text{Quasi-linear PDE} \subsetneq \text{PDE}$$

Example

- $xu_x + yu_y = u$ (**linear**)
- $xu_x + yu_y = u^2$ (**semi-linear**)
- $u_x + (x+y)u_y = xy$ (**linear**)
- $uu_x + u_y = 0$ (**quasi-linear**)
- $xu_x^2 + yu_y^2 = 2$ (**nonlinear**)

How first-order PDEs occur?

Example

The equation

$$x^2 + y^2 + (u - c)^2 = r^2, \tag{5}$$

where r and c are arbitrary constants, represents the set of all spheres whose centers lie on the u-axis. Differentiating (5) with respect to x, we obtain

$$x + (u - c)\frac{\partial u}{\partial x} = 0. \tag{6}$$

Differentiating (5) with respect to y to have

$$y + (u - c)\frac{\partial u}{\partial y} = 0. \tag{7}$$

Eliminating the arbitrary constant c from (6) and (7), we obtain the first-order PDE

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0. \tag{8}$$

How first-order PDEs occur?

- **Two-parameter family of surfaces:** Let

$$f(x, y, u, a, b) = 0 \tag{9}$$

represent two parameters family of surfaces in $\mathbb{R}^3$, where a and b are arbitrary constants.Differentiating (9) with respect to x and y yields a relations

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial u} = 0, \tag{10}$$

$$\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial u} = 0. \tag{11}$$

Eliminating a and b from (9), (10) and (11), we get a relation of the form

$$F(x, y, u, p, q) = 0, \tag{12}$$

which is a PDE for the unknown function u of two independent variables.

- The applications of conservation principles often yield a first-order PDEs.

Unknown function of known functions

Example

Consider a surface described by an equation of the form

$$u = f(x^2 + y^2), \tag{13}$$

where f is an arbitrary function of a known function $g(x, y) = x^2 + y^2$.
All surfaces of revolution about u-axis are characterized by these equations.
Differentiating (13) with respect to x and y, it follows that

$$u_x = 2xf'(g); \qquad u_y = 2yf'(g),$$

where $f'(g) = \dfrac{df}{dg}$. Eliminating $f'(g)$ from the above two equations, we obtain a first-order PDE

$$yu_x - xu_y = 0.$$

Unknown function of known functions

- *Unknown function of a single known function*

  Let
  $$u = f(g), \tag{14}$$
  where f is an unknown function and g is a known function of two independent variable x and y.

  Differentiating (14) with respect to x and y yield the equations
  $$u_x = f'(g)g_x \tag{15}$$
  and
  $$u_y = f'(g)g_y, \tag{16}$$
  respectively. Eliminating $f'(g)$ from (19) and (20), we obtain
  $$g_y u_x - g_x u_y = 0,$$
  which is a first-order linear PDE for u.

Unknown functions of two known functions

A generalization of $u = f(g)$ is

$$F(u, v) = 0, \tag{17}$$

where u and v are known functions of x, y and z, and F is any function of u and v.

Differentiating (17) w.r.t. x and y, respectively, we get

$$\begin{aligned} \frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right\} + \frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right\} &= 0, \\ \frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right\} + \frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right\} &= 0. \end{aligned}$$

If we, now, eliminate $\partial F/\partial u$ and $\partial F/\partial v$ from these equations, we obtain the equation

$$p\ \frac{\partial(u, v)}{\partial(y, z)} + q\ \frac{\partial(u, v)}{\partial(z, x)} = \frac{\partial(u, v)}{\partial(x, y)}, \tag{18}$$

which is a linear first-order pde.

- *Unknown functions of two known functions*

  Let

$$u = f(x - ay) + g(x + ay), \tag{19}$$

where $a > 0$ is a constant. With $v(x, y) = x - ay$ and $w(x, y) = x + ay$, we write (19) as

$$u = f(v) + g(w). \tag{20}$$

Differentiating (20) w. r. t. x and y yields

$$\begin{aligned} p = u_x &= f'(x - ay) + g'(x + ay), \\ q = u_y &= -af'(x - ay) + ag'(x + ay) \end{aligned}$$

Eliminating $f'(v)$ and $g'(w)$, we get

$$q_y = a^2 p_x.$$

In terms of u, the above PDE is the well-known wave equation

$$u_{yy} = a^2 u_{xx}.$$

Example (Geometrical problem)

Suppose $u = u(x, y)$ be a surface with the following property: At any point $P(x_0, y_0, u(x_0, y_0))$ on the surface, the plane tangent to the surface passes through the origin. What differential equation the surface should satisfy?

The equation of the tangent plane at $(x_0, y_0, u(x_0, y_0))$ is

$$u_x(x_0, y_0)(x - x_0) + u_y(x_0, y_0)(y - y_0) - \Big(u - u(x_0, y_0)\Big) = 0.$$

Since this plane passes through the origin $(0, 0, 0)$, we have

$$-u_x(x_0, y_0)x_0 - u_y(x_0, y_0)y_0 + u(x_0, y_0) = 0. \tag{21}$$

For equation (21) to hold for all (x_0, y_0) in the domain of u, u must satisfy

$$xu_x + yu_y - u = 0,$$

which is a first-order PDE.

Cauchy's problem or IVP for first-order PDEs: If

(a) $x_0(\mu), y_0(\mu)$ and $z_0(\mu)$ are functions which, together with their first derivatives, are continuous in the interval M defined by $\mu_1 < \mu < \mu_2$; and

(b) $F(x, y, z, p, q)$ is a continuous function of x, y, z, p and q in a certain region U of the $xyzpq$ space,

then it is required to establish the existence of a function $\phi(x, y)$ with the following properties:

1. $\phi(x, y)$ and its partial derivatives with respect to x and y are continuous functions of x and y in a region R of the xy space.
2. For all values of x and y lying in R, the point $\Big(x, y, \phi(x, y), \phi_x(x, y), \phi_y(x, y)\Big)$ lies in U and
$$F\left(x, y, \phi(x, y), \phi_x(x, y), \phi_y(x, y)\right) = 0.$$
3. For all μ belonging to the interval M, the point $\left(x_0(\mu), y_0(\mu)\right)$ belongs to the region R, and
$$\phi\Big(x_0(\mu), y_0(\mu)\Big) = z_0(\mu).$$

Cauchy's problem or IVP for first-order PDEs:

From the point of view of geometry: whether there exists a surface $z = \phi(x, y)$ which passes through the curve Γ whose parametric equation is

$$x = x_0(\mu), \quad y = y_0(\mu), \quad z = z_0(\mu), \quad \mu \in M,$$

and at every point of which the direction $(p, q, -1)$ of the normal is such that

$$F(x, y, z, p, q) = 0.$$

This is one of the various forms of Cauchy's problem.

The curve $\Gamma : \big(x_0(\mu), y_0(\mu), z_0(\mu)\big), \mu \in M$, is called the initial curve of the problem and the equation

$$\phi\Big(x_0(\mu), y_0(\mu)\Big) = z_0(\mu).$$

is called the initial condition (or side condition) of the problem.

One Existence Theorem:

Theorem

If $g(y)$ and all its derivatives are continuous for $|y - y_0| < \delta$, if x_0 is a given number, and $z_0 = g(y_0),\ q_0 = g'(y_0)$, and if $F(x, y, z, q)$ and all its partial derivatives are continuous in a region S defined by

$$|x - x_0| < \delta,\ |y - y_0| < \delta,\ |q - q_0| < \delta,$$

then there exists a unique function $\phi(x, y)$ such that
(a) $\phi(x, y)$ and all its partial derivatives are continuous in a region R defined by $|x - x_0| < \delta_1,\ |y - y_0| < \delta_1$;
(b) For all $(x, y) \in R,\ \ z = \phi(x, y)$ is a solution of the equation

$$\frac{\partial z}{\partial x} = F\left(x, y, z, \frac{\partial z}{\partial y}\right);$$

(c) For all values of y in the interval $|y - y_0| < \delta_1$, $\phi(x_0, y) = g(y)$.

The above theorem is due to Sonia Kowalewski and is also known as Cauchy-Kowalewski theorem.

- **Well-posed Problem** (In the sense of Hadamard)

The Cauchy's problem (PDE + side condition) is said to be well-posed if it satisfies the following criteria:

1. The solution must exist.
2. The solution should be unique.
3. The solution should depend continuously on the initial and/or boundary data.

If one or more of the conditions above does not hold, we say that the problem is ill-posed.