MA 201: Integral Transforms
Slides - 15

Fourier Integrals and Transforms

Fourier Integral

Recall that a periodic function f of period $2L$ has a Fourier series:

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi t}{L} + B_n \sin \frac{n\pi t}{L} \right] \tag{1}$$

where

$$\begin{aligned} A_n &= \frac{1}{L} \int_{-L}^{L} f(s) \cos \frac{n\pi s}{L}\, ds, \quad n = 0, 1, 2, 3, \ldots, \\ B_n &= \frac{1}{L} \int_{-L}^{L} f(s) \sin \frac{n\pi s}{L}\, ds, \quad n = 1, 2, 3, \ldots. \end{aligned}$$

(1) can be written in complex form as:

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{\mathrm{i}n\pi t/L}, \tag{2}$$

where $C_n = \dfrac{1}{2L} \displaystyle\int_{-L}^{L} f(s) e^{-\mathrm{i}n\pi s/L}\, ds, \quad n = 0, \pm 1, \pm 2, \ldots.$

Fourier Integral

For $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-L}^{L} f(s) e^{-i\left(\frac{n\pi}{L}\right)s} \, ds \right] e^{i\left(\frac{n\pi}{L}\right)t} \left(\frac{\pi}{L}\right) \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-L}^{L} f(s) e^{-i(n\Delta\sigma)s} \, ds \right] e^{i(n\Delta\sigma)t} \Delta\sigma \end{aligned}$$

where $\Delta\sigma = \pi/L$. Using integral as a limit of sums, i.e.,

$$\int_{-\infty}^{\infty} g(\sigma) d\sigma = \lim_{\Delta\sigma \to 0} \sum_{-\infty}^{\infty} g(n\Delta\sigma)\Delta\sigma.$$

we get

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[e^{i\sigma t} \int_{-\infty}^{\infty} f(s) e^{-i\sigma s} \, ds \right] d\sigma. \tag{3}$$

(3) is known as the (complex) Fourier integral representation of f.

Fourier Integral

Note: (3) is valid if

(a) $f(t)$ is defined and is piecewise continuous in every finite interval .
(b) the improper integral $\int_{-\infty}^{\infty} |f(t)| dt$ exists.

Fourier Cosine and Sine Integrals

Equation (3) gives

$$
\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\left(\int_{-\infty}^{\infty} e^{-i\sigma(s-t)}d\sigma\right) ds \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\left(\int_{-\infty}^{\infty} \cos\sigma(s-t)d\sigma\right) ds \\
&\qquad \text{(since } \sin\sigma(s-t) \text{ is an odd function of } \sigma\text{)} \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\left(\int_{0}^{\infty} \cos\sigma(s-t)d\sigma\right) ds. \\
&\qquad \text{(since } \cos\sigma(s-t) \text{ is an even function of } \sigma\text{)}
\end{aligned}
$$

Fourier cosine and sine integrals

Expanding $\cos\sigma(s-t)$ we get

$$\begin{aligned} f(t) &= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^0 f(s)\cos\sigma s\cos\sigma t\ ds + \int_0^\infty f(s)\cos\sigma s\cos\sigma t\ ds\right] d\sigma \\ &+ \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^0 f(s)\sin\sigma s\sin\sigma t\ ds + \int_0^\infty f(s)\sin\sigma s\sin\sigma t\ ds\right] d\sigma \end{aligned} \tag{4}$$

Now, consider f as a function defined on $[0,\infty)$ and extend f to $\mathbb{R}$ as an even function: $f(-a) = f(a)$ for $a > 0$. Then, for $t \geq 0$,

$$f(t) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(s)\cos\sigma s\cos\sigma t\ ds\ d\sigma. \tag{5}$$

Similarly, extending f to an odd function, we get

$$f(t) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(s)\sin\sigma s\sin\sigma t\ ds\ d\sigma. \tag{6}$$

Equation (5) (resp. (6)) is called the Fourier cosine (resp. sine) integral representation of f.

Complex Fourier Transform

Recalling equation (3):

$$f(t) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} e^{i\sigma t} \int_{-\infty}^{\infty} f(s) e^{-i\sigma s} \, ds \right] d\sigma,$$

The *Fourier transform* of a function $f \in L^1(\mathbb{R})$ is defined by

$$\mathcal{F}\{f(t)\} = g(\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\sigma t} f(t) \, dt. \tag{7}$$

The *inverse Fourier transform* is defined as

$$f(t) = \mathcal{F}^{-1}\{g(\sigma)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\sigma) e^{i\sigma t} \, d\sigma. \tag{8}$$

Thus, the Fourier integral representation of f is

$$f(t) = \mathcal{F}^{-1}\Big\{\mathcal{F}\{f(t)\}\Big\}.$$

Fourier cosine transform

Recalling equation (5):

$$f(t) = \frac{2}{\pi} \int_0^\infty \cos \sigma t \left(\int_0^\infty f(s) \cos \sigma s \, ds \right) \, d\sigma,$$

we can define the *Fourier cosine transform* of f as

$$\mathcal{F}_c\{f(t)\} = g_c(\sigma) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(s) \cos \sigma s \, ds. \tag{9}$$

The *inverse Fourier cosine transform* is defined as

$$f(t) = \mathcal{F}_c^{-1}\{g_c(\sigma)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty g_c(\sigma) \cos \sigma t \, d\sigma. \tag{10}$$

The cosine transforms are valid for the entire real line, if f is even.

Fourier sine transform

Recalling equation (6):

$$f(t) = \frac{2}{\pi} \int_0^\infty \sin \sigma t \left(\int_0^\infty f(s) \sin \sigma s \, ds \right) \, d\sigma,$$

we can define the *Fourier sine transform* of f as

$$\mathcal{F}_s\{f(t)\} = g_s(\sigma) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \sigma t \, dt. \qquad (11)$$

The *inverse Fourier sine transform* is defined as

$$f(t) = \mathcal{F}_s^{-1}\{g_s(\sigma)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(\sigma) \sin \sigma t \, d\sigma. \qquad (12)$$

The sine transforms are valid for the entire real line, if f is odd.

Some Properties of Fourier transform

Theorem (Linearity)

If $\mathcal{F}\{f_1(t)\} = g_1(\sigma), \mathcal{F}\{f_2(t)\} = g_2(\sigma)$, then

$$\begin{aligned}\mathcal{F}\{(c_1 f_1 \pm c_2 f_2)(t)\} &= c_1 \mathcal{F}\{f_1(t)\} \pm c_2 \mathcal{F}\{f_2(t)\} \\ &= c_1 g_1(\sigma) \pm c_2 g_2(\sigma),\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}^{-1}\{(c_1 g_1 \pm c_2 g_2)(\sigma)\} &= c_1 \mathcal{F}^{-1}\{g_1(\sigma)\} \pm c_2 \mathcal{F}^{-1}\{g_2(\sigma)\} \\ &= c_1 f_1(t) \pm c_2 f_2(t),\end{aligned}$$

where c_1 and c_2 are constants.

Some Properties of Fourier transform

Theorem (Shifting)

If $\mathcal{F}\{f(t)\} = g(\sigma)$, then

$$\mathcal{F}\{f(t-a)\} = e^{-\mathrm{i}\sigma a} g(\sigma).$$

Proof: By definition,

$$\begin{aligned}
\mathcal{F}\{f(t-a)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma t} f(t-a) dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma(\xi+a)} f(\xi) d\xi, \quad \text{by taking } t-a=\xi \\
&= e^{-\mathrm{i}\sigma a} g(\sigma)
\end{aligned}$$

$\square$

Some Properties of Fourier transform

Theorem (Scaling)

If $\mathcal{F}\{f(t)\} = g(\sigma)$, then

$$\mathcal{F}\{f(at)\} = \frac{1}{a} g(\sigma/a).$$

Proof: By definition,

$$\begin{aligned}
\mathcal{F}\{f(at)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma t} f(at) dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma(\xi/a)} f(\xi) d\xi/a, \quad \text{by taking } at = \xi \\
&= \frac{1}{a} g(\sigma/a).
\end{aligned}$$

$\square$

Some Properties of Fourier transform

Theorem (Translation)

If $\mathcal{F}\{f(t)\} = g(\sigma)$, then

$$\mathcal{F}\{e^{\mathrm{i}at} f(t)\} = g(\sigma - a).$$

Proof: By definition,

$$\begin{aligned}
\mathcal{F}\{e^{\mathrm{i}at} f(t)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma t} e^{\mathrm{i}at} f(t) dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}(\sigma - a)t} f(t) dt \\
&= g(\sigma - a)
\end{aligned}$$

$\square$

Theorem

If $\mathcal{F}\{f(t)\} = g(\sigma)$, $f(t)$ is continuously differentiable and $\lim\limits_{t\to\pm\infty} f(t) = 0$, then

$$\mathcal{F}\{f'(t)\} = \mathrm{i}\sigma g(\sigma).$$

Proof: By definition,

$$\begin{aligned} \mathcal{F}\{f'(t)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma t} f'(t) dt \\ &= \frac{1}{\sqrt{2\pi}} \left\{ [f(t)e^{-\mathrm{i}\sigma t}]_{-\infty}^{\infty} + i\sigma \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma t} f(t) dt \right\} \\ &= \mathrm{i}\sigma g(\sigma). \end{aligned}$$

Theorem (Extension of the above Theorem)

If $f(t)$ is continuously n-times differentiable and $f^{(k)}(t) \to 0$ as $|t| \to \infty$ for $k = 1, 2, \ldots, (n-1)$, then

$$\mathcal{F}\{f^{(n)}(t)\} = (\mathrm{i}\sigma)^n \mathcal{F}\{f(t)\} = (\mathrm{i}\sigma)^n g(\sigma).$$

Some examples of Fourier integrals

Example: Find the Fourier integral representation of the function

$$f(t) = \begin{cases} e^{at}, & t < 0, \\ e^{-at}, & t > 0. \end{cases} \quad a > 0$$

Solution. f is non-periodic. The complex Fourier transform of $f(t)$ is

$$\begin{aligned} \mathcal{F}\{f(t)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\sigma t}\, dt \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(a-i\sigma)t}\, dt + \int_{0}^{\infty} e^{-(a+i\sigma)t}\, dt \right] \\ &= \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+\sigma^2} = \sqrt{\frac{2}{\pi}}\, \frac{a}{a^2+\sigma^2}. \end{aligned}$$

Some examples of Fourier integrals

Taking the inverse

$$
\begin{aligned}
f(t) &= \mathcal{F}^{-1}\left\{\sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+\sigma^2}\right\} = \sqrt{\frac{2}{\pi}}\,\frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{i\sigma t}}{a^2+\sigma^2}d\sigma \\
&= \frac{2a}{\pi}\int_0^{\infty}\frac{\cos\sigma t}{a^2+\sigma^2}\,d\sigma.
\end{aligned}
$$

Some examples of Fourier integrals

Example. Find the Fourier integral representation of the function

$$f(t) = \begin{cases} \sin t, & t^2 < \pi^2, \\ 0, & t^2 > \pi^2. \end{cases}$$

Solution. The function f is odd. We find its sine Fourier integral representation.

$$\begin{aligned} \mathcal{F}_s\{f(t)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \sigma t \; dt \;=\; \sqrt{\frac{2}{\pi}} \int_0^\pi \sin t \sin \sigma t \; dt \\ &= \sqrt{\frac{2}{\pi}} \frac{\sin \sigma\pi}{1-\sigma^2}. \end{aligned}$$

Taking the inverse

$$f(t) \;=\; \mathcal{F}_s^{-1}\{g(\sigma)\} \;=\; \frac{2}{\pi} \int_0^\infty \frac{\sin \sigma\pi \sin \sigma t}{1-\sigma^2} \; d\sigma.$$

Some examples of Fourier integrals

Exercise. Find the Fourier integral representation of the following non-periodic function

$$f(t) = \begin{cases} 0, & -\infty < t < -1, \\ -1, & -1 < t < 0, \\ 1, & 0 < t < 1, \\ 0, & 1 < t < \infty. \end{cases}$$

Answer. $f(t) = \dfrac{2}{\pi} \displaystyle\int_0^\infty \frac{(1 - \cos\sigma)}{\sigma} \sin \sigma t \, d\sigma.$ (Work out the details.)