Functions, Limit and Continuity

Functions of a complex variable

- Let $S \subseteq \mathbb{C}$. A **complex valued function** f on S is a function $f : S \to \mathbb{C}$.
- We write $w = f(z)$. The set S is called the **domain of** f and the set $\{f(z) : z \in S\}$ is called **range** of f.
- Suppose $z = x + iy$, that $f(z) = w = u + iv$. Then

$$f(z) = f(x + iy) = u(x, y) + iv(x, y),$$

  i.e., u and v are real valued functions of two real variables.
- **Ex.** If $w = z^2$, then

$$u(x, y) + iv(x, y) = (x + iy)^2 = (x^2 - y^2) + i \cdot 2xy,$$

  i.e., $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$.
- (In polar form): Suppose $z = re^{i\theta}$ and $f(z) = w = u + iv$. We can write

$$f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta).$$

- **Ex.** For $w = z^2$, $u(r, \theta) + iv(r, \theta) = z^2 = r^2 e^{i2\theta}$ so that

$$u(r, \theta) = r^2 \cos 2\theta, \ \ v(r, \theta) = r^2 \sin 2\theta.$$

Visualizing a complex function

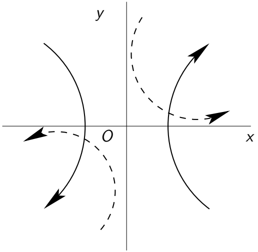
- A real valued function of a real variable is visualized with its graph. However, graph of a complex function is not a curve.
- **Example:** Consider $w = f(z) = \overline{z}$, defined on $\mathbb{C}$. Image of each point is the reflection about the real axis. What is the image of the set $\{z : |z - i| \leq 2\}$?
- For visualizing a complex function $w = f(z)$ we often need two planes.
- Take xy-plane as z-plane, and the domain is on this plane.
  Take uv-plane as w-plane, and the codomain is on this plane.
- $w = f(x)$ is visualized by the images of of sets and curves under the mapping.
- **Example:** Consider $w = f(z) = z^2$, defined on $\mathbb{C}$.
  - What is the image of a point z? Use $z = re^{i\theta}$.
  - image of the set $\{z = e^{i\theta} : 0 \leq \theta \leq \pi/2\}$?
  - of the set $\{z = re^{i\theta} : 0 \leq \theta \leq \pi/2\}$? If $r < 1$? $r > 1$?
  - of the set $\{z = re^{i\theta} : 0 < r \leq r_0,\ 0 \leq \theta \leq \pi/2\}$?
  - of the set $\{z = re^{i\theta} : 0 < r \leq r_0,\ 0 \leq \theta \leq \pi\}$?

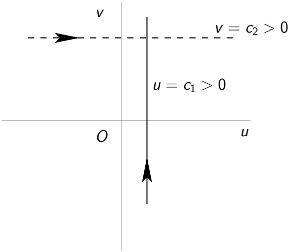
Example: Again consider $w = f(z) = z^2$, defined on $\mathbb{C}$. Note that

$$w = u(x,y) + iv(x,y) = (x+iy)^2 = (x^2 - y^2) + i \cdot 2xy,$$

i.e., $u(x,y) = x^2 - y^2, \;\; v(x,y) = 2xy.$

- What is the image of the hyperbola $x^2 - y^2 = c_1, \; c_1 > 0$?





- It is a vertical line in w-plane given by $u = c_1$.
- Note: $v = \pm 2y\sqrt{y^2 + c_1}, \;\; y \in \mathbb{R}$.
- Each branch of the hyperbola maps the line in one-one manner.
- As you travel upward (downward) on the branch right (left) to y-axis, you travel upward on the image.

Visualizing a complex function

Example (contd.): For $w = f(z) = z^2$,

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.$$

- What is the image of the hyperbola $2xy = c_2, \; x > 0, c_2 > 0$?

  - It is the horizontal line $v = c_2$.
  - Note: $u = x^2 - \frac{c_2^2}{x^2}$, $\lim_{x \to 0^+} u = -\infty$, $\lim_{x \to \infty} u = \infty$.
  - The hyperbola mapped to the line in one-one manner. In which orientation?

- What is the image of $\{z = x + iy : 0 < x^2 - y^2 \leq c\}$?
- What is the image of $\{z = x + iy : 0 < 2xy \leq c, x > 0\}$?

Limit of a function

- **Limit of a function:** Suppose f is a complex valued function defined on a deleted neighborhood of z_0. We say f has a **limit** a as $z \to z_0$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - a| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

  We then write

$$\lim_{z \to z_0} f(z) = a.$$

- **Example:** $$\lim_{z \to i} \frac{2i}{z} = 2.$$

  Take any ϵ and draw diagram and see:

  - $w = f(z) = \dfrac{2i}{z}$ is defined in a deleted neighborhood of i,
  - $|f(z) - 2| = \dfrac{2|z-i|}{|z|} < \epsilon$ if $|z - i| < \dfrac{\epsilon}{4}$ and $|z| > \frac{1}{2}$,
  - If $\delta = \min\{\frac{\epsilon}{4}, \frac{1}{2}\}$, then $|f(z) - 2| < \epsilon$ when $|z - i| < \delta$.
- The limit of a function $f(z)$ at a point z_0, if exists, is **unique**.

Limit of a function

- If $f(z) = u(x,y) + iv(x,y)$ and $z_0 = x_0 + iy_0$, pause then

$$\lim_{z\to z_0} f(z) = u_0 + iv_0 \iff \begin{cases} \lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 & \text{and} \\ \lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0. \end{cases}$$

- **Ex.** $\lim\limits_{z\to z_0} z^2 = z_0^2$.
- The point z_0 can be approached from **any direction**. If the limit $\lim\limits_{z\to z_0} f(z)$ exists, then $f(z)$ must approach a **unique** limit, no matter how z approaches z_0.
- If the limit $\lim\limits_{z\to z_0} f(z)$ is different for different path of approaches then $\lim\limits_{z\to z_0} f(z)$ does not exists.
- **Ex.** $\lim\limits_{z\to 0} \dfrac{z}{\bar{z}}$ does not exist.

  Take $z = (x,0) \to 0$ and $z = (0,y) \to 0$ separately and see.

Limit contd....

Let f, g be complex valued functions with $\lim_{z \to z_0} f(z) = \alpha$ and $\lim_{z \to z_0} g(z) = \beta$. Then,

- $\lim_{z \to z_0} [f(z) \pm g(z)] = \lim_{z \to z_0} f(z) \pm \lim_{z \to z_0} g(z) = \alpha \pm \beta$.
- $\lim_{z \to z_0} [f(z) \cdot g(z)] = \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} g(z) = \alpha\beta$.
- $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)} = \dfrac{\alpha}{\beta} \quad (\text{if} \quad \beta \neq 0)$.
- $\lim_{z \to z_0} Kf(x) = K \lim_{z \to z_0} f(z) = K\alpha \quad \forall \quad K \in \mathbb{C}$.

Continuous functions

- **Continuity at a point:** Let D be a domain or a region. A function $f : D \to \mathbb{C}$ is continuous at a point $z_0 \in D$ if for for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta.$$

  In other words, f is is continuous at a point z_0 in the domain if the following conditions are satisfied.

    - $\lim\limits_{z \to z_0} f(z) \quad$ exists,
    - $\lim\limits_{z \to z_0} f(z) = f(z_0)$.

- A function f is continuous on D if it is continuous at each and every point in D.
- A function $f : D \to \mathbb{C}$ is continuous at a point $z_0 \in D$ if and only if $u(x, y) = \text{Re } (f(z))$ and $v(x, y) = \text{Im } (f(z))$ are continuous at z_0.

Continuity

Let $f, g : D \subseteq \mathbb{C} \to \mathbb{C}$ be continuous functions at the point $z_0 \in D$. Then

- $f \pm g,\ fg,\ Kf\ (k \in \mathbb{C}), \dfrac{f}{g}\ (g(z_0) \neq 0)$ are continuous at z_0.
- Composition of continuous functions is continuous.
- $\overline{f(z)}, |f(z)|, \text{Re}\ (f(z))$ and $\text{Im}\ (f(z))$ are continuous.
- If a function $f(z)$ is continuous and nonzero at a point z_0, then there is a $\epsilon > 0$ such that $f(z) \neq 0, \ \ \forall\ z \in B(z_0, \epsilon)$.
- Continuous image of a compact set (closed and bounded set) is compact.