Department of mathematics

MA201 Mathematics III

Academic Year 2015 - 2016

Tutorial & Additional Problem Set - 4

Date of Tutorial Class: August 25, 2015

Complex Integration, Cauchy's Integral Theorem and its Consequences

SECTION - A (for Tutorial – 4)

1. Let $[z_1,\ z_2]$ denote the line segment joining z_1 and z_2 and oriented from z_1 to z_2. Compute $\displaystyle\int_{[z_1,\, z_2,\, z_3]} \overline{z}\, dz$, and $\displaystyle\int_{[z_1,\, z_3]} \overline{z}\, dz$, where $z_1 = -1$, $z_2 = 1$ and $z_3 = i$.

2. Evaluate $\displaystyle\int_C |z|\, \overline{z}\, dz$ where C is a positively oriented simple closed contour consists of the line segment from $-2i$ to $2i$ and the semi circle $|z| = 2$ in the second and third quadrants.

3. Without using the Cauchy integral formula, evaluate $\displaystyle\int_C (z - z_0)^{n-1}\, dz$ where C is any negatively oriented simple closed contour that encloses the point z_0 and n is any integer.

4. If C is the boundary of the triangle with vertices at the points 0, $3i$ and -4 oriented in the counterclockwise direction then show that $\left| \displaystyle\int_C (e^z - \overline{z})\ dz \right| \leq 60.$

5. Evaluate $\displaystyle\int_C \frac{z^2 - 4}{z^2 + 4}\, dz$ if C is a simple closed contour described in the counterclockwise direction and if
   (i) the point $2i$ lies inside C and $-2i$ lies outside C,
   (ii) the point $-2i$ lies inside C and $2i$ lies outside C,
   (iii) the points $\pm 2i$ lie outside C,
   (iv) the points $\pm 2i$ lie inside C.

6. Evaluate $\displaystyle\int_C \frac{\cosh z}{(z - i)^{2n+1}}\, dz$ where $C : |z - i| = 1$.

7. Let $g(z)$ be an analytic function in $|z| < 2$. Suppose f is a function of the form
$$f(z) = \frac{a_k}{z^k} + \frac{a_{k-1}}{z^{k-1}} + \cdots + \frac{a_1}{z} + a_0 + g(z) \qquad \text{for } z \in \mathbb{C}$$
where a_i's are complex constants. Compute $\displaystyle\int_{|z|=1} f(z)\, dz$.

8. Let f be an entire function such that $|f(z)| \leq A + B|z|^n$ for all $z \in \mathbb{C}$ where A and B are positive real constants and n is a fixed natural number. Show that f is a polynomial of degree at most n. (It is a generalization of Exercise Problem 1 of Section 50, Brown and Churchill, 7th edition)

9. Let $f(z) = (z + 1)^2$ for $z \in \mathbb{C}$. Let R be the closed triangular region with vertices at the points $z = 0$, $z = 2$ and $z = i$. Find points in R where $|f(z)|$ has its maximum and minimum values.

SECTION - B (Supplementary Problems)

(These are for your own practice. They will not be discussed in Tutorials.)

10. Does Cauchy's theorem hold separately for the real and the imaginary parts of an analytic function $f(z)$. If so, prove that it does, if not give a counter example. (Hint: $f(z) = z$ and $C: \ |z| = 1$).

11. Show that if C is positively oriented simple closed contour, then the area of the region enclosed by C can be written $\dfrac{1}{2i} \displaystyle\int_C \overline{z}\, dz$. (Hint: Apply Green's Theorem)

12. Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial of degree $n \geq 1$ with $a_n \neq 0$. Show that there exists $R > 0$ such that $\dfrac{|a_n|\,|z|^n}{2} \leq |P(z)| < \dfrac{3}{2}\,|a_n|\,|z|^n$ for $|z| \geq R$.

13. Let f be analytic in the disk $D = \{z \in \mathbb{C} \ : \ |z| < 1\}$. Suppose that $|f(z)| \leq 1$ for $z \in D$ and $f(0) = 0$. Show that $|f(z)| \leq |z|$ for $z \in D$ and $|f'(0)| \leq 1$. Moreover, if $|f'(0)| = 1$ or if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$ then show that there is a constant c, $|c| = 1$, such that $f(z) = cz$ for all z in D. (This result is known as *Schwarz's Lemma. See: Saff and Snider Book Page 299 or Conway Book Page 130*).

14. Show that an entire function satisfying $f(z+1) = f(z)$ and $f(z+i) = f(z)$ is a constant.