MA 201: Partial Differential Equations
Lecture - 13

The Laplace Equation (contd.)

Laplace Equation in Polar Coordinates

Two-dimensional Laplace equation in cartesian coordinates:

$$u_{xx} + u_{yy} = 0. \tag{1}$$

In cases when boundary is not rectangular, other curvilinear coordinates are appropriate in many cases. For example, for two dimensional problems if the boundary is a circle, then one uses polar coordinates: $x = r\cos\theta,\ y = r\sin\theta.$ Then,

$$r_x = \cos\theta,\ r_y = \sin\theta,\ \theta_x = -\frac{\sin\theta}{r},\ \theta_x = \frac{\cos\theta}{r}.$$

Assume $u = u(r, \theta)$. Then,

$$u_x = u_r r_x + u_\theta \theta_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}$$

$$u_y = u_r r_y + u_\theta \theta_y = u_r \sin\theta + u_\theta \frac{\sin\theta}{r}$$

Laplace Equation in Polar Coordinates

Therefore,

$$\begin{aligned} u_{xx} &= (u_x)_r r_x + (u_x)_\theta \theta_x \\ &= \frac{\partial}{\partial r}\left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right)\cos\theta + \frac{\partial}{\partial \theta}\left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right)\left(-\frac{\sin\theta}{r}\right) \\ &= \left(u_{rr}\cos\theta - u_{\theta r}\frac{\sin\theta}{r} + u_\theta \frac{\sin\theta}{r^2}\right)\cos\theta \\ &\quad + \left(u_{r\theta}\cos\theta - u_r \sin\theta - u_{\theta\theta}\frac{\sin\theta}{r} - u_\theta\frac{\cos\theta}{r}\right)\left(-\frac{\sin\theta}{r}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} u_{yy} &= \left(u_{rr}\sin\theta + u_{\theta r}\frac{\cos\theta}{r} - u_\theta\frac{\cos\theta}{r^2}\right)\sin\theta \\ &\quad + \left(u_{r\theta}\sin\theta + u_r\cos\theta + u_{\theta\theta}\frac{\cos\theta}{r} - u_\theta\frac{\sin\theta}{r}\right)\left(\frac{\cos\theta}{r}\right). \end{aligned}$$

Thus, $u_{xx} + u_{yy} = 0$ gives

$$\boxed{u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0} \tag{2}$$

Laplace Equation in Cylindrical Coordinates

Three-dimensional Laplace equation in cartesian coordinates:

$$u_{xx} + u_{yy} + u_{zz} = 0. \tag{3}$$

Cylindrical coordinates: (r, θ, z) are linked to the cartesian coordinates by

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z.$$

Exercise. Show that in cylindrical coordinates (3) transforms to

$$\boxed{u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0} \tag{4}$$

Laplace Equation in Spherical Coordinates

Spherical coordinates: (r, θ, ϕ) are linked to the cartesian coordinates by

$$x = r \sin\theta \cos\phi, \;\; y = r \sin\theta \sin\phi, \;\; z = r\cos\theta.$$

In spherical coordinates the three-dimensional Laplace equation (3) transforms to

$$\boxed{\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0.} \qquad (5)$$

Laplace equation in polar coordinates: Solutions

- For a problem involving circular disk, polar coordinates are more appropriate than rectangular coordinates.
- Let us formulate the steady-state heat flow problem in polar coordinates r, θ, where $x = r\cos\theta, y = r\sin\theta$.
- A circular plate of radius a can be simply represented by $r \leq a$ with $0 \leq \theta \leq 2\pi$.
- The unknown temperature inside the plate is now $u = u(r, \theta)$.
- The given temperature on the boundary of the plate is $u(a, \theta) = f(\theta)$, where f is a known function.

Laplace equation in polar coordinates: solutions

we have the following equation:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, r \geq 0, \ 0 \leq \theta \leq 2\pi, \tag{6}$$

There is a periodic boundary condition which is implicit in nature:

$$u(r,\theta) = u(r,\theta + 2\pi). \tag{7}$$

Using the separation of variables method, assume a solution:

$$u(r,\theta) = R(r)T(\theta).$$

Using this in equation (6),

$$R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'' = 0, \quad \text{i.e.,} \quad r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{T''}{T} = 0. \tag{8}$$

Laplace equation in polar coordinates: solutions

Separating the variables

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{T''}{T} = k \tag{9}$$

which give rise to the ODEs:

$$\begin{aligned} r^2R'' + rR' - kR &= 0, \quad (10) \\ T'' + kT &= 0. \quad (11) \end{aligned}$$

If k is negative, then the ODE in $T(\theta)$ has exponential solutions which cannot satisfy periodicity conditions.

Therefore, choose $k = \lambda^2, \ \lambda \geq 0$.
Note that $k = 0$ will produce linear solutions for (11) out of which the constant solutions are acceptable.

Laplace equation in polar coordinates: solutions

Hence the equations reduce to

$$\begin{aligned} r^2R'' + rR' - \lambda^2 R &= 0, \quad (12) \\ T'' + \lambda^2 T &= 0. \quad (13) \end{aligned}$$

(13) has the general solution

$$T(\theta) = A\cos\lambda\theta + B\sin\lambda\theta. \quad (14)$$

The periodic boundary condition (7) gives $T(2\pi+\theta) = T(\theta)$ which gives

$$2\pi\lambda = 2n\pi, \text{ i.e., } \lambda = n, \; n = 0, 1, 2, \ldots.$$

We get solutions for $T(\theta)$:

$$T_n(\theta) = A_n \cos n\theta + B_n \sin n\theta. \quad (15)$$

The Cauchy-Euler ODEs of second order

The second order Cauchy-Euler Equation is:

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0. \tag{16}$$

Substitution $x = e^t$ transforms it into a second order linear ODE with constant coefficients. Suppose α and β are the roots of its auxiliary (or characteristic) equation

$$a\Lambda^2 + (b-a)\Lambda + c = 0 \tag{17}$$

Then the solutions of (16) are:

$$y = \begin{cases} c_1x^\alpha + c_2x^\beta, & \text{if } \alpha, \beta \text{ are real and distinct,} \\ x^\alpha\big[c_1 + c_2\ln x\big], & \text{if } \alpha = \beta \in \mathbb{R}, \\ x^\gamma\big[c_1\cos(\delta\ln x) + c_2\sin(\delta\ln x)\big], & \text{if } \gamma \pm i\delta \text{ are the roots.} \end{cases}$$

Laplace equation in polar coordinates: solutions

With $\lambda = n$, (12) ia the Cauchy-Euler equation

$$r^2 R'' + rR' - n^2 R = 0 \tag{18}$$

with auxiliary equation $\Lambda^2 - n^2 = 0$.

The equation (18) therefore has solutions:

$$R_n = C_n r^{-n} + D_n r^n \tag{19}$$

Using superpositions we get the general solution of (6) as

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} (C_n r^{-n} + D_n r^n)(A_n \cos n\theta + B_n \sin n\theta).} \tag{20}$$

Interior Dirichlet problem for a circle

Here, we have the following BVP:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 \le r \le a, \; 0 \le \theta \le 2\pi, \tag{21}$$

$$u(a,\theta) = f(\theta), \; 0 \le \theta \le 2\pi. \tag{22}$$

Consider the general solution (20). To get a bounded solution in the circle $0 \le r \le a$, we must have $C_n = 0$ for $n \ge 1$. We write the solution as

$$\boxed{u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta).} \tag{23}$$

Interior Dirichlet problem for a circle

Using the given boundary condition (22),

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta), \qquad (24)$$

The coefficients are given by

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta, \; n = 0, 1, 2, 3, \ldots \qquad (25a)$$

$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta, \; n = 1, 2, 3, \ldots \qquad (25b)$$

(23) with the coefficients given by (25) is the solution of the Interior Dirichlet Problem.

Exterior Dirichlet problem for a circle

Here, we have the following BVP:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad a \leq r < \infty, \; 0 \leq \theta \leq 2\pi, \tag{26}$$

$$u(a,\theta) = f(\theta), \; 0 \leq \theta \leq 2\pi. \tag{27}$$

Consider the general solution (20). To get a bounded solution in the exterior of the circle $r = a$, we must have $D_n = 0$ for $n \geq 1$. We write the solution as

$$\boxed{u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^{-n}(A_n \cos n\theta + B_n \sin n\theta).} \tag{28}$$

Exterior Dirichlet problem for a circle

Using the given boundary condition (27),

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^{-n}(A_n \cos n\theta + B_n \sin n\theta), \qquad (29)$$

The coefficients are given by

$$A_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \; d\theta, \;\; n = 0, 1, 2, 3, \ldots \qquad (30a)$$

$$B_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \; d\theta, \;\; n = 1, 2, 3, \ldots \qquad (30b)$$

(28) with the coefficients given by (30) is the solution of the Exterior Dirichlet Problem.

Interior Neumann problem for a circle

Exercise. Solve the Neumann problem for the interior of a circle:

$$\text{PDE: } \nabla^2 u = 0,\ \ u = u(r,\theta), 0 \le r \le a,\ \ 0 \le \theta \le 2\pi,$$

$$\text{BC: } u_r(a,\theta) = g(\theta), \quad r = a.$$

Answer. $$u(r,\theta) = \sum_{n=0}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta),$$

where A_n and B_n are given by

$$na^{n-1} A_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta d\theta,$$

$$na^{n-1} B_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin n\theta d\theta.$$

Laplace equation in cylindrical coordinates

We have the equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0. \tag{31}$$

Any solution of (31) is said to be cylindrical harmonic.

We seek for a separable solution

$$u(r, \theta, z) = R(r)H(\theta)Z(z).$$

Then, (31) becomes

$$R''HZ + \frac{1}{r}R'HZ + \frac{1}{r^2}RH''Z + RHZ'' = 0. \tag{32}$$

We have from (32)

$$R''HZ + \frac{1}{r}R'HZ + \frac{1}{r^2}RH''Z + RHZ'' = 0$$

i.e., $$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{Z''}{Z} = -\frac{1}{r^2}\frac{H''}{H}$$

i.e., $$r^2\frac{R''}{R} + \frac{rR'}{R} + \frac{r^2Z''}{Z} = -\frac{H''}{H} = \mu^2$$

i.e., $$\boxed{r^2\frac{R''}{R} + \frac{rR'}{R} + \frac{r^2Z''}{Z} - \mu^2 = 0}, \quad \boxed{H'' + \mu^2 H = 0} \text{ (i)}$$

The first equation is further separable:

i.e., $$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{\mu^2}{r^2} = -\frac{Z''}{Z} = -\lambda^2$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{\mu^2}{r^2} = -\frac{Z''}{Z} = -\lambda^2$$

i.e., $$\boxed{r^2R'' + rR' + (\lambda^2r^2 - \mu^2)R = 0,} \text{ (ii)} \quad \boxed{Z'' - \lambda^2 Z = 0.} \text{ (iii)}$$

Equations (i) and (iii) have solutions

$$H(\theta) = A\cos\mu\theta + B\sin\mu\theta$$
$$Z(z) = Ce^{\lambda z} + De^{-\lambda z}.$$

Equation (ii) is of very special kind. It is called Bessel's Equation of order μ with parameter λ.

Bessel's Equation and Bessel's functions

The Bessel's equation of order n

$$x^2y'' + xy' + (x^2 - n^2)y = 0. \tag{33}$$

Assuming a series solution $y = x^n \sum_{k=0}^{\infty} b_k x^k$, a solution of (33) is obtained as

$$y_1(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{2k}. \tag{34}$$

For any $\nu \in \mathbb{C}$, the Bessel function of the first kind J_ν is defined by

$$\boxed{J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k}.} \tag{35}$$

Then, $y_1(x)$ in (34) is $J_n(x)$. The solution corresponding to $-n$ is $J_{-n} = (-1)^n J_n$. So, J_n and J_{-n} are not independent.

Bessel's Equation and Bessel's functions

The Bessel function of the second kind Y_ν is defined by

$$\boxed{Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu - J_{-\nu}(x)}{\sin(\nu\pi)},} \tag{36}$$

for $\nu \notin \mathbb{Z}$. For $n \in \mathbb{Z}$, define $Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$.

The general solution of the Bessel's equation (33) is given by

$$y(x) = c_1 J_n(x) + c_2 Y_n(x).$$

The general solution of the Bessel's equation

$$x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0$$

is given by

$$\boxed{y(x) = c_1 J_n(\lambda x) + c_2 Y_n(\lambda x).}$$

Thus, the general solution of the Laplace equation in cylindrical coordinates is

$$u(r,\theta,z) = (A\cos\mu\theta + B\sin\mu\theta)(Ce^{\lambda z} + De^{-\lambda z})(c_1 J_n(\lambda r) + c_2 Y_n(\lambda r)).$$

For specific problems, the coefficients are determined by the given boundary conditions, boundedness and periodicity of solutions.

Steady-state heat conduction in a circular cylinder

Consider a right circular cylinder of radius a and height l having
(a) its convex surface and base in the xy-plane at temperature 0^0C,
(b) the top end $z = l$ is kept at temperature $f(r)^0$C.
To find the steady-state temperature at any point of the cylinder.

The governing equation for this problem will be Laplace's equation in r, θ, z.

But assuming that the cylinder is symmetrical about its axis, Laplace's equation takes the form:

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0,\ 0 < r \leq a,\ 0 \leq z \leq l. \tag{37}$$

The boundary conditions are:

$$\begin{aligned} u(a,z) &= 0, 0 \leq z \leq l & (38a) \\ u(r,0) &= 0, 0 < r \leq a & (38b) \\ u(r,l) &= f(r), 0 < r \leq a. & (38c) \end{aligned}$$

Assume a solution in the form

$$u(r,z) = R(r)Z(z)$$

Applying it to the governing equation (37):

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{Z''}{Z} = 0.$$

Steady-state heat conduction in a circular cylinder

By separating the variables:

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{Z''}{Z} = k.$$

Observing that only the negative value of the separation constant will give rise to nontrivial solutions, we get the following ODEs by considering $k = -\lambda^2$:

$$Z'' - \lambda^2 Z = 0, \tag{39}$$

$$R'' + \frac{1}{r}R' + \lambda^2 R = 0, \tag{40}$$

The solutions of the above equations are, respectively, given by

$$Z(z) = A\cosh\lambda z + B\sinh\lambda z, \tag{41}$$

$$R(r) = CJ_0(\lambda r) + DY_0(\lambda r), \tag{42}$$

Steady-state heat conduction in a circular cylinder

The solution $u(r, z)$:

$$u(r, z) = (A \cosh \lambda z + B \sinh \lambda z)(CJ_0(\lambda r) + DY_0(\lambda r)) \qquad (43)$$

We are looking for a bounded solution in $0 \leq r \leq a$, we must take $D = 0$ since $Y_0 \to -\infty$ as $r \to 0$. Equation (43) can be written as

$$u(r, z) = J_0(\lambda r)(A \cosh \lambda z + B \sinh \lambda z). \qquad (44)$$

Now applying the boundary condition (38a), we get

$$J_0(\lambda a) = 0.$$

Steady-state heat conduction in a circular cylinder

Hence

$$\lambda_n a = \nu_n,$$

where ν_n are the zeros of J_0. The eigenvalues are given by

$$\lambda_n = \frac{\nu_n}{a}. \tag{45}$$

$$u_n(r,z) = A_n J_0\left(\frac{\nu_n}{a} r\right) \cosh \frac{\nu_n}{a} z + B_n J_0\left(\frac{\nu_n}{a} r\right) \sinh \frac{\nu_n}{a} z.$$

By superimposing all the solutions,

$$u(r,z) = \sum_{n=1}^{\infty} \left(A_n J_0\left(\frac{\nu_n}{a} r\right) A \cosh \frac{\nu_n}{a} z + B_n J_0\left(\frac{\nu_n}{a} r\right) \sinh \frac{\nu_n}{a} z \right). \tag{46}$$

Steady-state heat conduction in a circular cylinder

Using the boundary condition (38b), we get $A_n = 0$ thereby reducing the solution to

$$u(r,z) = \sum_{n=1}^{\infty} B_n J_0\left(\frac{\nu_n}{a} r\right) \sinh \frac{\nu_n}{a} z. \tag{47}$$

The coefficient B_n can be obtained by using the boundary condition (38c):

$$f(r) = \sum_{n=1}^{\infty} B_n J_0\left(\frac{\nu_n}{a} r\right) \sinh \frac{\nu_n}{a} l \tag{48}$$

i.e.,

$$\int_0^a f(r) r J_0\left(\frac{\nu_m}{a} r\right) dr = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{\nu_n}{a} l\right) \int_0^a r J_0\left(\frac{\nu_m}{a} r\right) J_0\left(\frac{\nu_n}{a} r\right) dr \tag{49}$$

Orthogonality Property:

$$\boxed{\int_0^a rJ_0\left(\frac{\nu_n}{a}r\right)J_0\left(\frac{\nu_n}{a}r\right)dr = \begin{cases} 0, & \text{if } m \neq n \\ \frac{a^2}{2}J_1^2\left(\frac{\nu_n}{a}\right) & \text{if } m = n \end{cases}.}$$

Thus,

$$\int_0^a r\,f(r)J_0\left(\frac{\nu_m}{a}r\right)dr = B_m \sinh\left(\frac{\nu_n}{a}l\right)\left(\frac{a^2}{2}J_1^2\left(\frac{\nu_n}{a}\right)\right)$$

i.e.,

$$B_m = \frac{2\int_0^a r\,f(r)J_0\left(\frac{\nu_m}{a}r\right)dr}{a^2\sinh\left(\frac{\nu_n}{a}l\right)J_1^2\left(\frac{\nu_n}{a}\right)}. \tag{50}$$