MA 201: Integral Transforms
Slides - 16

Fourier Transforms: Applications to PDE

An important use of Fourier series

Theorem (Parseval's Identity)

If $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),\ 0 \leq x \leq L$, then

$$\boxed{\frac{2}{L}\int_0^L [f(x)]^2 dx = \sum_{n=1}^{\infty} b_n^2.} \tag{1}$$

Proof. We have

$$\begin{aligned}
\int_0^L [f(x)]^2 dx &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_m b_n \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_m b_n \delta_{mn} \cdot \frac{L}{2} = \frac{L}{2}\sum_{n=1}^{\infty} b_n^2.
\end{aligned}$$

Parseval's Identity

Theorem

If for $-L \leq x \leq L$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right),$$

then

$$\boxed{\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right).} \quad (2)$$

Proof. Similar.

Applications: Examples

Let $f(x) = x,\ x \in [0, 2]$. Then, its sine Fourier series is

$$x = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{2}\right).$$

By Parseval's Identity, $\displaystyle \frac{2}{2} \int_0^2 x^2 dx = \left(\frac{4}{\pi}\right)^2 \sum_{n=1}^{\infty} \frac{1}{n^2}$, i.e.,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{16} \int_0^2 x^2 dx = \frac{\pi^2}{16} \cdot \frac{8}{3} = \frac{\pi^2}{6}.$$

This also gives

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(n)^2} - \sum_{n=0}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{6} - \frac{1}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8}.$$

Exercise

Consider $f(x) = x^2,\ x \in [-\pi, \pi]$. Show that the half range cosine Fourier series of f is

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx).$$

Deduce that $\dfrac{\pi^2}{12} = 4\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)^2}$. (Put $x = \pi/2$.)

Use Parseval's Identity to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Convolution of functions

Definition: The convolution of two functions $f_1(t)$ and $f_2(t)$, $-\infty < t < \infty$, is defined as

$$(f_1 * f_2)(t) = f_1(t) * f_2(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau,$$

provided the integral exists for each t.

The convolution is an integral that expresses the amount of overlap of one function f_2 as it is shifted over another function f_1. It "blends" one function with another.

Note that

$$\begin{aligned} f_1(t) * f_2(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(t-\nu) f_2(\nu)\, d\nu = f_2(t) * f_1(t). \end{aligned}$$

Convolution: Some properties

Commutativity: $f_1 * f_2 = f_2 * f_1$.

Associativity: $f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3$.

Distributivity: $(\alpha f_1 + \beta f_2) * f_3 = \alpha(f_1 * f_3) + \beta(f_2 * f_3)$.

Theorem

If $h(t)$ is the convolution of the functions $f_1(t)$ and $f_2(t)$, then

$$\mathcal{F}\{h(t)\} = \mathcal{F}\{(f_1 * f_2)(t)\} = \mathcal{F}\{f_1(t)\}\mathcal{F}\{f_2(t)\}.$$

Convolution theorem

Proof: By definition

$$\begin{aligned}
\mathcal{F}\{(f_1 * f_2)(t)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f_1(t) * f_2(t)) e^{-i\sigma t}\, dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau \right] e^{-i\sigma t}\, dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(\tau) \left[\int_{-\infty}^{\infty} f_2(t-\tau) e^{-i\sigma t}\, dt \right] d\tau.
\end{aligned}$$

Putting $t - \tau = \omega$ in the inner integral

$$\begin{aligned}
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(\tau) \left[\int_{-\infty}^{\infty} f_2(\omega) e^{-i\sigma(\omega+\tau)}\, d\omega \right] d\tau \\
&= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(\tau) e^{-i\sigma\tau}\, d\tau \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(\omega) e^{-i\sigma\omega}\, d\omega \right) \\
&= \mathcal{F}\{f_1(t)\} \mathcal{F}\{f_2(t)\}.
\end{aligned}$$

Example:

If f_1 and f_2 are given by

$$f_1(t) = \begin{cases} 0, & -\infty < t \le 0, \\ ae^{-at}, & 0 < t < \infty, \; a > 0 \end{cases}$$

$$f_2(t) = \begin{cases} 0, & -\infty < t \le -1, \\ 1, & -1 < t \le 1, \\ 0, & 1 < t < \infty, \end{cases}$$

then show that the convolution $h(t)$ of $f_1(t)$ and $f_2(t)$ is given by

$$h(t) = \frac{a}{\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin\sigma}{\sigma(a + i\sigma)} e^{i\sigma t} \, d\sigma$$

Solution: The Fourier transform of $f_1(t)$:

$$\begin{aligned}
\mathcal{F}\{f_1(t)\} &= g_1(\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(t) e^{-i\sigma t}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} a e^{-at}\, e^{-i\sigma t}\, dt = \frac{a}{\sqrt{2\pi}} \left| \frac{e^{-(a+i\sigma)t}}{-(a+i\sigma)} \right|_0^{\infty} \\
&= \frac{a}{\sqrt{2\pi}(a+i\sigma)}.
\end{aligned}$$

The Fourier transform of $f_2(t)$:

$$\begin{aligned}
\mathcal{F}\{f_2(t)\} &= g_2(\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(t) e^{-i\sigma t}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\sigma t}\, dt = \frac{1}{\sqrt{2\pi}} \left| \frac{e^{-(i\sigma)t}}{-i\sigma} \right|_{-1}^{1} \\
&= \frac{1}{\sqrt{2\pi}}\ \frac{e^{i\sigma} - e^{-i\sigma}}{i\sigma} = \sqrt{\frac{2}{\pi}} \frac{\sin\sigma}{\sigma}.
\end{aligned}$$

Solution contd.

Therefore,

$$\mathcal{F}\{h(t)\} = \mathcal{F}\{f_1(t)\}\mathcal{F}\{f_2(t)\} = \frac{1}{\pi}\frac{a\sin\sigma}{\sigma(a+i\sigma)}$$

Consequently,

$$h(t) = \mathcal{F}^{-1}\{\mathcal{F}\{f_1(t)\}\mathcal{F}\{f_2(t)\}\} = \mathcal{F}^{-1}\left\{\frac{1}{\pi}\ \frac{a\sin\sigma}{\sigma(a+i\sigma)}\right\}$$

which will give us

$$\begin{aligned} h(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\pi}\frac{a\sin\sigma}{\sigma(a+i\sigma)}e^{i\sigma t}\,d\sigma \\ &= \frac{a}{\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\sin\sigma}{\sigma(a+i\sigma)}e^{i\sigma t}\,d\sigma \end{aligned}$$

Fourier Transforms of partial derivatives

The Fourier transform of a function $U(x,t)$ with respect to x is given by

$$\mathcal{F}\{U(x,t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\sigma x} U(x,t)\, dx = \hat{U}(\sigma,t). \qquad (3)$$

The inverse Fourier transform $U(x,t)$ of $\hat{U}(\sigma,t)$ is given by

$$U(x,t) = \mathcal{F}^{-1}\{\hat{U}(\sigma,t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\sigma x} \hat{U}(\sigma,t)\, d\sigma.$$

Under the assumption that U, $\frac{\partial U}{\partial x}$ and $\frac{\partial^2 U}{\partial x^2}$ vanish as $x \to \pm\infty$, we obtain the following:

$$\mathcal{F}\left\{\frac{\partial U(x,t)}{\partial x}\right\} = i\sigma\hat{U}(\sigma,t), \qquad \mathcal{F}\left\{\frac{\partial^2 U(x,t)}{\partial x^2}\right\} = -\sigma^2\hat{U}(\sigma,t),$$

$$\mathcal{F}\left\{\frac{\partial U(x,t)}{\partial t}\right\} = \frac{d}{dt}\hat{U}(\sigma,t), \qquad \mathcal{F}\left\{\frac{\partial^2 U(x,t)}{\partial t^2}\right\} = \frac{d^2}{dt^2}\hat{U}(\sigma,t).$$

Fourier Transforms of partial derivatives

Result I:

$$\mathcal{F}\left\{\frac{\partial U(x,t)}{\partial x}\right\} = \mathrm{i}\sigma\hat{U}(\sigma,t). \tag{4}$$

Proof: Follows from $\mathcal{F}\{f'(x)\} = \mathrm{i}\sigma g(\sigma) = \mathrm{i}\sigma\mathcal{F}\{f(x)\}$.

Result II:

$$\mathcal{F}\left\{\frac{\partial^2 U(x,t)}{\partial x^2}\right\} = -\sigma^2\hat{U}(\sigma,t). \tag{5}$$

Proof:

$$\mathcal{F}\left\{\frac{\partial^2 U(x,t)}{\partial x^2}\right\} = \mathrm{i}\sigma\mathcal{F}\left\{\frac{\partial U(x,t)}{\partial x}\right\} = -\sigma^2\hat{U}(\sigma,t).$$

Fourier Transforms of partial derivatives

Result III:

$$\mathcal{F}\left\{\frac{\partial U(x,t)}{\partial t}\right\} = \frac{d}{dt}\hat{U}(\sigma,t). \tag{6}$$

Proof:
$$\begin{aligned}\mathcal{F}\left\{\frac{\partial U(x,t)}{\partial t}\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}\frac{\partial U(x,t)}{\partial t}dx \\ &= \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}U(x,t)dx \\ &= \frac{1}{\sqrt{2\pi}}\frac{d}{dt}\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}U(x,t)dx \\ &= \frac{d}{dt}\hat{U}(\sigma,t).\end{aligned}$$

Result IV:

$$\mathcal{F}\left\{\frac{\partial^2 U(x,t)}{\partial t^2}\right\} = \frac{d^2}{dt^2}\hat{U}(\sigma,t). \tag{7}$$

Example A. An infinitely long string extending in $-\infty < x < \infty$ under uniform tension is displaced into the curve $y = f(x)$ and let go from rest with velocity $g(x)$. To find the displacement $U(x,t)$ at any point at any subsequent time.

Solution: The boundary value problem is

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2} &= c^2 \frac{\partial^2 U}{\partial x^2}, \;\; -\infty < x < \infty, \; t > 0, & (8) \\ U(x,0) &= f(x) \;\; \text{(initial displacement)}, & (9) \\ \frac{\partial U}{\partial t}(x,0) &= g(x) \;\; \text{(initial velocity)}. & (10) \end{aligned}$$

Taking Fourier transform on both sides of PDE (8),

$$\frac{d^2}{dt^2}\hat{U}(\sigma,t) = -c^2\sigma^2\hat{U}(\sigma,t), \tag{11}$$

(11) can be written in standard form as

$$\frac{d^2}{dt^2}\hat{U}(\sigma,t) + c^2\sigma^2\hat{U}(\sigma,t) = 0. \tag{12}$$

On solving, we get

$$\hat{U}(\sigma,t) = A(\sigma)\cos(c\sigma t) + B(\sigma)\sin(c\sigma t). \tag{13}$$

Taking Fourier transforms on the initial conditions (9) and (10),

$$\hat{U}(\sigma,0) = \hat{f}(\sigma), \tag{14}$$

$$\frac{d}{dt}\hat{U}(\sigma,0) = \hat{g}(\sigma), \tag{15}$$

where $\hat{f}(\sigma)$ and $\hat{g}(\sigma)$, are the Fourier transforms of $f(x)$ and $g(x)$, respectively.

Using the initial conditions, $A(\sigma)$ and $B(\sigma)$ can be obtained as:

$$\hat{U}(\sigma,0) = A(\sigma) = \hat{f}(\sigma), \quad \frac{d}{dt}\hat{U}(\sigma,0) = c\sigma B(\sigma) = \hat{g}(\sigma).$$

Now

$$\hat{U}(\sigma,t) = \hat{f}(\sigma)\cos(c\sigma t) + \frac{1}{c\ \sigma}\,\hat{g}(\sigma)\sin(c\sigma t). \tag{16}$$

We get the solution using the inverse Fourier transform:

$$U(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\hat{f}(\sigma)\cos(c\sigma t) + \frac{1}{c\ \sigma}\,\hat{g}(\sigma)\sin(c\sigma t)\right]e^{i\sigma x}\,d\sigma. \tag{17}$$

It can be computed explicitly in terms of f and g to yield d'Alembert's form of the solution:

$$U(x,t) = \frac{1}{2}\Big[f(x-ct) + f(x+ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau. \tag{18}$$

From (16) we have

$$U(x,t) = \mathcal{F}^{-1}\Big\{\hat{f}(\sigma)\cos(c\sigma t)\Big\} + \frac{1}{c}\ \mathcal{F}^{-1}\left\{\hat{g}(\sigma)\frac{\sin(ct\sigma)}{\sigma}\right\} \quad (19)$$

Application of Fourier Transform to PDEs

$$
\begin{aligned}
&\mathcal{F}^{-1}\Big\{\hat{f}(\sigma)\cos(c\sigma t)\Big\} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\sigma)\cos(ct\sigma)e^{ix\sigma}\,d\sigma \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\rho)e^{-i\rho\sigma}\,d\rho\right)\left[\frac{e^{ict\sigma}+e^{-ict\sigma}}{2}\right]e^{ix\sigma}\,d\sigma \\
&= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{i\sigma(x+ct)}\int_{-\infty}^{\infty}f(\rho)e^{-i\sigma\rho}\,d\rho\right)d\sigma + \right. \\
&\qquad \left.\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{i\sigma(x-ct)}\int_{-\infty}^{\infty}f(\rho)e^{-i\sigma\rho}\,d\rho\right)d\sigma\right] \\
&= \frac{1}{2}\big[f(x+ct)+f(x-ct)\big],
\end{aligned}
$$

using Fourier integral formula.

Result: $\mathcal{F}^{-1}\left\{\sqrt{\frac{2}{\pi}}\frac{\sin(\alpha\sigma)}{\sigma}\right\} = h(x) = \begin{cases} 1, & \text{if } |x| < \alpha \\ 0, & \text{if } |x| > \alpha \end{cases}$

Proof. We have

$$\begin{aligned}\mathcal{F}\{h(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x)e^{-i\sigma x}dx = \frac{1}{\sqrt{2\pi}}\int_{-\alpha}^{\alpha} e^{-i\sigma x}dx \\ &= \frac{1}{\sqrt{2\pi}}\frac{-1}{i\sigma}e^{-i\sigma x}\Big]_{-a}^{a} = \sqrt{\frac{2}{\pi}}\frac{\sin(\alpha\sigma)}{\sigma}\end{aligned}$$

$$\begin{aligned}\text{Thus, } \mathcal{F}^{-1}\left\{\hat{g}(\sigma)\frac{\sin(ct\sigma)}{\sigma}\right\} &= g(x) * \mathcal{F}^{-1}\left\{\frac{\sin(ct\sigma)}{\sigma}\right\} \\ = \sqrt{\frac{\pi}{2}}g(x) * h(x) &= \sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\tau)h(x-\tau)d\tau \\ &= \frac{1}{2}\int_{x-ct}^{x+ct} g(\tau)d\tau.\end{aligned}$$

Example B. Consider the heat conduction in an infinite rod with thermal diffusivity α with initial temperature distribution $f(x)$. To find the temperature distribution $U(x,t)$ at any point at any subsequent time.

Solution: The boundary value problem is

$$\begin{aligned} \frac{\partial U}{\partial t} &= \alpha\frac{\partial^2 U}{\partial x^2}, \;\; -\infty < x < \infty, \; t > 0, && (20) \\ U(x,0) &= f(x) \;\; \text{(initial temperature distribution).} && (21) \end{aligned}$$

Proceed as in the previous example:

$$\hat{U}(\sigma,t) = \hat{f}(\sigma)e^{-\alpha\sigma^2 t}, \tag{22}$$

where $\hat{f}(\sigma)$ is the Fourier transform of $f(x)$.

We use the inverse Fourier transform to obtain

$$U(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\sigma)e^{-\alpha\sigma^2 t}e^{i\sigma x}\, d\sigma. \tag{23}$$

Fourier sine and cosine transforms

- The Fourier sine and cosine transforms can be employed to solve a partial differential equation when the range of the spatial variable extends from 0 to ∞.
- The choice of sine or cosine transform is decided by the form of the boundary conditions at $x = 0$.
- If the boundary condition is in terms of some value of $U(0, t)$ (i.e. Dirichlet boundary condition), then sine transform is to be used.
- When the boundary condition is in terms of some value of $\frac{\partial U}{\partial x}(0, t)$ (i.e. Neumann boundary condition), then cosine transform is to be used.

Exercises:

Let f be defined on $[0, \infty)$ be such that $f(x), f'(x)$ and f'' vanish as $x \to \infty$. If $\mathcal{F}_s$ and $\mathcal{F}_c$ stand for the sine and cosine transforms, respectively, prove that

- $\mathcal{F}_s\{f'(x)\} = -\sigma \mathcal{F}_c\{f(x)\}$.
- $\mathcal{F}_c\{f'(x)\} = -\sqrt{\dfrac{2}{\pi}}\, f(0) + \sigma \mathcal{F}_c\{f(x)\}$.
- $\mathcal{F}_s\{f''(x)\} = \sqrt{\dfrac{2}{\pi}}\, \sigma f(0) - \sigma^2 \mathcal{F}_s\{f(x)\}$.
- $\mathcal{F}_c\{f''(x)\} = -\sqrt{\dfrac{2}{\pi}}\, f'(0) - \sigma^2 \mathcal{F}_c\{f(x)\}$.

The *Fourier sine transform* of a function $U(x,t)$ with respect to x is

$$\mathcal{F}_s\{U(x,t)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty U(x,t) \sin \sigma x \, dx = \hat{U}_s(\sigma, t). \qquad (24)$$

Under the assumption that U, $\frac{\partial U}{\partial x}$ and $\frac{\partial^2 U}{\partial x^2}$ vanish as $x \to \infty$,

$$\mathcal{F}_s\left\{\frac{\partial U(x,t)}{\partial t}\right\} = \frac{d}{dt}\hat{U}_s(\sigma,t), \qquad (25)$$

$$\mathcal{F}_s\left\{\frac{\partial^2 U(x,t)}{\partial t^2}\right\} = \frac{d^2}{dt^2}\hat{U}_s(\sigma,t), \qquad (26)$$

$$\mathcal{F}_s\left\{\frac{\partial^2 U(x,t)}{\partial x^2}\right\} = \sqrt{\frac{2}{\pi}}\,\sigma U(0,t) - \sigma^2 \hat{U}_s(\sigma,t). \qquad (27)$$

The *Fourier cosine transform* of a function $U(x,t)$ with respect to x is

$$\mathcal{F}_c\{U(x,t)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} U(x,t) \cos \sigma x \, dx = \hat{U}_c(\sigma,t). \qquad (28)$$

Under the assumption that U and $\frac{\partial U}{\partial x}$ vanish as $x \to \infty$,

$$\begin{aligned}
\mathcal{F}_c\left\{\frac{\partial U(x,t)}{\partial t}\right\} &= \frac{d}{dt}\hat{U}_c(\sigma,t), && (29)\\
\mathcal{F}_c\left\{\frac{\partial^2 U(x,t)}{\partial t^2}\right\} &= \frac{d^2}{dt^2}\hat{U}_c(\sigma,t), && (30)\\
\mathcal{F}_c\left\{\frac{\partial^2 U(x,t)}{\partial x^2}\right\} &= -\sqrt{\frac{2}{\pi}}\frac{\partial U}{\partial x}(0,t) - \sigma^2 \hat{U}_c(\sigma,t). && (31)
\end{aligned}$$

Example C: If $U(x,t)$ is the temperature at time t and α the thermal diffusivity of a semi-infinite metal bar, find the temperature distribution in the bar at any point at any subsequent time if the initial temperature distribution is given as $f(x)$ and the boundary is kept at U_0 degrees.

Solution: The boundary value problem is the following:

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2}, \;\; x>0, \; t>0, \tag{32}$$

$$U(x,0) = f(x), \; t>0 \;\; U(0,t) = U_0, \; x>0. \tag{33}$$

The boundary condition suggests that we need to use Fourier sine transform. Taking the transform on (32),

$$\frac{d}{dt}\hat{U}_s(\sigma,t) = \alpha[\sqrt{\frac{2}{\pi}}\,\sigma U(0,t) - \sigma^2 \hat{U}_s(\sigma,t)]. \tag{34}$$

Using the boundary condition,

$$\frac{d}{dt}\hat{U}_s(\sigma,t) + \alpha\sigma^2\hat{U}_s(\sigma,t) = \sqrt{\frac{2}{\pi}}\,\alpha\sigma U_0. \tag{35}$$

On solving

$$\hat{U}_s(\sigma,t) = A(\sigma)e^{-\alpha\sigma^2 t} + \sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma}U_0. \tag{36}$$

Using the initial condition,

$$\hat{f}_s(\sigma) = A(\sigma) + \sqrt{\frac{2}{\pi}}\,\frac{U_0}{\sigma},$$

where $\hat{f}_s(\sigma)$ is the Fourier sine transform of $f(x)$, that is,

$$A(\sigma) = \hat{f}_s(\sigma) - \sqrt{\frac{2}{\pi}}\,\frac{U_0}{\sigma}$$

Now $\hat{U}_s(\sigma, t)$ is

$$\begin{aligned}
\hat{U}_s(\sigma, t) &= (\hat{f}_s(\sigma) - \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma})e^{-\alpha\sigma^2 t} + \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma} \\
&= \hat{f}_s e^{-\alpha\sigma^2 t} + \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma}(1 - e^{-\alpha\sigma^2 t}) \qquad (37)
\end{aligned}$$

The inversion gives

$$U(x, t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[\hat{f}_s e^{-\alpha\sigma^2 t} + \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma}(1 - e^{-\alpha\sigma^2 t})\right]\sin\sigma x \, d\sigma. \qquad (38)$$

Example D: Consider the same equation as in the previous example subject to the boundary conditions

$$\frac{\partial U}{\partial x}(0,t) = 0, \quad U(x,0) = f(x)$$

Taking Fourier cosine transform

$$\frac{d}{dt}\hat{U}_c(\sigma,t) + \alpha\sigma^2\hat{U}_c = 0, \quad \text{i.e.,} \quad \hat{U}_c(\sigma,t) = Ae^{-\alpha\sigma^2 t}.$$

Using the initial condition,

$$\hat{U}_c(\sigma,t) = \hat{f}_c(\sigma)e^{-\alpha\sigma^2 t}, \tag{39}$$

where $\hat{f}_c(\sigma)$ is the Fourier cosine transform of $f(x)$. Taking inverse, we get the solution

$$U(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\sigma)e^{-\alpha\sigma^2 t}\cos\sigma x\, d\sigma. \tag{40}$$