MA 201: Partial Differential Equations
Lecture - 2

Linear First-Order PDEs

For a PDE

$$f(x, y, z, p, q) = 0,$$

- a solution of the type

$$F(x, y, z, a, b) = 0 \tag{1}$$

  which contains two arbitrary constants a and b is said to be a *complete solution* or a *complete integral* of that equation.

- a solution of the type

$$F(u, v) = 0 \tag{2}$$

  involving an arbitrary function F connecting known functions u and v is called a *general solution* or a *general integral* of that equation.

A general integral provides a bigger set of solutions of the pde than does a complete integral.

Consider the (quasi)-linear pde (called Lagrange's equation)

$$Pp + Qq = R, \tag{3}$$

where P, Q and R are given functions of x, y and z (not involving derivatives), p and q respectively denote $\partial z/\partial x$ and $\partial z/\partial y$.

Let $z = f(x, y)$ be a solution (integral surface) of (3).
Let $\Gamma : (x(t), y(t), z(t))$ be a curve on the surface. Then $(f_x, f_y, -1)$ is normal to the surface and $(x'(t), y'(t), z'(t)) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ is along the tangent to the curve (and so to the surface). Thus, we get,

$$\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}-\frac{dz}{dt}=0, \text{ i.e. } p\frac{dx}{dt}+q\frac{dy}{dt}-\frac{dz}{dt}=0, \text{ i.e. } p\,dx+q\,dy-dz=0.$$

Since this is compatible with (3), we must have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{4}$$

In general, the curve Γ is given by the intersection of two surfaces satisfying (4).

Theorem

The general solution of the PDE

$$Pp + Qq = R, \tag{5}$$

is

$$F(u, v) = 0, \tag{6}$$

where F is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ form a solution of the equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{7}$$

Proof: The equations $u(x, y, z) = c_1, \;\; v(x, y, z) = c_2$ give

$$u_x dx + u_y dy + u_z dz = 0, \quad v_x dx + v_y dy + v_z dz = 0. \tag{8}$$

Therefore, if $u = c_1$ and $v = c_2$ satisfy equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, then the equations (8) and

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

must be compatible, i.e., we must have

$$Pu_x + Qu_y + Ru_z = 0.$$
$$Pv_x + Qv_y + Rv_z = 0.$$

Solving these equations for P, Q and R, we have

$$\frac{P}{\partial(u,v)/\partial(y,z)} = \frac{Q}{\partial(u,v)/\partial(z,x)} = \frac{R}{\partial(u,v)/\partial(x,y)}. \tag{9}$$

However, the relation $F(u, v) = 0$ leads to the pde

$$p\,\frac{\partial(u,v)}{\partial(y,z)} + q\,\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}. \tag{10}$$

Substituting (9) into (10), we see that $F(u, v) = 0$ is a solution of (5) for given u and v.

- A solution of the equation $Pp + Qq = R$ is called an integral surface. In the theorem $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are integral surfaces of $Pp + Qq = R$.
- A solution of the equations (ODE) $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$ is called an integral curve. Indeed, the intersection of the surfaces $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ give a two parameter family of integral curves of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.
- The equation $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$ is called the auxiliary equation of $Pp + Qq = R$.

Example. Find the general solution of the equation

$$(x+z)\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + y^2.$$

Solution: The integral surfaces of this equation are generated by the integral curves of the equations

$$\frac{dx}{x+z} = \frac{dy}{y} = \frac{dz}{z+y^2}.$$

The second equation gives $\dfrac{dz}{dy} - \dfrac{z}{y} = y, \quad$ i.e., $\quad \dfrac{d}{dy}\left(\dfrac{z}{y}\right) = 1$,yielding solution

$$u(x,y,z) = \frac{z}{y} - y = c_1.$$

The first equation gives $\dfrac{dx}{dy} = \dfrac{x}{y} + \dfrac{z}{y} = \dfrac{x}{y} + c_1 + y$, i.e., $\dfrac{d}{dy}\left(\dfrac{x}{y}\right) = \dfrac{c_1}{y} + 1$, yielding solution $\dfrac{x}{y} = c_1 \ln y + y + c_2$, i.e.,

$$v(x,y,z) = \frac{1}{y}\left[x + y^2 + (y^2 - z)\ln y\right] = c_2.$$

We now get the general solution of the given equation as $F(u,v) = 0$, where F is an arbitrary function.

We can generalize the previous theorem as follows:

Theorem

If $u_i(x_1, x_2, x_3, \ldots, x_n, z) = c_i, i = 1, 2, 3, \ldots, n$ are independent solutions of the equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{R},$$

then the relation $\phi(u_1, u_2, \ldots, u_n) = 0$, in which the function ϕ is arbitrary, is a general solution of the linear pde

$$P_1 \frac{\partial z}{\partial x_1} + P_2 \frac{\partial z}{\partial x_2} + \cdots + P_n \frac{\partial z}{\partial x_n} = R.$$

Integral surfaces passing through a given curve: Example

Example. Consider the quasi-linear PDE

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z. \tag{11}$$

Is there an integral surface of (11) which contains the line $x+y=0,\ z=1$?

Solution: The auxiliary equations are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}.$$

Taking

$$\frac{y\ dx + x\ dy}{xy^3+xyz-yx^3-yxz} = \frac{dz}{(x^2-y^2)z}$$

which will ultimately give us

$$\frac{d(xy)}{xy} = -\frac{dz}{z},$$

whose solution is

$$xyz = c_1. \tag{12}$$

Similarly taking

$$\frac{x\ dx + y\ dy}{x^2y^2 + x^2z - y^2x^2 - y^2z} = \frac{dz}{(x^2 - y^2)z},$$

gives us

$$x^2 + y^2 - 2z = c_2. \tag{13}$$

The line $x + y = 0,\ z = 1$, is given by $x = t, y = -t, z = 1$. We try to find $F(c_1, c_2) = 0$ such that $F(u, v) = 0$ contains $(t, -t, 1)$ for all values of t.

Substitute $x = t, y = -t, z = 1$ in (12) and (13) and get

$$-t^2 = c_1,\ 2t^2 - 2 = c_2.$$

Eliminating t from them, we obtain the relation

$$2c_1 + c_2 + 2 = 0,$$

showing that the desired integral surface is

$$x^2 + y^2 + 2xyz - 2z + 2 = 0.$$

Integral surfaces passing through a given curve

Problem. Using general solutions to determine the integral surface which passes through a given curve $x = x(t),\ y = y(t),\ z = z(t)$.
Solution. Find two solutions

$$u(x, y, z) = c_1, \quad v(x, y, z) = c_2 \tag{14}$$

of the auxiliary equations $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$. Then any solution of the corresponding linear equation is of the form $F(u, v) = 0$. Thus, we need to find a function $F = F_0$ such that

$$F_0\Big(u\big(x(t), y(t), z(t)\big), v\big(x(t), y(t), z(t)\big)\Big) = 0$$

for all values of t. Such a function F_0 may be obtained by putting

$$u\big(x(t), y(t), z(t)\big) = c_1, \quad v\big(x(t), y(t), z(t)\big) = c_2, \tag{15}$$

and getting a relation $F_0(c_1, c_2) = 0$ by eliminating t from (15).