MA 201: Laplace Transforms

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Inverse Laplace Transform

Applications of Laplace Transform in DE

Inverse Laplace Transform

If $\mathcal{L}\{F(t)\} = f(s)$, then $F(t)$ is said to be the inverse Laplace transform of $f(s)$. We then write $\mathcal{L}^{-1}\{f(s)\} = F(t)$.

Theorem (Linearity)

The inverse Laplace transform is linear, i.e.,

$$\mathcal{L}^{-1}\{a_1 f_1(s) \pm a_2 f_2(s)\} = a_1 \mathcal{L}^{-1}\{f_1(s)\} + a_2 \mathcal{L}^{-1}\{f_2(s)\}. \tag{1}$$

Theorem

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then $\mathcal{L}^{-1}\{f(s-a)\} = e^{at}F(t)$.

Theorem

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{e^{-as} f(s-a)\} = \begin{cases} F(t-a), & t \geq a, \\ 0, & t < a. \end{cases} \tag{2}$$

Examples:

1. $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at, \quad \mathcal{L}^{-1}\left\{\dfrac{a}{s^2-a^2}\right\} = \sinh at.$
2. $\mathcal{L}^{-1}\left\{\dfrac{s-2}{s^2+4s+13}\right\} = \dfrac{e^{-2t}}{3}[3\cos 3t - 4\sin 3t]:$

$$\frac{s-2}{s^2+4s+13} = \frac{s-2}{(s+2)^2+3^2} = \frac{s+2-4}{(s+2)^2+3^2}.$$

3. $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s+3)^3}\right\} =?$

Use the partial fractions to write

$$\frac{s^2}{(s+3)^3} = \frac{1}{s+3} - \frac{6}{(s+3)^2} + \frac{9}{(s+3)^3}.$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s+3)^3}\right\} = e^{-3t} - 6te^{-3t} + \frac{9}{2}t^2e^{-3t}.$$

Convolution Theorem

Theorem

If $F(t)$ and $G(t)$ are two functions of exponential order and given $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$, then

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(\tau)\, G(t-\tau)\, d\tau = F * G. \tag{3}$$

Convolution

Proof: By definition

$$\mathcal{L}\{F(t) * G(t)\} = \int_0^\infty e^{-st} \int_0^t F(\tau)G(t-\tau)\, d\tau\, dt.$$

The domain of this repeated integral takes the form of a wedge in the t, τ-plane. Write

$$\mathcal{L}\{F(t) * G(t)\} = \int_0^\infty \int_0^t e^{-st} F(\tau)G(t-\tau)\, d\tau\, dt.$$

Integrating with respect to t first

$$\begin{aligned}\mathcal{L}\{F(t) * G(t)\} &= \int_0^\infty \int_\tau^\infty e^{-st} F(\tau)G(t-\tau)\, d\tau\, dt \\ &= \int_0^\infty F(\tau) \left\{ \int_\tau^\infty e^{-st} G(t-\tau)\, dt \right\} d\tau.\end{aligned}$$

Convolution

In the inner integral above, put $u = t - \tau$ so that it can be written as

$$\begin{aligned}\int_{\tau}^{\infty} e^{-st} G(t-\tau)\, dt &= \int_{0}^{\infty} e^{-s(u+\tau)} G(u)\, du \\ &= e^{-s\tau} g(s).\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{L}\{F(t) * G(t)\} &= \int_{0}^{\infty} F(\tau) e^{-s\tau} g(s)\, d\tau \\ &= f(s) g(s),\end{aligned}$$

which gives us the following desired result:

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_{0}^{t} F(\tau)\, G(t-\tau)\, d\tau = F * G.$$

Convolution

Example: Find $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s-2)}\right\}$

The Laplace transforms of e^{2t} and $\cos t$ are, respectively,
$f(s) = \dfrac{1}{s-2}$ and $g(s) = \dfrac{s}{s^2+1}$.
Using the convolution theorem, we get

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s-2)}\right\} &= e^{2t} * \cos t \\
&= \int_0^t e^{2\tau}\cos(t-\tau)\, d\tau \\
&= \frac{2}{5}e^{2t} + \frac{1}{5}(\sin t - 2\cos t).
\end{aligned}$$

Inverse Laplace transform by method of residue

Theorem

When $\mathcal{L}\{F(x)\} = f(s)$ is considered on complex domain, i.e., $\mathfrak{Re}\,(s) > a$, the inverse Laplace transform $F(t)$ of $f(s)$ is given by

$$\mathcal{L}^{-1}\{f(s)\} = F(t) = \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} f(s) e^{st} ds.$$

Proof.

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left(\int_0^\infty F(t) e^{-st} dt \right) e^{sx} ds \\
= \;& \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\int_0^\infty F(t) e^{-(a+iw)t} dt \right) e^{(a+iw)x}\, i\, dw \\
= \;& \frac{a^{ax}}{2\pi} \int_{-\infty}^{\infty} \left(\int_0^\infty \left(F(t) e^{-at} \right) e^{-iwt} dt \right) e^{iwx} dw \\
= \;& e^{ax} \left(F(x) e^{-ax} \right) \; = \; F(x),
\end{aligned}$$

by Fourier integral formula, defining $F(x) = 0$ for $x < 0$.

Inverse Laplace transform by method of residue

Theorem

Suppose $f(s) = \mathcal{L}\{F(t)\}$ is analytic except at finitely many singular points (poles) c_i, $i = 1, \ldots, n$, each of which lies to the left of the vertical line $\mathfrak{Re}\,(s) = a$. If $sf(s)$ is bounded as s approaches infinity through the half-plane $\mathfrak{Re}\,(s) \leq a$, then

$$\begin{aligned}\mathcal{L}^{-1}\{f(s)\} &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} f(s)e^{st}\,ds \\ &= \sum_{i=1}^{n} \operatorname{Res}\,(f(s)e^{st}; c_i).\end{aligned}$$

Note: $\operatorname{Res}\,(f(s)e^{st}; c_i) = \lim_{s\to c_i}(s-c_i)f(s)e^{st}$, if c_i is a simple pole,

$$= \lim_{s\to c_i}\frac{1}{r!}\frac{d^{r-1}}{ds^{r-1}}\Big\{(s-c_i)^r f(s)e^{st}\Big\},$$

if c_i is a pole of order $r > 1$.

Inverse Laplace transform by method of residue

Example 1. Evaluate

$$\mathcal{L}^{-1}\left\{\frac{s^2-s+3}{s^3+6s^2+11s+6}\right\}.$$

Solution. We can see that $\lim_{s\to\infty} sf(s) = 1$ which is bounded. The poles are found to be $s=-1,-2,-3$ which are all simple poles. We have

$$\operatorname{Res}\left(f(s)e^{st};-1\right)=\frac{5}{2}e^{-t},\quad \operatorname{Res}\left(f(s)e^{st};-2\right)=-9e^{-2t},$$

$$\operatorname{Res}\left(f(s)e^{st};-3\right)=\frac{15}{2}e^{-3t}.$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{s^2-s+3}{s^3+6s^2+11s+6}\right\}=\frac{5}{2}e^{-t}-9e^{-2t}+\frac{15}{2}e^{-3t}.$$

Inverse Laplace transform by method of residue

Example 2. Evaluate

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)^2}\right\}.$$

Solution. Here $s=-1$ is a simple pole whereas $s=2$ is a double pole.

$$\begin{aligned}\operatorname{Res}\left(f(s)e^{st};-1\right) &= \frac{e^{-t}}{9},\\ \operatorname{Res}\left(f(s)e^{st};2\right) &= \left(\frac{t}{3}-\frac{1}{9}\right)e^{2t}\end{aligned}$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)^2}\right\} = \frac{e^{-t}}{9} + \left(\frac{t}{3}-\frac{1}{9}\right)e^{2t}.$$

Application of Laplace transform in solving ODEs

ODEs with constant coefficients

Example ODE1 (First order ODE):

$$\frac{dx}{dt} + 3x = 0, \; x(0) = 1.$$

By taking Laplace transform on both sides of the equation,

$$\begin{aligned}
&\mathcal{L}\left\{\frac{dx}{dt}\right\} + \mathcal{L}\{3x\} = 0 \\
\Rightarrow\ & s\mathcal{L}\{x\} - x(0) + 3\mathcal{L}\{x\} = 0 \\
\Rightarrow\ & (s+3)\mathcal{L}\{x\} = 1 \\
\Rightarrow\ & \mathcal{L}\{x\} = \frac{1}{s+3}
\end{aligned}$$

Taking inverse transform: $x = e^{-3t}$.

Application of Laplace transform in solving ODEs

$\boxed{\text{ODEs with constant coefficients}}$

Example ODE2 (Second order ODE):

$$\frac{d^2x}{dt^2} + x = t, \; x(0) = 1, \frac{dx}{dt}(0) = -2.$$

By taking Laplace transform on both sides of the equation,

$$\begin{aligned} & \mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} + \mathcal{L}\{x\} = \mathcal{L}\{t\} \\ \Rightarrow & \; s^2\mathcal{L}\{x\} - sx(0) - \dot{x}(0) + \mathcal{L}\{x\} = 1/s^2 \\ \Rightarrow & \; (s^2+1)\mathcal{L}\{x\} = \frac{1}{s^2} + s - 2 \\ \Rightarrow & \; \mathcal{L}\{x\} = \frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1} = \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}. \end{aligned}$$

Taking inverse transform $\quad x = t + \cos t - 3\sin t.$

Application of Laplace transform in solving ODEs

$\boxed{\text{ODEs with variable coefficients}}$

Example ODE3 (Second order ODE):

$$t\frac{d^2x}{dt^2} + 2(t-1)\frac{dx}{dt} + (t-2)x = 0.$$

By taking Laplace transform on both sides of the equation,

$$\mathcal{L}\{t\frac{d^2x}{dt^2}\} + 2\mathcal{L}\{(t-1)\frac{dx}{dt}\} + \mathcal{L}\{(t-2)x\} = 0$$

$$\Rightarrow -\frac{d}{ds}\mathcal{L}\{\ddot{x}\} - 2\frac{d}{ds}\mathcal{L}\{\dot{x}\} - 2\mathcal{L}\{\dot{x}\} - \frac{d}{ds}\mathcal{L}\{x\} - 2\mathcal{L}\{x\} = 0$$

which will ultimately lead to the differential equation

$$\frac{d}{ds}\mathcal{L}\{x\} + \frac{4s+4}{s^2+2s+1}\mathcal{L}\{x\} = \frac{3x_0}{(s+1)^2} \quad \text{(Here, } x_0 = x(0)).$$

Example ODE3 contd.

$$\frac{d}{ds}\mathcal{L}\{x\} + \frac{4s+4}{s^2+2s+1}\mathcal{L}\{x\} = \frac{3x_0}{(s+1)^2},$$

Giving us the solution (find the integrating factor as $(s+1)^4$)

$$\mathcal{L}\{x\} = \frac{x_0}{s+1} + \frac{C}{(s+1)^4}.$$

Taking inverse transform, $x = x_0\, e^{-t} + C\frac{t^3}{6}\, e^{-t}.$

Application of Laplace transform in solving ODEs

Simultaneous ODEs

Example ODE4 (First order):

$$\left.\begin{aligned} \tfrac{dx}{dt} &= 2x - 3y, \\ \tfrac{dy}{dt} &= y - 2x, \end{aligned}\right\} \quad x(0) = 8,\ y(0) = 3.$$

Taking Laplace transform on both sides of the first equation,

$$(s-2)\mathcal{L}\{x\} + 3\mathcal{L}\{y\} = 8. \tag{4}$$

Similarly, taking Laplace transform on both sides of the second equation,

$$2\mathcal{L}\{x\} + (s-1)\mathcal{L}\{y\} = 3. \tag{5}$$

By application of Cramer's rule in (4) and (5),

$$\mathcal{L}\{x\} = \frac{5}{s+1} + \frac{3}{s-4}, \quad \mathcal{L}\{y\} = \frac{5}{s+1} - \frac{2}{s-4}.$$

Application of Laplace transform in solving ODEs

$\boxed{\text{Example ODE4 contd.}}$

$$\begin{aligned}\mathcal{L}\{x\} &= \frac{5}{s+1}+\frac{3}{s-4},\\ \mathcal{L}\{y\} &= \frac{5}{s+1}-\frac{2}{s-4}.\end{aligned}$$

By taking the inverse transform,

$$x(t)=5e^{-t}+3e^{4t},\ \ y(t)=5e^{-t}-2e^{4t}.$$

Application of Laplace transform in solving ODEs

$\boxed{\text{Simultaneous ODEs}}$

Example ODE5 (Second order)

$$\left.\begin{array}{rl} \frac{d^2x}{dt^2} - x + 5\frac{dy}{dt} & = t, \\ \frac{d^2y}{dt^2} - 4y - 2\frac{dx}{dt} & = -2, \end{array}\right\} \quad x(0) = 0,\ \dot{x}(0) = 0,\ y(0) = 1,\ \dot{y}(0) = 0.$$

Taking Laplace transform on both sides of the equations,

$$\begin{aligned} (s^2-1)\mathcal{L}\{x\} + 5s\mathcal{L}\{y\} - 5 &= \frac{1}{s^2}, \\ -2s\mathcal{L}\{x\} + (s^2-4)\mathcal{L}\{y\} - s &= -\frac{2}{s}. \end{aligned}$$

Eliminating $\mathcal{L}\{x\}$ from the above equations,

$$\mathcal{L}\{y\} = \frac{1}{s} - \frac{2}{3}\frac{s}{s^2+4} + \frac{2}{3}\frac{s}{s^2+1},$$

Application of Laplace transform in solving ODEs

$\boxed{\text{Example ODE5 contd.}}$

$$\mathcal{L}\{y\} = \frac{1}{s} - \frac{2}{3}\frac{s}{s^2+4} + \frac{2}{3}\frac{s}{s^2+1},$$

On inversion it gives

$$y(t) = 1 - \frac{2}{3}\cos 2t + \frac{2}{3}\cos t.$$

Substituting back into the second original equation gives

$$x(t) = -t - \frac{5}{3}\sin t + \frac{4}{3}\sin 2t.$$

Application of Laplace transform in solving PDEs

The Laplace transform of a function $u(x, t)$ with respect to t is given by

$$\mathcal{L}\{u(x,t)\} = \overline{u}(x,s) = \int_0^\infty e^{-st} u(x,t)\, dt. \tag{6}$$

We get

$$\mathcal{L}\left\{\frac{\partial}{\partial t}\, u(x,t)\right\} = s\bar{u}(x,s) - u(x,0), \tag{7}$$

$$\mathcal{L}\left\{\frac{\partial^2}{\partial t^2}\, u(x,t)\right\} = s^2\bar{u}(x,s) - su(x,0) - u_t(x,0), \tag{8}$$

$$\mathcal{L}\left\{\frac{\partial}{\partial x}\, u(x,t)\right\} = \frac{d}{dx}\, \bar{u}(x,s), \tag{9}$$

$$\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\, u(x,t)\right\} = \frac{d^2}{dx^2}\, \bar{u}(x,s). \tag{10}$$

Application of Laplace transform in solving PDEs

Example PDE1: (First order) Find a bounded solution of the following problem

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{subject to} \quad u(x,0) = 6\ e^{-3x}.$$

Solution. Taking Laplace transform on both sides of the given PDE and using the initial condition,

$$\frac{d\bar{u}}{dx} - (2s+1)\bar{u} = -12e^{-3x}.$$

After finding the integrating factor,

$$\bar{u}(x,s) = \frac{6}{s+2}\ e^{-3x} + C\ e^{(2s+1)x}.$$

Application of Laplace transform in solving PDEs

Example PDE1 contd.

Now, $u(x,t)$ should be bounded when $x \to \infty$. Hence its Laplace transform $\bar{u}(x,s)$ should also be bounded as $s \to \infty$ and we take $C = 0$. Thus,

$$\bar{u}(x,s) = \frac{6}{s+2}\, e^{-3x}.$$

Taking the inverse transform

$$u(x,t) = 6e^{-(2t+3x)}.$$

Application of Laplace transform in solving PDEs

Example PDE2: (Second order) Consider the one-dimensional heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1,\ t > 0$$

subject to the following conditions:

$$u(0,t) = 1,\ u(1,t) = 1,\ t > 0;\ u(x,0) = 1 + \sin \pi x,\ 0 < x < 1.$$

Solution. Taking Laplace transform on both sides and applying the given initial condition,

$$\frac{d^2}{dx^2}\,\bar{u}(x,s) - s\bar{u}(x,s) = 1 + \sin \pi x.$$

Application of Laplace transform in solving PDEs

Example PDE2 contd.

$$\frac{d^2}{dx^2}\,\bar{u}(x,s) - s\bar{u}(x,s) = 1 + \sin \pi x.$$

The complementary function and the particular integral of the above equation can be derived as

$$\begin{aligned}\bar{u}_c(x,s) &= Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x},\\ \bar{u}_p(x,s) &= \frac{1}{s} + \frac{\sin \pi x}{s+\pi^2}.\end{aligned}$$

Thus,

$$\bar{u}(x,s) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x} + \frac{1}{s} + \frac{\sin \pi x}{s+\pi^2}. \tag{11}$$

Convert the boundary conditions in terms of $\bar{u}(x,s)$:

$u(0,t) = 1 \Rightarrow \bar{u}(0,s) = \frac{1}{s}, \;\; u(1,t) = 1 \Rightarrow \bar{u}(1,s) = \frac{1}{s}.$

Application of Laplace transform in solving PDEs

$\boxed{\text{Example PDE2 contd.}}$

Using the boundary conditions in (11)

$$\begin{aligned} \frac{1}{s} &= A + B + \frac{1}{s} \ \Rightarrow A + B = 0, \\ \frac{1}{s} &= Ae^{\sqrt{s}} + Be^{-\sqrt{s}} + \frac{1}{s} \ \Rightarrow Ae^{\sqrt{s}} + Be^{-\sqrt{s}} = 0. \end{aligned}$$

These imply $A = 0 = B$, i.e.,

$$\bar{u}(x,s) = \frac{1}{s} + \frac{\sin \pi x}{s + \pi^2}.$$

Solution is obtained by taking the inverse

$$u(x,t) = 1 + e^{-\pi^2 t} \sin \pi x. \tag{12}$$

Application of Laplace transform in solving PDEs

Example PDE3: (Second order)

$$\begin{aligned} U_{tt} &= c^2 U_{xx} + \sin\left(\frac{\pi x}{l}\right)\sin(\sigma t),\ 0 < x < l,\ t > 0, \\ U(0,t) &= 0,\ U(l,t) = 0,\ t > 0, \\ U(x,0) &= 0,\ U_t(x,0) = 0,\ 0 < x < l. \end{aligned}$$

Solution. Taking Laplace transform on the equation, it gets reduced to

$$\frac{d^2}{dx^2}\overline{u}(x,s) - \frac{s^2}{c^2}\overline{u}(x,s) = -\frac{\sigma \sin(\pi x/l)}{c^2(s^2+\sigma^2)}$$

the solution of which can be obtained as

$$\overline{u}(x,s) = A(s)e^{\sqrt{\frac{s}{c}}x} + B(s)e^{-\sqrt{\frac{s}{c}}x} + \frac{\sigma}{c^2}\frac{\sin(\pi x/l)}{(s^2+\sigma^2)(\frac{s^2}{c^2}+\frac{\pi^2}{l^2})}$$

Application of Laplace transform in solving PDEs

Example PDE3 contd.

The solution can be simplified to

$$\overline{u}(x,s) = A(s)e^{\sqrt{\frac{s}{c}}x} + B(s)e^{-\sqrt{\frac{s}{c}}x} + \frac{\sigma}{\left(\frac{c^2\pi^2}{l^2} - \sigma^2\right)}\left(\frac{1}{s^2+\sigma^2} - \frac{1}{s^2+\frac{c^2\pi^2}{l^2}}\right)$$

Taking transforms on the boundary conditions

$$\overline{u}(0,s) = 0,\ \overline{u}(l,s) = 0$$

which give $A(s) = 0 = B(s)$. Thus

$$\overline{u}(x,s) = \frac{l^2\sigma}{c^2\pi^2 - \sigma^2 l^2}\left(\frac{1}{s^2+\sigma^2} - \frac{1}{s^2+\frac{c^2\pi^2}{l^2}}\right)$$

Application of Laplace transform in solving PDEs

Example PDE3 contd.

Inverting we get

$$U(x,t) = \frac{l^2\sigma}{c^2\pi^2 - \sigma^2 l^2}\left[\frac{1}{\sigma}\sin\sigma t - \frac{l}{c\pi}\sin(\frac{\pi ct}{l})\right]$$

That is

$$U(x,t) = \frac{l^2}{c^2\pi^2 - \sigma^2 l^2}\left[\sin\sigma t - \frac{l\sigma}{c\pi}\sin(\frac{\pi ct}{l})\right]$$