Elementary functions

- Exponential Function: e^z
- Trigonometric functions: $\sin z,\ \cos z, \ldots$
- Hyperbolic functions: $\sinh z, \cosh x, \ldots$
- Logarithm function: $\log z,\ \operatorname{Log} z$
- Complex exponents / Power function: z^w
- Inverse trigonometric functions: $\sin^{-1} z, \cos^{-1} z, \ldots$

The Exponential Function

Recall:

- $e^x : \mathbb{R} \to \mathbb{R}$ and $e^{x+y} = e^x e^y$ for any $x, y \in \mathbb{R}$.
- Euler's Formula: For $\theta \in \mathbb{R}$, $e^{i\theta} = \cos\theta + i\sin\theta$.
- For $\theta, \phi \in \mathbb{R}$, $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$.

Definition: For $z = x + iy$ e^z or $\exp(z)$ is defined by the formula

$$e^z = e^{(x+iy)} := e^x(\cos y + i\sin y) = e^x e^{iy}.$$

$e^z : \mathbb{C} \to \mathbb{C}$ is called the exponential function.

Properties of Exponential Function

- $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$. For $z = x + iy$, $w = s + it$

$$\begin{aligned} e^{z+w} &= e^{(x+s)+i(y+t)} = e^{(x+s)}e^{i(y+t)} \\ &= (e^x e^s)(e^{iy}e^{it}) = (e^x e^{it})(e^s e^{iy}) = e^z e^w. \end{aligned}$$

- $|e^z| = e^{\Re(z)}$ and $\arg(e^z) = \Im(z) + 2n\pi, n \in \mathbb{Z}$.
- $e^z \neq 0$, for all $z \in \mathbb{C}$. Indeed $|e^z| = e^{\Re(z)} > 0$.

Properties of Exponential Function

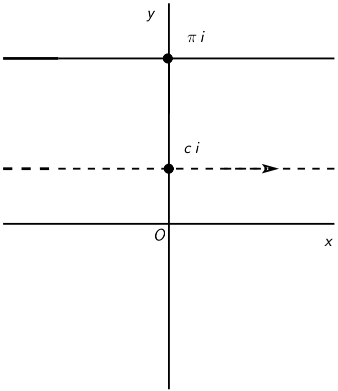
- $\frac{d}{dz}e^z = e^z,\ z \in \mathbb{C}$: e^z is an entire function. $\Big[e^z = e^x \cos y + ie^x \sin y$, i.e., $u = e^x \cos y,\ v = e^x \sin y$. Check: Everywhere in $\mathbb{C}$, $u_x = v_y, u_y = -v_x$ and the partial derivatives are continuous. Thus, e^z is entire and
$$\frac{d}{dz}e^z = u_x + iv_x = e^x \cos y + ie^x \sin y = e^z.\Big]$$

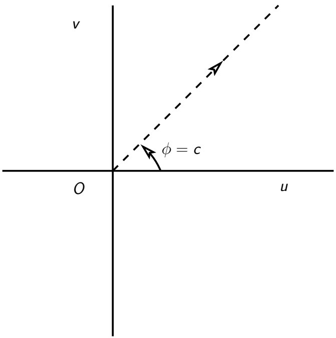
A function $f : \mathbb{C} \to \mathbb{C}$ is said to be **periodic** if there is a $w \in \mathbb{C}$ such that $f(z + w) = f(z)$ for all $z \in \mathbb{C}$. Then, w is called a **period** of f.

- e^z is periodic of period $2n\pi i$ for any $n \in \mathbb{Z}$. $[e^{z+2n\pi i} = e^z \ \ \forall z \in \mathbb{C}.]$
- e^z is not injective (one-one) in $\mathbb{C}$, unlike the real exponential. $[e^0 = e^{2n\pi i} = 1$ for any $n \in \mathbb{Z}.]$
- $\overline{e^z} = e^{\bar{z}}$. $\left[\overline{e^z} = \overline{e^x e^{iy}} = e^x\,\overline{e^{iy}} = e^x e^{-iy} = e^{x-iy} = e^{\bar{z}}.\right]$
- $|e^z| \le e^{|z|}$. $\left[x \le |z| \implies |e^z| = e^x \le e^{|z|}.\right]$
- $|e^z| = e^{|z|}$ if $z \ge 0$.

A horizontal line in $\mathbb{C}$ is mapped to an open ray (open at 0).

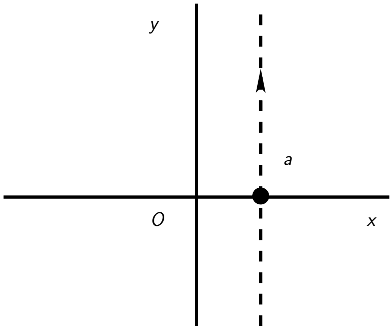
$\{x + ci : x \in \mathbb{R}\} \longmapsto \{(r, \phi) : r = e^x, \phi = c,\ x \in \mathbb{R}\}$





A vertical line in $\mathbb{C}$ is mapped to a circle centered at 0.

$\{a + iy : y \in \mathbb{R}\} \longmapsto \{(r, \theta) : r = e^a,\, \theta = y,\; y \in \mathbb{R}\}$



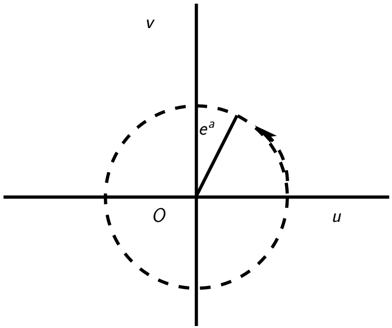
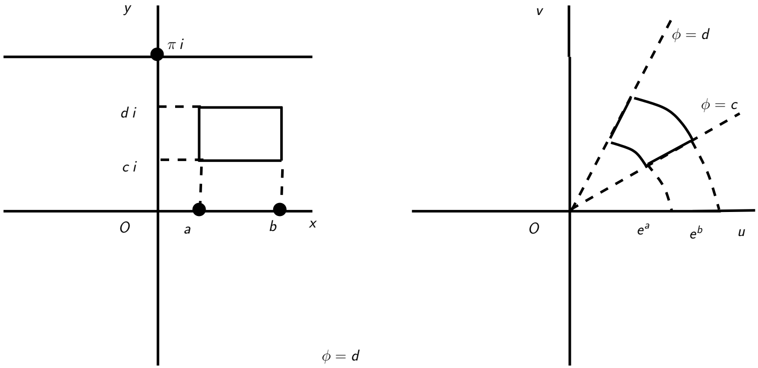


Image under the Exponential Function

A small rectangular region in $\mathbb{C}$ is mapped to an annular segment.

$$\{x+iy : a \le x \le b,\ c \le y \le d\} \longmapsto \{(r,\theta) : e^a \le r \le e^b,\ c \le \phi \le d\}.$$



Trigonometric Functions

Define for $z \in \mathbb{C}$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}); \quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz}).$$

Properties:

- $\sin^2 z + \cos^2 z = 1.$
- $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$,
  $\sin(z + 2k\pi) = \sin z$, $\cos(z + 2k\pi) = \cos z$,.
- $\sin z = 0 \iff z = n\pi$ and $\cos z = 0 \iff z = (n + \frac{1}{2})\pi, \ , \ n \in \mathbb{Z}.$
- $\sin z, \cos z$ are entire functions.
- $\frac{d}{dz}(\sin z) = \cos z$, $\frac{d}{dz}(\cos z) = -\sin z$.
- Is $\sin z$ a bounded function?
  Note $\sin(-iy) = \dfrac{1}{2i}(e^{y} - e^{-y}) \to \infty$ as $y \to \infty$.

We define $\quad \tan z = \dfrac{\sin z}{\cos z}, \ \sec z = \dfrac{1}{\cos z}$ for $z \neq (n + \frac{1}{2})\pi$,

$$\cot z = \frac{\cos z}{\sin z}, \ \csc z = \frac{1}{\sin z} \text{ for } z \neq n\pi, \ n \in \mathbb{Z}$$

These functions are analytic.

Trigonometric functions

- For $z \in \mathbb{C}$, $e^{iz} = \cos z + i \sin z$: Follows from definitions

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}); \quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz}).$$

- $\sin(z + w) = \sin z \cos w + \cos z \sin w$,
  $\cos(z + w) = \cos z \cos w - \sin z \sin w$.
- For real y, $\cosh y = \frac{1}{2}(e^y + e^{-y})$, $\sinh y = \frac{1}{2}(e^y - e^{-y})$. So,

$$\sin(iy) = \frac{1}{2i}(e^{-y} - e^{y}) = i \sinh y, \quad \cos(iy) = \frac{1}{2}(e^{y} + e^{-y}) = \cosh y.$$

Thus,

$$\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$
$$\cos z = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y.$$

Exercise. Prove: For any $z \in \mathbb{C}$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \quad |\cos z|^2 = \cos^2 x + \sinh^2 y.$$

Hint. Use $\cosh^2 y - \sinh^2 y = 1$.

Hyperbolic Trigonometric functions

Define for $z \in \mathbb{C}$

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}.$$

Properties:

- $\sinh z, \cosh z$ are **entire** functions.
- $\cosh^2 z - \sinh^2 z = 1$.
- $\sinh(-z) = -\sinh z$, $\cosh(-z) = \cosh z$,
- $\sinh(z + 2k\pi i) = \sinh z$, $\cosh(z + 2k\pi i) = \cosh z$, $k \in \mathbb{Z}$.
- $\sinh(iz) = i \sin z$ and $\cosh(iz) = \cos z$
- $\sinh z = 0 \iff z = n\pi i$ and $\cosh z = 0 \iff z = (n + \frac{1}{2})\pi i,$, $n \in \mathbb{Z}$.
- $\frac{d}{dz}(\sinh z) = \cosh z$, $\frac{d}{dz}(\cosh z) = \sinh z$.

We define $\tanh z = \dfrac{\sinh z}{\cosh z}$, $\operatorname{csc h} z = \dfrac{1}{\sinh z}$ for $z \neq (n + \frac{1}{2})\pi i$,

$$\coth z = \frac{\cosh z}{\sinh z}, \quad \operatorname{sec h} z = \frac{1}{\cosh z} \text{ for } z \neq n\pi i, \ n \in \mathbb{Z}$$

These functions are analytic.

A multiple valued function

- The function z^2 is not one-one: $z^2 = (-z)^2$. Can z^2 have an inverse? Is $w = z^{1/2}$ a function?
- If $0 \neq z = re^{i\theta}$ then $w^2 = z$ if
$$w = w_1 = \sqrt{r}e^{i\theta/2} \quad \text{or} \quad w = w_2 = \sqrt{r}e^{i(\pi+\theta/2)}.$$
We write $z^{1/2} = \{w_1, w_2\}$, and say $z^{1/2}$ is a multiple valued function.
- Consider the two functions
$$f_1(z) = \sqrt{|z|}e^{i\mathrm{Arg}(z)/2}, \quad f_2(z) = \sqrt{|z|}e^{i(\pi+\mathrm{Arg}(z)/2)}$$
for $z \neq 0$, and 0 at $z = 0$. Then, $(f_1(z))^2 = z = (f_2(z))^2$ for all z.
- f_1 maps $\mathbb{C} \setminus \{0\}$ into the right half plane and f_2 into the left half plane. Note: $f_2(z) = -f_1(z)$.
- $-\pi < \mathrm{Arg}(z) \leq \pi$, and f_1 and f_2 are discontinuous at every point on negative real axis (i.e. when $\mathrm{Arg}(z) = \pi$).
- f_1 and f_2 when restricted to the open set $\mathbb{C} \setminus \{z : z \leq 0\}$ are analytic. These analytic functions are two branches of $z^{1/2}$.
- The closed ray $\{z : z \leq 0\}$ is the branch cut for f_1 and f_2.
- In fact, for any $\alpha \in \mathbb{R}$ you can get a branch of $z^{1/2}$ with the ray $\theta = \alpha$ as the branch cut. Take $z = re^{i\theta}$ where $\alpha < \theta \leq \alpha + 2\pi$.

Branches of a multiple valued function

- **Branch:** Let F be a multiple valued function defined on a domain D. A single valued function f defined on a domain $D_0 \subset D$ is a branch of F, if f is analytic in D_0 and $f(z)$ takes a value of $F(z)$.

  Example. f_1 and f_2 just defined are branches of $z^{1/2}$.

- **Branch Cut:** The portion of a line or a curve introduced in order to define a branch f of a multiple valued function F is called the branch cut for the branch.

  Example. The closed ray $\{z : z \leq 0\}$ is the branch cut for f_1 and f_2.

- **Branch Point:** Any point that is common to all branch cuts for a multiple valued function F is called a branch point for F.

  Example. $z = 0$ is the branch point for $z^{1/2}$.

Logarithm function

- Given $z \in \mathbb{C}$ is there $w \in \mathbb{C}$ such that $e^w = z$? Yes, if $z \neq 0$. Let $z = re^{i\theta}$, $w = u + iv$. Then

$$e^w = z \implies e^u e^{iv} = re^{i\theta} \implies \begin{aligned} u &= \ln r, \\ v &= \theta + 2n\pi = \arg(z). \end{aligned}$$

  i.e., $w = \ln|z| + i\arg(z) = \{\ln|z| + i(\mathrm{Arg}(z) + 2n\pi) : n \in \mathbb{Z}\}$.
  For each value of w we have $e^w = z$.

- Define **complex logarithm** to be the multiple valued function on $\mathbb{C} \setminus \{0\}$ given by

$$\log z = \ln|z| + i\arg(z) = \{\ln|z| + i(\mathrm{Arg}(z) + 2n\pi) : n \in \mathbb{Z}\}.$$

- Consider the horizontal strip $H = \{z = x + iy : -\pi < y \leq \pi\}$. $e^z : H \to \mathbb{C} \setminus \{0\}$ is a bijection. The inverse of this function is the **principal branch** of $\log z$: $\mathrm{Log}\, z = \ln|z| + i\mathrm{Arg}(z)$.

Example. $\log i = i(\pi/2 + 2k\pi),\ k \in \mathbb{Z}, \quad \mathrm{Log}\, i = i\pi/2.$

$\log(-1) = i(\pi + 2k\pi),\ k \in \mathbb{Z},\ \ \mathrm{Log}\,(-1) = i\pi$

$\log(-1 - i) = \ln(\sqrt{2}) + i(\frac{-3\pi}{4} + 2k\pi),\ k \in \mathbb{Z},\ \ \mathrm{Log}\,(-1 - i) = \ldots$

Complex Logarithm

- $\operatorname{Log} z = \ln|z| + i\operatorname{Arg}(z)$. So, for $z = x > 0$, $\operatorname{Log} z = \ln z$, i.e., Log extends ln.
- For $z \neq 0$ $e^{\operatorname{Log} z} = e^{\ln|z| + i\operatorname{Arg}(z)} = e^{\ln|z|}e^{i\operatorname{Arg}(z)} = |z|e^{i\operatorname{Arg}(z)} = z$.
  Is $\operatorname{Log}(e^z) = z$? What is $\log(e^z)$?
- The identity $\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$ is not always valid. Give example. Valid if and only if $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 \in (-\pi, \pi]$ (Check).
- $\operatorname{Log} z$ is not continuous on the negative real axis $\mathbb{R}^- = \{z = x < 0\}$.

  To see this consider the point $z_0 = -\alpha$, $\alpha > 0$. Then,
  as $\theta \to \pi^-$ we have $z = \alpha e^{i\theta} \to z_0$ from above and $\operatorname{Log} z \to (\ln\alpha + i\pi)$,
  as $\theta \to -\pi^+$ we have $z = \alpha e^{i\theta} \to z_0$ from below and $\operatorname{Log} z \to (\ln\alpha - i\pi)$
  So, $\operatorname{Log} z$ is not continuous at z_0.
- $\operatorname{Log} z$ is analytic on $\mathbb{C} \setminus \mathbb{R}^-$: Let $z = re^{i\theta}$, $r > 0, \theta \in (-\pi, \pi)$. Then $\operatorname{Log} z = \ln r + i\theta = u(r,\theta) + iv(r,\theta)$. Now, $u_r = \dfrac{1}{r}v_\theta = \dfrac{1}{r}$, $v_r = -\frac{1}{r}u_\theta = 0$, and the partial derivatives are continuous. Thus,
  $$\frac{d}{dz}\operatorname{Log} z = e^{-i\theta}(u_r + iv_r) = \frac{1}{r}e^{-i\theta} = \frac{1}{z}.$$
- $\operatorname{Log} z$ on $\mathbb{C} \setminus \mathbb{R}^-$ is a branch of $\log z$. Branch cut: $\mathbb{R}^-$. What are other branches? What are the branch points?

Complex Exponents

Let $w \in \mathbb{C}$. For any $z \neq 0$, define

$$z^w = e^{w \log z} = exp(w \log z)$$

- $i^i = \exp[i \log i] = \exp[i(\log 1 + i\frac{\pi}{2})] = \exp(-\frac{\pi}{2})$.
- For fixed $a, c \in \mathbb{C}$ a^z and z^c are multiple valued functions.

Inverse trigonometric functions

Note: $\sin(z + 2n\pi) = \sin z$. So, $\sin z$ is not one-one. However, it is onto (Range of $\sin z$ is $\mathbb{C}$).

Suppose $w \in \mathbb{C}$. Then, $\sin w = z \Rightarrow \dfrac{e^{iw} - e^{-iw}}{2i} = z \Rightarrow e^{2iw} - 2ize^{iw} - 1 = 0$.

i.e., $e^{iw} = \dfrac{2iz + \left((-2iz)^2 - 4\right)^{1/2}}{2} = iz + (1 - z^2)^{1/2}$.

- $\text{arc } \sin(z) = \sin^{-1}(z) = -i \log\left[iz + (1 - z^2)^{1/2}\right]$.
- $\text{arc } \cos(z) = \cos^{-1}(z) = -i \log\left[z + i(1 - z^2)^{1/2}\right]$.
- $\text{arc } \tan(z) = \tan^{-1}(z) = \left(\dfrac{i}{2}\right) \log\left(\dfrac{i + z}{i - z}\right)$.
- $$\frac{d}{dx}\left(\sin^{-1} z\right) = \frac{1}{(1 - z^2)^{1/2}},$$
  $$\frac{d}{dx}\left(\cos^{-1} z\right) = \frac{-1}{(1 - z^2)^{1/2}}$$
  $$\frac{d}{dx}\left(\tan^{-1} z\right) = \frac{1}{(1 + z^2)}$$

Inverse hyperbolic functions

- $\sinh^{-1}(z) = \log\left[z + (z^2+1)^{1/2}\right]$.
- $\cosh^{-1}(z) = \log\left[z + (z^2-1)^{1/2}\right]$.
- $\tanh^{-1}(z) = \log\left[z + (z^2-1)^{1/2}\right]$.