MA 201: Partial Differential Equations
Lecture - 11

Heat Equation

Heat conduction in a thin rod

The IBVP under consideration consists of:

The governing equation:

$$u_t = \alpha u_{xx}, \tag{1}$$

where α is the thermal diffusivity.

The boundary conditions: for all $t > 0$

$$u(0, t) = 0, \tag{2a}$$

$$u(l, t) = 0. \tag{2b}$$

The initial condition: for $0 \le x \le l$

$$u(x, 0) = f(x). \tag{3}$$

Heat conduction in a thin rod (Contd.)

We use separation of variable technique: assume a solution

$$u(x,t) = X(x)T(t).$$

The governing equation is converted to the following ODEs:

$$\begin{aligned} X'' - kX &= 0, \\ T' - k\alpha T &= 0. \end{aligned}$$

where k is a separation constant. The boundary conditions for this IBVP are the same as those in the one-dimensional wave equation. We assume that a feasible nontrivial solution exists for the negative values of k. The above equations give

$$X'' + \lambda^2 X = 0, \tag{4}$$

$$T' + \lambda^2 \alpha T = 0. \tag{5}$$

Heat conduction in a thin rod (Contd.)

The solutions:

$$\begin{aligned} X(x) &= A\cos(\lambda x) + B\sin(\lambda x), & (6)\\ T(t) &= Ce^{-\alpha\lambda^2 t}. & (7) \end{aligned}$$

Solution for $u(x,t)$:

$$u(x,t) = [A\cos(\lambda x) + B\sin(\lambda x)]\,Ce^{-\alpha\lambda^2 t}. \qquad (8)$$

Using the boundary conditions (2a) and (2b) we get $A = 0$ and $\lambda_n = \frac{n\pi}{l},\ n = 1, 2, 3, \ldots$.
Solution corresponding to each n:

$$u_n(x,t) = B_n \sin\left(\frac{n\pi x}{l}\right)\, e^{-\alpha \frac{n^2\pi^2}{l^2}\, t}.$$

Heat conduction in a thin rod (Contd.)

The general solution:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha \frac{n^2\pi^2}{l^2} t}$$

where B_n is obtained (refer to the solution of the one-dimensional wave equation) by using the initial condition (3):

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx, \; n = 1, 2, 3, \ldots \tag{9}$$

Observe that what we have solved is a homogenous equation subject to homogenous (zero) conditions. Ideally the boundary conditions are very likely to be non-homogenous for diffusion problems. (It will be discussed later.)

Heat conduction problem with flux and radiation conditions

An insulation-type flux condition at, say, $x = 0$ is a condition on the derivative of the form $u_x(0, t) = 0, t > 0$; because the flux is proportional to the temperature gradient (Fourier's heat flow law states flux $= -\alpha u_x$, where α is the conductivity).

Heat flux is defined as the rate of heat transfer per unit cross-sectional area.

The insulation condition requires that no heat flow across the boundary $x = 0$.

A radiation condition, on the other hand, is a specification of how heat radiates from the end of the rod, say at $x = 0$, into the environment, or how the end absorbs heat from its environment.

Linear, homogeneous radiation condition take the form $-\alpha u_x(0, t) + bu(0, t) = 0, t > 0$, where b is a constant.

A typical problem in heat conduction may have a combination of Dirichlet, insulation and radiation boundary conditions.

Heat conduction problem with flux conditions

Consider the IBVP consisting of the following:

$$u_t = \alpha u_{xx} \ , \tag{10}$$

The boundary conditions for all $t > 0$

$$u_x(0, t) = 0, \tag{11a}$$

$$u(l, t) = 0. \tag{11b}$$

The initial condition for $0 \leq x \leq l$:

$$u(x, 0) = f(x). \tag{12}$$

The boundary conditions (11) tell us that the left end of the rod is insulated and the right end is kept at zero degrees.

Heat conduction problem with flux conditions

Assuming a solution of the form $u(x,t) = X(x)T(t)$ the PDE is converted to the following ODEs:

$$\begin{aligned} X'' - kX &= 0, \\ T' - k\alpha T &= 0. \end{aligned}$$

Taking $k = -\lambda^2$, the above equations become

$$X'' + \lambda^2 X = 0, \tag{13}$$
$$T' + \lambda^2 \alpha T = 0. \tag{14}$$

Giving us the solutions:

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{15}$$
$$T(t) = Ce^{-\alpha\lambda^2 t}. \tag{16}$$

The solution:

$$u(x,t) = [A\cos(\lambda x) + B\sin(\lambda x)]Ce^{-\alpha\lambda^2 t}. \tag{17}$$

Heat conduction problem with flux conditions

To use the boundary condition (11a) we have to differentiate (17) w.r.t. x to get

$$u_x = \lambda[-A\sin(\lambda x) + B\cos(\lambda x)]Ce^{-\alpha\lambda^2 t}. \tag{18}$$

Using boundary condition (11a) $B = 0$.

The boundary condition (11b) will give the eigenvalues as

$$\lambda_n = \left(\frac{2n+1}{2}\right)\frac{\pi}{l},\ n = 0, 1, 2, 3, \ldots.$$

Heat conduction problem with flux conditions

The solution to the IBVP will be given by

$$u(x,t) = \sum_{n=0}^{\infty} C_n \exp\left[-\alpha\left(\frac{2n+1}{2}\right)^2 \frac{\pi^2}{l^2} t\right] \cos(\frac{(2n+1)\pi x}{2l}), \tag{19}$$

where C_n can be obtained from IC (12):

$$f(x) = \sum_{n=0}^{\infty} C_n \cos(\frac{(2n+1)\pi x}{2l})$$

as

$$C_n = \frac{2}{l}\int_0^l f(x)\ \cos(\frac{(2n+1)\pi x}{2l})\, dx,\ \ n = 0, 1, 2, 3, \ldots \tag{20}$$

Note that here $n = 0$ also contributes to the solution

In other words, we can say that the eigenvalue λ_0, corresponding to $n = 0$ also contributes.

Heat conduction problem with different flux conditions at ends

Note the difference in solution when Dirichlet condition at both ends of the rod are changed to one Neumann (at $x = 0$) and one Dirichlet (at $x = l$) conditions

We may have two other combinations of pairs of conditions, viz.,

3. Dirichlet condition at $x = 0$ and Neumann condition at $x = l$
4. Neumann condition at both ends $x = 0$ and $x = l$.

Heat conduction problems with different conditions at ends

Problem I. Governing equation: $u_t = \alpha u_{xx}$

Boundary conditions: $u(0,t) = 0 = u(l,t)$,

Initial Condition: $u(x,0) = f(x)$

Solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l})\ e^{-\alpha \frac{n^2\pi^2}{l^2}\ t}.$$

with

$$A_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx.$$

Heat conduction problems with different conditions at ends

Problem II. Governing equation $u_t = \alpha u_{xx}$

Boundary conditions: $u_x(0,t) = 0 = u(l,t)$,

Initial Condition: $u(x,0) = f(x)$

The solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \exp\left[-\alpha\left(\frac{2n+1}{2}\right)^2 \frac{\pi^2}{l^2} t\right] \cos(\frac{(2n+1)\pi x}{2l}).$$

with

$$A_n = \frac{2}{l}\int_0^l f(x)\cos(\frac{(2n+1)\pi x}{2l})dx.$$

Heat conduction problems with different conditions at ends

Problem III. Governing equation $u_t = \alpha u_{xx}$

Boundary conditions: $u(0,t) = 0 = u_x(l,t)$,

Initial Condition: $u(x,0) = f(x)$

Solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \exp\left[-\alpha\left(\frac{2n+1}{2}\right)^2 \frac{\pi^2}{l^2} t\right] \sin\left(\frac{(2n+1)\pi x}{2l}\right)$$

with

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{(2n+1)\pi x}{2l}\right)dx.$$

Heat conduction problems with different conditions at ends

Problem IV. Governing equation $u_t = \alpha u_{xx}$
Boundary conditions: $u_x(0,t) = 0 = u_x(l,t)$,
Initial Condition: $u(x,0) = f(x)$ The solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l})\ e^{-\alpha \frac{n^2\pi^2}{l^2}\ t}$$

with

$$A_n = \frac{2}{l}\int_0^1 f(x)\cos(\frac{n\pi x}{l})dx.$$

In each of the four solutions $\boxed{\text{OBSERVE}}$ the function for x and also the eigenvalues. TRY solving Problems III and IV yourself.

A Non-homogeneous Heat Conduction Problem

Consider the non-homogeneous PDE

$$u_t = \alpha u_{xx} + ax, \tag{21}$$

where a is a nonzero constant.

Boundary conditions: $u(0,t) = 0 = u(l,t)$,

Initial Condition: $u(x,0) = f(x)$

Seek a solution in the form:

$$u(x,t) = v(x,t) + h(x), \tag{22}$$

where $h(x)$ is an unknown function of x alone.

Now substituting (22) into (21)

$$v_t = \alpha[v_{xx} + h''(x)] + ax, \tag{23}$$

subject to

$$v(0,t) + h(0) = 0, \tag{24a}$$
$$v(l,t) + h(l) = 0, \tag{24b}$$

and

$$v(x,0) + h(x) = f(x), \tag{25}$$

Equations (23)-(25) can be conveniently split into two problems:

Problem I:

$$\begin{aligned} v_t &= \alpha v_{xx}, \\ v(0,t) &= 0 = v(l,t), \\ v(x,0) &= f(x) - h(x). \end{aligned}$$

Problem II:

$$\begin{aligned} \alpha h''(x) &= -ax, \\ h(0) &= 0 = h(l). \end{aligned}$$

The solution of Problem I is known to us from the previous problem, which is: Solution is

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l})\, e^{-\alpha \frac{n^2\pi^2}{l^2}\, t}. \tag{26}$$

with

$$A_n = \frac{2}{l}\int_0^l [f(x) - h(x)] \sin(\frac{n\pi x}{l}) dx.$$

The solution for Problem II can be easily found by integrating $h''(x)$ twice and using the boundary conditions:

$$h(x) = \frac{-ax}{6\alpha}(x^2 - l^2). \tag{27}$$

The coefficients A_n in (26) can now be obtained by putting $h(x)$ as in and (27). Hence the solution $u(x, t)$ for our IBVP is given by the sum of (26) and (27).

Time-Independent Non-homogeneous BC

We now turn to the situation where the BC are not both homogeneous, but are independent of time variable t. The method of solution consists of the following steps:

- Step 1: Find a particular solution of the PDE and BC.
- Step 2: Find the solution of a related problem with homogeneous BC. Then, add this solution to that particular solution obtained in Step 1.

The procedure is illustrated in the following example:

$$\text{PDE:} \qquad u_t = \alpha u_{xx}, \quad 0 \le x \le l,\ t > 0, \tag{28}$$

$$\text{BC:} \qquad u(0,t) = a,\ \ u(l,t) = b, \quad t > 0, \tag{29}$$

$$\text{IC:} \qquad u(x,0) = f(x), \quad 0 \le x \le l, \tag{30}$$

where a and b are arbitrary constants and $f(x)$ is a given function.

Time-Independent Non-homogeneous BC (Contd.)

Solution. Seek a particular solution $u_p(x)$ of the form $u_p(x) = cx + d$, where c and d are to be chosen so that the BC are satisfied:

$$\begin{aligned} & a = u_p(0) = c \cdot 0 + d = d, \\ & b = u_p(l) = cl + d = cl + a. \\ \Longrightarrow \quad & d = a \quad \text{and} \quad c = (b-a)/l. \end{aligned}$$

Thus
$$u_p(x) = (b-a)x/l + a$$

solves the PDE with the BC satisfied. Consider the related homogeneous problem (i.e., with homogeneous PDE and BC)

$$\begin{aligned} \text{PDE:} \quad & v_t = \alpha v_{xx} \quad 0 \le x \le l, \ t > 0, \\ \text{BC:} \quad & v(0,t) = 0, \ \ v(l,t) = 0, \quad t > 0, \\ \text{IC:} \quad & v(x,0) = f(x) - u_p(x), \quad 0 \le x \le l. \end{aligned} \tag{31}$$

If $f(x) - u_p(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/l)$, then its solution is given by

$$v(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n\pi/l)^2 \alpha t} \sin(n\pi x/l).$$

Now, set $u(x,t) = v(x,t) + u_p(x)$. Then it is easy to verify that $u(x,t)$ solves (28). Indeed, $u(x,t)$ solves (28) by the superposition principle. Further, we have

$$\begin{aligned}
\text{BC:} \quad & u(0,t) = v(0,t) + u_p(0) = 0 + a = a \\
& u(l,t) = v(l,t) + u_p(l) = 0 + b = b \\
\text{IC:} \quad & u(x,0) = v(x,0) + u_p(x) = f(x) - u_p(x) + u_p(x) = f(x).
\end{aligned}$$

Example PDE: $$u_t = 2u_{xx} \quad 0 \le x \le 1,\ t > 0, \tag{32}$$

BC: $$u_x(0,t) = 1 \ \ u(1,t) = -1, \quad t > 0, \tag{33}$$

IC: $$u(x,0) = x + \cos^2(3\pi x/4) - 5/2. \tag{34}$$

Solution. Take $u_p(x) = cx + d$. The first BC $u_x(0,t) = 1$ yields $c = 1$, while $u_p(1) = 1 + d$ yields $d = -2$ by the second BC. Thus, $u_p(x) = x - 2$. The related homogeneous problem is

$$v_t = 2v_{xx} \quad 0 \le x \le 1,\ t > 0,$$

$$v_x(0,t) = 0 \ \ v(1,t) = 0, \quad t > 0$$

$$\begin{aligned} v(x,0) &= [x + \cos^2(3\pi x/4) - 5/2] - (x - 2) \\ &= \frac{1}{2} + \frac{1}{2}\cos(3\pi x/2) - 5/2 + 2 = \frac{1}{2}\cos(3\pi x/2). \end{aligned}$$

It is easy to obtain the solution of the related homogeneous problem as

$$v(x,t) = e^{-9\pi^2 t/2}[\frac{1}{2}\cos(3\pi x/2)].$$

From the above examples, we notice that the particular solution is time independent, or in steady-state.

Note: Any steady-state solution of the heat equation $u_t = \alpha u_{xx}$ is of the form $cx + d$. The solutions $u(x,t)$ are sums of a steady-state particular solution of the PDE and BC and the solution $v(x,t)$ of the related homogeneous problem which is transient in the sense that $v(x,t) \to 0$ as $t \to \infty$. Thus

$$\begin{array}{ccccccc} u(x,t) & = & u_p(x) & + & v(x,t) & \to & u_p(x), \text{ as } t \to \infty. \\ & & \text{(steady-state solution)} & & \text{(transient solution)} & & \end{array}$$

That is, the solution u approaches the steady-state solution as $t \to \infty$.

However, for some types of BC there are no steady-state particular solutions.

Example

$$\text{PDE:} \quad u_t = \alpha u_{xx}, \quad 0 \le x \le l, \ t > 0, \tag{35}$$

$$\text{BC:} \quad u_x(0,t) = a, \ \ u_x(l,t) = b, \tag{36}$$

$$\text{IC:} \quad u(x,0) = f(x), \tag{37}$$

where a and b are constants, and $f(x)$ is a given function.

Solution. Let $u_p(x) = cx + d$. Then, using BC, we obtain $c = a$ and $c = b$, which is impossible unless $a = b$.

NOTE: The boundary conditions state that heat is being drained out of the end $x = 0$ at a rate $u_x(0,t) = a$ and heat is flowing into the end $x = l$ at a rate $u_x(l,t) = b$. If $b > a$, then the heat energy is being added to the rod at a constant rate. If $b < a$, the rod loses heat at a constant rate. Thus, we cannot expect a steady-state solution of the PDE and BC, unless $a = b$.