DEPARTMENT OF MATHEMATICS

MA201 Mathematics III

Academic Year 2015 - 2016

Tutorial & Additional Problem Set - 3

Date of Tutorial Class: August 18, 2015

Elementary Analytic Functions and their Properties
SECTION - A (for Tutorial – 3)

1. Prove that $\sinh(\Im(z)) \leq |\sin(z)| \leq \cosh(\Im(z))$ where $\Im(z)$ denotes the imaginary part of z. Deduce that $|\sin(z)|$ tends to ∞ as $|\Im(z)| \to \infty$.
   (Thus, $\sin(z)$ is an unbounded complex-valued function of a complex variable, whereas $\sin(x)$ for $x \in \mathbb{R}$ is a bounded real-valued function of a real variable.).

2. Find all solutions of
   (i) $\exp(z) = 1$ (ii) $\exp(z) = i$ (iii) $\exp(z - 1) = 1$.

3. Describe the image of the following sets in the z-plane under the mapping $w = \sin z$.
   (a) $\{z = x + iy \in \mathbb{C} \;:\; x = \pi/2,\; -\infty < y < \infty\}$
   (b) $\{z = x + iy \in \mathbb{C} \;:\; x = -\pi/2,\; -\infty < y < \infty\}$
   (c) $\{z = x + iy \in \mathbb{C} \;:\; |x| \leq \pi/2,\; y = 0\}$
   (d) $\{z = x + iy \in \mathbb{C} \;:\; x = 0,\; -\infty < y < \infty\}$
   (e) $\{z = x + iy \in \mathbb{C} \;:\; x = a \text{ with } |a| < \pi/2,\; -\infty < y < \infty\}$
   (f) $\{z = x + iy \in \mathbb{C} \;:\; |x| < \pi/2,\; y > 0\}$
   (g) $\{z = x + iy \in \mathbb{C} \;:\; |x| < \pi/2,\; y = b,\; b \neq 0\}$
   (Hint. $\cos(z) = \sin(z + \frac{\pi}{2})$, $\sinh(z) = -i\sin(iz)$ and $\cosh(z) = \cos(iz)$)

4. Evaluate the following:
   (i) $\log(3 - 2i)$ (ii) $\text{Log}\, i$ (iii) $(i)^{(-i)}$

5. Find the principal branch of the multiple valued function $\log(2z - 1)$.

6. If $z \neq 0$, c is a complex number and n is an integer then we know that the identity $(z^c)^n = (z)^{cn}$ holds. Give an example to show that this identity does not hold if n is replaced with arbitrary complex number.

SECTION - B (Supplementary Problems)

(These are for your own practice. They will not be discussed in Tutorials.)

7. Find the values of z for which $\exp(z)$ is (a) real and (b) purely imaginary.

8. Find the image of the vertical line $x = c_1$ and the horizontal line $y = c_2$ where c_1 and c_2 are real constants under the following mappings:
   (i) $w = \exp(z)$ (ii) $w = (1/z)$ (Assume that c_1 and c_2 are non-zero.)

9. Show that the identity $\text{Log}\,(z_1 z_2) = \text{Log}\,(z_1) + \text{Log}\,(z_2)$ is not always valid. (Note that this identity holds true iff $-\pi < \text{Arg}\,(z_1) + \text{Arg}\,(z_2) \leq \pi$.)

10. Determine the domain of analyticity for the function $f(z) = \text{Log}\,(3z - i)$ and compute $f'(z)$.
