MA 201: MATHEMATICS III

End Semester Examination (July - November, 2014)

Duration: 3 hrs Date: 25/11/2014 **Total Marks: 50**

Important Instructions: This paper contains **SIX** questions. **Answer to each question should appear together. Please mention the answer page number for each question on the front page of the answer booklet.** Give appropriate reasons wherever necessary.

Wishing you all the best

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1. (a) By using residue theorem, evaluate the integral $\displaystyle\int_0^{2\pi} \frac{1}{1+3\cos^2 t}\,dt.$ [4]

   (b) Find a Möbius transformation that maps $z_1 = 1, z_2 = 0$ and $z_3 = -1$ onto the points $w_1 = i, w_2 = \infty$ and $w_3 = 1$. [3]

   (c) Find the image of $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ under the Möbius transform $S(z) = \dfrac{i-z}{i+z}$. [3]

$z = e^{it}$

$dz = it$

2. (a) Use the method of characteristics to find the solution of the following initial value problem (IVP): [3]

$$\frac{\partial u}{\partial y} + (1-2u)\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad y > 0,$$

$$u(x,0) = f(x) = \begin{cases} 0, & x \le 0, \\ x, & 0 < x \le 1, \\ 1, & 1 < x. \end{cases}$$

$q + p - 2up$

~~$1-2u$~~

   (b) Determine the form of the functions $f(x)$ for which the first order partial differential equation $\dfrac{\partial u}{\partial x}(x,y) - u(x,y) = 0$, with side condition $u(x,1) = f(x)$, has a solution.

$u_x - u = 0$

[2]

3. (a) Use d'Alembert's formula to find the solution to the following initial boundary value problem (IBVP): [3]

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0,$$
$$u(0,t) = 2t, \quad t > 0,$$
$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \le x < \infty,$$

   where $f(x)$ and $g(x)$ are given functions.

   (b) Show that the solution $u(x,t)$ of the following IBVP [3]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0,$$
$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0,$$
$$u(x,0) = \sin x + \frac{1}{2}\sin(2x), \quad 0 \le x \le \pi$$

   satisfies $0 \le u(x,t) \le \frac{3\sqrt{3}}{4}$ for all $0 \le x \le \pi, \ t \ge 0$.

$dz = ie^{i\theta} d\theta = iz\,d\theta$

$\cos t + i\sin\theta$

$dz = d\theta\, e^{i\theta}$

$\cos\theta + i\sin\theta + 1\ \cos\theta - i\sin\theta$

$z = e^{i\theta}$

$a\cos\theta$

4. (a) Given the Fourier series

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx), \quad -\pi < x < \pi,$$

$\frac{5\pi^4}{144}$

$\int x^2 dx$ $\frac{2\pi^3}{3}$

find the value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$. [3]

(b) Find the Fourier transform of [3]

$$f(x) = \begin{cases} x^2 e^{-x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

(c) Find a function $f(x)$ which solves the integral equation [3]

$$f(x) + \int_0^{\infty} f(x-t)e^{-t}dt = \frac{1}{1+x^2}.$$

5. The following three problems are to be solved with the help of separation of variables method.

(a) Solve the following nonhomogeneous wave equation [4]

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + Kx, \quad 0 < x < 1, \ t > 0, \quad \text{where } K \text{ is a constant,}$$

subject to $u(0,t) = 0 = u(1,t)$, $t > 0$; $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = 0$, $0 \leq x \leq 1$.

(b) Consider the heat conduction problem in a thin metal bar of length L and thermal diffusivity α with an initial temperature distribution $f(x)$ and no heat loss at both ends. Formulate and solve the IBVP for all x and $t > 0$. [4]

(c) Find the steady-state temperature $u(r,\theta)$ in a disk of radius 1 if the upper half of the circumference is kept at 100^0 and the lower half is kept at 0^0. What can you conclude about the temperature at the centre of the disk with respect to the temperature on the circumference? [4]

6. (a) Find the Laplace transform of the unit step function $H(\sin \pi t)$. [2]

(b) By using Laplace transform, find the solution of the following IVP:

$$\frac{d^2y}{dt^2} - y = t * \cos t, \quad y(0) = -1, \quad \frac{dy}{dt}(0) = 0,$$

where $*$ denotes convolution. [3]

(c) Formulate and solve the following IBVP by using Laplace transform method:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \sin \pi x, \quad 0 < x < 1, \ t > 0,$$

with all zero conditions, where the boundary conditions are of Dirichlet type. [3]

***** Paper Ends *****

$\frac{e^{-it}}{\alpha}$