

## FIT3139: Assignment 2

(Due by 11:59pm, Friday, 28 May 2021)

[Weight: 15 = 7 + 8 marks.]

This assignment has two parts. The objective of the first part is to investigate the dynamics of a deterministic and a stochastic Lotka-Volterra model; this part is worth 7 of 15 marks. The second part of the assignment is about using Markov Chain Theory to formulate and analyse a simple model of social segregation; this part is worth 8 of 15 marks.

### Follow these procedures to submit this assignment

The assignment must be submitted *online* via Moodle, and should follow the following procedure:

- Accept the Electronic Plagiarism Statement for this Assignment. All your scripts/program will be scanned using MOSS (a **plagiarism detection software**). Read Monash Student Academic Integrity policy for consequences of plagiarism.
- All your scripts and reports **MUST** contain your name and student ID.
  - **You are free to program the assignment in either MATLAB or Python.**
  - Your submitted archive must extract to a directory named as your student ID.
  - This directory should contain a subdirectory for each of the two questions in the assignment, named as q1/, and q2/.
  - Within each of those subdirectories the contents include MATLAB/Python A PDF report with references to the scripts you used (in Python or MATLAB). You should include the scripts as well.
  - When submitting scripts and reports, choose file names that identify the subquestion. (Eg. q1a\_script.py, or q1b\_report.pdf, or q2\_script\_driver.m, or q2\_output.txt)
- Submit your zipped file electronically via Moodle.

# Part 1

## Background

This exercise uses the Lotka-Volterra model to explore the sustainability of an ecosystem with rabbits, foxes and hunting. We start by exploring the effect of hunting rates rates in a deterministic model.

A stochastic extension further allow us to explore the sustainability of the ecosystem using a combination of Gillespie and Montecarlo simulations.

## Questions Part 1A [2 marks]

The system tracks the number of rabbits  $x$ , and foxes  $y$ , and the dynamics are given by:

$$\dot{x} = g(x, y)$$

$$\dot{y} = h(x, y)$$

In the absence of foxes and hunting, rabbits grow exponentially with a growth rate  $b$  and a death rate  $d$  ( $b > d$ ); the number of rabbits eaten by foxes is proportional to the product of the number of rabbits present and the number of foxes present by a factor  $\alpha$ . Foxes decline exponentially in isolation, with a birth and death rate  $b'$  and  $d'$ , respectively ( $d' > b'$ ). A parameter  $\beta$  is the proportional constant of foxes surviving by eating rabbits – thus foxes in the presence of rabbits increase at a rate  $b' + \beta x$  **proportional to the number of foxes**. Rabbits are hunted at a rate  $f$ , where as foxes are twice as hard to hunt.

- (a) Formulate  $g(x, y)$  and  $h(x, y)$  reflecting the assumptions above\*.
- (b) Using *your own* RK2 numerical integrator, inspect the dynamics of this system for different hunting rates  $f$ . Assume  $b = 0.8$ ,  $d = 0.1$ ,  $b' = 0.1$ ,  $d' = 0.6$ ,  $\alpha = 0.04$ , and  $\beta = 0.01$ . The initial population consist of 25 rabbits, and 15 foxes. You should study at least 3 different hunting rates, and present the time-evolution of the system as well as a phase plot of the dynamics for each parameter set.

## What should you submit for this question?

You will have to submit

- Your own implementation of RK2.
- For each of your chosen  $f$  rates, a time-evolution plot, a phase plot and an interpretation of the dynamics.

## Questions Part 1B [5 marks]

When populations are finite and small, stochastic effects can be significant. In the case of the population model above, a stochastic model can account for the extinction of very small populations.

To study these extinction events,

- (a) Formulate and implement a Gillespie algorithm of the model above, and do a similar exploration of the same parameters, now using the stochastic model. What features are now salient due to stochasticity?
- (b) We define the system to be sustainable if the probability of extinctions in a time horizon of 10 years is less than 0.2. Using Montecarlo as well as your Gillespie implementation, determine what is the largest hunting rate that guarantees a sustainable ecosystem. Explain how you set up your computational experiments and the reasoning behind your answer.

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\*Hint: This should be very similar to the Lotka-Volterra model studied in the lectures

## What should you submit for this question?

You will have to submit

- Description and implementation of the Gillespie algorithm.
- Discussion of the effects of stochasticity, comparing the deterministic and stochastic models.
- Explanation of how to estimate, using Montecarlo, the probability of extinctions happening in a time horizon up to  $t = 10$ . Reasoning and evidence of your chosen hunting rate to guarantee a sustainable system as defined.

## Part 2: A simplified Schelling model

### Background

The Schelling model was proposed by Thomas Schelling to explain segregation. We discussed the basic elements of the model in the first lecture of the unit. Because the model has a very large number of possible states it is hard to compute the quantities of interest exactly, mostly having to rely on Montecarlo simulations. A simple implementation of this simulation model can be found here: <http://www.natureincode.com/code/various/schelling.html>.

To make this model tractable we propose a simplified version of the model, using Markov chains. The model is described below <sup>†</sup>.

In the simplified Schelling model agents live on a cycle of finite size  $n$ . Agents can be of two types, say 0 and 1. There are no empty positions, thus, a cycle of size  $n$  also implies  $n$  agents.<sup>‡</sup> In this simplified version there are no thresholds. Instead, an agent is “happy” if at least one of her neighbours is of the same type.

Time is discrete, so  $t = 1, 2, 3, \dots$ . The dynamics go as follows: At each time-step, two individuals (residing in different slots, not necessarily adjacent) are matched to *potentially* trade places. Each encounter may result in the agents trading places or retaining their position. Agents will agree to trade places if and only if at least one of the two agents benefits<sup>§</sup>, *and* none of the two is worse off after the swap. The *matching* procedure is randomly uniform. It is easy to see that this process gives rise to an absorbing Markov Chain.

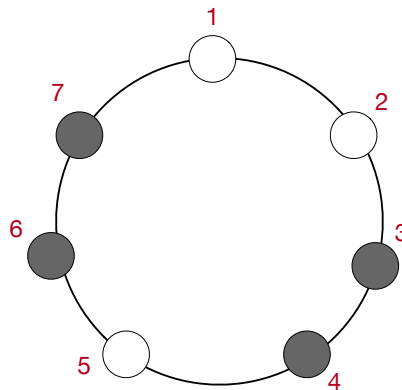


Figure 1: Example: A cycle of size 7. According to the rules above, all agents are “happy” except agent 5

<sup>†</sup>Please read carefully, perhaps aloud, and several times, being aware of the punctuation marks

<sup>‡</sup>This is a big difference with the standard Schelling model in which vacant spots are a fundamental feature.

<sup>§</sup>i.e., goes from unhappy to happy

## Questions Part 2 [8 marks]

(a) An absorbing Markov Chain:

- Specify explicitly the transition Matrix of the MC for  $n = 4$ . Explain how the transition probabilities are computed and how the states are labeled.
- Show the canonical form of the Markov chain for  $n = 4$ . Make sure to specify clearly how states are re-labeled or re-ordered if necessary.
- Using Montecarlo simulations show how the absorption time varies with  $n = 4, 5, \dots 10$ . ¶
- Numerically approximate the absorption times for  $n = 4$  and  $n = 5$  and show that they agree with the Montecarlo simulations. ¶

### What should you submit for this question?

Your submission must include a report for part 2A, answering the questions and referencing any scripts used to perform the calculations/simulations. The scripts should also be included as part of your submission. Alternatively, if using Python, you can submit a Jupyter notebook containing descriptive text and code.

- (b) We now turn to a model in which agent may swap places “by mistake”. This means with a probability  $\epsilon$  they will fail to swap places when they intend to, or will fail to stay put when they should. This small change results in a new chain that is ergodic.\*\*
- Specify the full transition matrix for  $n = 4$ , compute the stationary distribution numerically and show that it is in agreement with Montecarlo simulations. What can you conclude from this model?

### What should you submit for this question?

Your submission must include a report for part 2B, answering the questions and referencing any scripts used to perform the calculations/simulations. The scripts should also be included as part of your submission. Alternatively, if using Python, you can submit a Jupyter notebook containing descriptive text and code.

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¶ Assume you have  $n/2$  type 0's and  $n/2$  type 1's for even  $n$ ; or  $(\lfloor n/2 \rfloor + 1)$  type 1's for odd  $n$ . Note that for this larger  $n$  you do not necessarily need to formulate the whole transition matrix of the Markov chain. You can simply simulate the events that transform one state into another.

¶ For the numerical calculations in the case of  $n=5$ , it may not be necessary to specify a full transition matrix.

\*\*For numerical and simulation results assume a small  $\epsilon$