COXETER GRAPHS

We start with the notion of level for a Coxeter graph.

Definition 1 (Level)**.** *Let Γ be the Coxeter graph of an (irreducible) Coxeter system. Its level is the smallest non-negative integer r such that every choice of r vertices in Γ, when removed, leaves a (possibly disconnected) Coxeter graph of level 0 (finite or affine type).*

We denote level of a graph Γ by L(Γ).

It'll be useful to define a set of all possible values of level of a Coxeter graph with n nodes.

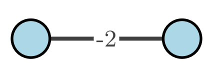
$$L_n \;=\; \left\{\, L(\Gamma) \mid \Gamma \text{ is a Coxeter graph with } n \text{ nodes} \right\}.$$

Corollary 1. *$n-1$ is an upper bound for L_n*

Lemma 1. *$n-1$ is in fact a maximum for L_n.*

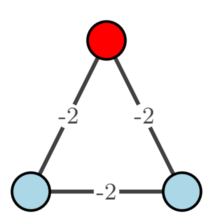
Proof. From the corollary above we know that n-1 is an upper bound for L_n. Hence, it suffices to show that $\exists$ a Coxeter graph with n nodes, say H, such that $L(H) = n-1$.

Consider the following graph H^2:

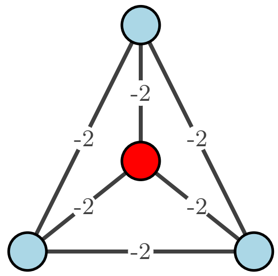


Notice that $L(H^2) = 1$ since $L(H^2) > 0$ and $L(H^2) \leq 1$.

Finally consider the following construction with H^2: After adding a node we get H^3, where $L(H^3) = 2$:



H^4, where $L(H^4) = 3$



and so on....

We claim that $L(H^n) = n - 1$.

Note that by symmetry on removing any node of H^n we get the graph H^{n-1}. In fact, on removing any choice of $n-2$ nodes we get H^2. Hence, $L(H^n) > n-2$ and we also have $L(H^n) \leq n-1$. And so, $L(H^n) = n-1$ $\square$

Therefore, a complete graph of n nodes with -2 on all edges gives us what we need.