Reflection & Coxeter Groups

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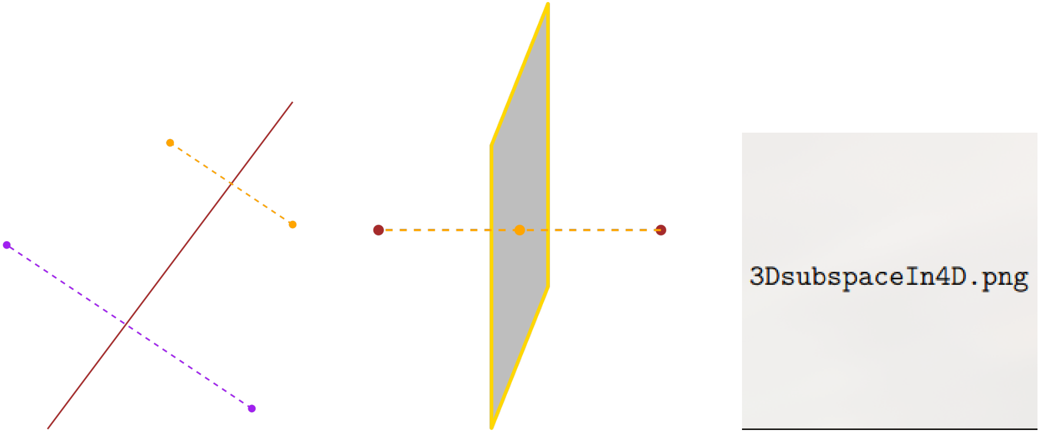
Overview

1. Mirrors and Reflections
2. Coxeter Groups

Hyperplanes and Mirrors

Hyperplane

In an n-dimensional vector space, an $n-1$ dimensional subspace is called a hyperplane.



(a) Mirror in $\mathbb{R}^2$ (b) Mirror in $\mathbb{R}^3$ (c) Mirror in $\mathbb{R}^4$

A precise definition

Reflections in a Vector space

Reflections are orthogonal transformations. Hence they preserve distance and angles. So if s is a reflection and $\alpha, \beta \in \mathbb{R}^n$ then, $||s(\alpha)|| = ||\alpha||$ and $s(\alpha).s(\beta) = \alpha.\beta$

More Precisely

Reflection in a Real Euclidean Vector space endowed with a positive definite symmetric bilinear form (λ, μ) is a linear operator that sends some non-zero vector α (normal to the hyperplane H_α) to its negative while fixing everything in the hyperplane H_α.

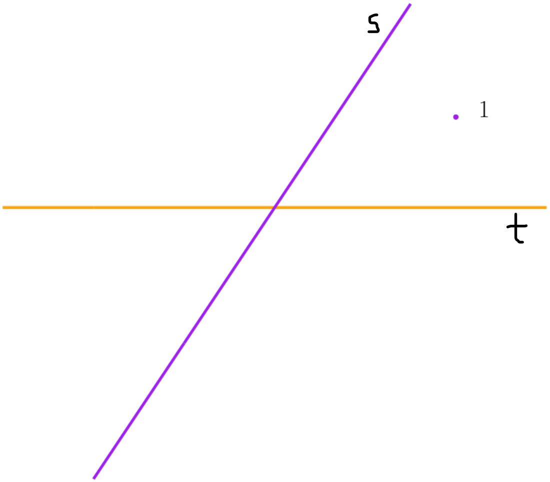
Reflection as an orthogonal transformation

Fact

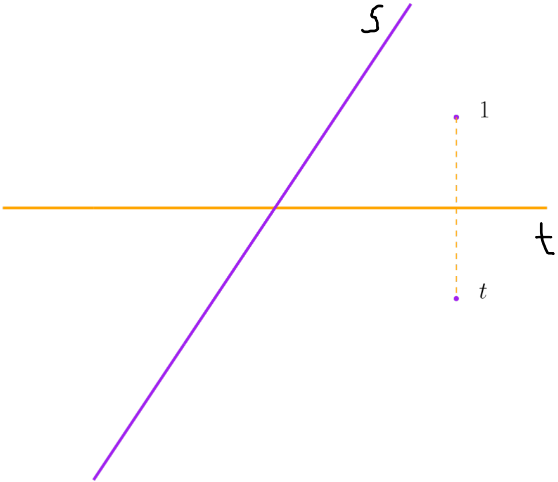
The set of all orthogonal transformations of a vector space V forms a group, $O(V)$.

And so, a finite group generated by a system of mirrors (multiple reflections) is just a subgroup of O(V).

Some Examples



Some Examples



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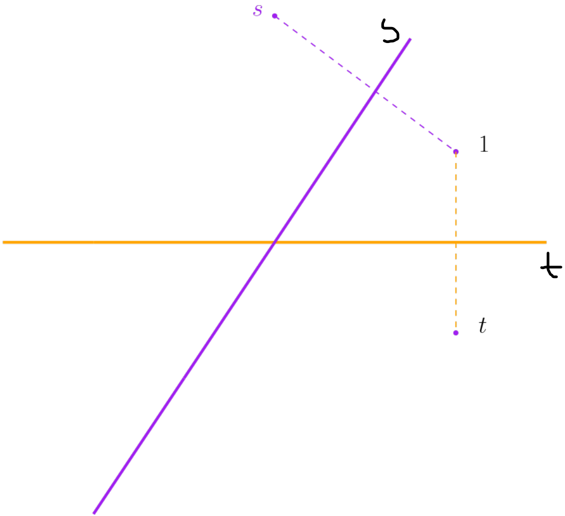

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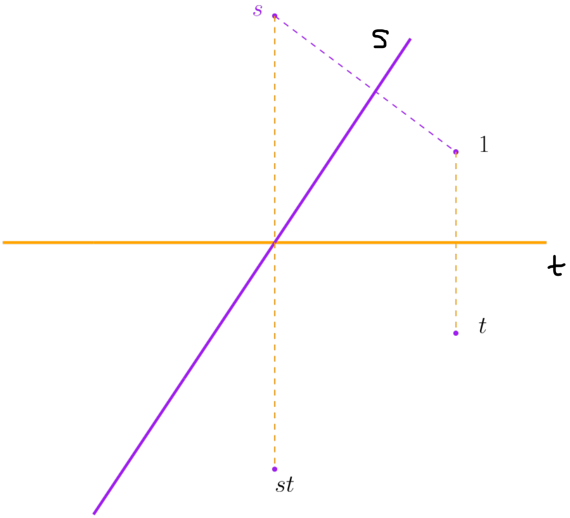

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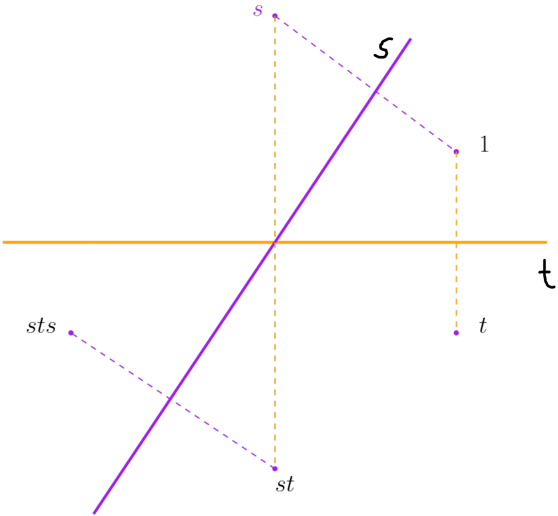

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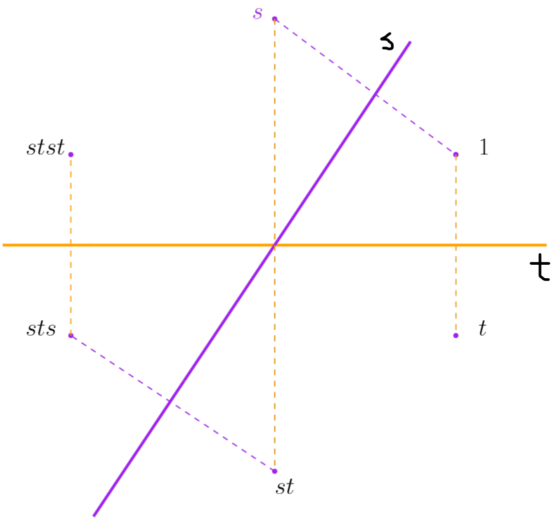

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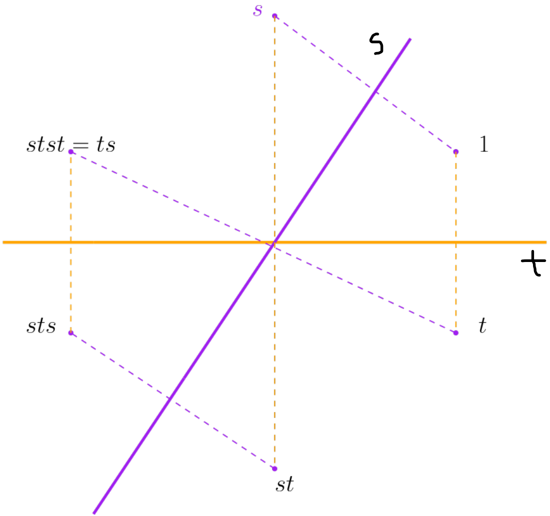

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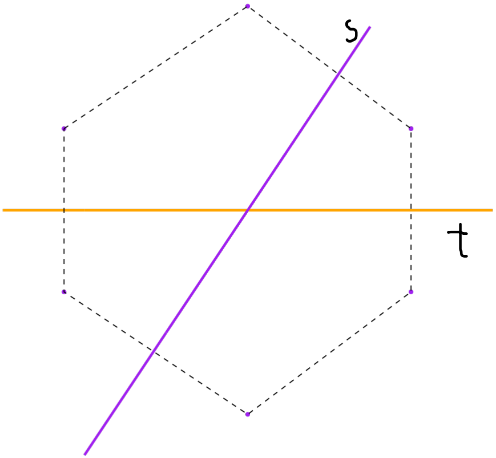


Conclusion

- $s^2 = t^2 = 1$ for any reflection s, t
- We get only 6 (finite) number of "images" or elements of the group.
- We get this mysterious relation: $sts = tst$
- We get the group: $\langle s, t \mid s^2 = t^2 = 1,\ sts = tst \rangle$

Another Example

Figure: $\tilde{A}_2$ in Inception

Another Example

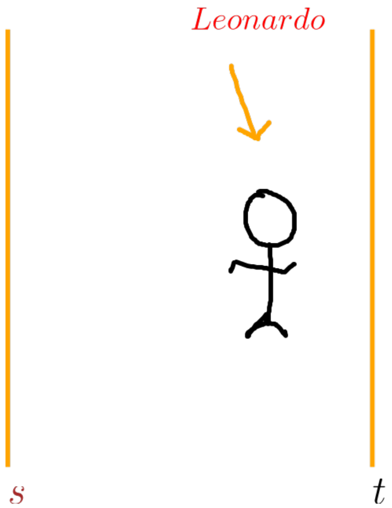


Figure: Inception Sketch

How to find all the Finite Configurations?

Theorem

The group of all isometries of $\mathbb{AR}^n$ which fix a point o coincides with the orthogonal group $\mathbb{O}_n$.

Consequence

The finite groups generated by reflections, i.e, Finite Reflection Groups, fix a point.

And so, all the hyperplanes in a system of mirrors must have a point in common in order to generate finite images.

Enter: Coxeter Systems

Coxeter system

A Coxeter system is a group generated by a finite set of generators

$$S = \{s_1, s_2, ..., s_n\}$$

under the relations:

$$s_i^2 = 1 \text{ for all } 1 \leq i \leq n$$

$$\underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ terms}} = \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ terms}} \text{ for all } 1 \leq i, j \leq n$$

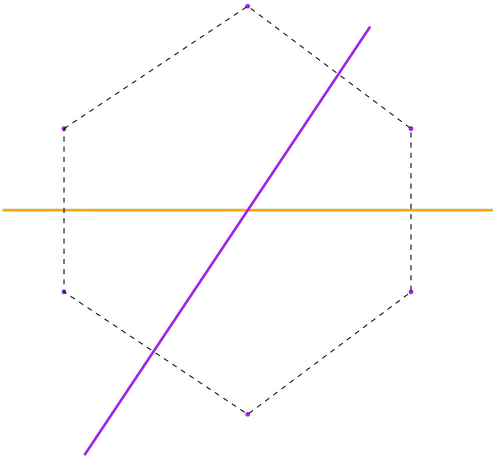
An Example

$$S = \{s, t\} \text{ with } m_{st} = 3$$

$$\implies W = \langle s, t | s^2 = t^2 = 1, sts = tst \rangle$$

An Example

And we know $W =$



Equivalence

The Connection

Each finite reflection group can be represented as a coxeter system.

And so we have turned a geometrical problem to a combinatorial one!

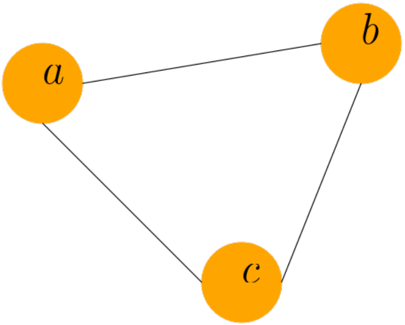
Coxeter Graphs

Representation of Coxeter systems

Each Coxeter system can be represented as a graph where each vertex is a generator of the corresponding Coxeter group, and there's an edge between vertices if $m_{ij} \geq 3$ with edge label$= m_{ij}$.

A combinatorial approach

We saw before how $\langle s, t \rangle$ with $m_{st} = 3$ was the finite group D_6
What about?



$$a^2 = 1$$
$$b^2 = 1$$
$$c^2 = 1$$
$$aba = bab$$
$$bcb = cbc$$
$$aca = cac$$

Figure: Enter Caption

Frame Title

Notice, $(abc)^n \in W \ \forall n \in \mathbb{N}$ The group generated is infinite!

A more elegant way

Let $S = \{s_1, s_2, ..., s_n\}$ be the generator set of our Coxeter system. Encode the Coxeter graph into a Coxeter matrix, say A, known as the Cartan Matrix such that

$$(a_{ij}) = (a_{ji}) = -cos(\pi/m_{ij}) \text{ for } s_i, s_j \in S$$

Note

A is a symmetric matrix so all of its eigenvalues $\in \mathbb{R}$

Theorem

If A is positive definite, i.e all eigenvalues of A are strictly positive then the Coxeter system associated to A is finite.
If A is positive semi-definite, i.e all eigenvalues of A are non-negative (zeroes are allowed) then the associated Coxeter System is affine.

All that remains is to see all possible Coxeter systems that are possible and compute its eigenvalues...

2.7 Classification of graphs of positive type

Theorem *The graphs in Figure 1 of 2.4 and Figure 2 of 2.5 are the only connected Coxeter graphs of positive type.*

Proof. Suppose there were a connected Coxeter graph Γ of positive type not pictured in either Figure 1 or Figure 2. We proceed in 20 easy steps

Figure: Humphreys: pg 36

20 Easy Steps

(1) All Coxeter graphs of rank 1 or 2 are clearly of positive type (A_1, $I_2(m)$, $\widetilde{A}_1$), so we must have $n \geq 3$.

(2) Since $\widetilde{A}_1$ cannot be a subgraph of Γ, we must have $m < \infty$.

(3) Since $\widetilde{A}_n (n \geq 2)$ cannot be a subgraph of Γ, Γ contains no circuits.

Suppose for the moment that $m = 3$.

(4) Γ must have a branch point, since $\Gamma \neq A_n$.

(5) Γ contains no $\widetilde{D}_n, n > 4$, so it has a unique branch point.

(6) Γ does not contain $\widetilde{D}_4$, so exactly three edges meet at the branch point (with $a \leq b \leq c$ further vertices lying in these three directions).

(7) Since $\widetilde{E}_6$ is not a subgraph of Γ, $a = 1$.

(8) Since $\widetilde{E}_7$ is not a subgraph of Γ, $b \leq 2$.

(9) Since $\Gamma \neq D_n$, b cannot be 1, so $b = 2$.

(10) Since $\widetilde{E}_8$ is not a subgraph of Γ, $c \leq 4$.

(11) Since $\Gamma \neq E_6, E_7, E_8$, the case $m = 3$ is impossible. Thus $m \geq 4$.

(12) Γ does not contain $\widetilde{C}_n$, so only one edge has a label > 3.

(13) Γ does not contain $\widetilde{B}_n$, so Γ has no branch point.

Now consider what happens if $m = 4$.

(14) Since $\Gamma \neq B_n$, the two extreme edges of Γ are labelled 3.

(15) Since Γ does not contain $\widetilde{F}_4$, n must be 4.

(16) But $\Gamma \neq F_4$, so the case $m = 4$ is impossible. Thus $m \geq 5$.

(17) Since Γ does not contain $\widetilde{G}_2$, we must have $m = 5$.

(18) Γ does not contain the nonpositive graph Z_4 in 2.5, so the edge labelled 5 must be an extreme edge.

(19) Γ does not contain the nonpositive graph Z_5, so $n \leq 4$.

(20) Now Γ must be either H_3 or H_4, which is absurd. So we have eliminated all possibilities. $\square$

Figure: Humphreys: 20 Easy steps

Finite Classification

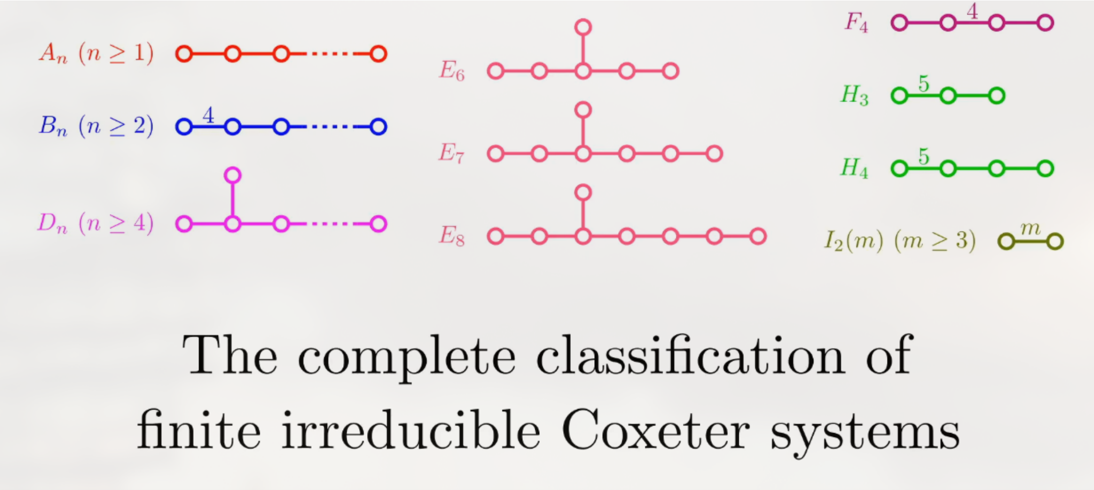


Figure: All possible ways you can orient mirrors in n-dimensions so that you get finite images.
Credits for the Image: Joseph Newton

References

- Alexandre V. Borovik, & Anna Borovik. (2010). *Mirrors and Reflections: The Geometry of Finite Reflection Groups*. Springer.
- James E. Humphreys. (1990). *Reflection Groups and Coxeter Groups*. Cambridge University Press.
- Newton, J. [Joseph Newton]. (n.d.). *The Coxeter Classification* [Video]. YouTube. https://www.youtube.com/watch?v=BV5mYjh8m4E&t=402s

Thank you!